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Extinction of a rumor model based on discrete-time control

Zuyu Zhao*, **Huikun Hao, An Yan and Zonghe Guo**

School of Engineering Science, Shandong Xiehe University, Jinan 250107, China

* **Correspondence:** Email: zhaozuyu2026@163.com.

Abstract: With the emergence of new media, the spread of rumors has become increasingly easy. Quickly controlling the spread of rumors has become an urgent problem to be solved. In this paper, we consider a rumor model and establish sufficient conditions for the extinction of the rumor based on discrete-time observation. By constructing an auxiliary system and estimating the error, we obtain the upper bound of the time delay. Then, the extinction of the rumor is established by using the perturbation method. Both deterministic and stochastic cases are considered. Finally, numerical examples verify our theoretic results.

Keywords: rumor model; discrete-time observation; extinction; feedback control
Mathematics Subject Classification: 93D50, 93D15, 34H15, 60H10

1. Introduction

Rumors are very annoying, and each of us wants them to disappear as soon as possible. However, in the real world, without extra control, it is difficult to stop them. Thus, we should add suitable control. In this paper, we first assume that a rumor is popular, and then we set a controller based on discrete-time observation to stop the spread of said rumor.

We first recall some known results about rumor models. The earliest research on rumor propagation models can be traced back to the 1960s. Daley and Kendall [5] established a stochastic rumor propagation model based on the infectious disease research method, dividing the model's population into three classes: Ignorant, spreaders, and stiflers. Maki and Thompson [20] studied rumor spread based on a Markov chain. Zanette [32] introduced a rumor propagation model in small world networks. Moreover, rumor models have been considered by using social networks [16, 38]. Zhao et al. [35] developed a new susceptible-infected-hibernator-removed model by adding a direct link from ignorant to stiflers and a new group, hibernator. The effect of media has also been added in rumor model development [9, 27]. Fractional rumor spreading dynamical models were studied in [24, 26]. Recently, the optimal control of a rumor model has been studied widely by many authors. Optimal control for a susceptible-infectious-recovered (SIR) model with limited hopitalized patients was studied in [1]. A rumor model with an Ornstein–Uhlenbeck process, a delayed rumor propagation model, and a fractional-order 2I2SR rumor spreading model were considered in [8, 28, 31], respectively. A reaction-diffusion rumor propagation model and a susceptible-infectious (SI) reaction–diffusion propagation model were studied in [29, 37], respectively. With the development of multimedia information dissemination, the spread of rumors is subject to various disturbances. To accurately describe the spread of rumors, we introduce random disturbances. On the other hand, in the rumor model, β denotes the transmission rate from spreading individuals to susceptible individuals; see Table 1. Thus, it is suitable to make the following assertion:

$$\beta dt \longrightarrow \beta dt + \sigma dB(t).$$

In this paper, we will consider the extinction of rumor by feedback control. In recent years, there are many researchers studying rumor propagation by using epidemic models, including Cheng et al. [2], Choi [3, 4], Jia et al. [10–12], and Zhu et al. [36]. Shi and Zhu [23] introduced a more complex rumor model. Generally speaking, the existence, uniqueness, and stationary distribution are the main points of rumor propagation models. Following the idea of Jia et al. [10–12] and Jiang et al. [13], in this paper, we consider the following stochastic rumor model:

$$\begin{cases} dS(t) = [A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t)]dt - \sigma S(t)I(t)dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu + \eta)I(t) - \alpha I^2(t)]dt + \sigma S(t)I(t)dB(t), \end{cases} \tag{1.1}$$

with initial data

$$\begin{cases} S(0) = s_0, \\ I(0) = i_0. \end{cases}$$

The meanings of the parameters in the model are shown in Table 1.

Table 1. Parameters and their meanings in model (1.1).

Parameters	Meaning
$S(t)$	The density of susceptible individuals at time t
$I(t)$	The density of infectious individuals at time t
A	The entering number of susceptible individuals per unit of time
μ	The emigration rate of individuals per unit of time
β	The transmission rate from spreading individuals to susceptible individuals
η	The recovery rate of susceptible individuals
α	The rate at which a rumor-infected user may transform into a rumor-susceptible user
σ	The tensity of noise

In this paper, our aim is to study the extinction of rumor based on discrete-time observation with or without noise. In other words, for the case $\sigma_i = 0$ $(i = 1, 2)$, assume that

$$R_0 = \frac{\beta A}{\mu(\mu + \eta)} > 1,$$

that is, the rumor is popular (that is to say, the number of people spread by a person who firmly believes in rumors is larger than one). We then consider the feedback control problem

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\eta)I(t) - \alpha I^2(t) - kI(\delta(t)), \end{cases} \tag{1.2}$$

where $\delta(t) = [\frac{t}{\tau}]\tau$, τ is the observation time interval, and $[a]$ denotes the integer part of a. The meaning of R_0 is similar to epidemic models [10]. The method used here is similar to [7, 14, 22], where the authors stabilized an unstable (stochastic) ordinary differential equation by using discrete-time observation. The method also can be used to consider data assimilation [18,19]. There are a lot of different methods to study nonlinear systems; see [6,30]. We remark that they have added the feedback control to the entire system, whereas we only add the feedback control to some parts of the system, which is a big difference from [7,14,22,33]. Generally speaking, in order to make rumors disappear, we would exert control over the system, that is, exert control over both susceptible individuals and infected individuals. However, we have found that exerting control only on infected individuals can achieve the desired effect. Hence, this strategy will enhance computational efficiency in the real world. Moreover, our calculations are easier than the full-state continuous control. In fact, the result is obviously right. The reason is that if we consider

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\eta)I(t) - \alpha I^2(t) - kI(t), \end{cases} \tag{1.3}$$

then we know that the basic reproduction number is expressed as

$$\hat{R}_0 = \frac{\beta A}{\mu(\mu+\eta+k)}.$$

Therefore, with $k > 0$ large enough, we have $\hat{R}_0 < 1$, so the rumor will disappear. By using perturbation theory, we know that if the rumor in system (1.3) disappears, then for small enough τ, the rumor in system (1.2) will also disappear. In this paper, we will give the upper bounded of τ. In other words, we will provide a strategy to make a rumor disappear. Using a similar idea, we can deal with the stochastic case (1.1). However, there is a big difference from the deterministic case. We can not get the uniform bounded in the stochastic case (see Lemmas 2.2). The reason why we use this control strategy is that the strategy is easy to implement and aligns with reality. Our contributions in this paper are as follows:

(1) The delay feedback control is added as a part of the system;

(2) the rumor model satisfies the local Lipschitz condition, which is different from the systems in [22, 33];

(3) both deterministic case and stochastic case are studied, and the generalization is not trivial. More precisely, for the stochastic case, there is another parameter p to determine the upper bound of τ; see Theorem 2.1 and Lemma 3.2.

In all, letting rumors disappear quickly can not only enable people to learn the truth of the matter more quickly, but it also reduces the economic losses caused by the existence of rumors.

This paper is organized as follows. In Section 2, the deterministic case is considered. Section 3 is concerned with the stochastic case. In Section 4, some examples are given in order to verify our results.

2. Deterministic case

In this section, we consider the deterministic case

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\eta)I(t) - \alpha I^2(t) - kI(\delta(t)), \end{cases} \tag{2.1}$$

where $k > 0$ will be determined later.

In order to establish the extinction of the rumor of (2.1), we first consider the auxiliary system

$$\begin{cases} \frac{d\tilde{S}(t)}{dt} = A - \beta \tilde{S}(t)\tilde{I}(t) - \mu \tilde{S}(t) + \alpha \tilde{I}^2(t), \\ \frac{d\tilde{I}(t)}{dt} = \beta \tilde{S}(t)\tilde{I}(t) - (\mu+\eta)\tilde{I}(t) - \alpha \tilde{I}^2(t) - k\tilde{I}(t). \end{cases} \tag{2.2}$$

It follows from the result of [34] that Eq (2.2) admits a unique positive solution. Denote

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)ds.$$

We immediately get the following result.

Lemma 2.1. *Let $k > 0$ satisfy $\beta A \le \mu(\mu + \eta + k)$. Then the solution of (2.2) satisfies*

$$\lim_{t\to 0}\frac{\tilde{S}(t)}{t} = \lim_{t\to 0}\frac{\tilde{I}(t)}{t} = 0,\ \lim_{t\to\infty}\langle \tilde{S}(t)\rangle = \frac{A}{\mu},\ \lim_{t\to\infty}\tilde{I}(t) = 0.$$

More precisely, for large enough t, $\tilde{I}(t) \le e^{(\mu+\eta+k)(\hat{R}_0-1)t}$.

Proof. Let $\tilde{Z}(t) = \tilde{S}(t) + \tilde{I}(t)$ so that we have

$$\begin{aligned} \frac{d\tilde{Z}(t)}{dt} &= A - \mu\tilde{Z}(t) - (\eta+k)\tilde{I}(t) \\ &\le A - \mu\tilde{Z}(t), \end{aligned}$$

which implies that

$$\tilde{Z}(t) \le \tilde{Z}(0) + \frac{A}{\mu}(1 - e^{-\mu t}) < \infty,\ \ \forall t \ge 0.$$

Consequently, we get

$$\frac{\tilde{S}(t)}{t} \vee \frac{\tilde{I}(t)}{t} \le \frac{\tilde{S}(t)+\tilde{I}(t)}{t} \le \frac{s_0+i_0}{t} + \frac{A}{\mu t}(1-e^{-\mu t}) \to 0. \tag{2.3}$$

Integrating with respect to t on both sides of the two equations of (2.2), we get

$$\begin{aligned} \frac{\tilde{S}(t)+\tilde{I}(t)}{t} - \frac{s_0+i_0}{t} &= \frac{\int_0^t (A - \mu\tilde{S}(r) - (\mu+\eta+k)\tilde{I}(r))dr}{t} \\ &= A - \frac{\mu}{t}\int_0^t \tilde{S}(r)dr - \frac{\mu+\eta+k}{t}\int_0^t \tilde{I}(r)dr \end{aligned}$$

$$= \quad A - \mu\langle \tilde{S}(t)\rangle - (\mu+\eta+k)\langle \tilde{I}(t)\rangle.$$

By using (2.3), we obtain

$$\langle \tilde{S}(t)\rangle = \frac{A}{\mu} - \frac{\mu+\eta+k}{\mu}\langle \tilde{I}(t)\rangle - \frac{1}{\mu}\varphi(t), \tag{2.4}$$

where $\varphi(t) = \frac{\tilde{S}(t)+\tilde{I}(t)}{t} - \frac{s_0+i_0}{t}$. On the other hand, we have

$$(\ln \tilde{I}(t))' = \beta\tilde{S}(t) - (\mu+\eta+k) - \alpha\tilde{I}(t). \tag{2.5}$$

Integrating with respect to t on both sides of (2.5) and then using (2.4), we get

$$\begin{aligned} \frac{\ln \tilde{I}(t)}{t} &= \beta\langle \tilde{S}(t)\rangle - \alpha\langle \tilde{I}(t)\rangle - (\mu+\eta+k) + \frac{\ln i_0}{t} \\ &= \frac{\beta A}{\mu} - \left(\frac{\beta(\mu+\eta+k)}{\mu} + \alpha\right)\langle \tilde{I}(t)\rangle - (\mu+\eta+k) + \frac{\ln i_0}{t} - \frac{\beta}{\mu}\varphi(t) \\ &\leq (\mu+\eta+k)(\hat{R}_0 - 1) + \frac{\ln i_0}{t} - \frac{\beta}{\mu}\varphi(t). \end{aligned}$$

Note that $\lim_{t\to\infty}\left(\frac{\ln i_0}{t} - \frac{\beta}{\mu}\varphi(t)\right) = 0$, so we immediately have that

$$\limsup_{t\to\infty} \frac{\ln \tilde{I}(t)}{t} \leq (\mu+\eta+k)(\hat{R}_0 - 1) < 0,$$

which implies that

$$\tilde{I}(t) \leq e^{(\mu+\eta+k)(\hat{R}_0-1)t}, \quad for\ t \gg 1 \Rightarrow \lim_{t\to\infty} \tilde{I}(t) = 0. \tag{2.6}$$

Inserting (2.6) into (2.4), we obtain $\lim_{t\to\infty}\langle \tilde{S}(t)\rangle = \frac{A}{\mu}$. Together with (2.3), we get the desired results. The proof is complete.

Now, we consider Eq (2.1). The existence and uniqueness of a positive solution of (2.1) is obtained by [15]. It follows from the first equation of (2.1) that if $0 \leq s_0 < \frac{A}{\mu}$,

$$S'(t) \leq A - \mu S(t) \Rightarrow S(t) \leq s_0 e^{-\mu t} + \frac{A}{\mu}(1 - e^{-\mu t}) \leq \frac{A}{\mu}.$$

Then, we have the following relationship between the solutions of (2.1) and (2.2).

Lemma 2.2. *Assume that the initial data are positive, satisfying* $0 \leq s_0 + i_0 \leq \frac{A}{\mu}$. *It holds that*

$$|I(t) - \tilde{I}(t)| \leq \frac{k\vartheta_2\vartheta_1}{\vartheta_3(\vartheta_2 - \vartheta_3)} I(0)\tau\left[e^{\vartheta_2 t} - 1\right],$$

where

$$\vartheta_1 = |\frac{(\beta-\alpha)A}{\mu} - (\mu+\eta+k)|, \quad \vartheta_2 = \mu+\eta+\frac{(2\beta+4\alpha)A}{\mu}, \quad \vartheta_3 = \frac{\beta A}{\mu}.$$

Proof. It follows from (2.1) that

$$\begin{aligned}[S(t)+I(t)]' &= A-\mu S(t)-(\mu+\eta)I(t)-kI(\delta(t))\\ &\le A-\mu[S(t)+I(t)],\end{aligned}$$

which yields that

$$S(t)+I(t)\le (s_0+i_0)e^{-\mu t}+\frac{A}{\mu}(1-e^{-\mu t})\le \frac{A}{\mu},$$

where we use the condition $0\le s_0+i_0\le \frac{A}{\mu}$. Note that $S(t), I(t)\ge 0$, so we have $S(t), I(t)\le \frac{A}{\mu}$. Next, we need to obtain the upper bound of $I(t)$, which depends on the initial data. It follows from the second equation of (2.1) that

$$\begin{aligned}I(t) &= I(0)e^{-(\mu+\eta)t}+\beta\int_0^t e^{-(\mu+\eta)(t-\tau)}S(r)I(r)dr\\ &\quad -\alpha\int_0^t e^{-(\mu+\eta)(t-r)}I^2(r)dr-k\int_0^t e^{-(\mu+\eta)(t-r)}I(\delta(r))dr\\ &\le I(0)e^{-(\mu+\eta)t}+\frac{\beta A}{\mu}\int_0^t e^{-(\mu+\eta)(t-r)}I(r)dr\\ &\quad -k\int_0^t e^{-(\mu+\eta)(t-r)}I(\delta(r))dr,\end{aligned}$$

which yields that

$$\begin{aligned}\max_{0\le r\le t} I(r) &\le I(0)e^{-(\mu+\eta)t}+\frac{\beta A}{\mu}\int_0^t e^{-(\mu+\eta)(t-\varsigma)}\max_{0\le s\le\varsigma} I(s)d\varsigma\\ &\le I(0)+\frac{\beta A}{\mu}\int_0^t \max_{0\le s\le\varsigma} I(s)d\varsigma.\end{aligned}$$

By using Grönwall's inequality, we get

$$\max_{0\le r\le t} I(r)\le I(0)e^{\frac{\beta A}{\mu}t}. \tag{2.7}$$

Consequently, we have

$$\begin{aligned}|I(t)-I(\delta(t))| &= |\int_{\delta(t)}^t dI(r)|\\ &= |\int_{\delta(t)}^t \Big[\beta S(r)I(r)-(\mu+\eta)I(r)-\alpha I^2(r)-kI(\delta(r))\Big]dr\\ &\le I(0)\tau\vartheta_1 e^{\frac{\beta A}{\mu}t},\end{aligned} \tag{2.8}$$

where $\vartheta_1=|\frac{(\beta-\alpha)A}{\mu}-(\mu+\eta+k)|$. It follows from (2.1) and (2.2) that

$$[S(t)-\tilde{S}(t)]'=-\beta(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))-\mu(S(t)-\tilde{S}(t))+\alpha(I^2(t)-\tilde{I}^2(t))$$

and

$$
\begin{aligned}
[I(t)-\tilde{I}(t)]' &= \beta(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))-(\mu+\eta)(I(t)-\tilde{I}(t)) \\
&\quad -\alpha(I^2(t)-\tilde{I}^2(t))-k(I(\delta(t))-\tilde{I}(t)),
\end{aligned}
$$

which yield that

$$
\begin{aligned}
& |S(t)-\tilde{S}(t)|+|I(t)-\tilde{I}(t)| \\
\leq & \left(\mu+\eta+(2\beta+4\alpha)\frac{A}{\mu}\right)\int_0^t |S(r)-\tilde{S}(r)|+|I(r)-\tilde{I}(r)|dr \\
& +k\int_0^t |I(r)-I(\delta(r))|dr.
\end{aligned}
$$

Grönwall's inequality yields that

$$
\begin{aligned}
& |S(t)-\tilde{S}(t)|+|I(t)-\tilde{I}(t)| \\
\leq & \frac{k\mu}{\beta A}I(0)\tau\vartheta_1\left[e^{\frac{\beta A}{\mu}t}-1\right]+\frac{\mu+\eta+\frac{(2\beta+4\alpha)A}{\mu}}{\mu+\eta+\frac{(\beta+4\alpha)A}{\mu}}\frac{k\mu}{\beta A}I(0)\tau\vartheta_1\left[e^{(\mu+\eta+\frac{(2\beta+4\alpha)A}{\mu})t}-e^{\frac{\beta A}{\mu}t}\right] \\
\leq & \frac{\mu+\eta+\frac{(2\beta+4\alpha)A}{\mu}}{\mu+\eta+\frac{(\beta+4\alpha)A}{\mu}}\frac{k\mu}{\beta A}I(0)\tau\vartheta_1\left[e^{(\mu+\eta+\frac{(2\beta+4\alpha)A}{\mu})t}-1\right],
\end{aligned}
$$

where (2.8) is used. The proof is complete.

Remark 2.1. *In the proof of Lemma 2.2, we obtain the estimate (2.8). Indeed, we can use the information* $S(t), I(t) \leq \frac{A}{\mu}$ *to deduce that*

$$
\begin{aligned}
|I(t)-I(\delta(t))| &= \left|\int_{\delta(t)}^t dI(r)\right| \\
&= \left|\int_{\delta(t)}^t \left[\beta S(r)I(r)-(\mu+\eta)I(r)-\alpha I^2(r)-kI(\delta(r))\right]dr\right. \\
&\leq \left[(\alpha+\beta)\left(\frac{A}{\mu}\right)+(\mu+\eta+k)\right]\frac{A}{\mu}\tau.
\end{aligned}
$$

However, in the above estimate, the upper bound does not depend on the initial data, and thus, we cannot use the homogeneous property of (2.1) to derive the large time estimate.

Next, we will give the main result.

Theorem 2.1. *Let the assumptions of Lemma 2.2 hold. Then, there exists a* $\tau^* > 0$ *such that for any* $\tau \in (0, \tau^*)$*, the solution of (2.1) will decay exponentially, where* τ^* *is the unique root of*

$$
\frac{k\vartheta_2\vartheta_1}{\vartheta_3(\vartheta_2-\vartheta_3)}I(0)\tau\left[e^{\vartheta_2\left(\tau+\log(\frac{1}{\varepsilon})/\gamma\right)}-1\right]=1-\varepsilon, \tag{2.9}
$$

and $\varepsilon \in (0,1)$*,* $\gamma = (\mu+\eta+k)(1-\hat{R}_0)$*.*

Proof. It is easy to see that the left-hand side of (2.9) is a continuously increasing function of $\tau \geq 0$ and equals to zero when $\tau = 0$. Thus, (2.9) must have a unique root τ^*. Let $\tau \in (0, \tau^*)$. Choose a positive integer $\bar{k}$ such that

$$\frac{\log(\frac{1}{\varepsilon})}{\gamma\tau} \leq \bar{k} < 1 + \frac{\log(\frac{1}{\varepsilon})}{\gamma\tau},$$

where γ is defined as in Theorem 2.1. Thus, we have $e^{-\gamma\bar{k}\tau} \leq \varepsilon$. By Lemma 2.1, we have

$$|\tilde{I}(\bar{k}\tau)| \leq I(0)e^{-\gamma\bar{k}\tau}, \quad t \geq 0.$$

Note that

$$\begin{aligned} |I(\bar{k}\tau)| &\leq |\tilde{I}(\bar{k}\tau)| + |I(\bar{k}\tau) - \tilde{I}(\bar{k}\tau)| \\ &\leq I(0)\left[\varepsilon + \frac{k\vartheta_1\vartheta_2}{\vartheta_3(\vartheta_2 - \vartheta_3)}\tau\left(e^{\vartheta_2\bar{k}\tau} - 1\right)\right]. \end{aligned}$$

It follows from the definition of $\bar{k}$ that

$$\varepsilon + \frac{k\vartheta_1\vartheta_2}{\vartheta_3(\vartheta_2 - \vartheta_3)}\tau\left(e^{\vartheta_2\bar{k}\tau} - 1\right) \leq \varepsilon + \frac{k\vartheta_1\vartheta_2}{\vartheta_3(\vartheta_2 - \vartheta_3)}\tau\left(e^{\vartheta_2\left(\tau + \log(\frac{1}{\varepsilon})/\gamma\right)} - 1\right) < 1.$$

We may therefore write

$$\varepsilon + \frac{k\vartheta_1\vartheta_2}{\vartheta_3(\vartheta_2 - \vartheta_3)}\tau\left(e^{\vartheta_2\left(\tau + \log(\frac{1}{\varepsilon})/\gamma\right)} - 1\right) = e^{-\lambda\bar{k}\tau}.$$

Consequently, we get

$$|I(\bar{k}\tau)| \leq e^{-\lambda\bar{k}\tau}.$$

Due to the time-homogeneous property of (2.1), we therefore see easily that

$$|I(i\bar{k}\tau)| \leq |I((i-1)\bar{k}\tau)||e^{-\lambda\bar{k}\tau} \leq e^{-\lambda i\bar{k}\tau}, \quad \forall i = 1, 2, \ldots.$$

It follows from (2.7) that

$$\max_{0 \leq r \leq \bar{k}\tau} |I(s)| \leq NI(0), \quad t \geq 0,$$

where $N = e^{\frac{\beta A}{\mu}\bar{k}\tau}$. Similarly, we can get

$$\max_{i\bar{k}\tau \leq r \leq (i+1)\bar{k}\tau} |I(r)| \leq N|I(i\bar{k}\tau)|, \quad t \geq 0.$$

Therefore, for any $t > 0$, there exists i such that $i\bar{k}\tau \leq t \leq (i+1)\bar{k}\tau$ and

$$|I(t)| \leq Ne^{-\lambda t}, \quad t \geq 0.$$

The proof is complete.

Remark 2.2. *In (2.9), τ^* exists uniquely. In fact, let*

$$g(\tau) = \frac{k\vartheta_2\vartheta_1}{\vartheta_3(\vartheta_2-\vartheta_3)} I(0)\tau\left[e^{\vartheta_2\left(\tau+\log(\frac{1}{\varepsilon})/\gamma\right)}-1\right]-(1-\varepsilon).$$

It is easy to check that $g(0) = -(1-\varepsilon) < 0$, $g(+\infty) = +\infty$, and

$$g'(\tau) = \frac{k\vartheta_2\vartheta_1}{\vartheta_3(\vartheta_2-\vartheta_3)} I(0)\left[e^{\vartheta_2\left(\tau+\log(\frac{1}{\varepsilon})/\gamma\right)}+\vartheta_2\tau-1\right] > 0,\ \forall\tau > 0.$$

Consequently, by using the continuous of $g(\tau)$, we obtain that there exists uniquely a solution τ^* of Eq (2.9).

3. Stochastic case

In this section, we will consider the stochastic rumor model

$$\begin{cases} dS(t) = [A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t)]dt - \sigma S(t)I(t)dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\eta)I(t) - \alpha I^2(t) - kI(\delta(t))]dt + \sigma S(t)I(t)dB(t), \end{cases} \tag{3.1}$$

where $\delta(t)$ is defined as in (1.2), $B(t)$ is Brownian motion, and $k > 0$. Our aim is to let the rumor disappear based on discrete-time observation . More precisely, we assume that if $k = 0$, then the basic reproduction number $R_0 > 1$, and if $k > 0$, we take some k to satisfy

$$\hat{R}_0 = \frac{\beta A}{\mu(\mu+\eta+k)} < 1.$$

Then, similar to [10, Theorems 3.1 and 4.1] and [17, Theorems 3.1 and 4.1], the rumor will exist if $k = 0$ and disappear if $k > 0$ and $\delta(t) = t$. In other words, we will first consider the system

$$\begin{cases} d\tilde{S}(t) = [A - \beta\tilde{S}(t)\tilde{I}(t) - \mu\tilde{S}(t) + \alpha\tilde{I}^2(t)]dt - \sigma\tilde{S}(t)\tilde{I}(t)dB(t), \\ d\tilde{I}(t) = [\beta\tilde{S}(t)\tilde{I}(t) - (\mu+\eta)\tilde{I}(t) - \alpha\tilde{I}^2(t) - k\tilde{I}(t)]dt + \sigma\tilde{S}(t)\tilde{I}(t)dB(t). \end{cases} \tag{3.2}$$

For system (3.2), it is easy to obtain the following result; see [17, Theorems 3.1 and 4.1].

Proposition 3.1. *Assume that $\mu > \frac{\sigma^2}{2}$. Let $(\tilde{S}(t), \tilde{I}(t))$ be the solution of system (3.2) with any initial value $S(0) > 0$ and $I(0) > 0$. If $\hat{R}_0 < 1$, then*

$$\lim_{t\to\infty} \frac{\ln\tilde{I}(t)}{t} \le (\mu+\eta)(\hat{R}_0 - 1) < 0 \ \ a.s.$$

In addition,

$$\lim_{t\to\infty}\langle\tilde{S}(t)\rangle = \frac{A}{\mu},\ \ a.s.$$

That is to say, $I(t)$ tends to zero exponentially almost surely, that is, the density of the rumor-infected users disappear with probability one.

Now, we will use the same idea as in Section 2 to prove the rumor will become extinct almost surely for system (3.1). In order to do that, we first need the existence and uniqueness of the solution to system (3.1). If we let

$$\varpi(t) = t - k\tau, \quad \text{for } t \in [k\tau, (k+1)\tau), \;\; k = 0, 1, \ldots,$$

then $\delta(t) = t - \varpi(t)$. Therefore, system (3.1) becomes

$$\begin{cases} dS(t) = [A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t)]dt - \sigma S(t)I(t)dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\eta)I(t) - \alpha I^2(t) - kI(t-\varpi(t))]dt + \sigma S(t)I(t)dB(t), \end{cases} \tag{3.3}$$

where $\varpi(t) \in [0, \tau]$. We have the following theorem.

Theorem 3.1. *For any initial value $S(0) > 0$ and $I(\varsigma) \geq 0$ for all $\varsigma \in [-\tau, 0)$ with $I(0) > 0$, system (3.3) has a unique positive solution (S(t),I(t)) on $t > 0$, and the solution will remain in $\mathbb{R}^2_+$ with probability one. That is to say, $(S(t), I(t)) \in \mathbb{R}^2_+$ for all $t > 0$ almost surely (a.s.).*

Proof. The proof is similar to that of [10, Theorem 3.1], and we only give the outline. We assume the initial value $S(0) > 0$ and $I(t) \geq 0$ for all $t \in [-\tau, 0)$ with $I(0) > 0$. Note that system (3.3) satisfies the local Lipschitz condition; then, a unique local positive solution $(S(t), I(t))$ exists on $t \in [-\tau, \tau_e)$, where τ_e represents the explosion time [21]. To prove the solution is global, we only need to show $\tau_e = \infty$ a.s. Define the following stopping time

$$\tau_* = inf\{t \in [0, \tau_e) | S(t) \leq 0 \; or \; I(t) \leq 0\},$$

where throughout this paper, we set $inf\emptyset = \infty$ (as usual $\emptyset$ represents the empty set). Obviously, $\tau_* \leq \tau_e$ a.s.. If $\tau_* = \infty$ a.s. is true, then $\tau_e = \infty$ a.s., and $(S(t), I(t)) \in \mathbb{R}^2_+$ a.s. for all $t > 0$. That is to say, to complete the proof, we only need to show $\tau_* = \infty$ a.s. If this assertion is false, then there exists a positive constant $T > 0$ such that $\mathbb{P}\{\tau_* \leq T\} > 0$. Define $V(S, I) = \ln(SI)$. Then, Itô's formula implies that

$$\begin{aligned} dV(S,I) &= \left[\frac{A}{S(t)} - \beta I(t) - \mu + \alpha\frac{I^2(t)}{S(t)}\right]dt \\ &\quad + \left[\beta S(t) - (\mu+\eta) - \alpha I(t) - k\frac{I(t-\varpi(t))}{I(t)})\right]dt + \sigma(S(t) - I(t))dB(t) \\ &\geq LV(S,I)dt + \sigma(S(t) - I(t))dB(t), \end{aligned}$$

where LV is defined by

$$LV(S,I) = -\beta I(t) - \mu - (\mu+\eta) - \alpha I(t) - k\frac{I(t-\varpi(t)}{I(t)}).$$

Therefore, we have

$$V(S(t), I(t)) \geq V(S(0), I(0)) + \int_0^t LV(S(r), I(r))dr + \sigma\int_0^t (S(r) - I(r))dB(r).$$

Notice that some components of $(S(\tau_*), I(\tau_*))$ equal 0, and thus, $\lim_{t\to\tau_*} V(S(t), I(t)) = -\infty$. Consequently, we obtain

$$-\infty \geq V(S(0), I(0)) + \int_0^{\tau_*} LV(S(r), I(r))dr + \sigma \int_0^{\tau_*} (S(r) - I(r))dB(r) > \infty,$$

which leads to a contradict. Consequently, we have $\tau_* = \infty$. This completes the proof.

Next, similar to Lemma 2.2, we consider the difference between $I(t)$ and $\tilde{I}(t)$. Because we cannot get the uniform bound for $S(t)$ almost surely, the method used here is different from that of Lemma 2.2.

Lemma 3.1. *Let the assumptions of Theorem 3.1 hold, and assume that* $0 \leq s_0 + i_0 \leq \frac{A}{\mu}$. *For* $0 < p < 2$, *we have*

$$\mathbb{E}\left[|I(t) - \tilde{I}(t)|^p\right] \leq \tilde{k}^p e^{(K+1)tp} \tau^p, \tag{3.4}$$

where $K = \max\{A_1, A_2\} + 2$, $\tilde{k} = c_k k$, *and*

$$\begin{aligned}
c_k &= \frac{A}{\mu}\left(\frac{\sigma^2 A^3}{\mu^3} + \frac{A}{\mu}(\alpha + \beta) + \mu + \eta + k\right) \\
A_1 &= \frac{2A}{\mu}(\alpha + 2\beta) + \frac{4\sigma^2 A^2}{\mu^2} + \frac{A}{\mu} - \mu - \beta, \\
A_2 &= \frac{4A}{\mu} + \frac{6A\alpha}{\mu} + \frac{4\sigma^2 A^2}{\mu^2} + \frac{4A}{\mu} + \frac{2A\beta}{\mu} + k - \mu - \eta.
\end{aligned}$$

Proof. It follows from (3.1) that

$$\begin{aligned}
d[S(t) + I(t)] &= A - \mu[S(t) + I(t)] - \eta I(t) - kI(\delta(t)) \\
&\leq A - \mu[S(t) + I(t)],
\end{aligned}$$

almost surely, which implies that

$$S(t) + I(t) \leq (s_0 + i_0)e^{-\mu t} + \frac{A}{\mu}(1 - e^{-\mu t}) \leq \frac{A}{\mu}, \ a.s.$$

Note that $S(t), I(t) \geq 0$, so we have $S(t), I(t) \leq \frac{A}{\mu}$. Similarly, we can get $\tilde{S}(t), \tilde{I}(t) \leq \frac{A}{\mu}$. Then, the Itô formula yields that

$$\begin{aligned}
d|S(t) - \tilde{S}(t)|^2 &= 2(S(t) - \tilde{S}(t))[-\beta(S(t)I(t) - \tilde{S}(t)\tilde{I}(t)) - \mu(S(t) - \tilde{S}(t)) \\
&\quad + \alpha(I(t) + \tilde{I}(t))(I(t) - \tilde{I}(t))]dt + \sigma^2(S(t)I(t) - \tilde{S}(t)\tilde{I}(t))^2 dt \\
&\quad - 2\sigma(S(t) - \tilde{S}(t))(S(t)I(t) - \tilde{S}(t)\tilde{I}(t))dB(t) \\
&\leq 2\left(\frac{A}{\mu}(\alpha + 2\beta) + \frac{\sigma^2 A^2}{\mu^2} - \mu - \beta\right)(S(t) - \tilde{S}(t))^2 dt \\
&\quad + \frac{2A}{\mu}\left(\alpha + \beta + \frac{\sigma^2 A}{\mu}\right)(I(t) - \tilde{I}(t))^2 dt \\
&\quad - 2\sigma(S(t) - \tilde{S}(t))(S(t)I(t) - \tilde{S}(t)\tilde{I}(t))dB(t), \\
d|I(t) - \tilde{I}(t)|^2 &= -2\beta(I(t) - \tilde{I}(t))(S(t)I(t) - \tilde{S}(t)\tilde{I}(t))dt - 2(\mu + \eta)(I(t) - \tilde{I}(t))^2 dt
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha(I(t)+\tilde{I}(t))(I(t)-\tilde{I}(t))^2dt-2k(I(t)-\tilde{I}(t))(I(\delta(t))-\tilde{I}(t))\\
&+\sigma^2(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))^2dt+2\sigma(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))dB(t)\\
\leq\ & \frac{A}{\mu}\left(1+\frac{2A\sigma^2}{\mu}\right)(S(t)-\tilde{S}(t))^2dt+k(I(t)-I(\delta(t)))^2dt\\
&+2\left(\frac{2A}{\mu}+\frac{2A\alpha}{\mu}+\frac{\sigma^2A^2}{\mu^2}+k-\mu-\eta\right)(I(t)-\tilde{I}(t))^2dt\\
&+2\sigma(S(t)-\tilde{S}(t))(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))dB(t),
\end{aligned}
$$

where we used the fact that $S(t), I(t), \tilde{S}(t), \tilde{I}(t) \leq \frac{A}{\mu}$. Adding the above two inequalities and taking expectation, we get

$$
\begin{aligned}
\mathbb{E}\left[|S(t)-\tilde{S}(t)|^2+|I(t)-\tilde{I}(t)|^2\right] \leq\ & K_1\int_0^t \mathbb{E}\left[|S(r)-\tilde{S}(r)|^2+|I(r)-\tilde{I}(r)|^2\right]dr\\
&+k\int_0^t \mathbb{E}\left[|I(r)-I(\delta(r))|^2\right]dr,
\end{aligned}
$$

where

$$
K_1=\left(\frac{2A}{\mu}(\alpha+2\beta)+\frac{4\sigma^2A^2}{\mu^2}+\frac{A}{\mu}-\mu-\beta\right)\vee\left(\frac{4A}{\mu}+\frac{6A\alpha}{\mu}+\frac{4\sigma^2A^2}{\mu^2}+\frac{4A}{\mu}+\frac{2A\beta}{\mu}+k-\mu-\eta\right).
$$

Grönwall's inequality implies that

$$
\mathbb{E}\left[|S(t)-\tilde{S}(t)|^2+|I(t)-\tilde{I}(t)|^2\right]\leq ke^{(K_1+1)t}\int_0^t \mathbb{E}\left[|I(r)-I(\delta(r))|^2\right]dr. \tag{3.5}
$$

On the other hand, the Itô isometry implies that

$$
\begin{aligned}
&\mathbb{E}\left[|I(t)-I(\delta(t))|^2\right]\\
=\ & \mathbb{E}\left|\int_{\delta(t)}^t [\beta S(r)I(r)-(\mu+\eta)I(r)-\alpha I^2(r)-kI(\delta(r))]dr+\sigma S(r)I(r)dB(r)\right|^2\\
=\ & \mathbb{E}\left|\int_{\delta(t)}^t [\beta S(r)I(r)-(\mu+\eta)I(r)-\alpha I^2(r)-kI(\delta(r))]dr\right|^2\\
&+\mathbb{E}\left|\sigma\int_{\delta(t)}^t S(r)I(r)dB(r)\right|^2\\
=\ & \mathbb{E}\left|\int_{\delta(t)}^t [\beta S(r)I(r)-(\mu+\eta)I(r)-\alpha I^2(r)-kI(\delta(r))]dr\right|^2\\
&+\mathbb{E}\left|\sigma^2\int_{\delta(t)}^t S^2(r)I^2(r)dt\right|\\
\leq\ & \frac{A}{\mu}\left(\frac{\sigma^2A^3}{\mu^3}+\frac{A}{\mu}(\alpha+\beta)+\mu+\eta+k\right)\tau.
\end{aligned}
$$

Submitting the above inequality into (3.5), we get

$$
\mathbb{E}\left[|I(t)-\tilde{I}(t)|^2\right]\leq c_k k e^{(K_1+2)t}\tau,
$$

where $c_k = \frac{A}{\mu}\left(\frac{\sigma^2 A^3}{\mu^3} + \frac{A}{\mu}(\alpha+\beta) + \mu + \eta + k\right)$. Then, the Hölder inequality yields (3.4). The proof is complete.

Let $I(j\bar{m}\tau; i_0)$ be the density of infectious individuals in system (3.1) at time $j\bar{m}\tau$ with the initial data i_0 and $\tilde{I}(j\bar{m}\tau; i_0)$ be the density of infectious individuals in system (3.2) at time $j\bar{m}\tau$ with the initial data i_0. Note that

$$\mathcal{F}_{j\bar{m}\tau} = \sigma\{B(t),\ 0 \le t \le j\bar{m}\tau\}.$$

Next, we state out the key lemma.

Lemma 3.2. *Let the assumptions of Theorem 3.1 hold, and let $\tau^* > 0$ be the unique root of the following equation:*

$$2^p \tilde{k}^p e^{(K+1)\left(p\tau + \ln\frac{2^p}{\varepsilon}/\gamma\right)} \tau^p = 1 - \varepsilon, \tag{3.6}$$

where K is as defined in Lemma 3.1. For any $\tau \in (0, \tau^*)$, there exists a pair of positive constants $\bar{m}$ and λ such that for any initial value $0 \le s_0 + i_0 \le \frac{A}{\mu}$ and $\varepsilon \in (0,1)$, the solution of system (3.1) satisfies

$$\mathbb{E}\,|I(j\bar{m}\tau; i_0)|^p \le |s_0 + i_0|^p (1 + i\lambda\bar{m}\tau)^{-2}, \quad \forall\ j = 1, 2, \cdots.$$

Proof. When p is fixed, the left side of Eq (3.6) is a continuous increasing function with respect to $\tau \ge 0$, and when $\tau = 0$, it is equal to 0. Therefore, Eq (3.6) has a unique positive root, which can be denoted as $\tau^* > 0$. For any fixed $\tau \in (0, \tau^*)$ and initial value $0 \le s_0 + i_0 \le \frac{A}{\mu}$, it follows from Proposition 3.1 that there exists a positive number m_* such that $m \ge m_*$, and it holds that

$$\left|\tilde{I}(m\tau; i_0)\right| \le |i_0 + s_0| e^{-\gamma m\tau}, \tag{3.7}$$

where $\gamma = \frac{(\mu+\eta)(1-\tilde{R}_{0s})}{2}$. Letting $\varepsilon \in (0,1)$, we take a large positive integer $\bar{m}$ satisfying (3.7) and

$$\frac{\ln\frac{2^p}{\varepsilon}}{\gamma p\tau} \le \bar{m} \le \frac{\ln\frac{2^p}{\varepsilon}}{\gamma p\tau} + 1, \tag{3.8}$$

which implies that

$$2^p e^{-\gamma p\bar{m}\tau} \le \varepsilon. \tag{3.9}$$

The Cauchy–Schwarz inequality yields that

$$\mathbb{E}|I(\bar{m}\tau; i_0)|^p \le 2^p \mathbb{E}|\tilde{I}(\bar{m}\tau; i_0)|^p + 2^p \mathbb{E}|I(\bar{m}\tau; i_0) - \tilde{I}(\bar{m}\tau; i_0)|^p.$$

Combining (3.7)–(3.9) and using Lemma 3.1, we have

$$\begin{aligned} \mathbb{E}|I(\bar{m}\tau; i_0)|^p &\le 2^p |i_0 + s_0|^p e^{-\gamma p\bar{m}\tau} + 2^p |i_0 + s_0|^p \tilde{k}^p e^{(K+1)\bar{m}\tau p} \tau^p \\ &\le |i_0 + s_0|^p \left(\varepsilon + 2^p \tilde{k}^p e^{(K+1)\bar{m}\tau p} \tau^p\right). \end{aligned} \tag{3.10}$$

Using (3.8), we have

$$e^{(K+1)\bar{m}\tau p} \le e^{(K+1)\left(p\tau + \ln\frac{2^p}{\varepsilon}/\gamma\right)}.$$

Consequently, we get

$$\varepsilon + 2^p\tilde{k}^p e^{(K+1)\bar{m}\tau p}\tau^p \le \varepsilon + 2^p\tilde{k}^p e^{(K+1)\left(p\tau+\ln\frac{2^p}{\varepsilon}/\gamma\right)}\tau^p < 1.$$

Thus, there exists $\lambda > 0$ such that

$$\varepsilon_1 + 2^p\tilde{k}^p e^{(K+1)\left(p\tau+\ln\frac{2^p}{\varepsilon}/\gamma\right)}\tau^p = e^{-\lambda\bar{m}\tau}.$$

Submitting the above inequality into (3.10), we get

$$\mathbb{E}|I(\bar{m}\tau; i_0)|^p \le |i_0 + s_0|^p e^{-\lambda\bar{m}\tau}.$$

Now, we investigate the solution $I(t)$ of system (3.1) on $t \ge \bar{m}\tau$, which can be seen as the solution of system (3.1) starting from time $t = \bar{m}\tau$ and state $I_{\bar{m}\tau}$. Therefore, from the time homogeneity of system (3.1), we obtain

$$\mathbb{E}\left(|I(2\bar{m}\tau; i_0)|^p|\mathcal{F}_{\bar{m}\tau}\right) \le |I(\bar{m}\tau; i_0)|^p e^{-\lambda\bar{m}\tau}.$$

Consequently,

$$\mathbb{E}|I(2\bar{m}\tau; i_0)|^p \le |s_0 + i_0|^p e^{-2\lambda\bar{m}\tau}.$$

By repeating the above process, we can deduce

$$\begin{aligned}\mathbb{E}\,|I(j\bar{m}\tau; i_0)|^p &\le \mathbb{E}\,|I\left((j-1)\bar{m}\tau; i_0\right)|^p\, e^{-\lambda\bar{m}\tau}\\ &\le |s_0 + i_0|^p e^{-\lambda j\bar{m}\tau}, \quad \forall\, j = 1, 2, \cdots.\end{aligned}$$

It is noted that the existence and uniqueness of τ^* of (3.6) is similar to that in Remark 2.2. Now, we get the following main result.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. There exists a positive constant τ^* such that for any initial value $0 \le s_0 + i_0 \le \frac{A}{\mu}$, if $\tau \in (0, \tau^*)$, the solution of (3.1) satisfies,*

$$\limsup_{t\to\infty} \frac{\log|I(t)|^p}{t} < 0 \quad \text{a.s.} \tag{3.11}$$

Proof. For any fixed $\tau \in (0, \tau^*)$ and initial value $0 \le s_0 + i_0 \le \frac{A}{\mu}$, we denote $I(t; i_0) = I(t)$. Let $\bar{m}$ be defined as in Lemma 3.2. For $t \in [0, \bar{m}\tau]$, from (3.1), we have

$$I(t) = i_0 + \int_0^t \left[\beta S(r)I(r) - (\mu+\eta)I(r) - \alpha I^2(r)\right]dr - k\int_0^t I(\delta(r))dr + \sigma\int_0^t S(r)I(r)dB(r).$$

By using Hölder inequality and Doob's inequality combined with the fact that $0 < S(t) + I(t) \le \frac{A}{\mu}$, we have

$$\mathbb{E}\left(\sup_{0\le r\le t}|I(r)|^2\right)$$

$$
\begin{aligned}
&\le 3|i_0|^2 + 3\sigma^2\mathbb{E}\left(\sup_{0\le u\le t}\left|\int_0^u S(r)I(r)dB(r)\right|^2\right)\\
&\quad +3\mathbb{E}\left(\sup_{0\le u\le t}\left|\int_0^u \left[\beta S(r)I(r)-(\mu+\eta)I(r)-\alpha I^2(r)\right]dr\right|^2\right) \text{ (by using Hölder inequality)}\\
&\le 3|i_0|^2 + \left[2\left(\frac{A}{\mu}(\alpha+\beta)+\mu+\eta\right)^2+\frac{12\sigma^2A^2}{\mu^2}\right]\int_0^t \mathbb{E}\left(\sup_{0\le u\le r}|I(r)|^2\right)dr. \text{ (by using Doob's inequality)}.
\end{aligned}
$$

The Grönwall inequality shows that

$$\mathbb{E}\left(\sup_{0\le t\le \bar{m}\tau}|I(t)|^2\right)\le 3|i_0|^2e^{C_0\bar{m}\tau},$$

where

$$C_0 = 2\left(\frac{A}{\mu}(\alpha+\beta)+\mu+\eta\right)^2+\frac{12\sigma^2A^2}{\mu^2}.$$

The Hölder inequality yields that

$$\mathbb{E}\left(\sup_{0\le t\le \bar{m}\tau}|I(t)|^p\right)\le 3^{\frac{p}{2}}|i_0|^pe^{C_0p\bar{m}\tau/2}\equiv C_1|i_0|^p. \tag{3.12}$$

Now, we consider the solution $I(t)$ of (3.1) on the interval $t\in[j\bar{m}\tau,(j+1)\bar{m}\tau](j=1,2,\cdots)$, which can be regarded as the solution $(I(t); j\bar{k}\tau)$ of (3.1) starting from time $t = j\bar{m}\tau$ and state $I(j\bar{k}\tau)$. It follows from the time homogeneity of (3.1) and (3.12) that

$$\mathbb{E}\left(\sup_{j\bar{m}\tau\le t\le(j+1)\bar{m}\tau}|I(t)|^p|\mathcal{F}_{j\bar{m}\tau}\right)\le C_1\mathbb{E}|I(j\bar{m}\tau;i_0)|^p.$$

By combining Lemma 3.2 for $j\ge 1$, we get

$$\mathbb{E}\left(\sup_{j\bar{m}\tau\le t\le(j+1)\bar{m}\tau}|I(t)|^p\right)\le C_1\mathbb{E}|I(j\bar{m}\tau;i_0)|^p\le C_1|s_0+i_0|^p. \tag{3.13}$$

Using the Chebyshev inequality, for any $j\ge 1$, $\gamma_1>0$, we have

$$\mathbb{P}\left(\sup_{j\bar{m}\tau\le t\le(j+1)\bar{m}\tau}|I(t)|^p\ge e^{-0.5\lambda j\bar{m}\tau}\right)\le C_1|i_0|^pe^{-0.5\lambda j\bar{m}\tau}.$$

According to the Borel–Cantelli lemma, for almost all $\omega\in\Omega$, there exists an integer $j_0 = j_0(\omega), \forall j\ge j_0(\omega)$ such that

$$\sup_{j\bar{m}\tau\le t\le(j+1)\bar{m}\tau}|I(t)|^p< e^{-0.5\lambda j\bar{m}\tau}$$

holds. Let $t\to\infty$, so we have

$$\limsup_{t\to\infty}\frac{\log|I(t,\omega)|^p}{t}\le -\frac{\lambda}{2p}<0.$$

The proof is complete.

4. Examples

In this section, we will use Matlab to simulate the systems (2.1) and (3.1), which verify the theoretical results.

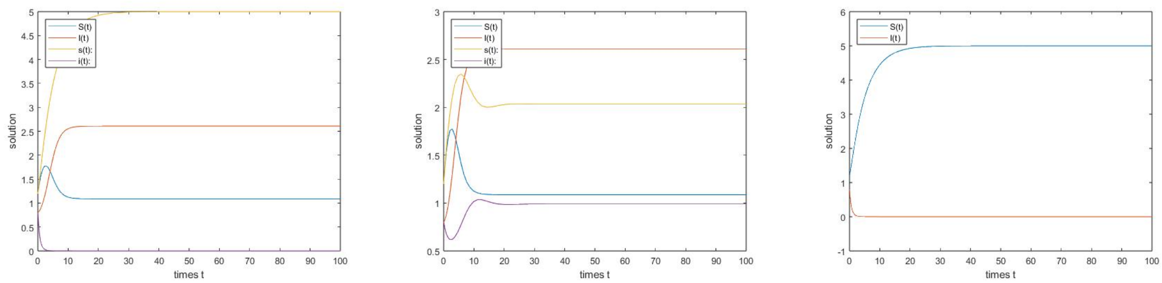
Let $A = 1, \beta = 0.3, \mu = 0.2, \eta = 0.1$, and $\alpha = 0.01$. Set $s_0 = 1.2$ and $i_0 = 0.8$. Then, we have

$$R_0 = \frac{\beta A}{\mu(\mu + \eta)} = 5 > 1.$$

It follows from the classical results of [5] that the rumor will be popular. Then, let $k = 5/3$, so we get

$$\hat{R}_0 = \frac{\beta A}{\mu(\mu + \eta + k)} = \frac{90}{119} < 1.$$

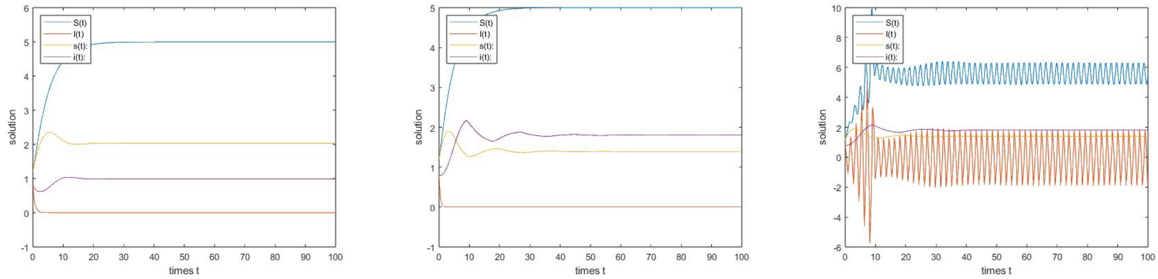
Consequently, it follows from Lemma 2.1 that the rumor will disappear exponentially; see Figure 1(a). In Figure 1(a), $(S(t), I(t))$ and $(s(t), i(t))$ are the solutions of (2.2) with $k = 0$ and $k = 5/3$, respectively. If we take $k = 0.3$, it is easy to get that $\hat{R}_0 = 5/2 > 1$, and thus, we have Figure 1(b), where $(s(t), i(t))$ is the solution of (2.2) with $k = 0.3$. Set $\varepsilon = 0.8$; then, it is not difficult to get $\tau^* > 0.024$. Let $\tau = 0.024$; then, it is easy to check that the assumptions of Theorem 2.1 hold. It follows from Theorem 2.1 that the rumor will disappear exponentially; see Figure 1(c).



(a) (b) (c)

Figure 1. (a)–(c) denote the solutions of system (2.1) with $k = 0$, system (2.2) and system (2.1) with $\tau = 0.024$, respectively.

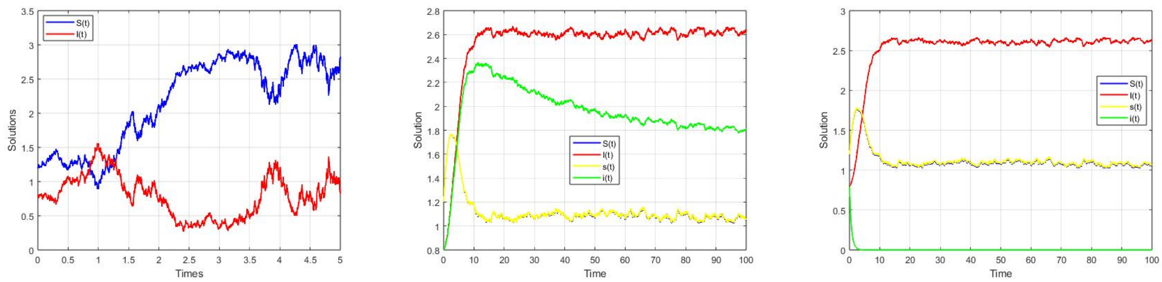
If we let $k = 0.3$ and $k = 5/3$, under the same conditions $\tau = 0.024$ of system (2.1), the solutions will have different behaviors; see Figure 2(a), where $S(t)$ and $I(t)$ are the solutions of system (2.1) with $k = 5/3$, and $s(t)$ and $i(t)$ are the solutions of system (2.1) with $k = 0.3$. Furthermore, under the condition $\tau = 0.024$, we get the solutions of (2.1) with $k = 5/2$ and $k = 0.1$; see Figure 2(b). If we let τ be large, the result of Theorem 2.1 will not hold. For example, let $\tau = 0.9$: The rumor will be popular; see Figure 2(c).



(a) (b) (c)

Figure 2. (a)–(c) denote the solutions of system (2.1) with different k.

For the stochastic case, let $\sigma = 0.1$, and keep the same parameters as in deterministic case. If $k = 0$, then the solution of (3.1) will be like Figure 3(a). If we take $k = 0.01$, it is easy to get $\hat{R}_0 > 1$, and hence, we obtain Figure 3(b), where $(S(t), I(t))$, and $(s(t), i(t))$ are the solutions of system (3.1) with $k = 0$ and $k = 0.01$, respectively. Similar to the deterministic case, let $k = 5/3$; then, the solution of (3.1) will be like Figure 3(c). Figure 3 verifies Theorem 3.2. In Figures 3(b),(c), the profiles of $S(t)$ and $s(t)$ mostly overlap.



(a) (b) (c)

Figure 3. (a)–(c) denote the solutions of system (3.1) with different k.

5. Conclusions

In this paper, in order to control rumors more effectively, we introduced a new algorithm to make rumor disappear. The new algorithm is easy to implement and aligns with reality. More precisely, the delay feedback control is added to a component of the system; then, we consider an auxiliary system and prove that the rumor disappears in the auxiliary system. By using perturbation method, we prove the rumor also disappears in the system with delay feedback control. Moreover, the upper bound of the time-delay is obtained, which is a solution of an algebraic equation.

For the stochastic rumor model, by using the Borel–Cantelli lemma and moments estimations, we can get similar results. The results obtained in this paper show that in the real world, in order to make a rumor disappear quickly, we can isolate rumor spreaders. In this paper, we provided a quantitative description: Given the rate of rumor extinction, we can determine the proportion of rumor spreaders that need to be isolated. It follows from the examples in Section 4 that the algorithm is effective.

With the development of artificial intelligence, we can study the issue of rumor control from the data perspective, for example, by utilizing reinforcement learning to devise strategies for rumor control. However, our primary focus is on studying and proposing strategies from a mechanistic perspective. Data learning emphasizes algorithms, whereas our emphasis here lies in models, and there is a significant difference between the two. Of course, combining the two would be even better. Furthermore, we did not use the actual data to study rumor. In the further work, we will study the phenomenon similarly to [39].

Author contributions

Zuyu Zhao: Conceptualization, method, investigation, writing-original draft; Huikun Hao: Conceptualization, investigation, software, writing-review and editing; An Yan: Validation, software, writing-review and editing; Zonghe Guo: Validation, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

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