Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 14763–14770.
DOI: 10.3934/math.2026607
Received: 21 December 2025
Revised: 22 April 2026
Accepted: 21 May 2026
Published: 26 May 2026

Research article

Embedding reverse Carleson measure for derivatives on Fock space

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Abstract: The paper establishes a full characterisation of the embedding of Carleson measures and reverse Carleson measures for derivatives on Fock space. The intrinsic relationship between (s, r)-averaging transform and embedding derivatives Carleson measures on Fock spaces is revealed. This improves existing results.

Keywords: reverse derivatives Carleson measure; Fock spaces
Mathematics Subject Classification: Primary 47B37, 30H20

1. Introduction and main results

Let $\mathbb{C}$ be the complex plane. We denote by $H(\mathbb{C})$ the space of all entire functions on $\mathbb{C}$. For parameters $\alpha > 0$ and $0 < p < \infty$, the Fock spaces F^p_α is defined as the collection of all $f \in H(\mathbb{C})$ satisfying

$$\|f\|^p_{p,\alpha} = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} \left| f(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p dA(\xi) < \infty,$$

where $dA(\xi) = dxdy$, $\xi = x + iy$. F^2_α forms a Hilbert space. Furthermore, it possesses the reproducing kernel $K^\alpha_\xi(w) = e^{\alpha\bar{\xi}w}$ with corresponding normalized reproducing kernels $k_\xi(w) = e^{\alpha\bar{\xi}w - \frac{\alpha|\xi|^2}{2}}$. It is evident that each k^α_ξ constitutes a unit vector in F^p_α for $0 < p < \infty$. The reader is directed to [12] for a comprehensive treatment for Fock spaces.

A (nonnegative) measure μ defined on the Borel subsets of $\mathbb{C}$ is called a (p, k)-Fock–Carleson measure for F^p_α if the embedding $f \mapsto f^{(k)}(\xi)e^{-\frac{\alpha|\xi|^2}{2}}$ acts boundedly from F^p_α into $L^p(\mu)$; that is, there exists $C > 0$ such that the given inequality

$$\int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p d\mu(\xi) \le C\|f\|^p_{p,\alpha}$$

holds for all $f \in F^p_\alpha$. In particular, the $(p,0)$-FC measure, known as the classical Fock–Carleson measure, has proven to be an essential tool for studying operators on Fock spaces and analytic function spaces on the unit disk, see [3, 9, 6] as well as the citations in this paper. In the reverse setting, the extension is readily established. Specifically, a $\mu > 0$ is called the reverse (p,k)-FC measure if one can find a $C > 0$ for which

$$\|f\|^p_{p,\alpha} \le C \int \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p d\mu(\xi)$$

holds for every $f \in F^p_\alpha$.

Let $D(\xi, r)$ stand for the Euclidean disk in $\mathbb{C}$ centered at ξ with radius $r > 0$. The area of a measurable set $E \subset \mathbb{C}$ is denoted by $|E|$. Define (s,r)-averaging transform

$$\widehat{\mu}_{s,r}(\xi) = (1 + |\xi|)^s \frac{\mu(D(\xi, r))}{|D(\xi, r)|}$$

for every $s, r > 0$ and $\xi \in \mathbb{C}$. To simplify notation, let us write $\widehat{\mu}_r(\xi) = \widehat{\mu}_{0,r}(\xi)$. As shown in [12, Theorem 3.29] and [7, Theorem 2.3], for a given $r > 0$ (or, equivalently, for every $r > 0$), the condition

$$\sup_{\xi \in \mathbb{C}} \widehat{\mu}_r(\xi) < \infty$$

is necessary and sufficient for μ to be a $(p,0)$-FC measure. For the general case $k > 0$, the necessary and sufficient condition for μ to be a (p,k)-FC measure is that for some (or any) $r > 0$,

$$\sup_{\xi \in \mathbb{C}} \widehat{\mu}_{kp,r}(\xi) < \infty, \tag{1.1}$$

as shown by Hu-Li-Qu [5]. A natural question is whether reverse (p,k)-FC measures can be characterized by replacing the conditions (1.1) with the analogous “reverse” conditions

$$\inf_{\xi \in \mathbb{C}} \widehat{\mu}_{kp,r}(\xi) > 0.$$

Indeed, the case of $k = 0$ has been considered in the context of several analytic function spaces, such as the Hardy, Bergman, Fock, de Branges-Rovnyak, certain harmonically weighted Dirichlet, and Paley-Wiener spaces. However, the results obtained to date are limited to sufficient conditions; see [4, 13]. For the case $k > 0$, the investigation of reverse Carleson inequalities for Bergman function derivatives was pioneered by Luecking [8, Theorem 4.5], who employed Taylor expansions with high-order remainder terms to establish sufficient conditions. This methodology was subsequently extended to invariant-weighted Bergman spaces in [10], where analogous sufficient criteria were developed through similar approximation techniques.

Despite these advancements, as of this writing, a complete answer has not been established for measures satisfying such reverse Carleson inequalities in both Fock and Bergman space settings.

Let us now focus on the setting of Fock spaces. However, for a measure μ of the form $d\mu = \chi_G dA$, where $G \subset \mathbb{C}$ is measurable and χ_G is defined by $\chi_G(z) = 1$ if $z \in G$ and 0 otherwise, the following characterization of it being a reverse $(p,0)$-FC measure was given by Wang and Zhao [11] as follows.

Theorem 1.1. *For a measurable set $G \subset \mathbb{C}$, the condition that there exist constants $\sigma, \gamma > 0$ with*

$$|G \cap D(z,\gamma)| \ge \sigma |D(z,\gamma)| \quad \textit{for all } z \in \mathbb{C}$$

is equivalent to the measure $\chi_G(z)\, dA(z)$ being a reverse $(p,0)$-FC measure for F^p_α.

This result has also been extended to some weighted Fock functions F^p_ϕ, see [1] for example. In this paper, we present a complete characterization of a (p,k)-FC measure as reverse (p,k)-FC measure for F^p_α as follows.

Theorem 1.2. *For parameters* $0 < p < \infty$, $k \in \mathbb{N}$, $R > 0$, *and a* (p,k)*-FC measure* μ *for* F^p_α, *the following statements are equivalent:*

(i) The measure μ *is a reverse* (p,k)*-FC measure for* F^p_α.

(ii) There exists a constant $C_R > 0$ *such that the inequality*

$$\|f\|^p_{p,\alpha} \le C_R \int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p \widehat{\mu}_r(\xi) dA(\xi)$$

holds uniformly for all $0 < r < R$ *and all* $f \in F^p_\alpha$ *satisfying* $f^{(j)}(0) = 0$ *for* $0 \le j \le k-1$.

(iii) There exist $\varepsilon, \kappa > 0$ *(depending only on R) such that for all* $0 < r < R$ *and all* $f \in F^p_\alpha$ *satisfying* $f^{(j)}(0) = 0$ *for* $0 \le j \le k-1$,

$$\int_{\Omega} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p \widehat{\mu}_r(\xi) dA(\xi) \ge \kappa \|f\|^p_{p,\alpha},$$

where $\Omega = \{\xi \in \mathbb{C} : \widehat{\mu}_{pk,r}(\xi) > \varepsilon\}$.

This result is an extension of Theorem 1.1 from a weighted measure $\chi_G(z)\, dA(z)$ to a much more general case. In contrast to $\chi_G\, dA$, the condition that

$$\inf_{z\in\mathbb{C}} \widehat{\mu}_{pk,r}(z) > 0$$

does not, in general, guarantee that a measure μ to be a reverse (p,k)-FC measure. In fact, a measure μ is a reverse (p,k)-FC measure only if the family of measures $\widehat{\mu}_r dA$ satisfies reverse (p,k)-FC inequalities with a uniform constant C for all sufficiently small r, as ensured by Theorem 1.2.

Throughout this paper, we adopt the following conventions for mathematical notation: α and p are the fixed weight parameters. The symbols c and C represent finite positive constants whose values may differ between instances. The subscripted forms c_a or C_a indicate constants depending exclusively on the parameter a. We say that $V(z) \simeq U(z)$ if there exist positive constants $c, C > 0$ satisfying that $cU(z) \le V(z) \le CU(z)$ holds for all $z \in \mathbb{C}$.

2. Preliminaries

To prepare for the proof of our main result, we first recall some essential properties of Fock spaces. We begin with a fundamental estimate for integral averages of Fock functions, as found in [12, Lemma 2.32].

Lemma 2.1. *Let* $0 < p < \infty$ *and* $0 < r < R$. *Then there exists a constant* $C_R > 0$ *with the property that for all* $f \in H(\mathbb{C})$ *and all* $\varsigma \in \mathbb{C}$, *the following inequality holds:*

$$\left| f(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right|^p \le \frac{C_R}{|D(\varsigma, r)|} \int_{D(\varsigma,r)} \left| f(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p dA(\xi).$$

Based on the lemma above, Hu-Li-Qu [5, Lemma 2.3] gives the higher order derivative version as follows.

Lemma 2.2. *Let* $0 < p < \infty$, $0 < r < R$ *and* $k \in \mathbb{N}$. *Then there exists a constant* $C_R > 0$ *with the property that for all* $f \in H(\mathbb{C})$ *and all* $\varsigma \in \mathbb{C}$, *the following inequality holds:*

$$\left|f^{(k)}(\varsigma)e^{-\frac{\alpha|\varsigma|^2}{2}}\right|^p \le \frac{C_R(1+|\varsigma|)^{kp}}{|D(\varsigma,r)|}\int_{D(\varsigma,r)}\left|f(\xi)e^{-\frac{\alpha|\xi|^2}{2}}\right|^p\,dA(\xi).$$

With the help of Fubini's theorem, Lemma 2.1 easily yields the following result.

Lemma 2.3. *Let* $0 < p < \infty$ *and* $0 < r < R$. *For a positive Borel measure* μ *on* $\mathbb{C}$, *then there exists a constant* $C_R > 0$ *with the property that for all* $f \in H(\mathbb{C})$, *the following inequality holds:*

$$\int_{\mathbb{C}}|f(\xi)|^p\,e^{-\frac{p\alpha|\xi|^2}{2}}\,d\mu(\xi) \le C_R\int_{\mathbb{C}}|f(\xi)|^p\,e^{-\frac{p\alpha|\xi|^2}{2}}\widehat{\mu}_r(\xi)\,dA(\xi).$$

Proof. For any $f \in H(\mathbb{C})$, Lemma 2.1 implies

$$\left|f(\varsigma)e^{-\frac{\alpha}{2}|\varsigma|^2}\right|^p \le \frac{C_R}{r^2}\int_{D(\varsigma,r)}\left|f(\xi)e^{-\frac{\alpha}{2}|\xi|^2}\right|^p\,dA(\xi).$$

Together with Fubini's theorem, we see that

$$\begin{aligned}\int_{\mathbb{C}}\left|f(\varsigma)e^{-\frac{\alpha|\varsigma|^2}{2}}\right|^p\,d\mu(\varsigma) &\le \frac{C_R}{r^2}\int_{D(\varsigma,r)}\left|f(\xi)e^{-\frac{\alpha}{2}|\xi|^2}\right|^p\,dA(\xi)\,d\mu(\varsigma)\\ &= \frac{C_R}{r^2}\int_{\mathbb{C}}\left|f(\xi)e^{-\frac{\alpha|\xi|^2}{2}}\right|^p\,dA(\xi)\int_{\mathbb{C}}\chi_{D(\xi,r)}(\varsigma)\,d\mu(\varsigma)\\ &= C_R\int_{\mathbb{C}}\left|f(\xi)e^{-\frac{\alpha|\xi|^2}{2}}\right|^p\widehat{\mu}_r(\xi)\,dA(\xi).\end{aligned}$$

□

The following estimate will be needed for our main result; see Lemma 4.6 of [12] for details.

Lemma 2.4. *The following estimate holds for all* $\xi \in \mathbb{C}$ *and* $\varsigma \in D(\xi, r)$:

$$\left|\left|f(\xi)e^{-\frac{\alpha|\xi|^2}{2}}\right| - \left|f(\varsigma)e^{-\frac{\alpha|\varsigma|^2}{2}}\right|\right|^p \le C_r|\xi-\varsigma|^p\int_{D(\xi,3r)}\left|f(\omega)e^{-\frac{\alpha|\omega|^2}{2}}\right|^p\,dA(\omega),$$

provided that $0 < p < \infty$, $f \in H(\mathbb{C})$, $r > 0$, *and* $C_r > 0$ *is a constant depending on* r.

In addition to the pointwise estimate above, we require a characterization of Fock spaces in terms of higher-order derivatives. Another fundamental tool is the following result, which can be found in [2].

Lemma 2.5. *For* $0 < p < \infty$ *and* $k \in \mathbb{N}$. *Then* $f \in F^p_\alpha$ *if and only if*

$$\frac{f^{(k)}(\xi)e^{-\frac{\alpha|\xi|^2}{2}}}{(1+|\xi|)^k} \in L^p(\mathbb{C}, dA).$$

Moreover, we have the norm equivalence

$$\|f\|^p_{p,\alpha} \simeq \sum_{n=0}^{k-1}|f^{(n)}(0)|^p + \int_{\mathbb{C}}\frac{\left|f^{(k)}(\xi)e^{-\frac{\alpha|\xi|^2}{2}}\right|^p}{(1+|\xi|)^{pk}}dA(\xi).$$

3. Proof of Theorem 1.2

To proceed with the proof, we first fix a positive parameter r and introduce the associated square lattice

$$r\mathbb{Z}^2 = \{rm + irn : m, n \in \mathbb{Z}\}.$$

This lattice will be used in the following argument.

Proof of Theorem 1.2. We recall a fundamental result from [5, Theorem 3.1]: For any μ (p,k)-FC measure on F^p_α,

$$\|\widehat{\mu}_{kp,r}\|_\infty < \infty \tag{3.1}$$

for some (or any) $r > 0$. That (i) implies (ii) is a direct consequence of Lemma 2.3.

(ii) ⇒ (iii). Assume condition (ii) holds: there exists a constant $C_R > 0$, uniform over $0 < r < R$, such that

$$C_R \int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p \widehat{\mu}_r(\xi)\, dA(\xi) \geq \|f\|^p_{p,\alpha}. \tag{3.2}$$

An application of Lemma 2.5 provides a constant $C > 0$, which is independent of r and R, such that for every function f satisfying $f^{(j)}(0) = 0$ for $0 \leq j \leq k-1$, the following holds.

$$C \int_{\mathbb{C}} \frac{\left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p}{(1+|\xi|)^{pk}}\, dA(\xi) \leq \|f\|^p_{p,\alpha}.$$

For any $a \in \mathbb{C}$, there exists $\{\lambda_i\}_{i=1}^N \subset r\mathbb{Z}^2$ such that $D(a,R) \subset \cup_{i=1}^N D(\lambda_i, r)$, where $N \simeq \frac{R^2}{r^2}$ counts r-lattice points in R-disks, which is the basic facts about $r\mathbb{Z}^2$ in $\mathbb{C}$. Therefore

$$\begin{aligned}(1+|a|)^{pk}\widehat{\mu}_R(a) &= \frac{(1+|a|)^{pk}\int_{D(a,R)} d\mu}{\pi R^2} \leq \frac{(1+|a|)^{pk}\sum_{i=1}^N \int_{D(\lambda_i,r)} d\mu}{\pi R^2} \\ &\leq c_R \frac{r^2 N \|\widehat{\mu}_{kp,r}\|_\infty}{R^2} \leq c_R \|\widehat{\mu}_{kp,r}\|_\infty,\end{aligned}$$

and we obtain that $\|\widehat{\mu}_{kp,R}\|_\infty \leq c_R \|\widehat{\mu}_{kp,r}\|_\infty$ with some $c_R > 0$.

Without loss of generality, we henceforth assume that the C_R appearing in inequality (3.2) additionally satisfies the lower bound

$$C_R \geq \frac{c_R C}{\|\widehat{\mu}_{kp,R}\|_\infty},$$

thus $\frac{C}{2C_R} \leq \frac{\|\widehat{\mu}_{kp,r}\|_\infty}{2}$. Define the exceptional set $M = \{z \in \mathbb{C} : \widehat{\mu}_{kp,r}(z) > \frac{C}{2C_R}\}$. Through strategic splitting of the integral:

$$\begin{aligned}\|f\|^p_{p,\alpha} &\leq C_R \int_M \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p \widehat{\mu}_r(\xi)\, dA(\xi) + C_R \int_{\mathbb{C}\setminus M} \frac{C\left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p}{2C_R(1+|\xi|)^{pk}}\, dA(\xi) \\ &\leq C_R \int_M \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p \widehat{\mu}_r(\xi)\, dA(\xi) + \frac{1}{2}\|f\|^p_{p,\alpha}.\end{aligned}$$

we derive the critical estimate:

$$\int_M \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p \widehat{\mu}_r(\xi)\, dA(\xi) \geq \frac{1}{2C_R} \|f\|_{p,\alpha}^p.$$

(iii) ⇒ (i). Assume condition (iii) holds with parameters $R > 6r$, $\varepsilon > 0$, and $\kappa > 0$. We begin by applying Lemmas 2.4 and 2.2 to deduce the existence of a $C_R > 0$ such that for all $\varsigma \in D(\xi, r)$ and $\xi \in \mathbb{C}$, the following inequality holds:

$$\begin{aligned}
&\left\| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right| - \left| f^{(k)}(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right\|^p \\
&\leq r^p C_R \int_{D(\xi,\frac{R}{2})} \left| f^{(k)}(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right|^p dA(\varsigma) \\
&\leq r^p C_R \int_{D(\xi,R)} \left| f(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right|^p (1+|\varsigma|)^{pk} dA(\varsigma).
\end{aligned}$$

After multiplying both sides by $\chi_{D(\xi,r)}(\varsigma)$ and integrating over $\varsigma \in \mathbb{C}$, we have

$$\begin{aligned}
\int_{\mathbb{C}} \frac{\chi_{D(\xi,r)}(\varsigma)}{|D(\xi,r)|} &\left\| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right| - \left| f^{(k)}(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right\|^p dA(\varsigma) \\
&\leq r^p C_R \int_{D(\xi,R)} \left| f(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right|^p (1+|\varsigma|)^{pk} dA(\varsigma).
\end{aligned}$$

We then integrate both sides over $\xi \in \mathbb{C}$ with respect to $d\mu(\xi)$ and apply Fubini's theorem, which gives

$$\begin{aligned}
\int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\chi_{D(\xi,r)}(\varsigma)}{|D(\xi,r)|} &\left\| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right| - \left| f^{(k)}(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right\|^p dA(\varsigma) d\mu(\xi) \\
&\leq r^p C_R \int_{\mathbb{C}} \left| f(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right|^p (1+|\varsigma|)^{pk} \widehat{\mu}_R(\varsigma) dA(\varsigma) \\
&\leq r^p C_R \|f\|_{p,\alpha}^p. \qquad (3.3)
\end{aligned}$$

The final bound is a consequence of (3.1).

When $0 < p < 1$, we apply the reverse triangle inequality together with condition (iii) to obtain

$$\begin{aligned}
&\kappa \|f\|_{p,\alpha}^p - \int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p d\mu(\xi) \\
\leq\ & \int_{\mathbb{C}} \left| f^{(k)}(\varsigma) e^{-\frac{\alpha|\varsigma|^2}{2}} \right|^p \widehat{\mu}_r(\varsigma)\, dA(\varsigma) - \int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p d\mu(\xi) \\
\leq\ & r^p C_R \|f\|_{p,\alpha}^p,
\end{aligned}$$

where κ depends only on R. Therefore,

$$(\kappa - C_R r^p) \|f\|_{p,\alpha}^p \leq \int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p d\mu(\xi).$$

For $p \geq 1$, we proceed analogously. An application of Minkowski's inequality to the left-hand side after raising (3.3) to the power $1/p$ leads to

$$(\kappa^{1/p} - C_R^{1/p} r) \|f\|_{p,\alpha} \leq \left(\int_{\mathbb{C}} \left| f^{(k)}(\xi) e^{-\frac{\alpha|\xi|^2}{2}} \right|^p d\mu(\xi) \right)^{1/p}.$$

Bearing in mind the fact that C_R and κ are irrelevant to r. For sufficiently small r satisfying $\kappa > C_R r^p$, we obtain that μ is a reverse (p, k)-FC for F^p_α, completing the proof.

□

4. Conclusions

Based on the introduction, the paper presents a complete characterization of reverse (p, k)-Fock–Carleson measures for Fock spaces F^p_α (Theorem 1.2). It shows that a (p, k)-FC measure is a reverse (p, k)-FC measure if and only if it satisfies a uniform reverse inequality involving the averaging transform $\widehat{\mu}_r$, or equivalently, a localization condition on the set where $\widehat{\mu}_{pk,r}$ exceeds a positive threshold. This generalizes previous sufficient conditions, covers arbitrary derivative order $k \geq 1$, and handles general measures beyond characteristic functions, revealing that the simple condition $\inf_\xi \widehat{\mu}_{pk,r}(\xi) > 0$ is not sufficient.

Author contributions

Zhengyuan Zhuo: Writing-original draft, investigation, formal analysis; Feifei Zhao: Writing-review and editing, formal analysis, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence(AI) tools in the creation of this article.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (Grant No. 12301093), Guangdong Key Discipline Construction and Research Capacity Enhancement Project(Grant No. 2024ZDJS023), Guangdong Polytechnic Normal University Science Foundation (Grant No. 2026SD-KYA083) and Hebei Natural Science Foundation (Grant No. A2022409005).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. G. Cao, L. He, Y. Zhang, Reverse Carleson measures on generalized Fock spaces, *Acta Math. Sci. Ser. B (Engl. Ed.)*, **43** (2023), 655–667. https://doi.org/10.1007/s10473-023-0211-7
2. O. Constantin, A Volterra-type integration operator on Fock spaces, *Proc. Amer. Math. Soc.*, **140** (2012), 4247–4257. https://doi.org/10.1090/S0002-9939-2012-11541-2
3. H. Chen, S. Ye, A generalized Hilbert operator acting on Hardy spaces, *Acta Math. Sci. Ser. B (Engl. Ed.)*, **46** (2026), 145–163. https://doi.org/10.1007/s10473-026-0110-9

4. E. Fricain, A. Hartmann, W. Ross, A survey on reverse Carleson measures, *Theta Ser. Adv. Math.*, **19** (2017), 91–123.
5. J. Hu, X. Li, D. Qu, Embedding derivatives of Fock spaces and generalized weighted composition operators, *J. Nonlinear Var. Anal.*, **5** (2021), 589–613. http://dx.doi.org/10.23952/jnva.5.2021.4.07
6. H. Hu, S. Ye, Norm of the Hilbert matrix operator between some spaces of analytic functions, *J. Geom. Anal.*, **35** (2025), Paper No. 184, 21 pp. https://doi.org/10.1007/s12220-025-02023-2
7. J. Isralowitz, K. Zhu, Toeplitz operators on the Fock space, *Integr. Equat. Operat. Theor.*, **66** (2010), 593–611. http://dx.doi.org/10.1007/s00020-010-1768-9
8. D. Luecking, Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, *Amer. J. Math.*, **107** (1985), 85–111. http://dx.doi.org/10.2307/2374458
9. T. Mengestie, Volterra type and weighted composition operators on weighted Fock spaces, *Integr. Equat. Oper. Th.*, **76** (2013), 81–94. http://dx.doi.org/10.1007/s00020-013-2050-8
10. C. Tong, J. Li, H. He, H. Arroussi, Reverse Carleson measures on the weighted Bergman spaces with invariant weight, *Ann. Funct. Anal.*, **12** (2021), Paper No. 56, 28 pp. http://dx.doi.org/10.1007/s43034-021-00139-4
11. Z. Wang, X. Zhao, Invertibility of Fock Toeplitz operators with positive symbols, *J. Math. Anal. Appl.*, **435** (2015), 1335–1351. http://dx.doi.org/10.1016/j.jmaa.2015.11.020
12. K. Zhu, *Analysis on Fock Spaces*, New York: Springer, 2012. http://dx.doi.org/10.1007/978-1-4419-8801-0
13. Z. Zhuo, Z. Lou, Sampling measure on doubling Fock spaces, *J. Geom. Anal.*, **33** (2023), Paper No. 313. http://dx.doi.org/10.1007/s12220-023-01380-0