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Research article

Block LS–poset codes over $\mathbb{Z}_m^n$: Perfect codes, Singleton bound, and MDS characterization

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Abstract: We introduce a block LS–poset (partially ordered set) metric on $\mathbb{Z}_m^n$ constructed from a block decomposition of $\mathbb{Z}_m^n$, a poset structure on the block indices, and the lattice of subgroups of $\mathbb{Z}_m$ arising from the prime factorization of m. Using a multiset representation associated with this subgroup lattice, we define the block LS–poset weight and show that the induced distance is a metric on $\mathbb{Z}_m^n$. We investigate the geometry of r-balls and $\mathcal{I}$-balls and establish their fundamental properties, including linearity, translation invariance, and duality. These structural results lead to characterizations of $\mathcal{I}$-perfect block LS–poset codes for ideals with full count and partial count. We further derive a Singleton-type bound for block LS–poset codes and introduce the notions of maximum distance separable (MDS) and partial-MDS block LS–poset codes. Connections among perfect codes, MDS codes, and r-perfect codes are also examined for certain classes of posets.

Keywords: block LS–poset metric; poset block code; perfect code; r-perfect code; $\mathcal{I}$-perfect code; MDS code; partial-MDS code; Singleton bound; subgroup lattice; multiset
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1. Introduction

Coding theory over finite rings and modules has developed into an active area of research, motivated both by its intrinsic algebraic structure and by applications to error detection and correction. The classical Hamming metric, introduced by Hamming [6], provides the foundational framework for measuring the number of corrupted coordinates in a codeword. Later, Lee [10] introduced the Lee metric, which is particularly suitable for codes over residue class rings and has led to an extensive theory of perfect codes and related packing problems; see also the work of Golomb and Welch [5].

Beyond these classical metrics, partially ordered set (poset) metrics were introduced by Brualdi, Graves, and Lawrence [2] as a generalization of the Hamming metric in which the support of a vector is governed by an underlying partial order. Later, Ahn et al. [1] classified perfect linear codes with crown poset structure, while Kim and Krotov [9] characterized poset metrics admitting binary perfect codes of codimension m. D'Oliveira and Firer [4] investigated packing radii for poset metrics, and Hyun and Kim [7] studied maximum distance separable (MDS) poset codes. Further structural aspects of poset metrics and their linear isometries were investigated in [13, 14].

A further extension of this direction replaces ordinary posets by more general ordered structures adapted to metric constructions over finite alphabets. In this context, partially ordered multiset (pomset) metrics and their coding-theoretic properties have been investigated in [17,18]. More recently, block versions of these metrics have received increasing attention, as in many coding models, the coordinates are naturally grouped into blocks rather than treated individually. Such block structures are especially relevant when different groups of coordinates represent correlated information or are subjected to different error mechanisms. Block codes in pomset metrics were studied in [12, 15], and weighted poset block metrics and related Singleton-type bounds were investigated in [11, 16]. In the setting of poset block codes, the work of Dass, Sharma, and Verma [3] established important connections between MDS codes and I-perfect codes.

Parallel to these developments, the subgroup structure of $\mathbb{Z}_m$ provides another natural source of metric information. Because the lattice of subgroups of a cyclic group is isomorphic to the divisor lattice of m, it is possible to encode algebraic information about coordinates through subgroup data rather than merely by their zero or nonzero status. This perspective was recently explored in [8], where weighted poset metrics based on the lattice of subgroups of $\mathbb{Z}_m$ were introduced. That construction shows that the arithmetic of m and the combinatorics of the divisor lattice can be combined effectively to produce new metric structures and new classes of perfect codes.

The aim of the present paper is to develop a block version of this lattice-subgroup approach by combining three ingredients: a poset on the block index set, a block decomposition of $\mathbb{Z}_m^n$, and the multiset structure arising from the prime factorization of m. This leads to the notion of a block LS–poset metric, which simultaneously generalizes subgroup-based weighted metrics and block poset-type metrics. The resulting distance reflects not only which blocks are nonzero, but also the subgroup complexity of the entries appearing inside each block. Consequently, the associated balls and packing conditions exhibit features that do not occur in the classical Hamming or ordinary poset settings.

The main contributions of this paper are as follows. First, we introduce the block LS–poset weight and prove that the induced distance on $\mathbb{Z}_m^n$ is a metric. Next, we study the structure of r-balls and $\mathcal{I}$-balls, obtaining explicit formulas and algebraic descriptions that depend on the corresponding ideals in the product poset $P \times \mathscr{P}(M)$ equipped with the partial order $\preccurlyeq_P$ defined below. We then characterize $\mathcal{I}$-perfect block LS–poset codes in both the full-count and partial-count cases, and establish a Singleton-type bound for block LS–poset codes. This bound motivates the notions of MDS and partial-MDS block LS–poset codes, and we show how these notions are related to perfectness with respect to suitable ideals. Finally, for certain special posets such as chains and star posets, we derive consequences for r-perfect codes and describe situations in which MDS, $\mathcal{I}$-perfect, and r-perfect notions coincide.

2. Preliminaries

2.1. Multisets and the power set of a multiset

To formulate the theory precisely, we begin with the multiset language associated with the prime decomposition of m, which provides a convenient combinatorial model for the lattice of subgroups of $\mathbb{Z}_m$. The notion of a multiset used here follows the standard definition and basic operations described in [17].

Let X be a finite nonempty set. A multiset (in short, mset) M over X is a pair $M = (X, \mathfrak{c}_M)$, where $\mathfrak{c}_M : X \to \mathbb{N}_0$ is a function, called the multiplicity function, which assigns to each $x \in X$ a non-negative integer $\mathfrak{c}_M(x)$. The value $\mathfrak{c}_M(x)$ represents the number of times x appears in the multiset M.

An mset M, drawn from $X = \{x_1, \ldots, x_t\}$, is commonly written as $M = \{k_1/x_1, k_2/x_2, \ldots, k_t/x_t\}$, or equivalently, in compact form [8],

$$M = x_1^{[k_1]} x_2^{[k_2]} \cdots x_t^{[k_t]},$$

where $\mathfrak{c}_M(x_i) = k_i > 0$ for $i = 1, \ldots, t$. If $k/a \in M$, then $r/a \in M$ for all $1 \le r \le k$. The cardinality of an mset $M = (X, \mathfrak{c}_M)$ is defined as $|M| = \sum_{x \in X} \mathfrak{c}_M(x)$. The root set of M is defined by $M^* := \{x \in X \ : \ \mathfrak{c}_M(x) > 0\}$.

Given two msets M_1 and M_2, we say that M_1 is a submultiset (or submset) of M_2, written as $M_1 \Subset M_2$, if $M_1^* \subseteq M_2^*$ and $\mathfrak{c}_{M_1}(x) \le \mathfrak{c}_{M_2}(x)$ for all $x \in M_1^*$. For example, $b^{[2]} = a^{[0]} b^{[2]} \Subset a^{[1]} b^{[2]}$. Here, $a^{[0]}$ indicates the absence of a from the root set. The relation $\Subset$ induces a partial order on the collection of multisets.

The power set of the mset $M = x_1^{[k_1]} x_2^{[k_2]} \cdots x_t^{[k_t]}$ is denoted by $\mathscr{P}(M)$, and is defined as

$$\mathscr{P}(M) := \{x_1^{[l_1]} x_2^{[l_2]} \cdots x_t^{[l_t]} \ : \ 0 \le l_i \le k_i \text{ for all } i = 1, \ldots, t\}.$$

Here, the empty multiset $x_1^{[0]} x_2^{[0]} \cdots x_t^{[0]}$ is denoted by $\emptyset_M$. Then, $(\mathscr{P}(M), \Subset)$ is a poset.

Recall some operations in $\mathscr{P}(M)$, where $M = x_1^{[k_1]} x_2^{[k_2]} \cdots x_t^{[k_t]}$. Let $\alpha_1, \alpha_2 \in \mathscr{P}(M)$, in which $\alpha_1 = x_1^{[l_1]} x_2^{[l_2]} \cdots x_t^{[l_t]}$, and $\alpha_2 = x_1^{[h_1]} x_2^{[h_2]} \cdots x_t^{[h_t]}$. Then,

$$\begin{array}{lll}
\text{(Addition)} & \alpha_1 + \alpha_2 = x_1^{[\mathfrak{a}_1(x_1)]} x_2^{[\mathfrak{a}_2(x_2)]} \cdots x_t^{[\mathfrak{a}_t(x_t)]}, & \text{where } \mathfrak{a}_i(x_i) = \min\{k_i, l_i + h_i\}; \\
\text{(Subtraction)} & \alpha_1 - \alpha_2 = x_1^{[\mathfrak{s}_1(x_1)]} x_2^{[\mathfrak{s}_2(x_2)]} \cdots x_t^{[\mathfrak{s}_t(x_t)]}, & \text{where } \mathfrak{s}_i(x_i) = \max\{l_i - h_i, 0\}; \\
\text{(Union)} & \alpha_1 \cup \alpha_2 = x_1^{[\mathfrak{u}_1(x_1)]} x_2^{[\mathfrak{u}_2(x_2)]} \cdots x_t^{[\mathfrak{u}_t(x_t)]}, & \text{where } \mathfrak{u}_i(x_i) = \max\{l_i, h_i\}; \\
\text{(Intersection)} & \alpha_1 \cap \alpha_2 = x_1^{[\mathfrak{i}_1(x_1)]} x_2^{[\mathfrak{i}_2(x_2)]} \cdots x_t^{[\mathfrak{i}_t(x_t)]}, & \text{where } \mathfrak{i}_i(x_i) = \min\{l_i, h_i\},
\end{array}$$

for all $i = 1, \ldots, t$. For each $\alpha \in \mathscr{P}(M)$, the smallest $\beta \in \mathscr{P}(M)$ such that $\alpha + \beta = M$ is called the dual of α, denoted by $\widehat{\alpha}$.

2.2. Ideals in the product poset $P \times \mathscr{P}(M)$

Given a poset $(Q, \preceq)$, a subset I of Q is called an order ideal (or simply an ideal) of Q if it satisfies the following: If $x \in I$ and $y \preceq x$ in Q, then $y \in I$. For $S \subseteq Q$, we denote $\langle S \rangle$ the smallest ideal containing S.

Let $P = ([n], \preceq_P)$ be a poset on the set $[n] := \{1, 2, \ldots, n\}$. We define a partial order $\preccurlyeq_P$ on $[n] \times \mathscr{P}(M)$ by

$$(i, \alpha) \preccurlyeq_P (j, \beta) \quad \Leftrightarrow \quad \begin{cases} i = j \text{ and } \alpha \Subset \beta, \\ i \preceq_P j \text{ where } i \neq j. \end{cases}$$

An ideal $\mathcal{I}$ of the poset $P \times \mathscr{P}_M = ([n] \times \mathscr{P}(M), \preccurlyeq_P)$ is called an ideal with full count if $(i, \alpha) \in \mathcal{I} \Rightarrow (i, M) \in \mathcal{I}$; otherwise, it is called an ideal with partial count. The set of all ideals in $P \times \mathscr{P}_M$ is denoted by $\mathfrak{I}(P \times \mathscr{P}_M)$. For $\mathcal{I} \in \mathfrak{I}(P \times \mathscr{P}_M)$, we let $\mathbb{M}_f(\mathcal{I}) = \{i \in [n] \ : \ (i, M) \in \mathcal{I}\}$, $\mathbb{M}_{f*}(\mathcal{I}) = \{i \in \mathbb{M}_f(\mathcal{I}) \ : \ (i, M) \text{ is maximal}\}$, and $\mathbb{M}_p(\mathcal{I}) = \{i \in [n] \ : \ (i, \alpha) \in \mathcal{I} \text{ but } \alpha \neq \emptyset_M, M\}$.

For $\alpha = x_1^{[h_1]} x_2^{[h_2]} \cdots x_t^{[h_t]} \in \mathscr{P}(M)$, for which $M = x_1^{[k_1]} x_2^{[k_2]} \cdots x_t^{[k_t]}$, we define $\lfloor \cdot \rceil : \mathscr{P}(M) \to \mathbb{N}_0$ by

$$\lfloor \alpha \rceil := \begin{cases} 0, & \text{if } \alpha = \emptyset_M, \\ |\alpha|, & \text{if } h_j = k_j = 1 \text{ for some } j \in \{1, \ldots, t\}, \\ |\alpha| + 1, & \text{otherwise.} \end{cases}$$

For example, $\lfloor x_1^{[1]} x_2^{[3]} x_3^{[1]} \rceil = 6 = \lfloor x_1^{[2]} x_2^{[3]} x_4^{[1]} \rceil$, where the mset $M = x_1^{[3]} x_2^{[3]} x_3^{[2]} x_4^{[1]}$.

We obtain the following result directly from the definition of $\lfloor \cdot \rceil$.

Proposition 2.1. *Let* $M = x_1^{[k_1]} x_2^{[k_2]} \cdots x_t^{[k_t]}$ *and* $\alpha = x_1^{[h_1]} x_2^{[h_2]} \cdots x_t^{[h_t]} \in \mathscr{P}(M)$. *Then, there exists* $\beta \in \mathscr{P}(M)$ *such that* $\beta \Subset \alpha$ *and* $\lfloor \beta \rceil = \lfloor \alpha \rceil - 1$ *except for the case* $\alpha = x_i^{[1]}$, *for which* $k_i > 1$.

Let $\mathfrak{I}_\circ(P \times \mathscr{P}_M)$ denote for the set containing $\mathcal{I} \in \mathfrak{I}(P \times \mathscr{P}_M)$ in which for each $i \in \mathbb{M}_{f*}(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})$, $\exists!\ \alpha_i \in \mathscr{P}(M)$ such that $\lfloor \alpha_i \rceil \geq \lfloor \beta \rceil$ for all $(i, \beta) \in \mathcal{I}$.

The dual poset with respect to P of $P \times \mathscr{P}_M$ is the poset $\widetilde{P} \times \mathscr{P}_M$, where $\widetilde{P}$ is the dual poset of P. For $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$, the complement of $\mathcal{I}$, denoted by $\mathcal{I}^{\mathfrak{C}}$, is the ideal of $\widetilde{P} \times \mathscr{P}_M$, which satisfies: (i) $\mathbb{M}_f(\mathcal{I}^{\mathfrak{C}}) = [n] \backslash (\mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I}))$, (ii) $\mathbb{M}_p(\mathcal{I}^{\mathfrak{C}}) = \mathbb{M}_p(\mathcal{I})$, and (iii) $\forall j \in \mathbb{M}_p(\mathcal{I}^{\mathfrak{C}})$. If (j, α) is the maximal element of $\mathcal{I}$, then $(j, \widehat{\alpha})$ is the maximal element of $\mathcal{I}^{\mathfrak{C}}$.

2.3. Block LS-poset support and weight

Consider $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ and the mset $M = p_1^{[k_1]} p_2^{[k_2]} \cdots p_t^{[k_t]}$, where m is written as $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ with distinct primes p_i and positive integers k_i. Let $\mathcal{L}(\mathbb{Z}_m)$ denote the lattice of subgroups of $\mathbb{Z}_m$. Define a map $\psi : \mathcal{L}(\mathbb{Z}_m) \to \mathscr{P}(M)$ by $\psi(H) = \alpha_{_H} := p_1^{[l_1]} p_2^{[l_2]} \cdots p_t^{[l_t]}$, where $|H| = p_1^{l_1} p_2^{l_2} \cdots p_t^{l_t}$. It is clear that ψ is an order-isomorphism.

Let $\pi : [s] \to \mathbb{N}$ be a label map such that $\sum\limits_{i=1}^{s} \pi(i) = n$. We consider the space $\mathbb{Z}_m^n$ as the direct sum of modules $\mathbb{Z}_m^{\pi(1)}, \mathbb{Z}_m^{\pi(2)}, \ldots, \mathbb{Z}_m^{\pi(s)}$. That is,

$$\mathbb{Z}_m^n = \mathbb{Z}_m^{\pi(1)} \oplus \mathbb{Z}_m^{\pi(2)} \oplus \cdots \oplus \mathbb{Z}_m^{\pi(s)}.$$

Each $\mathbf{v} \in \mathbb{Z}_m^n$ can be written uniquely as $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_s$, where $\mathbf{v}_i = (v_{i_1}, \ldots, v_{i_{\pi(i)}}) \in \mathbb{Z}_m^{\pi(i)}$. For each $i \in [s]$, there exists a unique element of $\{\alpha_{\langle v_{i_1} \rangle}, \ldots, \alpha_{\langle v_{i_{\pi(i)}} \rangle}\}$, denoted by $\alpha_{\mathfrak{m}_i(\mathbf{v})}$, for which $\lfloor \alpha_{\mathfrak{m}_i(\mathbf{v})} \rceil \geq \lfloor \gamma \rceil$ for all $\gamma \in \{\alpha_{\langle v_{i_1} \rangle}, \ldots, \alpha_{\langle v_{i_{\pi(i)}} \rangle}\}$, and $\psi^{-1}(\alpha_{\mathfrak{m}_i(\mathbf{v})})$ has maximal cardinality among elements $\psi^{-1}(\alpha_{\langle v_{i_1} \rangle}), \ldots, \psi^{-1}(\alpha_{\langle v_{i_{\pi(i)}} \rangle})$.

For $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_s \in \mathbb{Z}_m^n$, the $\pi_{_{LS}}$-support of $\mathbf{v}$ is defined as

$$\operatorname{supp}_{LS,\pi}(\mathbf{v}) := \{(i, \alpha_{\mathfrak{m}_i(\mathbf{v})}) \ : \ i \in [s] \text{ and } \alpha_{\mathfrak{m}_i(\mathbf{v})} \neq \emptyset_M\}.$$

The LS-poset block weight of $\mathbf{v} \in \mathbb{Z}_m^n$ is defined as

$$w_{LS,\pi}(\mathbf{v}) = |\mathbb{M}_f(\langle \operatorname{supp}_{LS,\pi}(\mathbf{v}) \rangle)| \cdot \lfloor M \rceil + \sum_{j \in \mathbb{M}_p(\langle \operatorname{supp}_{LS,\pi}(\mathbf{v}) \rangle)} \lfloor \alpha_{\mathfrak{m}_j(\mathbf{v})} \rceil.$$

The LS–poset block distance between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_m^n$ is defined as

$$d_{\pi_{LS}}(\mathbf{u}, \mathbf{v}) = w_{LS,\pi}(\mathbf{u} - \mathbf{v}).$$

In the next theorem, we prove that $d_{\pi_{LS}}$ is a metric on $\mathbb{Z}_m^n$.

3. Block LS–poset metric

Lemma 3.1. *Let* $M = p_1^{[k_1]} p_2^{[k_2]} \cdots p_t^{[k_t]}$. *For* $\alpha, \beta \in \mathscr{P}(M)$, *the following statements hold:*

(1) If $\alpha \Subset \beta$, *then* $\lfloor \alpha \rceil \leq \lfloor \beta \rceil$.

(2) $\lfloor \alpha + \beta \rceil \leq \lfloor \alpha \rceil + \lfloor \beta \rceil$.

Proof. Suppose $\alpha \Subset \beta$. That is, each multiplicity in α is less than or equal the corresponding multiplicity in β. By the definition of $\lfloor \cdot \rceil$, we have $\lfloor \alpha \rceil \leq \lfloor \beta \rceil$.

As the mset sum, let $\alpha + \beta = p_1^{[\mathfrak{a}_1(p_1)]} p_2^{[\mathfrak{a}_2(p_2)]} \cdots p_t^{[\mathfrak{a}_t(p_t)]}$, $\mathfrak{a}_i(p_i) = \min\{k_i, l_i + h_i\}$, where $\alpha = p_1^{[l_1]} p_2^{[l_2]} \cdots p_t^{[l_t]}$ and $\beta = p_1^{[h_1]} p_2^{[h_2]} \cdots p_t^{[h_t]}$. It is clear that

$$|\alpha + \beta| = \sum_i \mathfrak{a}_i(p_i) \leq \sum_i (l_i + h_i) = |\alpha| + |\beta|.$$

Then, we have $\lfloor \alpha + \beta \rceil \leq |\alpha + \beta| + 1 \leq (|\alpha| + 1) + (|\beta| + 1)$. Up to a root with multiplicity 1 and $k_j = 1$ in each of $\alpha, \beta, \alpha + \beta$, it can be shown directly that $\lfloor \alpha + \beta \rceil \leq \lfloor \alpha \rceil + \lfloor \beta \rceil$. □

Lemma 3.2. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_m^n$. *Then,* $w_{LS,\pi}(\mathbf{x} + \mathbf{y}) \leq w_{LS,\pi}(\mathbf{x}) + w_{LS,\pi}(\mathbf{y})$.

Proof. For each $i \in [s]$, we write $\mathbf{x}_i = (x_{i_1}, \ldots, x_{i_{\pi(i)}})$, $\mathbf{y}_i = (y_{i_1}, \ldots, y_{i_{\pi(i)}}) \in \mathbb{Z}_m^{\pi(i)}$. Fix a block i. For each coordinate j in this block, the cyclic subgroup $\langle x_{i_j} + y_{i_j} \rangle$ has an order dividing the least common multiple of the orders of $\langle x_{i_j} \rangle$ and $\langle y_{i_j} \rangle$. Hence, under the order-isomorphism ψ, we obtain

$$\alpha_{\langle x_{i_j} + y_{i_j} \rangle} \Subset \alpha_{\langle x_{i_j} \rangle} \cup \alpha_{\langle y_{i_j} \rangle} \Subset \alpha_{\langle x_{i_j} \rangle} + \alpha_{\langle y_{i_j} \rangle}.$$

By the definition of the maximal element $\alpha_{\mathfrak{m}_i(\cdot)}$ in block i, it follows that

$$\alpha_{\mathfrak{m}_i(\mathbf{x}+\mathbf{y})} \Subset \alpha_{\mathfrak{m}_i(\mathbf{x})} \cup \alpha_{\mathfrak{m}_i(\mathbf{y})} \Subset \alpha_{\mathfrak{m}_i(\mathbf{x})} + \alpha_{\mathfrak{m}_i(\mathbf{y})}.$$

Therefore, the ideal generated by the support of $\mathbf{x} + \mathbf{y}$ is contained in the ideal generated by the blockwise sums coming from $\mathbf{x}$ and $\mathbf{y}$. In particular, every full-count or partial-count contribution appearing in $\langle \mathrm{supp}_{LS,\pi}(\mathbf{x} + \mathbf{y}) \rangle$ is dominated by the corresponding contribution coming from $\langle \mathrm{supp}_{LS,\pi}(\mathbf{x}) \rangle$ and $\langle \mathrm{supp}_{LS,\pi}(\mathbf{y}) \rangle$.

Now, Lemma 3.1 gives

$$\lfloor \alpha_{\mathfrak{m}_i(\mathbf{x}+\mathbf{y})} \rceil \leq \lfloor \alpha_{\mathfrak{m}_i(\mathbf{x})} + \alpha_{\mathfrak{m}_i(\mathbf{y})} \rceil \leq \lfloor \alpha_{\mathfrak{m}_i(\mathbf{x})} \rceil + \lfloor \alpha_{\mathfrak{m}_i(\mathbf{y})} \rceil.$$

Summing over all blocks, we conclude that the total contribution of $\mathbf{x} + \mathbf{y}$ to the LS–poset block weight cannot exceed the sum of the contributions of $\mathbf{x}$ and $\mathbf{y}$. Hence, $w_{LS,\pi}(\mathbf{x} + \mathbf{y}) \leq w_{LS,\pi}(\mathbf{x}) + w_{LS,\pi}(\mathbf{y})$. □

Theorem 3.1. *The* LS*–poset block distance* $d_{\pi_{LS}}(\cdot, \cdot)$ *is a metric on* $\mathbb{Z}_m^n$.

Proof. Clearly, $d_{\pi_{LS}}(\mathbf{x},\mathbf{y}) \geq 0$ for $\mathbf{x},\mathbf{y} \in \mathbb{Z}_m^n$. Because $\langle x\rangle = \langle -x\rangle$ for $x \in \mathbb{Z}_m$, we have that $d_{\pi_{LS}}(\cdot,\cdot)$ is a symmetry. Using the inequality as in Lemma 3.2, $d_{\pi_{LS}}(\cdot,\cdot)$ satisfies the triangle inequality. □

The metric $d_{\pi_{LS}}(\cdot,\cdot)$ on $\mathbb{Z}_m^n$ is called the LS–poset block metric. A submodule C of a space $(\mathbb{Z}_m^n, d_{\pi_{LS}})$ is called an $(n,K,d)_{\pi_{LS}}$-code, where K is the cardinality of C, and d is the minimum distance of C, denoted by $d_{\pi_{LS}}(C)$,

$$d_{\pi_{LS}}(C) = \min\{w_{LS,\pi}(\mathbf{v}) \ : \ \mathbf{0} \neq \mathbf{v} \in C\}.$$

The dual of an $(n,K,d)_{\pi_{LS}}$-code C is defined as

$$C^{\perp} = \{\mathbf{v} \in \mathbb{Z}_m^n \ : \ \mathbf{u}\cdot\mathbf{v} = u_1v_1 + \cdots + u_nv_n = 0 \text{ for all } \mathbf{u} \in C\}.$$

Example 3.1. Let $P_1 = \{1,2,3,4,5\}$ be a subposet of the poset P as given in Figure 1, and let $M = 2^{[3]}3^{[1]}5^{[2]}$. Consider $\mathbb{Z}_{600}^{13} = \mathbb{Z}_{600}^2 \oplus \mathbb{Z}_{600}^5 \oplus \mathbb{Z}_{600}^3 \oplus \mathbb{Z}_{600}^1 \oplus \mathbb{Z}_{600}^2$, that is, $\pi(1) = 2, \pi(2) = 5, \pi(3) = 3, \pi(4) = 1$, and $\pi(5) = 2$.

Let $S = \{\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus (0,24,60) \oplus 0 \oplus (8,12) \ : \ \mathbf{v}_1 \in \mathbb{Z}_{600}^2, \mathbf{v}_2 \in \mathbb{Z}_{600}^5\}$. Then, for any $\mathbf{v} \in S$,

$$\langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle = \langle\{(3,5^{[2]}),(5,2^{[1]}5^{[2]})\}\rangle.$$

Hence, $w_{LS,\pi}(\mathbf{v}) = 2\cdot 6 + \lfloor 5^{[2]}\rceil + \lfloor 2^{[1]}5^{[2]}\rceil = 19$.

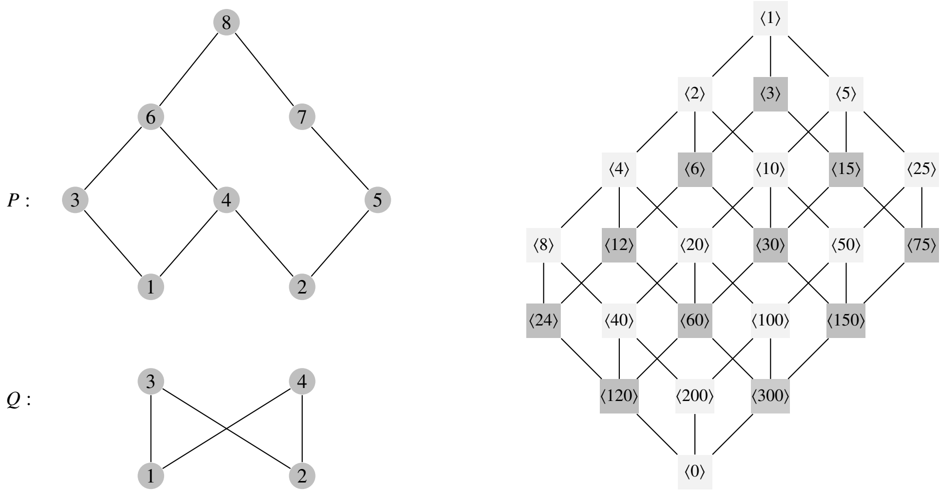

Figure 1. Two posets $P = ([8], \leq_P)$, $Q = ([4], \leq_Q)$, and $\mathcal{L}(\mathbb{Z}_{600})$.

Given $\mathcal{I} \in \Im_\circ(P \times \mathscr{P}_M)$, there is a codeword $\mathbf{u}$ of $\mathbb{Z}_m^n$, $\langle \mathrm{supp}_{LS,\pi}(\mathbf{u})\rangle = \mathcal{I}$. So, we can define a map ζ_M, (see [8]), from $\Im_\circ(P \times \mathscr{P}_M)$ to $\mathbb{N}$, by

$$\zeta_M(\mathcal{I}) = |\mathbb{M}_f(\mathcal{I})| \cdot \lfloor M\rceil + \sum_{j\in\mathbb{M}_p(\mathcal{I})} \lfloor \Omega_j\rceil, \tag{3.1}$$

where $(i,\Omega_i) \in \max(\mathcal{I})$, the set of maximal elements of $\mathcal{I}$.

Proposition 3.1. *Let* $m = p_1^{k_1}\cdots p_t^{k_t}$, *and let* $0 \leq r = k\lfloor M\rceil < s\lfloor M\rceil$ *for some* $k \in \mathbb{N}_0$. *Then, the following statements hold:*

(1) There exists an ideal $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ *with* $\zeta_M(\mathcal{I}) = r$*, and an element* $e \in [s]$ *such that either* $(e, \emptyset_M)$ *is an upper bound of* $\mathcal{I}$*, or* $(e, \emptyset_M)$ *is incomparable with every element of* $\mathcal{I}$*.*

(2) If there exist distinct indices i and j such that $k_i = k_j = 1$*, then there exist at least two distinct ideals* $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ *such that*

$$\zeta_M(\mathcal{I}_1) = \zeta_M(\mathcal{I}_2) = r + 1.$$

Proof. (1) Because $0 \le r = k\lfloor M\rceil < s\lfloor M\rceil$, Proposition 2.1 guarantees the existence of an ideal $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ with $\zeta_M(\mathcal{I}) = r$. In the construction of such an ideal, one proceeds inside the product poset $P \times \mathscr{P}_M$ by successively replacing elements of the form (i, Ω) with (i, Ω') where $\Omega \Subset \Omega'$ or $\Omega' \Subset \Omega$ in $\mathscr{P}_M$, while preserving downward closure in $P \times \mathscr{P}_M$. This process yields some $e \in [s]$ for which either $(e, \emptyset_M)$ lies above every element of $\mathcal{I}$, or $(e, \emptyset_M)$ remains incomparable with every element of $\mathcal{I}$. Thus, the claim in (1) follows.

(2) Suppose there exist distinct indices i and j such that $k_i = k_j = 1$. Then, $p_i^{[1]}$ and $p_j^{[1]}$ are two distinct minimal nontrivial elements of $\mathscr{P}(M) \setminus \{\emptyset_M\}$ corresponding to the prime divisors p_i and p_j. Let $\mathcal{I}$ be an ideal as in (1) with $\zeta_M(\mathcal{I}) = r$. By Proposition 2.1, the ideal $\mathcal{I}$ can be extended in the second coordinate by adjoining either $(e, p_i^{[1]})$ or $(e, p_j^{[1]})$, while preserving the ideal property in $P\times\mathscr{P}_M$. These two distinct extensions yield ideals $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{I}_\circ(P\times\mathscr{P}_M)$. To see that each extension increases the value of ζ_M by one, note that $p_i^{[1]}$ and $p_j^{[1]}$ are minimal nonempty elements of $\mathscr{P}(M)$ because $k_i = k_j = 1$. By the definition of the LS–weight on $\mathscr{P}(M)$, every such minimal element satisfies

$$\lfloor p_i^{[1]}\rceil = \lfloor p_j^{[1]}\rceil = 1.$$

Because $\zeta_M(\mathcal{I})$ is defined as the sum of the $\lfloor\cdot\rceil$-values corresponding to the maximal counted elements of $\mathcal{I}$, adjoining $(e, p_i^{[1]})$ or $(e, p_j^{[1]})$ contributes exactly one to this sum. Consequently,

$$\zeta_M(\mathcal{I}_1) = \zeta_M(\mathcal{I}_2) = r + 1.$$

□

3.1. r-balls

The (LS, π)-*ball* and the (LS, π)-*sphere* with center $\mathbf{u}$ and radius r are defined, respectively, as

$$B_{LS,\pi}(\mathbf{u}, r) = \{\mathbf{v} \in \mathbb{Z}_m^n \ : \ d_{\pi_{LS}}(\mathbf{u}, \mathbf{v}) \le r\}, \quad S_{LS,\pi}(\mathbf{u}, r) = \{\mathbf{v} \in \mathbb{Z}_m^n \ : \ d_{\pi_{LS}}(\mathbf{u}, \mathbf{v}) = r\}.$$

They are also called as an *r-ball* and *r-sphere* centered at $\mathbf{u}$ and denoted by $B_r(\mathbf{u})$ and $S_r(\mathbf{u})$, respectively.

Definition 3.1. In a space $(\mathbb{Z}_m^n, d_{\pi_{LS}})$, a code C over $\mathbb{Z}_m$ is said to be r-perfect if the (LS, π)-balls of radius r centered at the codewords of C are pairwise disjoint, and their union covers the entire space $\mathbb{Z}_m^n$.

Remark 3.1. The map ζ_M plays a fundamental role in the definition of the block LS–poset weight. Indeed, for any vector $\mathbf{v} \in \mathbb{Z}_m^n$, let $\mathrm{supp}_{LS,\pi}(\mathbf{v})$ denote its block LS–support and $\langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle$ the ideal generated by this support in $P \times \mathscr{P}_M$. By definition of the block LS–poset weight, we have

$$w_{LS,\pi}(\mathbf{v}) = \zeta_M\Big(\langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle\Big).$$

Thus, the weight of a vector is obtained by evaluating the function ζ_M on the ideal generated by its support. Consequently, the r-ball centered at the origin can be described in terms of ideals of $P \times \mathscr{P}_M$ satisfying $\zeta_M(\mathcal{I}) \leq r$.

Because a code C is a submodule of $\mathbb{Z}_m^n$ over $\mathbb{Z}_m$, all balls of the same radius have the same cardinality. That is, the cardinality of an r-ball centered at $\mathbf{0}$ represents the cardinality of any other ball with radius r. We then formulate this cardinality in order to characterize a code C that is r-perfect. It is clear that

$$B_r(0) = \bigcup_{q=0}^{r} S_q(0),$$

so we have

$$|B_r(0)| = 1 + \sum_{q=1}^{r} |S_q(0)|.$$

To compute $|S_q(0)|$, we partition $S_q(0)$ according to the ideals

$$\mathfrak{I}^q := \{\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M) : \zeta_M(\mathcal{I}) = q\}.$$

Indeed, for every $\mathbf{v} \in S_q(0)$, the ideal $\langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle$ belongs to $\mathfrak{I}^q$, as $w_{LS,\pi}(\mathbf{v}) = \zeta_M(\langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle) = q$. Hence,

$$S_q(0) = \bigsqcup_{\mathcal{I} \in \mathfrak{I}^q} \{\mathbf{v} \in \mathbb{Z}_m^n : \langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle = \mathcal{I}\}.$$

Given $d \in \mathbb{N}_0$, let

$$A_d = \{\alpha \in \mathscr{P}(M) : \lfloor \alpha \rceil = d\}$$

and define

$$U_d = \bigcup_{e=0}^{d} \bigcup_{\alpha \in A_e} \psi^{-1}(\alpha) \subseteq \mathbb{Z}_m.$$

Thus, U_d consists of all elements of $\mathbb{Z}_m$ whose associated multiset level does not exceed d. Consequently, the number of block vectors $\mathbf{v}_i \in \mathbb{Z}_m^{\pi(i)}$ whose level equals d is

$$N_{\mathrm{level}}(i, d) = |U_d|^{\pi(i)} - |U_{d-1}|^{\pi(i)},$$

where $N_{\mathrm{level}}(i, 0) = 1$, and

$$|U_d| = 1 + \sum_{e=1}^{d} \sum_{\alpha \in A_e} \phi(|\psi^{-1}(\alpha)|),$$

with ϕ denoting Euler's totient function.

Now, fix $\mathcal{I} \in \mathfrak{I}^q$. For each maximal block $i \in \mathbb{M}_{f*}(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})$, the block $\mathbf{v}_i$ must have a level exactly $\lfloor \Omega_i \rceil$, and the number of such block vectors must equal $N_{\mathrm{level}}(i, \lfloor \Omega_i \rceil)$. For each nonmaximal full-count block $j \in \mathbb{M}_f(\mathcal{I}) \setminus \mathbb{M}_{f*}(\mathcal{I})$, the block $\mathbf{v}_j$ may be chosen arbitrarily from $\mathbb{Z}_m^{\pi(j)}$, giving $m^{\pi(j)}$ possibilities. Therefore,

$$\left|\{\mathbf{v} \in \mathbb{Z}_m^n : \langle \mathrm{supp}_{LS,\pi}(\mathbf{v})\rangle = \mathcal{I}\}\right| = \left(\prod_{i \in \mathbb{M}_{f*}(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})} N_{\mathrm{level}}(i, \lfloor \Omega_i \rceil) \right) m^{\sum_{j \in \mathbb{M}_f(\mathcal{I}) \setminus \mathbb{M}_{f*}(\mathcal{I})} \pi(j)}.$$

Summing over all $\mathcal{I} \in \mathfrak{J}^q$ and then over $q = 1, \dots, r$, we obtain

$$|B_r(0)| = 1 + \sum_{q=1}^{r} \sum_{\mathcal{I}\in\mathfrak{J}^q} \left(\prod_{i\in\mathbb{M}_{f*}(\mathcal{I})\cup\mathbb{M}_p(\mathcal{I})} N_{\text{level}}(i, \lfloor\Omega_i\rceil) \right) m^{\sum_{j\in\mathbb{M}_f(\mathcal{I})\setminus\mathbb{M}_{f*}(\mathcal{I})} \pi(j)}. \tag{3.2}$$

3.1.1. The chain poset

Assume that $P = ([s], \preceq_P)$ is the chain, $1 \prec_P \cdots \prec_P s$. For $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_s$, $\mathbf{v}_i \in \mathbb{Z}_m^{\pi(i)}$, let $\lambda(\mathbf{v}) := \max\{i \in [s] \ : \ \mathbf{v}_i \neq \mathbf{0}\}$. For $k \in [s]$, $V_k(r)$ stands for the set containing all $\mathbf{v}$ in $B_r(\mathbf{0})$ such that $\lambda(\mathbf{v}) = k$. This implies that the sets $\{\mathbf{0}\}, V_1(r), \dots, V_s(r)$ are pairwise disjoint, and their union is $B_r(\mathbf{0})$. Because P is a chain, it follows that for $\mathbf{v} \in V_k(r)$, $\langle \text{supp}_{LS,\pi}(\mathbf{v})\rangle$ contains all pairs (i, M) for $i < k$, and its only possible partial contribution occurs in block k. Consequently, $w_{LS,\pi}(\mathbf{v}) = (k-1)\lfloor M\rceil + \lfloor \alpha_{\mathfrak{m}_k(\mathbf{v})}\rceil$. We have that the condition $\mathbf{v} \in B_r(\mathbf{0})$ is equivalent to $(k-1)\lfloor M\rceil + d \leq r$, where $1 \leq d \leq \lfloor M\rceil$. Letting

$$V_{k,d} = \{\mathbf{v} \in V_k(r) \ : \ \lfloor \alpha_{\mathfrak{m}_k(\mathbf{v})}\rceil = d\},$$

it follows that $V_k(r)$ can be partitioned into $V_{k,d}$ for $1 \leq d \leq \lfloor M\rceil$. Then, $|V_{k,d}| = m^{\sum_{j=1}^{k-1}\pi(j)} N_{level}(k, d)$, that is, the cardinality of the r-ball centered at $\mathbf{0}$ of $\mathbb{Z}_m^n$ is

$$\text{(Chain poset)} \qquad 1 + \sum_{k=1}^{s} m^{\sum_{j=1}^{k-1}\pi(j)} \sum_{\substack{1\leq d\leq\lfloor M\rceil \\ (k-1)\lfloor M\rceil+d\leq r}} N_{level}(k, d).$$

The cardinalities of the balls $B_r(\mathbf{0})$ for several values of m in the chain poset case are listed in Table 1.

Table 1. The cardinality of $B_r(\mathbf{0})$ in $\mathbb{Z}_m^6 = \mathbb{Z}_m^2 \oplus \mathbb{Z}_m^1 \oplus \mathbb{Z}_m^2 \oplus \mathbb{Z}_m^1$ with the chain $P = ([4], \preceq_P)$.

r	1	2	3	4	5	6	7	8
$m = 2^3$	1	4	16	64	64	128	256	512
$m = 2^3 3^2$	1	16	196	784	2304	5184	5184	20736
$m = 2^3 3^2 5$	25	400	4900	19600	57600	129600	648000	2592000

Observe that there is no submset α with $\lfloor\alpha\rceil = 1$, in either $\mathscr{P}(2^{[3]})$ or $\mathscr{P}(2^{[3]}3^{[2]})$. So, $|B_4(\mathbf{0})| = |B_5(\mathbf{0})|$ in $\mathbb{Z}_8^6$, and $|B_6(\mathbf{0})| = |B_7(\mathbf{0})|$ in $\mathbb{Z}_{72}^6$.

3.1.2. The star poset

Assume that $P = ([s], \preceq_P)$ is a star $\vee$-poset, $1 \prec_P i$ for all $i \in \{2, \dots, s\}$, and $2, \dots, s$ are pairwise incomparable. For $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_s \in \mathbb{Z}_m^n$, if $\mathbf{v}_j = \mathbf{0}$ for all $j > 1$, $w_{LS,\pi}(\mathbf{v}) \leq \lfloor M\rceil$. So, $|B_r(\mathbf{0})|$ is

$$\text{(Star} \vee \text{-poset)} \qquad \begin{cases} 1 + \displaystyle\sum_{d=1}^{r} N_{level}(1, d) = |U_r|^{\pi(1)}, & \text{if } 0 \leq r \leq \lfloor M\rceil, \\ m^{\pi(1)} + m^{\pi(1)} \displaystyle\sum_{\emptyset\neq J\subseteq\{2,\dots,s\}} \sum_{\substack{(d_j)_{j\in J}\in\{1,\dots,\lfloor M\rceil\}^J \\ \lfloor M\rceil+\sum_{j\in J} d_j\leq r}} \prod_{j\in J} N_{level}(j, d_j), & \text{if } r > \lfloor M\rceil. \end{cases}$$

Consider $\widetilde{P}$ as the dual poset of P. Then, $\widetilde{P}$ is a star $\wedge$-poset. Clearly, if $\mathbf{v}_1 \neq \mathbf{0}$, $w_{LS,\pi}(\mathbf{v}) > (s-1)\lfloor M \rceil$. A general counting formula for $|B_r(\mathbf{0})|$ is

$$\text{(Star} \wedge \text{-poset)} \qquad \begin{cases} \displaystyle\sum_{\substack{(d_2,\ldots,d_s)\in\{0,\ldots,\lfloor M\rceil\}^{s-1} \\ \sum_{j=2}^{s} d_j \le r}} \prod_{j=2}^{s} N_{level}(j,d_j), & \text{if } 0 \le r \le (s-1)\lfloor M \rceil, \\ m^{\sum_{j=2}^{s}\pi(j)} \left(1 + \displaystyle\sum_{d_1=1}^{\min(\lfloor M\rceil,\, r-(s-1)\lfloor M\rceil)} N_{level}(1,d_1) \right), & \text{if } r > (s-1)\lfloor M \rceil. \end{cases}$$

3.2. $\mathcal{I}$-balls

Following the definition of $\mathcal{I}$-balls introduced for poset block codes over finite fields (see Definition 4.1 in [3]), we extend this notion to the block LS–poset setting.

Let $\mathcal{I}$ be an ideal in $P \times \mathscr{P}_M$. We denote the $\mathcal{I}$-ball (resp. $\mathcal{I}$-sphere) centered at $\mathbf{u}$ by $B_{LS,\pi}(\mathbf{u},\mathcal{I})$ (resp. $S_{LS,\pi}(\mathbf{u},\mathcal{I})$), where

$$\mathbf{v} \in B_{LS,\pi}(\mathbf{u},\mathcal{I}) \iff \langle \mathrm{supp}_{LS,\pi}(\mathbf{u}-\mathbf{v})\rangle \subseteq \mathcal{I} \qquad (\ \mathbf{v} \in S_{LS,\pi}(\mathbf{u},\mathcal{I}) \iff \langle \mathrm{supp}_{LS,\pi}(\mathbf{u}-\mathbf{v})\rangle = \mathcal{I}\ ).$$

For notational convenience, $B_{\mathcal{I}}(\mathbf{u})$ and $S_{\mathcal{I}}(\mathbf{u})$ will be used for $B_{LS,\pi}(\mathbf{u},\mathcal{I})$ and $S_{LS,\pi}(\mathbf{u},\mathcal{I})$, respectively.

Definition 3.2. In a space $(\mathbb{Z}_m^n, d_{\pi_{LS}})$, a code C over $\mathbb{Z}_m$ is said to be $\mathcal{I}$-perfect if $\mathcal{I}$-balls centered at the codewords of C are pairwise disjoint, and their union is $\mathbb{Z}_m^n$.

The following proposition establishes the linearity of the $\mathcal{I}$-ball centered at the zero vector and describes several fundamental properties of $\mathcal{I}$-balls under the block LS–poset metric $d_{\pi_{LS}}$. The proof follows arguments analogous to those used in Propositions 8 and 9 for the classical LS–poset metric d_{LS} in [8]. Indeed, the block LS–poset metric differs from the LS–poset metric only in that each coordinate is replaced by a block $\mathbb{Z}_m^{\pi(i)}$, whereas the order structure of the poset and the definition of the weight remain unchanged. Hence, the arguments in [8] extend to the present block setting with only minor notational modifications.

Proposition 3.2. *Let $\mathcal{I} \in \Im_\circ(P \times \mathscr{P}_M)$. In the block LS–poset metric $d_{\pi_{LS}}$, the following properties hold:*

(1) (Full count $\mathcal{I}$):

$$B_{\mathcal{I}}(\mathbf{0}) = \bigoplus_{i\in\mathbb{M}_f(\mathcal{I})} \mathbb{Z}_m^{\pi(i)} \oplus \{\mathbf{0}\},$$

where the zero component corresponds to blocks $i \notin \mathbb{M}_f(\mathcal{I})$.

(2) (Partial count $\mathcal{I}$):

$$B_{\mathcal{I}}(\mathbf{0}) = \left(\bigoplus_{i\in\mathbb{M}_f(\mathcal{I})} \mathbb{Z}_m^{\pi(i)} \right) \oplus \left(\bigoplus_{j\in\mathbb{M}_p(\mathcal{I})} H_j^{\pi(j)} \right) \oplus \{\mathbf{0}\},$$

where for each $j \in \mathbb{M}_p(\mathcal{I})$, $(j,\Omega_j) \in \max(\mathcal{I})$, and $H_j = \psi^{-1}(\Omega_j)$, the zero component corresponds to blocks $i \notin \mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})$.

(3) (Translation invariance): For any $\mathbf{u} \in Z_m^n$, $B_{\mathcal{I}}(\mathbf{u}) = \mathbf{u} + B_{\mathcal{I}}(\mathbf{0})$.

(4) *(Coset partition property): For any* $\mathbf{u}, \mathbf{v} \in Z_m^n$*,* $B_{\mathcal{I}}(\mathbf{u})$ *and* $B_{\mathcal{I}}(\mathbf{v})$ *are either disjoint or identical. Moreover,*

$$B_{\mathcal{I}}(\mathbf{u}) = B_{\mathcal{I}}(\mathbf{v}) \Leftrightarrow \mathbf{u} - \mathbf{v} \in B_{\mathcal{I}}(\mathbf{0}).$$

(5) *(Duality):* $B_{\mathcal{I}}(\mathbf{0})^{\perp}$ *is the* $\mathcal{I}^{\mathfrak{C}}$*-ball centered at* $\mathbf{0}$ *under the poset* $\widetilde{P} \times \mathscr{P}_M$*.*

4. $\mathcal{I}$-Perfect block LS–poset codes

4.1. Ideals with full count

The following theorem establishes the existence of an $\mathcal{I}$-perfect block LS–poset code in $\mathbb{Z}_m^n$ for any ideal $\mathcal{I}$ of $P \times \mathscr{P}_M$ which satisfies the full count condition $\mathbb{M}_p(\mathcal{I}) = \emptyset$.

Theorem 4.1. *Let* $\mathcal{I} \in \Im_{\circ}(P \times \mathscr{P}_M)$ *be such that* $\mathbb{M}_p(\mathcal{I}) = \emptyset$*. Then, there exists an* $\mathcal{I}$*-perfect block* LS*–poset code* $C \subseteq \mathbb{Z}_m^n$ *with cardinality* $m^{n-\sum_{i\in\mathbb{M}_f(\mathcal{I})}\pi(i)}$*. Moreover, the dual code* $C^{\perp}$ *is an* $\mathcal{I}^{\mathfrak{C}}$*-perfect code under the poset* $\widetilde{P} \times \mathscr{P}_M$*, and its cardinality is given by* $|C^{\perp}| = m^{\sum_{i\in\mathbb{M}_f(\mathcal{I})}\pi(i)}$*.*

Proof. Because $\mathbb{M}_p(\mathcal{I}) = \emptyset$, Proposition 3.2(1) yields $B_{\mathcal{I}}(\mathbf{0}) = \bigoplus_{i\in\mathbb{M}_f(\mathcal{I})} \mathbb{Z}_m^{\pi(i)} \oplus \{\mathbf{0}\}$. Hence, $B_{\mathcal{I}}(\mathbf{0})$ is a submodule of $\mathbb{Z}_m^n$ of size $|B_{\mathcal{I}}(\mathbf{0})| = m^{\sum_{i\in\mathbb{M}_f(\mathcal{I})}\pi(i)}$. Define $C = \bigoplus_{j\in[s]\setminus\mathbb{M}_f(\mathcal{I})} \mathbb{Z}_m^{\pi(j)} \oplus \{\mathbf{0}\}$. Then, C is a submodule of $\mathbb{Z}_m^n$ satisfying $|C| = m^{n-\sum_{i\in\mathbb{M}_f(\mathcal{I})}\pi(i)}$. By construction, $\mathbb{Z}_m^n = C \oplus B_{\mathcal{I}}(\mathbf{0})$, so every vector in $\mathbb{Z}_m^n$ can be uniquely written as $\mathbf{x} = \mathbf{c} + \mathbf{b}$, for $\mathbf{c} \in C$, $\mathbf{b} \in B_{\mathcal{I}}(\mathbf{0})$. Using the translation invariance and coset properties from Proposition 3.2(3) and (4), the family of balls $\{\, B_{\mathcal{I}}(\mathbf{c}) : \mathbf{c} \in C \,\}$ forms a partition of $\mathbb{Z}_m^n$. Therefore, C is an $\mathcal{I}$-perfect block LS–poset code.

Next, from the orthogonal decomposition $\mathbb{Z}_m^n = C \oplus B_{\mathcal{I}}(\mathbf{0})$, it follows that $C^{\perp} = B_{\mathcal{I}}(\mathbf{0})$. By Proposition 3.2(5), under the dual poset $\widetilde{P} \times \mathscr{P}_M$, we have $B_{\mathcal{I}}(\mathbf{0})^{\perp} = B_{\mathcal{I}^{\mathfrak{C}}}(\mathbf{0})$, which implies that $C^{\perp}$ is an $\mathcal{I}^{\mathfrak{C}}$-perfect block LS–poset code. Finally, $|C|\,|C^{\perp}| = m^n$, and the stated cardinality of $C^{\perp}$ follows immediately. □

4.2. Ideals with partial count

Define

$$\mathbb{E}_M = \{\alpha \in \mathscr{P}(M) \;:\; \alpha^* \cap \widehat{\alpha}^* = \emptyset\},$$

that is, the set of all elements of $\mathscr{P}(M)$ whose root set does not intersect the root set of their dual.

To analyze $\mathcal{I}$-perfect block LS–poset codes in the partial count case, we require the following structural property. The argument is analogous to that of [8, Lemma 1]. The key idea is to use the set $\mathbb{E}_M$ to construct complementary subgroups H, K of $\mathbb{Z}_m$ which satisfy $\mathbb{Z}_m = H + K$ and $H \cap K = \{0\}$, and then extend this decomposition blockwise to each component $\mathbb{Z}_m^{\pi(i)}$ in the block decomposition $\mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$. Because the block structure only replaces each coordinate of $\mathbb{Z}_m$ by a block $\mathbb{Z}_m^{\pi(i)}$, the argument of [8, Lemma 1] applies with only notational modifications. Hence, the proof is omitted.

Proposition 4.1. *Let* $\mathcal{I} \in \Im_{\circ}(P \times \mathscr{P}_M)$ *be an ideal with partial count, that is,* $\mathbb{M}_p(\mathcal{I}) \neq \emptyset$*. If* $C \subseteq Z_m^n$ *is an* $\mathcal{I}$*-perfect block* LS*–poset code, then for each* $i \in \mathbb{M}_p(\mathcal{I})$ *with* $(i, \Omega_i) \in \max(\mathcal{I})$*, we have* $\Omega_i \in \mathbb{E}_M$*.*

From Proposition 3.2(2) and Proposition 4.1, the following result follows directly. The condition $\Omega_i \in \mathbb{E}_M$ guarantees that the subgroup corresponding to Ω_i admits a complementary subgroup in $\mathbb{Z}_m$. This allows the block component $\mathbb{Z}_m^{\pi(i)}$ to be decomposed into complementary submodules, which is crucial for constructing the corresponding $\mathcal{I}$-ball and its complementary submodule forming an $\mathcal{I}$-perfect code.

Theorem 4.2. *Let $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ be an ideal with partial count. Then, an $\mathcal{I}$-perfect block LS–poset code $C \subseteq \mathbb{Z}_m^n$ exists if and only if $C = B_{\mathcal{I}}(\mathbf{0})^\perp$, where*

$$B_{\mathcal{I}}(\mathbf{0}) = \left(\bigoplus_{i \in \mathbb{M}_f(\mathcal{I})} \mathbb{Z}_m^{\pi(i)} \right) \oplus \left(\bigoplus_{j \in \mathbb{M}_p(\mathcal{I})} H_j^{\pi(j)} \right) \oplus \{\mathbf{0}\},$$

with $\mathbb{Z}_m = H_j + K_j,\ H_j \cap K_j = \{0\}$. Equivalently,

$$C = \left(\bigoplus_{i \in [s] \setminus \left(\mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})\right)} \mathbb{Z}_m^{\pi(i)} \right) \oplus \left(\bigoplus_{j \in \mathbb{M}_p(\mathcal{I})} K_j^{\pi(j)} \right) \oplus \{\mathbf{0}\}.$$

Observe that $\mathbb{Z}_{p^k}$ is a cyclic p-group whose subgroups are totally ordered by inclusion. Consequently, any two nontrivial subgroups of $\mathbb{Z}_{p^k}$ have nontrivial intersection. Hence, the following corollary follows immediately.

Corollary 4.1. *Let $m = p^k$ be a prime power, and let $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ be an ideal with partial count, that is, $\mathbb{M}_p(\mathcal{I}) \neq \emptyset$. Then, no $\mathcal{I}$-perfect block LS–poset code exists in $\mathbb{Z}_{p^k}^n$.*

Example 4.1. Consider the poset $P = ([8], \leq_P)$ as given in Figure 1 and the space $\mathbb{Z}_{120}^{12}$ with block structure $\pi = (1, 1, 1, 1, 2, 2, 2, 2)$. In $\mathbb{Z}_{120}$, the following pairs of nontrivial subgroups form an internal direct sum of $\mathbb{Z}_{120}$: $(\langle 15\rangle, \langle 8\rangle)$, $(\langle 40\rangle, \langle 3\rangle)$, $(\langle 24\rangle, \langle 5\rangle)$, that is, in each case, $\mathbb{Z}_{120} = H + K$, and $H \cap K = \{0\}$.

(1) The code

$$C = \{0\}^3 \times \langle 15\rangle \times \langle 3\rangle \times \langle 3\rangle \times \mathbb{Z}_{120}^6 \subseteq \mathbb{Z}_{120}^{12}$$

is an $\mathcal{I}$-perfect block LS–poset code, where

$$\mathcal{I} = \langle \{(3, M), (4, 3^{[1]}5^{[1]}), (5, 3^{[1]})\} \rangle \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$$

is an ideal with partial count.

(2) Its dual

$$C^\perp = \mathbb{Z}_{120}^3 \times \langle 8\rangle \times \langle 40\rangle \times \langle 40\rangle \times \{0\}^6$$

is an $\mathcal{I}^{\mathfrak{C}}$-perfect block LS–poset code under $\widetilde{P} \times \mathscr{P}_M$, where

$$\mathcal{I}^{\mathfrak{C}} = \langle \{(4, 2^{[3]}), (5, 2^{[3]}5^{[1]})\} \rangle \in \mathfrak{I}_\circ(\widetilde{P} \times \mathscr{P}_M).$$

5. MDS block LS–poset codes

Theorem 5.1 (Singleton bound)**.** *Let* $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ *be a block LS–poset code with minimum distance* $d_{\pi_{LS}}(C)$*. Then,*

$$n - \lceil \log_m |C| \rceil \ \geq\ \max_{\mathcal{I}} \sum_{i \in \mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})} \pi(i),$$

where the maximum is taken over all ideals $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ *which satisfy* $\zeta_M(\mathcal{I}) \leq d_{\pi_{LS}}(C) - 1$*, and* $\left|\mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})\right| = \left\lfloor \frac{d_{\pi_{LS}}(C)-1}{\lfloor M \rceil} \right\rfloor$*.*

Proof. Let $k = \left\lfloor \frac{d_{\pi_{LS}}(C)-1}{\lfloor M \rceil} \right\rfloor$. Then, $k\lfloor M \rceil \leq d_{\pi_{LS}}(C) - 1 < (k+1)\lfloor M \rceil$. Because each full-count block contributes exactly $\lfloor M \rceil$ to the value of ζ_M, the integer k is the largest possible number of blocks whose total contribution to ζ_M does not exceed $d_{\pi_{LS}}(C) - 1$.

By Proposition 3.1, there exists an admissible ideal whose ζ_M-value is a multiple of $\lfloor M \rceil$. Hence, for the present argument, it is enough to choose a full-count ideal $\mathcal{I}$ with $|\mathbb{M}_f(\mathcal{I})| = \left\lfloor \frac{d_{\pi_{LS}}(C)-1}{\lfloor M \rceil} \right\rfloor$. Then,

$$\zeta_M(\mathcal{I}) = \left|\mathbb{M}_f(\mathcal{I})\right| \cdot \lfloor M \rceil = \left\lfloor \frac{d_{\pi_{LS}}(C)-1}{\lfloor M \rceil} \right\rfloor \cdot \lfloor M \rceil \ \leq\ d_{\pi_{LS}}(C) - 1.$$

Now, let $\mathbf{c} \in C$ be any codeword. If $\mathbf{c}_i = \mathbf{0}$ for all $i \in [s] \setminus \mathbb{M}_f(\mathcal{I})$, then the block LS–poset weight of $\mathbf{c}$ satisfies $w_{LS,\pi}(\mathbf{c}) \leq \left\lfloor \frac{d_{\pi_{LS}}(C)-1}{\lfloor M \rceil} \right\rfloor \cdot \lfloor M \rceil \leq d_{\pi_{LS}}(C) - 1$. Therefore, no nonzero codeword of C can have all its nonzero blocks contained in $\mathbb{M}_f(\mathcal{I})$; otherwise, its weight would be strictly smaller than the minimum distance. Equivalently, for every nonzero $\mathbf{c} \in C$, there exists $i \in [s] \setminus \mathbb{M}_f(\mathcal{I})$ with $\mathbf{c}_i \neq \mathbf{0}$. It follows that $|C| \leq m^{n - \sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i)}$. That is, $n - \lceil \log_m |C| \rceil \geq \sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i)$. □

Remark 5.1. Theorem 5.1 shows that the minimum block LS–poset distance $d_{\pi_{LS}}(C)$ controls how many block coordinates may be deleted while preserving the injectivity of the puncturing map. In particular, if $\pi(i) = 1$ for all $i \in [s]$, then the block structure coincides with the coordinate structure, that is, Theorem 5.1 reduces to

$$n - \lceil \log_m |C| \rceil \ \geq\ \left\lfloor \frac{d_{\pi_{LS}}(C) - 1}{\lfloor M \rceil} \right\rfloor.$$

Definition 5.1 (MDS block LS–poset code)**.** Let $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ be a block LS–poset code with minimum distance $d_{\pi_{LS}}(C)$. We say that C is maximum distance separable (MDS) with respect to the block LS–poset metric if it attains the Singleton bound.

Example 5.1. Consider the space $\mathbb{Z}_{10}^6$ with block structure $\pi = (2, 2, 1, 1)$ and the poset $Q = ([4], \leq_Q)$ given by the relations $1 \prec_Q 3$, $1 \prec_Q 4$, $2 \prec_Q 3$, and $2 \prec_Q 4$ (see Figure 1). Let $C \subseteq \mathbb{Z}_{10}^6$ be the block code generated by

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then, $|C| = 10^2$, and $d_{\pi_{LS}}(C) = 5$. Hence,

$$\left\lfloor \frac{d_{\pi_{LS}}(C) - 1}{\lfloor M \rceil} \right\rfloor = \left\lfloor \frac{4}{\lfloor M \rceil} \right\rfloor = 2,$$

where $M = 2^{[1]}5^{[1]}$ corresponds to $\mathbb{Z}_{10}$. Now, consider ideals $\mathcal{I} \in \Im_\circ(Q \times \mathscr{P}_M)$ which satisfy

$$\zeta_M(\mathcal{I}) \le d_{\pi_{LS}}(C) - 1 \quad \text{and} \quad \left|\mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})\right| = 2.$$

Because Q has two minimal elements $\{1, 2\}$, such an ideal may be generated by $(1, \alpha_1)$ and $(2, \alpha_2)$ for some $\alpha_1, \alpha_2 \in \mathscr{P}(M) \backslash \{\emptyset_M\}$. Therefore, the Singleton-type bound yields

$$\max_{\mathcal{I}} \sum_{i \in \mathbb{M}_f(\mathcal{I}) \cup \mathbb{M}_p(\mathcal{I})} \pi(i) = \pi(1) + \pi(2) = 2 + 2 = 4.$$

On the other hand,

$$n - \left\lceil \log_{10} |C| \right\rceil = 6 - 2 = 4.$$

Thus, equality holds in the Singleton bound, and consequently, C is MDS.

Theorem 5.2. *Let* $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ *be a block LS–poset code with* $|C| = m^t$ *and minimum distance* $d = d_{\pi_{LS}}(C)$. *Then, C is MDS if and only if C is an* $\mathcal{I}$*-perfect block LS–poset code for some ideal* $\mathcal{I} \in \Im_\circ(P \times \mathscr{P}_M)$ *with full count and* $|\mathbb{M}_f(\mathcal{I})| = \left\lfloor \frac{d-1}{\lfloor M \rceil} \right\rfloor$.

Proof. Set $r := \left\lfloor \frac{d-1}{\lfloor M \rceil} \right\rfloor$.

($\Rightarrow$) Assume that C is MDS. Choose an ideal $\mathcal{J} \in \Im_\circ(P \times \mathscr{P}_M)$ with full count such that $|\mathbb{M}_f(\mathcal{J})| = r$. By the Singleton-type bound (Theorem 5.1), we have $n - t = \sum_{i \in \mathbb{M}_f(\mathcal{J})} \pi(i)$. (For a full-count ideal, $\mathbb{M}_p(\mathcal{J}) = \emptyset$; hence, the sum above is of the same form as in the Singleton maximization.) Because $\mathcal{J}$ has full count, and $|\mathbb{M}_f(\mathcal{J})| = r$, we have $\zeta_M(\mathcal{J}) = r \cdot \lfloor M \rceil \le d - 1$. Therefore, for any distinct $\mathbf{c}_1, \mathbf{c}_2 \in C$, the $\mathcal{J}$-balls are disjoint: $B_{\mathcal{J}}(\mathbf{c}_1) \cap B_{\mathcal{J}}(\mathbf{c}_2) = \emptyset$. By Proposition 3.2(1)–(4), the $\mathcal{J}$-ball centered at $\mathbf{0}$ is a subgroup, and $|B_{\mathcal{J}}(\mathbf{0})| = m^{\sum_{i \in \mathbb{M}_f(\mathcal{J})} \pi(i)}$. Hence, the packing inequality gives $|C|\,|B_{\mathcal{J}}(\mathbf{0})| \le m^n$. Using $|C| = m^t$, we obtain equality: $|C|\,|B_{\mathcal{J}}(\mathbf{0})| = m^t\, m^{n-t} = m^n$. Thus, the $\mathcal{J}$-balls around codewords of C partition $\mathbb{Z}_m^n$, and C is $\mathcal{J}$-perfect.

($\Leftarrow$) Conversely, assume that C is $\mathcal{I}$-perfect for some full-count ideal $\mathcal{I} \in \Im_\circ(P \times \mathscr{P}_M)$ with $|\mathbb{M}_f(\mathcal{I})| = r$. By Proposition 3.2(1)–(4), we have $|B_{\mathcal{I}}(\mathbf{0})| = m^{\sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i)}$. Perfectness gives

$$m^n = |C|\,|B_{\mathcal{I}}(\mathbf{0})| = m^t\, m^{\sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i)},$$

so $n - t = \sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i)$. Moreover, because $|\mathbb{M}_f(\mathcal{I})| = r$, we have $\zeta_M(\mathcal{I}) = r \lfloor M \rceil \le d - 1$, so $\mathcal{I}$ is admissible in the Singleton bound (Theorem 5.1). Therefore,

$$n - t \ge \max_{\mathcal{J}} \sum_{i \in \mathbb{M}_f(\mathcal{J}) \cup \mathbb{M}_p(\mathcal{J})} \pi(i) \ge \sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i) = n - t,$$

and equality holds throughout. Hence, C meets the Singleton-type bound with equality, that is, C is MDS. □

Remark 5.2. Because $\mathbb{Z}_m$ is generally not a field when m is composite, block LS–poset codes over $\mathbb{Z}_m^n$ may exhibit behaviors that do not occur in the classical theory over finite fields. In particular, extremal codes attaining equality in the Singleton bound may involve ideals with partial count.

For a code $C \subseteq \mathbb{Z}_m^n$, the minimum block LS–poset distance $d_{\pi_{LS}}(C)$ restricts the admissible ideals used in the packing argument. Namely, if $\mathcal{J} \in \Im_\circ(P \times \mathscr{P}_M)$ satisfies

$$\zeta_M(\mathcal{J}) \le d_{\pi_{LS}}(C) - 1,$$

then the balls $B_{\mathcal{J}}(\mathbf{x})$ and $B_{\mathcal{J}}(\mathbf{y})$ are disjoint for distinct $\mathbf{x}, \mathbf{y} \in C$.

Motivated by Theorem 5.1, one may enlarge a full-count ideal by adjoining partial elements in several blocks simultaneously. This leads naturally to the construction below.

Definition 5.2. Let

$$C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$$

be a block LS–poset code with minimum distance $d = d_{\pi_{LS}}(C) = k\lfloor M \rceil + r,\ 0 \le r < \lfloor M \rceil$.

Let $\mathcal{I} \in \mathfrak{I}_{\circ}(P \times \mathscr{P}_M)$ be a maximal ideal with full count satisfying $|\mathbb{M}_f(\mathcal{I})| = k$.

Let $\mathcal{D}_M := \{\emptyset_M\} \cup \min(\mathscr{P}(M) \setminus \{\emptyset_M\})$. For a map $\chi : [s] \setminus \mathbb{M}_f(\mathcal{I}) \to \mathcal{D}_M$, define the enlarged ideal

$$\mathcal{I}(\chi) = \Big\langle \mathcal{I} \cup \{(i, \chi(i)) :\ i \in [s] \setminus \mathbb{M}_f(\mathcal{I})\} \Big\rangle.$$

Assume that $\displaystyle\sum_{i \in [s] \setminus \mathbb{M}_f(\mathcal{I})} \pi(i)\lfloor \chi(i) \rceil = r$. Then, C is called a partial-MDS block LS–poset code if

$$\log_m |C| = n - \max_{\substack{\chi \in \mathcal{D}_M^{[s] \setminus \mathbb{M}_f(\mathcal{I})},\ \sum_i \pi(i)\lfloor \chi(i) \rceil = r \\ \mathcal{I}(\chi) \in \mathfrak{I}_{\circ}(P \times \mathscr{P}_M) \\ C \cap B_{\mathcal{I}(\chi)}(\mathbf{0}) = \{\mathbf{0}\}}} \log_m \left| B_{\mathcal{I}(\chi)}(\mathbf{0}) \right|.$$

Remark 5.3. The set $\mathcal{D}_M$ consists of the trivial element $\emptyset_M$ together with the minimal nontrivial elements of $\mathscr{P}(M)$. Thus, $\chi(i) = \emptyset_M$ corresponds to no partial contribution in block i, whereas $\chi(i) \in \min(\mathscr{P}(M) \setminus \{\emptyset_M\})$ produces a partial contribution of size $\pi(i)\lfloor \chi(i) \rceil$ to the block LS-weight.

The decomposition $d = k\lfloor M \rceil + r$ separates the contribution of full-count blocks from the remaining partial contribution. The map χ distributes the residual term r among the remaining blocks, allowing the remainder to be realized through several partially counted blocks.

Example 5.2. We illustrate the notions of MDS and partial-MDS block LS–poset codes.

(1) MDS and nonpartial-MDS examples. Consider the space $\mathbb{Z}_{12}^6$ with block structure $\pi = (3, 2, 1)$ and the star $\vee$-poset $P = ([3], \preceq_P)$ defined by $1 \prec_P 2$ and $1 \prec_P 3$. Let

$$C_1 = \{0\}^3 \oplus \mathbb{Z}_{12}^2 \oplus \mathbb{Z}_{12}, \qquad C_2 = \{0\}^3 \oplus (3\mathbb{Z}_{12})^2 \oplus (4\mathbb{Z}_{12}).$$

Because $M = 2^{[2]}3^{[1]}$, we have $\lfloor M \rceil = 3$.

For C_1, any nonzero codeword has the form $\mathbf{v} = (0, 0, 0, v_4, v_5, v_6),\ v_4, v_5, v_6 \in \mathbb{Z}_{12}$. The smallest nonzero support occurs when either

$$(v_4, v_5) \in (4\mathbb{Z}_{12})^2 \setminus \{(0, 0)\} \ \text{ or } \ v_6 \in 4\mathbb{Z}_{12} \setminus \{0\}.$$

In this case, the generated subgroup corresponds to the multiset $3^{[1]}$, whose weight is $\lfloor 3^{[1]} \rceil = 1$. Because block 1 is minimal in the poset and $\pi(1) = 3$, the ideal generated by the support contributes $\zeta_M(\langle \mathrm{supp}_{LS,\pi}(\mathbf{v}) \rangle) = 4$. Hence, $d_{\pi_{LS}}(C_1) = 4$. Thus,

$$\left\lfloor \frac{d_{\pi_{LS}}(C_1) - 1}{\lfloor M \rceil} \right\rfloor = \left\lfloor \frac{4 - 1}{3} \right\rfloor = 1.$$

The maximal full-count ideal satisfying this constraint is $\mathcal{I} = \langle\{(1, M)\}\rangle$. Because $n = 6$, and $\pi(1) = 3$, we obtain

$$\log_{12} |C_1| = 3 = 6 - \pi(1).$$

Hence, C_1 attains the Singleton bound and is an MDS block LS–poset code.

For C_2, every codeword has the form $\mathbf{v} = (0, 0, 0, 3a, 3b, 4c)$, $a, b, c \in \mathbb{Z}_{12}$. The minimum block LS–poset weight again yields $d_{\pi_{LS}}(C_2) = 4$. Hence,

$$\left\lfloor \frac{d_{\pi_{LS}}(C_2) - 1}{\lfloor M \rceil} \right\rfloor = \left\lfloor \frac{4 - 1}{3} \right\rfloor = 1.$$

Thus, the maximal full-count ideal is still $\mathcal{I} = \langle\{(1, M)\}\rangle$. However, for the remainder $r = 1$, the only admissible map χ satisfies

$$\chi(2) = \emptyset_M, \qquad \chi(3) = 3^{[1]},$$

as $r = \pi(2) \cdot 0 + \pi(3) \cdot 1$. The resulting extensions $\mathcal{I}(\chi)$ do not produce equality in the partial Singleton bound. Therefore, C_2 is not a partial-MDS block LS–poset code.

(2) A partial-MDS example. Consider $\mathbb{Z}_{12}^6$ with block structure $\pi = (2, 2, 2)$ and the star $\wedge$-poset $Q = ([3], \preceq_Q)$ defined by $2 \prec_Q 1$ and $3 \prec_Q 1$. Let

$$C_3 = (3\mathbb{Z}_{12})^2 \oplus \{0\}^2 \oplus \{0\}^2.$$

To determine the minimum distance, we observe that $6\mathbb{Z}_{12}$ is the smallest nontrivial subgroup of $3\mathbb{Z}_{12}$. Choose a nonzero codeword $\mathbf{v} = (6, 0, 0, 0, 0, 0) \in C_3$. Because $\pi = (2, 2, 2)$, the vector $\mathbf{v}$ is nonzero only in block 1. Under the map ψ, the subgroup $\langle 6 \rangle = 6\mathbb{Z}_{12}$ corresponds to the multiset $2^{[1]}$. Because $M = 2^{[2]}3^{[1]}$, the definition of the multiset weight yields $\lfloor 2^{[1]} \rceil = 2$. Because blocks 2 and 3 are predecessors of block 1 in the $\wedge$-poset, the ideal generated by the support of $\mathbf{v}$ also contains the full-count elements $(2, M)$ and $(3, M)$. Therefore,

$$\zeta_M(\langle \mathrm{supp}_{LS,\pi}(\mathbf{v}) \rangle) = \lfloor 2^{[1]} \rceil + \lfloor M \rceil + \lfloor M \rceil = 2 + 3 + 3 = 8.$$

Consequently, $d_{\pi_{LS}}(C_3) = 8$. Hence,

$$\left\lfloor \frac{d_{\pi_{LS}}(C_3) - 1}{\lfloor M \rceil} \right\rfloor = \left\lfloor \frac{8 - 1}{3} \right\rfloor = 2.$$

Thus, the maximal full-count ideal satisfying the Singleton constraint is

$$\mathcal{J} = \langle\{(2, M), (3, M)\}\rangle.$$

Next, we consider the map

$$\chi : \{1\} \to \mathcal{D}_M, \qquad \chi(1) = 3^{[1]}.$$

Because $\lfloor 3^{[1]} \rceil = 1$, we obtain $\pi(1)\lfloor \chi(1) \rceil = 2$. With the extension $\mathcal{J}(\chi)$, we have

$$\log_{12} |C_3| = \log_{12}(4^2) = 6 - \log_{12}\left(12^{\pi(2)+\pi(3)} \cdot 3^{\pi(1)}\right) = 6 - \log_{12} |B_{\mathcal{J}(\chi)}(\mathbf{0})|.$$

Hence, C_3 attains equality in the partial Singleton bound and is a partial-MDS block LS–poset code.

Proposition 5.1. *Let* $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ *be a block* LS*–poset code with minimum distance*

$$d = d_{\pi_{LS}}(C) = k\lfloor M \rceil.$$

Then, C *is a partial-MDS block* LS*–poset code if and only if* C *is an MDS block* LS*–poset code.*

Proof. If $d_{\pi_{LS}}(C) = k\lfloor M \rceil$, then the remainder in the definition of partial-MDS is $r = 0$. Hence,

$$\sum_{i} \pi(i)\lfloor \chi(i) \rceil = 0,$$

forces $\chi(i) = \emptyset_M$ for every i, and so $\mathcal{I}(\chi) = \mathcal{I}$. Thus, the partial-MDS equality reduces exactly to the MDS equality for a full-count ideal $\mathcal{I}$ with $|\mathbb{M}_f(\mathcal{I})| = k$. Therefore, C is partial-MDS if and only if it is MDS. □

Theorem 5.3. *Let* $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ *be a block* LS*–poset code with minimum distance*

$$d = d_{\pi_{LS}}(C) = k\lfloor M \rceil + r, \qquad 0 \leq r < \lfloor M \rceil.$$

Let $\mathcal{I} \in \mathfrak{I}_{\circ}(P \times \mathscr{P}_M)$ *be a maximal ideal with full count such that* $|\mathbb{M}_f(\mathcal{I})| = k$*, and let*

$$\chi : [s] \setminus \mathbb{M}_f(\mathcal{I}) \to \mathcal{D}_M$$

satisfy

$$\sum_{i \in [s] \setminus \mathbb{M}_f(\mathcal{I})} \pi(i)\lfloor \chi(i) \rceil = r.$$

Set $\mathcal{J} = \mathcal{I}(\chi)$*. Then,* C *is a partial-MDS block* LS*–poset code, and the defining maximum is attained at* $\mathcal{J}$ *if and only if* C *is a* $\mathcal{J}$*-perfect block* LS*–poset code.*

Proof. ($\Rightarrow$) Assume that C is partial-MDS and that the defining maximum is attained at $\mathcal{J} = \mathcal{I}(\chi)$. Then, $\log_m |C| = n - \log_m |B_{\mathcal{J}}(\mathbf{0})|$, so $|C| \cdot |B_{\mathcal{J}}(\mathbf{0})| = m^n$. Moreover, because $\mathcal{J}$ is admissible in the definition of partial-MDS, we have $C \cap B_{\mathcal{J}}(\mathbf{0}) = \{\mathbf{0}\}$. Hence, the translates $\{\mathbf{c} + B_{\mathcal{J}}(\mathbf{0}) : \mathbf{c} \in C\}$ are pairwise disjoint. Because their total cardinality is

$$|C| \cdot |B_{\mathcal{J}}(\mathbf{0})| = m^n = |\mathbb{Z}_m^n|,$$

they form a partition of $\mathbb{Z}_m^n$. Therefore, C is a $\mathcal{J}$-perfect block LS–poset code.

($\Leftarrow$) If C is a $\mathcal{J}$-perfect block LS–poset code, then $|C| \cdot |B_{\mathcal{J}}(\mathbf{0})| = m^n$, and $C \cap B_{\mathcal{J}}(\mathbf{0}) = \{\mathbf{0}\}$. Thus,

$$\log_m |C| = n - \log_m |B_{\mathcal{J}}(\mathbf{0})|.$$

Because $\mathcal{J} = \mathcal{I}(\chi)$ is admissible in the definition of partial-MDS, it follows that C is partial-MDS and that the defining maximum is attained at $\mathcal{J}$. □

Corollary 5.1. *Every* $\mathcal{J}$*-perfect block* LS*–poset code arises from an ideal* $\mathcal{J} = \mathcal{I}(\chi)$*, where* $\mathcal{I}$ *has full count, and*

$$\sum_{i \in [s] \setminus \mathbb{M}_f(\mathcal{I})} \pi(i)\lfloor \chi(i) \rceil = r,$$

is partial-MDS.

Remark 5.4. The assumption that $\mathcal{J} = \mathcal{I}(\chi)$ with $\chi(i) \in \mathcal{D}_M$ is essential. In Example 5.2, the code

$$C_2 = \{0\}^3 \oplus (3\mathbb{Z}_{12})^2 \oplus (4\mathbb{Z}_{12}) \subseteq \mathbb{Z}_{12}^6$$

is $\mathcal{J}$-perfect for

$$\mathcal{J} = \langle \{(2, 3^{[1]}), (3, 2^{[2]})\} \rangle,$$

as

$$B_{\mathcal{J}}(\mathbf{0}) = \mathbb{Z}_{12}^3 \oplus (4\mathbb{Z}_{12})^2 \oplus (3\mathbb{Z}_{12}),$$

and

$$|C_2| \cdot |B_{\mathcal{J}}(\mathbf{0})| = 12^6.$$

Nevertheless, C_2 is not partial-MDS, because $2^{[2]} \notin \mathcal{D}_M$. Hence, $\mathcal{J}$ is not admissible in the definition of partial-MDS.

6. r-perfect block LS–poset codes under special posets

Theorem 6.1. *Let $\mathbb{Z}_m^n = \mathbb{Z}_m^{\pi(1)} \oplus \cdots \oplus \mathbb{Z}_m^{\pi(s)}$, and let $P = ([s], \preceq_P)$ be a poset. Suppose that for some integer $0 < k < s$, the set*

$$\mathfrak{J}^{k\lfloor M \rceil} = \{\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M) : \zeta_M(\mathcal{I}) = k\lfloor M \rceil\}$$

contains exactly one ideal, say $\mathcal{I}$.

Then, for a block LS–poset code $C \subseteq \mathbb{Z}_m^n$, the following statements are equivalent:

(1) C is an $\mathcal{I}$-perfect block LS–poset code;

(2) C is $k\lfloor M \rceil$-perfect;

(3) C is an MDS block LS–poset code with $|C| = m^{n - \sum_{i \in \mathbb{M}_f(\mathcal{I})} \pi(i)}$.

Proof. Because $\mathfrak{J}^{k\lfloor M \rceil} = \{\mathcal{I}\}$, the unique ideal with M-count $k\lfloor M \rceil$ is $\mathcal{I}$. Hence, the ball of radius $k\lfloor M \rceil$ coincides with the $\mathcal{I}$-ball:

$$B_{k\lfloor M \rceil}(\mathbf{0}) = B_{\mathcal{I}}(\mathbf{0}).$$

Thus, C is $\mathcal{I}$-perfect if and only if C is $k\lfloor M \rceil$-perfect.

On the other hand, by Theorem 5.2, a block LS–poset code is MDS if and only if it is $\mathcal{I}$-perfect for some ideal $\mathcal{I}$ with full count satisfying

$$|\mathbb{M}_f(\mathcal{I})| = \left\lfloor \frac{d_{\pi_{LS}}(C) - 1}{\lfloor M \rceil} \right\rfloor.$$

Because such an ideal is unique by assumption, the three statements are equivalent. □

Remark 6.1. The assumption that the set $\mathfrak{J}^{k\lfloor M \rceil}$ contains exactly one ideal reflects a structural property of the poset P. Indeed, ideals in $\mathfrak{I}_\circ(P \times \mathscr{P}_M)$ are determined by their block indices in P together with the subgroup structure encoded in $\mathscr{P}_M$. When the poset P is sufficiently ordered (for example, when P is a chain), the M-count of an ideal uniquely determines the set of blocks in $\mathbb{M}_f(\mathcal{I})$. Consequently, there is exactly one ideal whose M-count equals $k\lfloor M \rceil$. In contrast, if P contains incomparable elements (such as in star or tree posets), different ideals may have the same M-count. In that case, the set $\mathfrak{J}^{k\lfloor M \rceil}$ may contain several ideals, and the equivalence in Theorem 6.1 need not hold.

When the poset structure admits a unique ideal with M-count $k\lfloor M\rceil$, the notions of $\mathcal{I}$-perfectness, $k\lfloor M\rceil$-perfectness, and the MDS property coincide.

Remark 6.2. Suppose that the poset P has a unique ideal $I \subseteq P$ with $|I| = t < |P|$. Then, every element $e \in P \setminus I$ must be comparable with the maximal elements of I in such a way that e cannot replace any maximal element of I while preserving the ideal property. Indeed, if there exists $e \in P \setminus I$ that can replace a maximal element $a \in \max(I)$ (that is, $(I \setminus \{a\}) \cup \{e\}$ remains an ideal), then we obtain another ideal of size t, contradicting the uniqueness of I.

This observation shows that the uniqueness of ideals depends strongly on the structure of the poset. For example, in a star $\vee$-poset, several ideals may have the same cardinality, whereas in a chain, the ideal of each size is unique. Moreover, if I is the unique ideal of size t in P, then the complement $P \setminus I$ is the unique filter of size $|P| - t$, which corresponds to a unique ideal in the dual poset $\widetilde{P}$.

Corollary 6.1. *Let C be a code satisfying the assumption of Theorem 6.1. Then, under the dual poset $\widetilde{P} \times \mathscr{P}_M$, the following statements are equivalent:*

(1) $C^{\perp}$ is an $\mathcal{I}^{\mathfrak{C}}$-perfect block LS–poset code;

(2) $C^{\perp}$ is $(s-k)\lfloor M\rceil$-perfect;

(3) $C^{\perp}$ is an MDS block LS–poset code with $|C^{\perp}| = m^{\sum_{i\in\mathbb{M}_f(\mathcal{I})}\pi(i)}$.

Corollary 6.2. *If P is a chain, then for every k, the set $\mathfrak{I}^{k\lfloor M\rceil}$ contains a unique ideal, say $\mathcal{I}_k$. Hence, by Theorem 6.1, a block LS–poset code C is MDS if and only if it is $k\lfloor M\rceil$-perfect, equivalently, $\mathcal{I}_k$-perfect.*

Theorem 6.2. *Let $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ be a block LS–poset code with*

$$d_{\pi_{LS}}(C) = k\lfloor M\rceil + r, \qquad 0 \le r < \lfloor M\rceil.$$

Let $\mathcal{I} \in \mathfrak{I}_\circ(P \times \mathscr{P}_M)$ be the unique ideal with full count such that $\zeta_M(\mathcal{I}) = k\lfloor M\rceil$. Assume that there exists a unique index $i \in [s] \setminus \mathbb{M}_f(\mathcal{I})$ which is closest to $\mathbb{M}_f(\mathcal{I})$ in the sense that $(j, M) \prec (i, \emptyset_M)$ for some $j \in \mathbb{M}_f(\mathcal{I})$, and no element lies strictly between (j, M) and $(i, \emptyset_M)$ in $P \times \mathscr{P}_M$.

Assume, moreover, that m has a unique prime divisor p with exponent one, and define χ by

$$\chi(i) = p^{[1]}, \qquad \chi(\ell) = \emptyset_M \text{ for } \ell \neq i.$$

Set $\mathcal{J} = \mathcal{I}(\chi)$. If C is a partial-MDS block LS–poset code, and the defining maximum is attained at $\mathcal{J}$, then C is $(k\lfloor M\rceil + 1)$-perfect.

Proof. Because p is the unique prime divisor of m with exponent one, we have $\lfloor p^{[1]}\rceil = 1$. Hence,

$$\zeta_M(\mathcal{J}) = \zeta_M(\mathcal{I}) + \lfloor p^{[1]}\rceil = k\lfloor M\rceil + 1.$$

By the uniqueness of $\mathcal{I}$, of the index i, and of the prime p, the ideal $\mathcal{J}$ is the unique ideal in $\mathfrak{I}^{k\lfloor M\rceil+1}$. Because C is partial-MDS, and the defining maximum is attained at $\mathcal{J}$, Theorem 5.3 yields that C is a $\mathcal{J}$-perfect block LS–poset code. Because $\mathfrak{I}^{k\lfloor M\rceil+1} = \{\mathcal{J}\}$, we have

$$B_{\mathcal{J}}(\mathbf{0}) = B_{k\lfloor M\rceil+1}(\mathbf{0}).$$

Therefore, C is $(k\lfloor M\rceil + 1)$-perfect. □

Corollary 6.3. *Let* $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ *be a block LS–poset code, where*

$$m = p_1 p_2^{k_2} \cdots p_t^{k_t}, \qquad k_2, \ldots, k_t > 1.$$

Suppose that $P_1 = ([s], \preceq)$ *is the chain poset,* $1 \prec 2 \prec \cdots \prec s$*. Then, for every* $0 \leq k < s$*, the set* $\mathfrak{J}^{k\lfloor M \rceil}$ *contains a unique ideal. Consequently, if C is a partial-MDS block LS–poset code whose defining maximum is attained at* $\mathcal{J} = \mathcal{I}(\chi)$ *with* $\chi(i) = p_1^{[1]}$*, then C is* $(k\lfloor M \rceil + 1)$*-perfect.*

Corollary 6.4. *Let* $C \subseteq \mathbb{Z}_m^n = \bigoplus_{i=1}^{s} \mathbb{Z}_m^{\pi(i)}$ *be a block LS–poset code, and suppose that* $P_2 = ([s], \preceq)$ *is the star* $\wedge$*-poset defined by* $2, \ldots, s \prec 1$*. Then, the set* $\mathfrak{J}^{k\lfloor M \rceil}$ *contains a unique ideal only when* $k = s - 1$*. In this case, if C is a partial-MDS block LS–poset code whose defining maximum is attained at* $\mathcal{J} = \mathcal{I}(\chi)$ *with* $\chi(1) = p_1^{[1]}$*, then C is* $((s-1)\lfloor M \rceil + 1)$*-perfect.*

7. Conclusions

In this paper, we introduced the block LS–poset metric on $\mathbb{Z}_m^n$ by combining a block decomposition, a poset structure on the block set, and the lattice-of-subgroups viewpoint arising from the prime factorization of m. This construction extends several previously studied metrics in coding theory and provides a unified framework for analyzing block codes over $\mathbb{Z}_m$ in which both order structure and subgroup structure play essential roles.

We established the basic metric properties of the block LS–poset distance and developed the corresponding theory of r-balls and $\mathcal{I}$-balls. In particular, we obtained explicit descriptions of $\mathcal{I}$-balls centered at the zero vector, together with translation, coset, and duality properties. These structural results allowed us to characterize $\mathcal{I}$-perfect block LS–poset codes in both the full-count and partial-count settings. In the partial-count case, the existence of perfect codes is closely related to complementary subgroup decompositions of $\mathbb{Z}_m$, which highlights a distinctive feature of the ring setting.

We further proved a Singleton-type bound for block LS–poset codes and used it to define MDS and partial-MDS block LS–poset codes. A main outcome is that, under suitable hypotheses, MDS codes are equivalent to $\mathcal{I}$-perfect codes for full-count ideals, whereas partial-MDS codes correspond to perfect codes associated with enlarged ideals. For special classes of posets, such as chains and star posets, we also derived consequences for r-perfect codes and clarified when r-perfectness, $\mathcal{I}$-perfectness, and the MDS property coincide.

Several interesting directions remain for future research. One direction is to determine the weight distribution of block LS–poset codes and to establish MacWilliams-type identities which relate a code and its dual under this metric. Another direction is the development of efficient decoding algorithms adapted to the block LS–poset structure, which could lead to practical implementations. A further direction is to extend the present theory to more general poset structures or other finite rings and to classify perfect and partial-MDS block LS–poset codes in these broader settings.

Author contributions

Thitarie Rungratgasame and Phichet Jitjankarn: Conceptualization, methodology, investigation and writing–original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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References

1. J. Ahn, H. K. Kim, J. S. Kim, M. Kim, Classification of perfect linear codes with crown poset structure, *Discrete Math.*, **268** (2003), 21–30. https://doi.org/10.1016/S0012-365X(02)00679-9
2. R. A. Brualdi, J. S. Graves, K. M. Lawrence, Codes with a poset metric, *Discrete Math.*, **147** (1995), 57–72. https://doi.org/10.1016/0012-365X(94)00228-B
3. B. K. Dass, N. Sharma, R. Verma, MDS and I-perfect poset block codes, *Finite Fields Appl.*, **62** (2020), 101620. http://dx.doi.org/10.1016/j.ffa.2019.101620
4. R. G. L. D'Oliveira, M. Firer, The packing radius of a code and partitioning problems: The case for poset metrics on finite vector spaces, *Discrete Math.*, **338** (2015), 2143–2167. https://doi.org/10.1016/j.disc.2015.05.011
5. S. W. Golomb, L. R. Welch, Perfect codes in the Lee metric and the packing of polyominos, *SIAM J. Appl. Math.*, **18** (1970), 302–317. https://doi.org/10.1137/0118025
6. R. W. Hamming, Error detecting and error correcting codes, *Bell Syst. Tech. J.*, **29** (1950), 147–160. https://doi.org/10.1002/j.1538-7305.1950.tb00463.x
7. J. Y. Hyun, H. K. Kim, Maximum distance separable poset codes, *Des. Codes Cryptogr.*, **48** (2008), 247–261. https://doi.org/10.1007/s10623-008-9204-8
8. P. Jitjankarn, Codes with weighted poset metrics based on the lattice of subgroups of $\mathbb{Z}_m$, *Symmetry*, **16** (2024), 1614. http://dx.doi.org/10.3390/sym16121614
9. H. K. Kim, D. S. Krotov, The poset metrics that allow binary codes of codimension m to be m-, $(m-1)$-, or $(m-2)$-perfect, In: *2007 IEEE international symposium on information theory*, France: IEEE, 2007, 1371–1375. https://doi.org/10.1109/ISIT.2007.4557414
10. C. Y. Lee, Some properties of nonbinary error-correcting codes, *IRE Trans. Inform. Theory*, **4** (1958), 77–82. http://dx.doi.org/10.1109/TIT.1958.1057446
11. W. Ma, J. Luo, Codes with respect to weighted poset block metric, *Des. Codes Cryptogr.*, **92** (2024), 341–363. http://dx.doi.org/10.1007/s10623-023-01311-8
12. W. Ma, J. Luo, Block codes in pomset metric over $\mathbb{Z}_m$, *Des. Codes Cryptogr.,* **91** (2023), 3263–3284. http://dx.doi.org/10.1007/s10623-023-01249-x
13. L. Panek, M. Firer, H. K. Kim, J. Y. Hyun, Groups of linear isometries on poset structures, *Discrete Math.*, **308** (2008), 4116–4123. https://doi.org/10.1016/j.disc.2007.08.001

14. L. Panek, J. A. Pinheiro, General approach to poset and additive metrics, *IEEE Trans. Inform. Theory*, **66** (2020), 6823–6834. https://doi.org/10.1109/TIT.2020.2983710
15. A. K. Shriwastva, R. S. Selvaraj, Block codes on pomset metric, *Asian-Eur. J. Math.*, **16** (2023), 2350171. https://doi.org/10.1142/S1793557123501711
16. A. K. Shriwastva, R. S. Selvaraj, Weight distribution of the weighted coordinates poset space and the Singleton bound, *J. Algebra Appl.*, **25** (2026), 2650035. http://dx.doi.org/10.1142/S0219498826500350
17. I. G. Sudha, R. S. Selvaraj, Code with a pomset metric and constructions, *Des. Codes Cryptogr.*, **86** (2018), 875–892.
18. G. S. Irrinki, R. S. Selvaraj, MDS and I-perfect codes in pomset metric, *IEEE Trans. Inform. Theory*, **67** (2021), 1622–1629. https://doi.org/10.1109/TIT.2020.3037782