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*Research article*

# On generalized quaternion Sylow 2-subgroups and 2-nilpotence of finite groups

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**Abstract:** In 2020, Mousavi proved that a finite group $G$ with a generalized quaternion Sylow 2-subgroup $S$ is 2-nilpotent if $3 \nmid |G|$ or if $G$ is solvable and $|S| > 16$. In this note, we generalized the result of Mousavi and provided a simpler proof. In detail, we showed that a finite group with a generalized quaternion Sylow 2-subgroup is 2-nilpotent if, and only if, it is $SL_2(3)$-free and that, for a finite group with a generalized quaterion Sylow 2-subgroup of order strictly greater than 16, the properties of being 2-nilpotent, solvable, and 2-constrained are equivalent.

**Keywords:** finite group; $p$-nilpotent; Sylow 2-subgroup; quaternion group; $p$-constrained
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## 1. Introduction

Let $G$ be a finite group, $p$ a prime, and $P$ a Sylow $p$-subgroup of $G$. Then, $G$ is called *p-nilpotent* if $G = N \rtimes P$, where $N$ is a normal subgroup of $G$. This concept plays an important role in finite group theory, and the search for $p$-nilpotency criteria is a traditional but still very active area of research.

A fruitful approach in obtaining new $p$-nilpotency criteria is to consider a finite group $G$ and assume that some subgroups of a Sylow $p$-subgroup of $G$ satisfy a certain group-theoretic property or a certain embedding property with respect to $G$. For example, under the assumption that $N_G(P)$ is $p$-nilpotent, Li et al. [13] gave criteria of this kind for the $p$-nilpotency of $G$; their results provide generalizations of the famous Burnside $p$-nilpotency criterion (see [12, Theorem 7.2.1]). In [15], the authors introduced a certain commutator condition on a $p$-subgroup of $G$, and they used it to establish a necessary and sufficient criterion for the $p$-nilpotency of $G$. Ballester-Bolinches et al. [2] studied the structure of a minimal counterexample among the non-$p$-nilpotent finite groups with $p$-nilpotent $p$-Sylow normalizers and used this to deduce several $p$-nilpotency criteria of the above kind. Building on this line of research, Asaad et al. [1] established a characterization of $p$-nilpotency in finite groups in terms of weakly $\mathcal{H}$-embedded subgroups. They showed that if $p$ is the smallest prime divisor of $|G|$ and $P$ is a non-cyclic Sylow $p$-subgroup of $G$, then $G$ is $p$-nilpotent if, and only if, there is a $p$-power $d$ with $1 < d < |P|$ such that all subgroups of $P$ of orders $d$ and $pd$ are weakly $\mathcal{H}$-embedded in $G$. Recall that a subgroup $S \leq G$ is a *p-sylowizer* of a $p$-subgroup $R$ in $G$ if $R \in \mathrm{Syl}_p(S)$ and $S$ is maximal with respect to the this property, i.e., if $S \leq T \leq G$ and $R \in \mathrm{Syl}_p(T)$, then $S = T$ (see [6]). Continuing the study of previous $p$-nilpotency criteria in finite groups, Y. Gao et al. [4] obtained new characterizations of $p$-nilpotent finite groups by investigating intersections between $p$-sylowizers of certain normal subgroups of a Sylow $p$-subgroup of $G$ with $O^p(G)$ (see [4, Theorems 3.1 and 3.4]). The $p$-nilpotency of finite groups has also been studied via permutability conditions on subgroups of Sylow $p$-subgroups. In this direction, [8] obtained new criteria ensuring that $G$ is $p$-nilpotent by assuming suitable conditions on certain subgroups associated with a Sylow $p$-subgroup.

In 2020, Mousavi [14] proved some 2-nilpotency criteria for finite groups with generalized quaternion Sylow 2-subgroups. For convenience, let us recall the definition of the generalized quaternion 2-groups.

**Definition 1.1.** *Let $n \geq 3$ be a natural number and*

$$Q_{2^n} := \langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1} \rangle.$$

We call $Q_{2^n}$ the generalized quaternion group of order $2^n$.

Note that the generalized quaternion groups, in the sense of the above definition, include the ordinary quaternion group of order 8. Finite groups with generalized quaternion Sylow 2-subgroups have been the object of very relevant research. In particular, Brauer and Suzuki proved a highly influential result on this class of groups, stating that a finite group $G$ has a center of order 2 if it has generalized quaternion Sylow 2-subgroups and no nontrivial normal subgroups of odd order (see [10, §45]). Mousavi [14] proved the following interesting result.

**Theorem 1.2.** *Let $G$ be a finite group with a generalized quaternion Sylow 2-subgroup. Then, $G$ is 2-nilpotent if either:*

1) $G$ is a $3'$-group;

2) or $G$ is solvable with a Sylow 2-subgroup of order strictly greater than 16.

The proof of Theorem 1.2 uses some deep results, such as that all non-abelian finite simple $3'$-groups are Suzuki groups. The aim of this note is to provide a simpler proof of Theorem 1.2, using only quite elementary results from standard textbooks.

We will, in fact, establish more general results, with Theorem 1.2 appearing as a corollary. To present our results, we begin by recalling some definitions. Let $G$ and $H$ be finite groups. If $K \leq G$, $N \trianglelefteq K$, then $K/N$ is said to be a *section* of $G$. We say that $H$ is *involved* in $G$ if $G$ has a section isomorphic to $H$. We say that $G$ is *H-free* if $H$ is not involved in $G$. If $p$ is a prime number, we say that $G$ is *of characteristic p* if $C_G(O_p(G)) \leq O_p(G)$. Furthermore, $G$ is said to be *p-constrained* if the quotient $G/O_{p'}(G)$ is of characteristic $p$.

As usual, we use $\mathrm{SL}_2(3)$ to denote the special linear group of degree 2 over the field with 3 elements. We will prove the following results.

**Theorem 1.3.** *Let G be a finite group with a generalized quaternion Sylow* 2*-subgroup. Then, the following statements are equivalent:*

1) $G$ is 2-nilpotent;

2) $G$ is $SL_2(3)$-free.

**Theorem 1.4.** *Let G be a finite group with a generalized quaternion Sylow* 2*-subgroup S such that* $|S| > 16$*. Then, the following statements are equivalent:*

1) $G$ is 2-nilpotent;

2) $G$ is solvable;

3) $G$ is 2-constrained.

Since any finite $3'$-group is $SL_2(3)$-free, Theorem 1.3 implies Theorem 1.2 (1). Obviously, Theorem 1.2 (2) is covered by Theorem 1.4.

We finish the introduction with two examples illustrating our results.

**Example 1.5.** *Let* $q \geq 5$ *be an odd prime power. Then, the special linear group* $SL_2(q)$ *has generalized quaternion Sylow* 2*-subgroups (see [9, Kapitel II, Satz 8.10 a)]). Since the projective special linear group* $PSL_2(q)$ *is non-abelian simple,* $SL_2(q)$ *is certainly not* 2*-nilpotent. So Theorem 1.3 shows that* $SL_2(q)$ *has a section isomorphic to* $SL_2(3)$*. This is in accordance with a result of Dickson (see [7, Chapter 2, Theorem 8.4]).*

**Example 1.6.** *Let G be the group indexed in the computer algebra system GAP [5] as SmallGroup(48,28). Then, using GAP [5], one can easily check that G has generalized quaternion Sylow* 2*-subgroups of order* 16 *and that G is not* 2*-nilpotent, but solvable. This shows that the condition* $|S| > 16$ *in Theorem 1.4 cannot be replaced by the condition* $|S| \geq 16$.

## 2. Preliminaries

In this section, we collect some results needed for the proofs of our main theorems.

**Lemma 2.1.** *Let n be a natural number.*

1) ([12, 5.3.3]) $\mathrm{Aut}(Q_8) \cong S_4$.

2) ([3, Proposition 4.53]) If $n \geq 4$, then $\mathrm{Aut}(Q_{2^n})$ is a 2-group.

**Lemma 2.2.** *Let C be a finite cyclic* 2*-group. Then,* $\mathrm{Aut}(C)$ *is a* 2*-group.*

*Proof.* This follows from [12, 2.2.5]. □

**Lemma 2.3.** *Let* $n \geq 3$ *be a natural number and H be a subgroup of the generalized quaternion group* $Q_{2^n}$*. Then, either H is a cyclic* 2*-group, or H is a generalized quaternion group.*

*Proof.* This follows from [7, Chapter 5, Theorem 4.3]. □

Given a normal subgroup $N$ of a finite group $G$, we write $\mathrm{Aut}_G(N)$ for the group of all automorphisms of $N$ induced by conjugation in $G$.

**Lemma 2.4.** *Let $p$ be a prime and $G$ be a finite group of characteristic $p$. If $\mathrm{Aut}_G(O_p(G))$ is a $p$-group, then $G$ is a $p$-group.*

*Proof.* Since $G$ is of characteristic $p$, we have $C_G(O_p(G)) = Z(O_p(G))$. Note that $G/Z(O_p(G)) = G/C_G(O_p(G)) \cong \mathrm{Aut}_G(O_p(G))$. Hence, $G/Z(O_p(G))$ is a $p$-group, and this immediately implies that $G$ is a $p$-group. □

**Proposition 2.5.** *Let $G$ be a finite group with a generalized quaternion Sylow 2-subgroup $S$ such that $G$ is of characteristic 2 and $|S| > 16$. Then, $G$ is a 2-group.*

*Proof.* By Lemma 2.3, $O_2(G)$ is either cyclic or a generalized quaternion group.

Assume $O_2(G) \cong Q_8$. Set $Z := Z(O_2(G)) \cong C_2$, and note that $G/Z = G/C_G(O_2(G)) \cong \mathrm{Aut}_G(O_2(G))$. By Lemma 2.1, $\mathrm{Aut}(O_2(G)) \cong S_4$, and so $G/Z$ is isomorphic to a subgroup of $S_4$. Hence, a 2-subgroup of $G/Z$ cannot be of order greater than 8. Thus, $|S/Z| \leq 8$, and so $|S| \leq 16$, a contradiction.

Consequently, $O_2(G)$ is cyclic or generalized quaternion of order at least 16. Hence, by Lemmas 2.1 and 2.2, $\mathrm{Aut}(O_2(G))$ is a 2-group. Then, Lemma 2.4 yields that $G$ is a 2-group, as required. □

## 3. Proofs of the main results

*Proof of Theorem 1.3.* (1) $\implies$ (2) Assume that $G$ is 2-nilpotent. Since subgroups and quotients of 2-nilpotent finite groups are again 2-nilpotent, we have that any section of $G$ is 2-nilpotent. Hence, since $\mathrm{SL}_2(3)$ is not 2-nilpotent, there is no section of $G$ which is isomorphic to $\mathrm{SL}_2(3)$. Consequently, $G$ is $\mathrm{SL}_2(3)$-free.

(2) $\implies$ (1) Assume that $G$ is $\mathrm{SL}_2(3)$-free. Pick a Sylow 2-subgroup $S$ of $G$, and let $Q \leq S$. By Frobenius' p-nilpotency criterion (see [11, Theorem 5.26]), it is enough to show that $N_G(Q)/C_G(Q)$ is a 2-group. Since $S \cong Q_{2^n}$ for some $n \geq 3$, we either have that $Q$ is a cyclic 2-group, or $Q \cong Q_{2^m}$ for some $3 \leq m \leq n$ (see Lemma 2.3). Then, by Lemmas 2.1 and 2.2, if $Q \ncong Q_8$, $\mathrm{Aut}(Q)$ is a 2-group, which implies that $N_G(Q)/C_G(Q)$ is also a 2-group.

Suppose now that $Q \cong Q_8$. By Lemma 2.1 (1), we have $\mathrm{Aut}(Q) \cong S_4$. Assume that $N_G(Q)/C_G(Q)$ is not a 2-group. Then, since $N_G(Q)/C_G(Q)$ is isomorphic to a subgroup of $S_4$, we have that 3 divides $|N_G(Q)/C_G(Q)|$ . Let $x \in N_G(Q)$ such that $xC_G(Q)$ has order 3 in $N_G(Q)/C_G(Q)$, and let $y \in \langle x \rangle$ such that $\langle y \rangle$ is the Sylow 3-subgroup of $\langle x \rangle$. Note that $\langle xC_G(Q) \rangle = \langle x \rangle C_G(Q)/C_G(Q)$. Hence, $\langle x \rangle/(\langle x \rangle \cap C_G(Q)) \cong \langle xC_G(Q) \rangle$ is of order 3. Since $\langle y \rangle$ is the 3-Sylow subgroup of $\langle x \rangle$, it follows that $\langle x \rangle = \langle y \rangle(\langle x \rangle \cap C_G(Q))$. Consequently, $\langle y \rangle/(\langle y \rangle \cap C_G(Q)) \cong \langle y \rangle(\langle x \rangle \cap C_G(Q))/(\langle x \rangle \cap C_G(Q)) = \langle x \rangle/(\langle x \rangle \cap C_G(Q))$ so that $\langle y \rangle/(\langle y \rangle \cap C_G(Q))$ is of order 3. Now, let $K := Q\langle y \rangle$ and $Z := \langle y \rangle \cap C_G(Q)$. Since $Z$ is normalized by both $Q$ and $\langle y \rangle$, $Z \trianglelefteq K$, and since $Z$ is a 3-group, $Q_8 \cong Q \cong QZ/Z \trianglelefteq K/Z$. Also, $\langle y \rangle/Z \cong C_3$, and $K/Z = QZ/Z \cdot \langle y \rangle/Z$. Thus, $K/Z = QZ/Z \rtimes \langle y \rangle/Z$. Since $1 \neq [Q, \langle y \rangle] \leq Q$ and $Q \cap Z = 1$, we have $[Q, \langle y \rangle] \nleq Z$. Therefore, $QZ/Z$ is not centralized by $\langle y \rangle/Z$, and so $K/Z \neq QZ/Z \times \langle y \rangle/Z$. Using [12, 8.6.10], we can conclude that $K/Z \cong \mathrm{SL}_2(3)$, which contradicts the assumption that $G$ is $\mathrm{SL}_2(3)$-free. It follows that $N_G(Q)/C_G(Q)$ is a 2-group.

We have shown that $N_G(Q)/C_G(Q)$ is a 2-group for each $Q \leq S$, and so $G$ is 2-nilpotent. □

*Proof of Theorem 1.4.* (1) $\implies$ (2) This follows from the odd order theorem.
(2) $\implies$ (3) This is true by [11, Theorem 3.21].
(3) $\implies$ (1) Assume that $G$ is 2-constrained. Set $\overline{G} := G/O_{2'}(G)$. Then, $\overline{S} := SO_{2'}(G)/O_{2'}(G)$ is a Sylow 2-subgroup of $\overline{G}$. Since $\overline{S} \cong S$, $\overline{S}$ is a generalized quaternion group with $\left|\overline{S}\right| = |S| > 16$. Since $G$ is 2-constrained, we have that $\overline{G}$ of characteristic 2. Applying Proposition 2.5, we conclude that $\overline{G}$ is a 2-group. This implies that $G$ is 2-nilpotent, and so the implication (3) $\implies$ (1) is true. □

## 4. Conclusions

In this note, we established two main results: First, a finite group with a generalized quaternion Sylow 2-subgroup is 2-nilpotent if, and only if, it is $SL_2(3)$-free. Second, for a finite group with a generalized quaternion Sylow 2-subgroup of order strictly greater than 16, the properties of being solvable, 2-nilpotent, and 2-constrained are equivalent. Our work generalizes a result of Mousavi stating that a finite group $G$ with a generalized quaternion Sylow 2-subgroup $S$ is 2-nilpotent if $G$ is a $3'$-group or if $G$ is solvable and $|S| > 16$. Our proofs are short and basic, thus simplifying the proof of Mousavi's result.

## Author contributions

The authors contributed equally to this work. All authors have read and aproved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.


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## Conflict of interest

The authors declare no conflict of interest in this paper.

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