AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 14715–14734.
DOI: 10.3934/math.2026604
Received: 31 December 2025
Revised: 11 March 2026
Accepted: 17 April 2026
Published: 25 May 2026

Research article

h-Almost conformal η-Ricci–Bourguignon solitons on Bulk viscous fluid string spacetimes with anti-torqued vector fields

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Abstract: The present paper investigates the role of anti-torqued vector fields in bulk viscous fluid string (BVFS) space-times characterized by the total pressure α, energy density ρ, and string tension β, admitting an h-almost conformal η-Ricci-Bourguignon soliton (h-ACERBS). The analysis is carried out within the framework of BVFS spacetimes incorporating bulk viscosity together with an anti-torqued vector field, where the underlying metric satisfies the h-ACERBS condition. Particular emphasis is placed on the physical significance of the conformal pressure $\bar{p}$ arising in the presence of such solitons and vector fields. Within this geometric and physical setting, the validity of the standard energy conditions is examined, along with criteria related to black-hole formation and the implications of Penrose's singularity theorem. In addition, generalized Liouville and Poisson equations associated with the h-ACERBS structure are derived and analyzed. The harmonic properties of h-ACERBS on BVFS space-times endowed with anti-torqued vector fields are also investigated, providing further insight into the interplay between the underlying geometric structures and their physical interpretations.

Keywords: Lorentzian spacetime manifold; Bulk viscous fluid string space-times; anti-torqued vector fields; Ricci–Bourguignon solitons
Mathematics Subject Classification: 53B30, 53C25, 53C44, 53C50, 53C80

1. Introduction

The general theory of relativity (GTR) furnishes the fundamental and most complete theoretical framework for investigating the large-scale geometry and dynamical evolution of the universe. Einstein's gravitational field equations [24, 28] form the cornerstone of modern cosmology, providing the mathematical foundation for constructing a wide class of cosmological solutions, including the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which serves as the basis of the standard cosmological paradigm. In order to reconcile theoretical models with observational data, contemporary cosmology postulates the presence of a non-luminous and non-baryonic component referred to as dark matter (DM) [23]. In addition, the universe is believed to be permeated by dark energy (DE) [37], which is widely recognized as the dominant factor responsible for the observed accelerated expansion of the universe and plays a crucial role in determining its large-scale matter-energy content. Collectively, these two components account for the majority of the cosmic energy density and significantly influence the formation, evolution, and ultimate fate of the universe.

The geometric configuration of spacetime is determined by the Einstein field equations (EFE), which, in the absence of a cosmological constant, are expressed as [22]

$$\mathrm{K}(t_1, t_2) - \frac{\tau}{2} g(t_1, t_2) = \kappa \, \mathrm{T}(t_1, t_2), \tag{1.1}$$

where g represents the pseudo-Riemannian metric tensor and T denotes the energy-momentum tensor associated with the matter content of the universe. The Ricci tensor K and the scalar curvature τ encapsulate the intrinsic curvature properties of spacetime. The universal gravitational constant $\mathcal{G}$ enters the formulation through the gravitational coupling constant $\kappa = 8\pi\mathcal{G}$.

An intriguing phenomenon in quantum optics [20] demonstrates that the motion of atoms interacting with light, particularly within the context of optical molasses, closely resembles the dynamics of particles moving through a viscous fluid. By analogy, one can envisage a form of gravitational molasses, arising from quantum fluctuations in the bulk viscosity of spacetime. These fluctuations are hypothesized to be linked to the cosmological constant (CC) and may contribute significantly to the emergence of this effect.

An alternative viewpoint, frequently explored in the literature, regards the CC as arising from the energy density associated with quantum vacuum fluctuations, which can be expressed as

$$\Lambda = \frac{8\pi\mathcal{G}\mathcal{H}_0}{c^4}, \tag{1.2}$$

where $\mathcal{H}_0$ represents the vacuum expectation value. From this perspective, a non-vanishing CC gives rise to a dissipative mechanism analogous to matter-energy interactions, wherein the bulk viscosity of spacetime is regulated by a constant scalar curvature. Conversely, the absence of a cosmological constant leads to a Ricci-flat spacetime configuration [29].

Within the theoretical framework of the GTR and contemporary cosmology, spacetime is commonly modeled as a four-dimensional, connected, and time-oriented Lorentzian manifold. Such pseudo-Riemannian manifolds are equipped with a Lorentzian metric of signature $(-, +, +, +)$, which provides a clear distinction between temporal and spatial directions [21, 22]. The geometric formulation of Lorentzian manifolds constitutes the mathematical foundation for describing vector fields, causal structures, and geodesic motion in curved spacetime, thereby playing a fundamental role in the formulation and analysis of GTR.

A quasi-Einstein Lorentzian manifold is said to represent a perfect fluid spacetime if its Ricci tensor K of type $(0,2)$ can be expressed in the form [13, 17]

$$\mathrm{K} = \gamma_1\, g(t_1, t_2) + \gamma_2\, \eta(t_1)\eta(t_2), \tag{1.3}$$

where γ_1 and γ_2 denote smooth scalar functions, and η is a 1-form associated with the velocity vector field ξ, satisfying the condition $\eta(\xi) = -1$. In this setting, the vector field ξ is unit time-like.

Following [9, 24, 28, 30], we recall the subsequent definition. A non-flat Riemannian or pseudo-Riemannian manifold $(\mathbb{N}^n, g)$, with dimension $n > 2$, is termed a generalized quasi-Einstein manifold (abbreviated as $\mathrm{G(QE)}_n$) if its Ricci tensor is non-vanishing and satisfies

$$\mathrm{K}(t_1, t_2) = \gamma_1 g(t_1, t_2) + \gamma_2 \eta(t_1)\eta(t_2) + \gamma_3 \theta(t_1)\theta(t_2),$$

where γ_1, γ_2, and γ_3 are non-zero scalar functions with $\gamma_2 \neq 0$ and $\gamma_3 \neq 0$. The 1-forms η and θ are non-vanishing and satisfy

$$\begin{aligned} g(t_1, \xi) &= \eta(t_1), \;\; \eta(\xi) = -1, \\ g(t_1, \zeta) &= \theta(t_1) \;\; \theta(\zeta) = 1, \\ g(\xi, \zeta) &= 0, \end{aligned}$$

for any vector field $t_1 \in \mathfrak{X}(\mathbb{N}^n, g)$. In this formulation, the 1-forms η and θ are associated with the unit time-like and space-like vector fields ξ and ζ, respectively, which are mutually orthogonal. Furthermore, the vector fields ξ and ζ act as generators of the manifold $(\mathbb{N}^n, g)$. The underlying geometric structure reduces to that of a perfect fluid spacetime in the special case when $\gamma_3 = 0$.

The notion of the Ricci flow was first introduced by R. S. Hamilton in the early 1980s as a natural extension of the pioneering work of Eells and Sampson on the harmonic map heat flow. Their study of geometric evolution under heat-type equations motivated Hamilton to apply similar principles to the intrinsic geometry of Riemannian manifolds. The Ricci flow [11, 14] describes the deformation of a Riemannian metric g_{ij} according to its Ricci curvature,

$$\frac{\partial}{\partial t} g = -2\,\mathrm{K}, \tag{1.4}$$

where K denotes the Ricci curvature tensor. This evolution equation acts as a geometric analogue of the heat equation, tending to regularize the curvature distribution over time. Hamilton's formulation introduced a fundamental analytical framework in differential geometry, which subsequently played a pivotal role in Perelman's resolution of the Poincaré and geometrization conjectures [11, 14].

The concept of a Ricci soliton naturally emerges as a generalization of Einstein metrics and can be interpreted as a self-similar solution to the Ricci flow [39]. On a pseudo-Riemannian manifold $(\mathbb{N}, g)$, a Ricci soliton is defined by

$$\mathrm{K}(t_1, t_2) + \tfrac{1}{2}\, \mathfrak{L}_Z g(t_1, t_2) = \pi_1\, g(t_1, t_2), \tag{1.5}$$

where K denotes the Ricci tensor associated with $(\mathbb{N}, g)$, $\mathfrak{L}_Z$ is the Lie derivative of the metric g along the vector field Z, and $\pi_1 \in \mathbb{R}$ is a real constant. In the particular case where $Z = \nabla\psi$ for a smooth function ψ on $(\mathbb{N}, g)$, the soliton is said to be of gradient type. According to the sign of π_1, Ricci solitons are classified as shrinking ($\pi_1 > 0$), steady ($\pi_1 = 0$), or expanding ($\pi_1 < 0$).

According to [3], a conformal Ricci soliton is defined by

$$\mathrm{K}(t_1, t_2) + \frac{1}{2}\mathfrak{L}_Z g(t_1, t_2) + \left[\pi_1 - \frac{1}{2}\left(\bar{p} + \frac{2}{n}\right)\right] g(t_1, t_2) = 0, \tag{1.6}$$

where $\bar{p}$ is a scalar, non-dynamical (time-dependent) field, and π_1 is a constant.

The Ricci-Bourguignon (RB) flow generalizes the Ricci flow by incorporating a scalar curvature term that modifies the evolution of the metric. It is defined as an evolution equation for metrics on $(\mathbb{N}, g)$ given by [6]

$$\frac{\partial g}{\partial t} = -2\left(\mathrm{K} - \tau\omega g\right), \tag{1.7}$$

where K denotes the Ricci tensor, τ represents the scalar curvature of the Riemannian metric g_{ij}, and $\omega \in \mathbb{R}$ is a constant parameter controlling the influence of the scalar curvature term. As shown in [10], for special values of ω, Eq 1.7 corresponds to different well-known soliton types: (i) $\omega = \frac{1}{2}$: Einstein soliton, $\mathrm{K} - \frac{\tau}{2}g$; (ii) $\omega = \frac{1}{n}$: traceless Ricci soliton, $\mathrm{K} - \frac{\tau}{n}g$; (iii) $\omega = \frac{1}{2(n-1)}$: Schouten soliton, $\mathrm{K} - \frac{\tau}{2(n-1)}g$; (iv) $\omega = 0$: Ricci soliton.

The notion of a RB soliton generalizes the classical Ricci soliton within the setting of the RB flow, describing self-similar solutions associated with this extended geometric evolution. A pseudo-Riemannian manifold $(\mathbb{N}, g)$ is said to admit a RB soliton if there exists a smooth vector field Z such that [10]

$$\mathrm{K}(t_1, t_2) + \tfrac{1}{2}\mathfrak{L}_Z g(t_1, t_2) = (\pi_1 + \tau\omega)\, g(t_1, t_2), \tag{1.8}$$

where K denotes the Ricci tensor of $(\mathbb{N}, g)$, τ represents the scalar curvature, $\omega \in \mathbb{R}$ is a constant, π_1 is a real parameter, and $\mathfrak{L}_Z$ stands for the Lie derivative of the metric g along the vector field Z.

If $Z = \nabla\Psi$ for some smooth function Ψ defined on $(\mathbb{N}, g)$, then the metric g is referred to as a gradient RB soliton (briefly, GRBS). In this case, Eq (1.8) reduces to

$$\mathrm{K}(t_1, t_2) + \nabla\nabla\Psi = (\pi_1 + \tau\omega) g(t_1, t_2). \tag{1.9}$$

Furthermore, a metric g_{ij} on $(\mathbb{N}, g)$ is said to define an h-almost conformal RB soliton (h-ACRBS) if there exists a smooth vector field Z satisfying

$$\mathrm{K}(t_1, t_2) + \frac{h}{2}\mathfrak{L}_Z g(t_1, t_2) = \left[\pi_1 - \frac{1}{2}\left(\bar{p} + \frac{2}{n}\right) + \tau\omega\right] g(t_1, t_2), \tag{1.10}$$

where π_1 and h are smooth functions, $\bar{p}$ denotes a scalar non-dynamical (time-dependent) field, τ represents the scalar curvature, and $\omega \in \mathbb{R}$ is a constant.

In particular, when $Z = \nabla\Psi$ for some smooth function Ψ defined on $(\mathbb{N}, g)$, Eq (1.10) characterizes an h-almost gradient conformal Ricci-Bourguignon soliton, which can be written as

$$\mathrm{K} + h\nabla\nabla\Psi = \left[\pi_1 - \frac{1}{2}\left(\bar{p} + \frac{2}{n}\right) + \tau\omega\right] g(t_1, t_2). \tag{1.11}$$

Analogously to Eqs 1.10 and 1.11, we recall the notions of an h-almost conformal η-Ricci-Bourguignon soliton (h-ACERBS) and a gradient h-ACERBS on $(\mathbb{N}, g)$, respectively:

$$\mathrm{K}(t_1, t_2) + \frac{h}{2}\mathfrak{L}_Z g(t_1, t_2) = \left[\pi_1 - \frac{1}{2}\left(\bar{p} + \frac{2}{n}\right) + \tau\omega\right] g(t_1, t_2) + \pi_2\eta(t_1)\eta(t_2), \tag{1.12}$$

$$K(t_1, t_2) + h\nabla\nabla\Psi = \left[\pi_1 - \frac{1}{2}\left(\bar{p} + \frac{2}{n}\right) + \tau\omega\right] g(t_1, t_2) + \pi_2\eta(t_1)\eta(t_2), \tag{1.13}$$

where π_2 is a real constant and η is a 1-form.

Anti-torqued vector fields play a significant role in modern differential geometry and cosmology, particularly in the study of anisotropic spacetimes. These vector fields are characterized by the property that their associated torque vanishes, which imposes specific geometric constraints on the underlying manifold. In the context of cosmological models, anti-torqued vector fields can influence the evolution of the spacetime structure, affecting the distribution of matter and energy, as well as the dynamics of cosmic strings and viscous fluids. Their incorporation into the study of solitonic structures, such as h-ACERBS, allows for a deeper understanding of the interplay between geometric flows and physical properties of the universe, providing new insights into both theoretical and observational aspects of general relativity.

The study of geometric solitons in relativistic fluid spacetimes has received considerable attention in recent years. In particular, Siddiqi and collaborators examined several classes of solitonic structures in relativistic fluid and string spacetimes [18, 31–36]. The notion of bulk viscous fluid string (BVFS) spacetime was introduced by Siddiqi and Al-Dayel [23]. Along similar lines, various authors have investigated the geometric and physical properties of spacetime through soliton structures (see [2, 26, 27]). Within this framework, the present work examines BVFS-spacetimes endowed with anti-torqued vector fields when the underlying metric admits h-ACERBS and gradient h-ACERBS, thereby extending the geometric analysis of such spacetime models.

2. Bulk viscous fluid string spacetime (BVFS-spacetime)

In this section, we explore the fundamental aspects of a spacetime model characterized by an energy-momentum tensor (EMT) that describes a bulk viscous fluid with string-like properties, hereafter referred to as a bulk viscous fluid string spacetime (BVFS-spacetime). The notion of BVFS spacetime was first introduced by Siddiqi and Al-Dayel [23], providing a framework for describing relativistic viscous fluids coupled with cosmic string distributions. Such models are of significant interest in cosmology, astrophysics, and theoretical physics due to their ability to describe early-universe conditions and exotic matter configurations.

A relativistic bulk viscous fluid is an imperfect fluid that arises in relativistic regimes involving high velocities or strong gravitational fields. It exhibits bulk viscosity, a property that accounts for the resistance to uniform expansion or contraction of the fluid. Unlike perfect fluids, such imperfect fluids incorporate dissipative effects, making them more realistic in modeling entropy-producing processes such as cosmic inflation or the late-time accelerated expansion of the universe.

String-like structures in spacetime arise naturally in various high-energy theories, including string theory and cosmological models involving topological defects. When coupled with bulk viscosity, these configurations provide a richer framework for analyzing anisotropies, structure formation, and the dynamics of the universe under non-equilibrium conditions.

Definition 2.1. *A four-dimensional spacetime manifold* $(\mathbb{N}^4, g)$ *characterized by a symmetric EMT* $\mathcal{T}$ *representing a bulk viscous fluid string is referred to as a BVFS-spacetime.*

The EMT corresponding to a BVFS-spacetime is defined as [23]

$$\mathcal{T}(t_1, t_2) = (\alpha + \rho)\eta(t_1)\eta(t_2) - \alpha g(t_1, t_2) - \beta\theta(t_1)\theta(t_2), \tag{2.1}$$

where $t_1, t_2 \in \mathfrak{X}(\mathbb{N}^4, g)$. In this expression, η denotes the four-velocity vector field, while θ is a space-like vector field indicating the anisotropic direction associated with the string. These vector fields satisfy the normalization and orthogonality conditions

$$\eta^i\eta_i = -\theta^i\theta_i = -1,\ \eta^i\theta_i = 0, \tag{2.2}$$

which guarantee that η is time-like and θ is space-like and orthogonal to η. Here, β represents the string tension density, and ρ denotes the rest energy density.

Moreover, the total pressure of the system, which accounts for both the isotropic pressure and the bulk viscous effects, is defined as

$$\alpha = \alpha^\circ - 3\zeta H, \tag{2.3}$$

where α° is the isotropic pressure, ζ is the bulk viscosity coefficient, and H is the Hubble parameter. Using the total pressure, the equation of state (EoS) relates α to the energy density of the fluid:

$$\alpha = \epsilon\rho\,, \quad \alpha^\circ = \epsilon_0\rho, \tag{2.4}$$

where ϵ_0 and ϵ are constants. This formulation ensures that all contributions to the pressure, including viscous effects, are correctly incorporated into the cosmological dynamics.

In cosmology, the equation-of-state parameter ϵ characterizes different evolutionary phases of the universe. Specifically, $\epsilon = -1$ corresponds to the vacuum-dominated era associated with a cosmological constant, $\epsilon = 0$ represents the matter-dominated era (dust), $\epsilon = 1/3$ describes the radiation-dominated era, and $\epsilon = 1$ signifies the stiff fluid era. These values of ϵ play a central role in determining the expansion behavior of the universe at different stages.

The EMT plays a fundamental role in describing the matter content of spacetime, particularly when the matter is modeled as a fluid endowed with properties such as energy density, pressure, and tension, as well as various dynamical and kinematic characteristics, including velocity, acceleration, shear, and expansion [21, 37]. In classical cosmological models, the matter content of the universe is typically treated as a fluid spacetime [18,33,35,41]. Recently, there has been growing interest in investigating the evolution of cosmological models from the early, high-energy universe to its current state by integrating concepts from string theory with dissipative processes such as bulk viscosity. This provides a more realistic framework for modeling anisotropic and non-equilibrium cosmological scenarios.

The passage from a conventional spacetime manifold to a BVFS spacetime constitutes a significant conceptual refinement in the modeling of the early universe, wherein the dynamics are governed simultaneously by topological defects, such as cosmic strings, and by dissipative fluid properties, including viscosity. The combined influence of these elements modifies the geometric structure of spacetime and yields a more comprehensive physical framework for the analysis of cosmological phenomena. In particular, this perspective facilitates a deeper understanding of large-scale structure formation, the observed accelerated expansion of the universe, and the anisotropies present in the cosmic microwave background radiation.

In a conventional spacetime manifold, events are localized using coordinate systems defined by vector fields, while the curvature of spacetime is determined by the distribution of mass and energy through the EFE. In contrast, a BVFS spacetime incorporates one-dimensional topological defects, namely cosmic strings, which are hypothesized to have formed during symmetry-breaking phase transitions in the early universe. These strings, possessing extremely high energy densities, are believed to act as gravitational seeds for the formation of galaxies and other large-scale structures. This transition involves adopting a cosmological model that blends the geometric foundation of general relativity with the thermodynamic behavior of a viscous fluid. The presence of bulk viscosity plays a critical role in modifying the universe's expansion rate, influencing matter distribution, and altering the overall spacetime geometry. Such dissipative effects make the model more realistic in describing entropy production and late-time cosmic acceleration.

To investigate such transitions, anisotropic cosmological models are commonly employed, including those based on Bianchi type-I and Kantowski-Sachs geometries, which offer sufficient flexibility to incorporate both viscous effects and the influence of cosmic strings [16, 23]. These models provide important insights into the evolutionary behavior of the early universe and serve as a bridge between high-energy theoretical physics and observational cosmology.

In recent years, a substantial body of work has been devoted to the analysis of diverse spacetime models within the framework of the GTR [31,33,35,40]. Such investigations have yielded meaningful contributions to the understanding of the geometric and physical properties of the universe, thereby establishing a robust foundation for further studies of BVFS spacetimes in the context of GTR. Inspired by these advances, it is both pertinent and timely to conduct a detailed examination of relativistic BVFS spacetimes, with the objective of clarifying their possible influence on the dynamical evolution of the universe.

3. Characteristics of BVFS-spacetime with anti-torqued vector field

A unit vector field ξ on a Riemannian manifold $(\mathbb{N}, g)$ is termed an anti-torqued vector field if its covariant derivative admits the decomposition

$$\nabla_{t_1}\xi = \psi(t_1 - \eta(t_1)\xi), \tag{3.1}$$

for any vector field t_1 on $\mathbb{N}^4$, where η is the one-form dual to the unit anti-torqued vector field ξ, and ψ is a non-zero smooth function. This structure imposes a highly rigid geometry on the flow of ξ; it is immediately geodesic, and its failure to be a Killing field is precisely quantified by the function ψ, which governs the shear and expansion of the distribution orthogonal to ξ [7]. The introduction of an anti-torqued vector field imposes a specific geometric structure on the spacetime manifold. In particular, it constrains the covariant derivative of the velocity field and modifies the behavior of curvature tensors. This structure influences the evolution of the soliton geometry and may lead to new classes of cosmological solutions characterized by anisotropic expansion and modified energy conditions.

Based on the above formulation, we can now establish the following results.

Proposition 3.1. *On a BVFS-spacetime coupled with an anti-torqued vector field ξ, the following relations hold:*

$$(\nabla_{t_1}\eta)(t_2) = \psi[g(t_1, t_2) - \eta(t_1)\eta(t_2)], \tag{3.2}$$

$$\eta(\nabla_{\xi}\xi) = -2\psi, \qquad \nabla_{\xi}\xi = 2\psi\xi, \tag{3.3}$$

$$
\begin{aligned}
\mathrm{R}(t_1,t_2)\xi &= \psi^2[\eta(t_1)t_2-\eta(t_2)t_1]+t_1(\psi)[t_2-\eta(t_2)\xi] \\
&\quad -t_2(\psi)[t_1-\eta(t_1)\xi], && (3.4)\\
\mathrm{R}(t_1,\xi)\xi &= \psi^2[t_1+\eta(t_1)\xi]+2t_1(\psi)\xi-\xi(\psi)[t_1-\eta(t_1)\xi], && (3.5)\\
\eta(\mathrm{R}(t_1,t_2)t_3) &= \psi^2[\eta(t_2)g(t_1,t_3)-\eta(t_1)g(t_2,t_3)] \\
&\quad -t_1(\psi)[g(t_2,t_3)-\eta(t_2)\eta(t_3)]+t_2(\psi)[g(t_1,t_3)-\eta(t_1)\eta(t_3)], && (3.6)\\
(\mathfrak{L}_\xi g)(t_1,t_2) &= 2\psi[g(t_1,t_2)-\eta(t_1)\eta(t_2)], && (3.7)
\end{aligned}
$$

where $\mathfrak{L}_\xi$ denotes the Lie derivative with respect to the vector field ξ.

Proof. First, we calculate

$$(\nabla_{t_1}\eta)(t_2)=t_1(\eta(t_2))-\eta(\nabla_{t_1}t_2)=t_1(g(t_2,\xi))-g(\nabla_{t_1}t_2,\xi).$$

Using the metric compatibility, we get

$$(\nabla_{t_1}\eta)(t_2)=g(t_2,\nabla_{t_1}\xi)=\psi[g(t_1,t_2)-\eta(t_1)\eta(t_2)],$$

which proves Eq 3.2. Moreover, since $(\nabla_\xi\eta)(t_1)=0$ and using Eq 3.1, the result follows.

Next, since

$$\mathrm{R}(t_1,t_2)\xi=\nabla_{t_1}\nabla_{t_2}\xi-\nabla_{t_2}\nabla_{t_1}\xi-\nabla_{[t_1,t_2]}\xi,$$

and applying Eq 3.1, after simplification we obtain Eqs 3.4, 3.5 and 3.6 follow directly from Eq 3.4 by specialization.

Finally, differentiating g with respect to ξ and using Eq 3.2, we derive

$$(\mathfrak{L}_\xi g)(t_1,t_2)=2\psi[g(t_1,t_2)-\eta(t_1)\eta(t_2)],$$

which proves Eq 3.7. Hence, the proof is complete. □

With the help of Eqs 1.1 and 2.1, the EFE for a relativistic BVFS spacetime can be written as

$$\mathrm{K}(v_1,v_2)=\left(\frac{\tau}{2}-\kappa\alpha\right)g(v_1,v_2)+\kappa(\alpha+\rho)\eta(v_1)\eta(v_2)-\kappa\beta\,\theta(v_1)\theta(v_2). \tag{3.8}$$

Contracting Eq 3.8 yields

$$\tau=\kappa(3\alpha-\rho-\beta). \tag{3.9}$$

By substituting Eq (3.9) into Eq (3.8), we obtain

$$\mathrm{K}(t_1,t_2)=\frac{\kappa}{2}[\alpha-\rho-\beta]g(t_1,t_2)+\kappa(\alpha+\rho)\eta(t_1)\eta(t_2)-\kappa\beta\,\theta(t_1)\theta(t_2), \tag{3.10}$$

which further yields

$$\mathrm{Q}t_1=\frac{\kappa}{2}[\alpha-\rho-\beta]t_1+\kappa(\alpha+\rho)\eta(t_1)\xi-\kappa\beta\,\theta(t_1)\zeta. \tag{3.11}$$

Theorem 3.1. *A BVFS-spacetime satisfying the EFE in the absence of a cosmological constant constitutes a $G(QE)$-spacetime with*

$$\gamma_1=\frac{\tau}{2}-\kappa\alpha,\ \gamma_2=\kappa(\alpha+\rho),\ \gamma_3=-\kappa\beta,$$

where α denotes the total pressure, ρ represents the energy density, and β corresponds to the string tension.

Corollary 3.1. *If a BVFS-spacetime obeys the EFE without a cosmological constant, then the scalar curvature τ is given by*

$$\tau = \kappa(3\alpha - \rho - \beta).$$

Here, α denotes the total pressure, ρ is the energy density, and β represents the string tension.

In view of [29], and by making use of Theorem 3.1 together with Corollary 3.1, we arrive at the following conclusion.

Corollary 3.2. *If a BVFS-spacetime satisfies the EFE without a cosmological constant, then the spacetime reduces to a bulk viscous fluid spacetime.*

Moreover, from Eq (3.9), we derive the relation

$$\alpha = \frac{\rho}{3} + \frac{1}{3\kappa}(\tau + \kappa\beta). \tag{3.12}$$

Consequently, we obtain the following result.

Theorem 3.2. *If a BVFS-spacetime obeys the EFE without a cosmological constant, then the is given by*

$$\alpha = \frac{\rho}{3} + \frac{1}{3\kappa}(\tau + \kappa\beta).$$

where α denotes the total pressure, ρ is the energy density, and β represents the string tension.

We now consider the case in which the source corresponds to radiation, that is, the EoS parameter satisfies $\epsilon = \frac{1}{3}$. Under this assumption, Equation (3.12) yields

$$\tau + \kappa\beta = 0. \tag{3.13}$$

Corollary 3.3. *If the source of a BVFS-spacetime is of radiation type, namely when the EoS parameter is $\epsilon = 1/3$, then the source quantities τ and β are not independent and are constrained by the relation $\tau + \kappa\beta = 0$. Under this condition, the EoS $\alpha = \rho/3$ is automatically satisfied, while the energy density ρ remains a free dynamical variable.*

In the vacuum-dominated epoch, the EoS takes the form $\rho = -\alpha$, which leads to

$$\alpha = -\rho = \frac{\tau + \kappa\beta}{4\kappa}. \tag{3.14}$$

equation of state

Corollary 3.4. *If a BVFS-spacetime is governed by a vacuum-dominated era, then the total pressure and energy density are determined by Eq* (3.14).

Corollary 3.5. *If the source of a BVFS-spacetime corresponds to dust matter, then the associated energy density is given by*

$$\rho = -\left(\frac{\tau}{\kappa} + \beta\right).$$

Corollary 3.6. *If the source of a BVFS-spacetime is of stiff fluid type, then the total pressure and energy density satisfy*

$$\alpha = \rho = \frac{\tau + \kappa\beta}{2\kappa}.$$

4. h-ACERBS on BVFS-spacetime admitting anti-torqued vector field

In this section, we investigate h-ACERBS of a BVFS-spacetime in which the conformal vector field ξ coincides with the time-like velocity vector field.

For a fixed $z = \xi$, Eq 1.12 becomes

$$\mathrm{K}(t_1, t_2) + \frac{h}{2}\mathfrak{L}_{\xi}g(t_1, t_2) = [\pi_1 - \frac{1}{2}(\bar{p} + \frac{2}{n}) + \tau\omega]g(t_1, t_2) + \pi_2\eta(t_1)\eta(t_2). \tag{4.1}$$

Using Eq 3.1, we obtain

$$\mathrm{K}(t_1, t_2) = [\pi_1 - \frac{1}{2}(\bar{p} + \frac{2}{n}) + \tau\omega - \psi h]g(t_1, t_2) + (\pi_2 + \psi h)\eta(t_1)\eta(t_2). \tag{4.2}$$

Substituting Eq 3.10 into Eq 4.1, we arrive at

$$\begin{aligned} 0 \;\; = \;\; & [\frac{\kappa}{2}(\alpha - \rho - \beta) - \pi_1 + \frac{1}{2}(\bar{p} + \tfrac{1}{2}) - \tau\omega + \psi h]g(t_1, t_2) \\ & + [\kappa(\alpha + \rho) - \pi_2 - \psi h]\eta(t_1)\eta(t_2) - \kappa\beta\theta(t_1)\theta(t_2). \end{aligned} \tag{4.3}$$

Now, plugging $t_1 = t_2 = \xi$ in Eq 4.3, we find

$$\pi_1 = -\frac{\kappa}{2}(\alpha + 3\rho + \beta) + \frac{1}{2}(\bar{p} + \tfrac{1}{2}) - \omega\tau + 2\psi h + \pi_2. \tag{4.4}$$

Thus, referring to Eqs 4.4 and 4.2, we may state the following result.

Theorem 4.1. *A BVFS-spacetime coupled with an anti-torqued vector field ξ admitting an h-ACERBS is a perfect fluid spacetime.*

Again, in view of Eq 4.4, we can articulate the following result:

Theorem 4.2. *Let a BVFS-spacetime be endowed with total pressure α, energy density ρ, and string tension β, and coupled with an anti-torqued vector field ξ admitting an ACERBS. Then the soliton is expanding, steady, or shrinking according to the following conditions:*

1. *Expanding if*
$$\frac{1}{2}\left(\bar{p} + \frac{1}{2}\right) < \frac{\kappa}{2}(\alpha + 3\rho + \beta) + \omega\tau - 2\psi h - \pi_2,$$
2. *Steady if*
$$\frac{1}{2}\left(\bar{p} + \frac{1}{2}\right) = \frac{\kappa}{2}(\alpha + 3\rho + \beta) + \omega\tau - 2\psi h - \pi_2,$$
3. *Shrinking if*
$$\frac{1}{2}\left(\bar{p} + \frac{1}{2}\right) > \frac{\kappa}{2}(\alpha + 3\rho + \beta) + \omega\tau - 2\psi h - \pi_2.$$

5. Physical significance of conformal pressure in BVFS-spacetimes

The time-dependent scalar field $\bar{p}$ is referred to as the conformal pressure, whereas in classical fluid mechanics the physically meaningful pressure is the quantity that ensures the incompressibility of the fluid. As noted in [3], the conformal pressure $\bar{p}$ assumes negative values away from equilibrium and vanishes precisely at an equilibrium state. Furthermore, the metric g corresponds to an equilibrium configuration, or equivalently an Einstein manifold, thereby providing a nonlinear restoring mechanism. Consequently, Equation (4.4) enables the explicit determination of the conformal pressure. This leads to the following result.

Theorem 5.1. *If a BVFS-spacetime endowed with total pressure α, energy density ρ, and string tension β admits an h-ACERBS such that ξ is an anti-torqued vector field, then the conformal pressure is given by*

$$\bar{p} = \kappa(\alpha + 3\rho + \beta) + 2[\pi_1 + \omega\tau - 2\psi h - \pi_2 - \frac{1}{4}].$$

Since the conformal pressure satisfies $\bar{p} = 0$ at equilibrium, the following consequence follows immediately.

Corollary 5.1. *If a BVFS-spacetime endowed with total pressure α, energy density ρ, and string tension β admits an h-ACERBS such that ξ is an anti-torqued vector field, then the metric g represents an equilibrium configuration (or equivalently, an Einstein metric) if and only if*

$$\alpha + 3\rho + \beta = -\frac{2}{\kappa}[\pi_1 + \omega\tau - 2\psi h - \pi_2 - \frac{1}{4}].$$

Moreover, the condition that the metric g is an equilibrium configuration is equivalent to it being an Einstein metric, which acts as a nonlinear restoring force [12]. This observation yields the following statement.

Corollary 5.2. *Let a BVFS-spacetime endowed with total pressure α, energy density ρ, and string tension β admit an h-ACERBS such that ξ is an anti-torqued vector field. Then the metric g is an equilibrium point and acts as a nonlinear restoring force.*

6. Energy conditions in BVFS-spacetime coupled with anti-torqued vector field and admits *h*-ACERBS

In this section, with reference to [27], let the Ricci tensor K in the BVFS-spacetime satisfy the restriction

$$\mathrm{K}(\xi, \xi) > 0, \tag{6.1}$$

where $\xi \in \mathfrak{X}(\mathbb{N}^4)$. In this case, Eq 6.1 is called the time-like convergence condition (briefly, TCC). From Eq 4.2, it follows that

$$\mathrm{K}(\xi, \xi) = \pi_2 - \pi_1 + \frac{1}{2}\left(\bar{p} + \frac{2}{n}\right) - \tau\omega + 2\psi h. \tag{6.2}$$

The BVFS-spacetime admits an h-ACRBS with an anti-torqued vector field ξ obeying the TCC if $\mathrm{K}(\xi, \xi) > 0$. Therefore, using Eqs 3.9 and 6.2, we obtain

$$\pi_1 < \pi_2 + \left(\bar{p} + \frac{2}{n}\right) - \kappa\omega\,(3\alpha - \rho - \beta) + 2\psi h. \tag{6.3}$$

Thus, from Eq 6.3, we can state the following result:

Theorem 6.1. *If a BVFS-spacetime admits an h-ACERBS with an anti-torqued vector field ξ and satisfies the time-like convergence condition (TCC), then the soliton is always shrinking.*

According to [15], the TCC implies the cosmological strong energy condition (SEC). Moreover, the TCC also implies the null convergence condition (NCC), while the SEC implies the null energy condition (NEC). Consequently, these relations establish that TCC implies NCC. Thus, together with this fact and Theorem 6.1, we can state the following result.

Theorem 6.2. *If a BVFS-spacetime admits a shrinking h-ACERBS with an anti-torqued vector field ξ, and if Eq 6.3 holds, then the spacetime satisfies the SEC.*

Corollary 6.1. *If a BVFS-spacetime admits a shrinking h-ACERBS with an anti-torqued vector field ξ, and if Eq 6.3 holds, then the spacetime satisfies the NCC.*

Corollary 6.2. *If a BVFS-spacetime admits an h-ACERBS with an anti-torqued vector field ξ and satisfies the SEC, then the Ricci tensor* K *is of the second Segre type [44].*

The relation between energy conditions, Ricci-type solitons, and Penrose singularity theorems has recently been investigated by Siddiqi and Fatima [34]. By employing Penrose's singularity theorem, Vilenkin and Wall [38] demonstrated that any spacetime satisfying the NCC necessarily admits the formation of black holes, together with the existence of trapped surfaces exterior to these black holes within the spacetime manifold.
Making use of this result in conjunction with Theorem 6.1 and Corollary 6.1, we arrive at the following conclusion.

Theorem 6.3. *If a BVFS-spacetime admits an h-ACERBS with an anti-torqued vector field ξ and satisfies the NCC, then the bulk viscous spacetime contains black holes with trapped surfaces lying outside these black holes.*

Corollary 6.3. *If a BVFS-spacetime admits a shrinking h-ACERBS with an anti-torqued vector field ξ and satisfies the NCC, then the bulk viscous spacetime contains black holes with trapped surfaces located outside these black holes.*

7. Generalized Liouville equation on BVFS-spacetime attached with anti-torqued vector field

Let a BVFS-spacetime be equipped with an anti-torqued vector field ξ and admit an h-ACERBS. Then, from Eqs 1.12 and 3.10, we have

$$0 = \left[\frac{\kappa}{2}(\alpha - \rho - \beta) - \pi_1 + \frac{1}{2}(\bar{p} + \frac{1}{2}) - \tau\omega\right] g(t_1, t_2)$$

$$+[\kappa(\alpha+\rho)-\pi_2]\eta(t_1)\eta(t_2)-\kappa\beta\,\theta(t_1)\theta(t_2)+\frac{h}{2}[g(\nabla_{t_1}\xi,t_2)+g(t_1,\nabla_{t_2}\xi)], \tag{7.1}$$

for any $t_1, t_2 \in \mathfrak{X}(\mathbb{N}^4)$. On contracting Eq 7.1, we obtain

$$h\,\mathrm{Div}(\xi) = -\kappa(3\alpha-\rho-3\beta)+4\pi_1+\pi_2-(2\bar{p}+1)+4\tau\omega. \tag{7.2}$$

Now, for a smooth function $\psi^\star \in C^\infty(\mathbb{N}^4, g)$ and a vector field t_1, we have

$$\mathrm{Div}(\psi^\star t_1) = t_1(d\psi^\star)+\psi^\star\,\mathrm{Div}(t_1), \tag{7.3}$$

where $\psi^\star$ is referred to as the last Lagrange multiplier of the vector field t_1 with respect to g if

$$\mathrm{Div}(\psi^\star t_1) = 0.$$

In this situation, Equation (7.3) simplifies to

$$\xi(d\log\psi^\star) = -\mathrm{Div}(\xi), \tag{7.4}$$

which is referred to as the generalized Liouville equation of the vector field ξ with respect to the metric g [25]. Consequently, by combining Eqs (7.2) and (7.4), we obtain the following result.

Theorem 7.1. *Let a BVFS-spacetime admit an h-ACERBS with an anti-torqued vector field ξ, and let $\psi^\star$ denote the last multiplier associated with ξ. The Liouville equation takes the form*

$$\xi(d\log\psi^\star) = \frac{1}{h}[\kappa(3\alpha-\rho-3\beta)-4\pi_1+\pi_2+(2\bar{p}+1)-4\tau\omega].$$

Corollary 7.1. *Let a BVFS-spacetime admit an h-ACERBS with an anti-torqued unit time-like vector field ξ, and let $\psi^\star$ be the corresponding last multiplier of ξ. If the vector field ξ is incompressible or Killing, then the soliton is classified as follows:*

(i) expanding, provided that

$$(2\bar{p}+1) > 4\tau\omega-\kappa(3\alpha-\rho-3\beta)-\pi_2,$$

(ii) steady, when

$$(2\bar{p}+1) = 4\tau\omega-\kappa(3\alpha-\rho-3\beta)-\pi_2,$$

(iii) shrinking, if

$$(2\bar{p}+1) < 4\tau\omega-\kappa(3\alpha-\rho-3\beta)-\pi_2.$$

The generalized Liouville equation associated with soliton structures in $f(R,T)$ has been studied previously in [32]. In the present work, we extend this approach to BVFS-spacetimes endowed with anti-torqued vector fields.

8. Harmonic characteristics of h-ACERBS on BVFS-spacetime

In this section, we establish the existence of an h-ACERBS on a BVFS-spacetime endowed with an anti-torqued vector field, under suitable conditions, when the g-dual 1-form η associated with ξ is either harmonic or Schrödinger–Ricci harmonic.

The vector field ξ is said to be a solution of the Schrödinger–Ricci equation if it satisfies

$$\mathrm{Div}(\mathfrak{L}_\xi g) = 0, \tag{8.1}$$

where $\mathfrak{L}_\xi g$ denotes the Lie derivative of the metric tensor g along the vector field ξ. According to [8], as shown by Chow et al., the divergence of the Lie derivative is expressed as

$$\mathrm{Div}(\mathfrak{L}_\xi g) = (\Theta + \mathrm{K})(\xi) + d(\mathrm{Div}(\xi)), \tag{8.2}$$

where Θ represents the Laplace–Hodge operator associated with the metric g, and K denotes the Ricci curvature tensor field.

From Eq 1.12, we have

$$\mathrm{K}(t_1, t_2) + \frac{h}{2}\mathfrak{L}_\xi g(t_1, t_2) - [\pi_1 - \frac{1}{2}(\bar{p} + \frac{1}{2}) + \tau\omega]g(t_1, t_2) - \pi_2\eta(t_1)\eta(t_2) = 0. \tag{8.3}$$

Taking the trace of Eq 8.3 and using Eq 3.9, we obtain

$$h\,\mathrm{Div}(\xi) + \kappa(3\alpha - \rho - \beta) - [4\pi_1 - 2\bar{p} - 1 + 4\tau\omega] - \pi_2|\xi|^2 = 0. \tag{8.4}$$

Moreover, by direct calculation,

$$\mathrm{Div}(\eta \otimes \eta) = \mathrm{Div}(\xi)\,\eta + \nabla_\xi\eta. \tag{8.5}$$

By taking the divergence of Eq (8.3) and making use of Eq (8.5), we obtain

$$h\,\mathrm{Div}(\mathfrak{L}_\xi g) + (1 + \omega)d(\tau) - 2\pi_2[\mathrm{Div}(\xi)\,\eta + \nabla_\xi\eta] = 0. \tag{8.6}$$

If the 1-form η is a solution of the Schrödinger–Ricci equation, then it satisfies

$$(\Theta + \mathrm{K})(\eta) + d(\mathrm{Div}(\eta)) = 0.$$

Consequently, we arrive at the following characterization.

Theorem 8.1. *Let (g, ξ, π_1, π_2) define an h-ACERBS on a BVFS-spacetime endowed with an anti-torqued vector field ξ, and let η be the g-dual 1-form associated with ξ. Then, η is a solution of the Schrödinger–Ricci equation if and only if*

$$d(\tau) = \frac{2\pi_2}{(1+\omega)h}\Big[\{(4\omega - 1)\kappa(3\alpha - \rho - \beta) + 4\pi_1 - (2\bar{p} + 1) - \pi_2\}\eta + h\,\nabla_\xi\eta\Big]. \tag{8.7}$$

As direct consequences of the preceding theorem, we obtain the following corollaries.

Corollary 8.1. *Let (g, ξ, π_1, π_2) be an Einstein soliton on a BVFS-spacetime associated with an anti-torqued vector field ξ, and let η be the g-dual 1-form of ξ. Then, η is a solution of the Schrödinger–Ricci equation if and only if*

$$d(\tau) = \frac{4\pi_2}{3h}[\{\kappa(3\alpha - \rho - \beta) + 4\pi_1 - (2\bar{p} + 1) - \pi_2\}\eta + h\,\nabla_\xi \eta].$$

Corollary 8.2. *Let (g, ξ, π_1, π_2) be an η-traceless soliton on a BVFS-spacetime associated with an anti-torqued vector field ξ, and let η be the g-dual 1-form of ξ. Then, η is a solution of the Schrödinger–Ricci equation if and only if*

$$d(\tau) = \frac{10\pi_2}{4h}[\{4\pi_1 - (2\bar{p} + 1) - \pi_2\}\eta + h\,\nabla_\xi \eta].$$

Corollary 8.3. *Let (g, ξ, π_1, π_2) be an η-Schouten soliton on a BVFS-spacetime associated with an anti-torqued vector field ξ, and let η be the g-dual 1-form of ξ. Then, η is a solution of the Schrödinger–Ricci equation if and only if*

$$d(\tau) = \frac{14\pi_2}{6h}[\{-3\kappa(3\alpha - \rho - \beta) + 4\pi_1 - (2\bar{p} + 1) - \pi_2\}\eta + h\,\nabla_\xi \eta].$$

Corollary 8.4. *Let (g, ξ, π_1, π_2) be a Ricci soliton on a BVFS-spacetime associated with an anti-torqued vector field ξ, and let η be the g-dual 1-form of ξ. Then, η is a solution of the Schrödinger–Ricci equation if and only if*

$$d(\tau) = \frac{2\pi_2}{h}[\{-\kappa(3\alpha - \rho - \beta) + 4\pi_1 - (2\bar{p} + 1) - \pi_2\}\eta + h\,\nabla_\xi \eta].$$

Finally, we consider the Schrödinger-Ricci harmonic form. Let the 1-form η be Schrödinger-Ricci harmonic [4], i.e.,

$$(\Gamma + \mathrm{K})(\eta) = 0.$$

In addition, if $\pi_2 = 0$, this yields an h-ACERBS. Otherwise, the following condition holds:

$$h\nabla_\xi \eta = -\Big\{\kappa(4\omega - 1)(3\alpha - \rho - \beta) + 4\pi_1 - (2\bar{p} + 1)\Big\}\eta.$$

Consequently, we obtain the following result.

Theorem 8.2. *Let (g, ξ, π_1, π_2) be an h-ACERBS on a BVFS-spacetime, where ξ is an anti-torqued vector field and η denotes the g-dual 1-form of ξ. Then, η is a solution of the Schrödinger–Ricci equation if and only if either $\pi_2 = 0$, in which case the structure reduces to an h-ACRBS, or*

$$h\nabla_\xi \eta = -\Big\{\kappa(4\omega - 1)(3\alpha - \rho - \beta) + 4\pi_1 - (2\bar{p} + 1)\Big\}\eta.$$

9. Gradient h-ACERBS on BVFS-spacetime coupled with anti-torqued vector field

Let the soliton vector $\xi = \mathcal{D}\Psi$, where Ψ is a smooth function and $\mathcal{D}$ stands for gradient operator of g on BVFS-spacetime attached with anti-torqued vector field. So from Eq 1.13 we have

$$\mathrm{K} + h\nabla\mathcal{D}\Psi = [\pi_1 - \frac{1}{2}(\bar{p} + \frac{1}{2}) + \tau\omega]g + \pi_2 \eta \otimes \eta,$$

which is equivalent to,

$$h\nabla\mathcal{D}\Psi = [\pi_1 - \frac{1}{2}(\bar{p} + \frac{1}{2}) + \tau\omega]I - \mathrm{Q} + \pi_2\eta \otimes \xi. \tag{9.1}$$

After, contracting Eq 9.1, and using Eq 3.9 we yield

$$\nabla\Psi = \frac{1}{h}[4\pi_1 - 2(\bar{p} + 1) + (4\omega - 1)\kappa(3\alpha - \rho - \beta) - \pi_2]. \tag{9.2}$$

Thus, we state the outcome

Theorem 9.1. *If a BVFS-spacetime coupled with anti-torqued vector field admits gradient h-ACERBS, then the potential function* Ψ *of the soliton satisfies Poisson's equation*

$$\nabla\Psi = \frac{1}{h}[4\pi_1 - 2(\bar{p} + 1) + (4\omega - 1)\kappa(3\alpha - \rho - \beta) - \pi_2].$$

Also from Eq 9.2, we conclude the following corollary

Corollary 9.1. *If a BVFS-spacetime coupled with anti-torqued vector field admits gradient h-almost conformal* η*- traceless RB solitons, then the potential function* Ψ *of the soliton satisfies Poisson's equation*

$$\nabla\Psi = \frac{1}{h}[4\pi_1 - 2(\bar{p} + 1) - \pi_2].$$

Corollary 9.2. *If a BVFS-spacetime coupled with anti-torqued vector field admits gradient h-almost conformal* η*-Schouten-Bourguignon solitons, then the potential function* Ψ *of the soliton satisfies Poisson's equation*

$$\nabla\Psi = \frac{1}{3h}[12\pi_1 - 6(\bar{p} + 1) - \kappa(3\alpha - \rho - \beta) - 3\pi_2].$$

It is noted that similar Poisson-type relations associated with Ricci-type solitons were previously derived in different geometric settings [1,5,19,36]. Here we adapt these relations to BVFS-spacetimes with anti-torqued vector fields. A smooth function χ on a Riemannian manifold $\mathbb{N}$ of dim $\mathbb{N} = n \geq 3$ is said to be harmonic, strictly super-harmonic and strictly sub-harmonic if $\nabla\chi = 0$, $\nabla\chi < 0$ and $\nabla\chi > 0$. So from above facts and Eq 9.2, we reveal the corollary

Corollary 9.3. *Let a BVFS-spacetime in coupled with anti-torqued vector field admit a gradient h-ACERBS. Then the solitons function* Ψ *is*

1. *harmonic if* $\pi_1 = \frac{1}{2}(\bar{p} + 1) - (\omega - \frac{1}{4})\kappa(3\alpha - \rho - \beta) + \frac{\pi_2}{4}$,
2. *strictly super-harmonic if* $\pi_1 < \frac{1}{2}(\bar{p} + 1) - (\omega - \frac{1}{4})\kappa(3\alpha - \rho - \beta) + \frac{\pi_2}{4}$,
3. *strictly sub-harmonic if* $\pi_1 > \frac{1}{2}(\bar{p} + 1) - (\omega - \frac{1}{4})\kappa(3\alpha - \rho - \beta) + \frac{\pi_2}{4}$.

Corollary 9.4. *Let a BVFS-spacetime coupled with anti-torqued vector field admit a gradient h-ACERBS. If the solitons function* Ψ *is harmonic, then the soliton is*

1. *steady if* $(\bar{p} + 1) = \frac{1}{2}[(4\omega - 1)\kappa(3\alpha - \rho - \beta) + \pi_2]$,
2. *expanding if* $(\bar{p} + 1) > \frac{1}{2}[(4\omega - 1)\kappa(3\alpha - \rho - \beta) + \pi_2]$,
3. *shrinking if* $(\bar{p} + 1) < \frac{1}{2}[(4\omega - 1)\kappa(3\alpha - \rho - \beta) + \pi_2]$.

10. Conclusions

This work has presented a comprehensive geometric and physical investigation of relativistic BVFS spacetimes in the presence of anti-torqued vector fields, within the framework of h-ACERBS. The analysis reveals several fundamental connections between the geometric structure of the spacetime and its physical properties.

It has been shown that a BVFS-spacetime satisfying the EFE without a cosmological constant is a generalized quasi-Einstein ($G(QE)$) spacetime. Explicit relationships among the scalar curvature, total pressure, energy density, and string tension were obtained, allowing the derivation of the corresponding equations of state for several cosmological phases, including radiation, vacuum-dominated, dust, and stiff fluid eras. Moreover, when the velocity field coincides with an anti-torqued vector field, the spacetime admitting an h-ACERBS is shown to reduce to a perfect fluid spacetime.

The classification of the soliton type, expanding, steady, or shrinking, was expressed in terms of the physical parameters of the model, including the total pressure, energy density, string tension, and the geometric soliton data. Within this framework, important physical viability conditions were examined. In particular, it was established that BVFS-spacetimes admitting shrinking h-almost conformal solitons necessarily satisfy the SEC as well as the NCC. By applying Penrose's singularity theorem, the analysis indicates the inevitable occurrence of trapped surfaces and black hole formation in such geometries.

From an analytical perspective, the generalized Liouville equation associated with the soliton structure was derived, and the classification of soliton behavior was linked to the incompressibility of the vector field through the last multiplier. Furthermore, the harmonic and Schrödinger-Ricci harmonic properties of the dual 1-form η were investigated. Necessary and sufficient conditions for η to satisfy the Schrödinger-Ricci equation were obtained, and the reduction to h-ACRBSs was identified.

In the gradient case, Poisson-type equations satisfied by the potential function were derived. The harmonic, strictly super-harmonic, and strictly sub-harmonic behaviors of the potential were classified, together with the corresponding constraints on the soliton parameters. These results illustrate how the underlying soliton geometry influences the analytic behavior of scalar and vector fields on the spacetime manifold.

Overall, the results demonstrate that BVFS-spacetimes with anti-torqued vector fields provide a consistent and rich geometric framework for modeling dissipative and anisotropic cosmological systems in general relativity. The interplay between bulk viscosity, string tension, and h-almost conformal soliton structures yields a mathematically robust setting that respects fundamental energy conditions and singularity theorems while generating novel harmonic and analytic structures. Future work may extend this framework to higher-dimensional manifolds, alternative gravitational theories, or numerical investigations addressing stability and cosmological evolution of BVFS-spacetimes admitting generalized soliton structures.

Author contributions

Conceptualization, S.S., and S.K.Y.; Methodology, S.S., S.K.Y., and C.M.; Software, A. A., S.S., and S.K.Y.; Validation, A. A., S.S., S.K.Y., and C.M.; Formal Analysis, A. A., S.S., S.K.Y., and C.M.; Investigation, A. A., S.S., S.K.Y., and C.M.; Resources, A. A., S.S., S.K.Y., and C.M.; Data Curation,

A. A., S.S., and S.K.Y.; Writing—Original Draft, A. A., S.K.Y., and S.S.; Supervision, A. A., S.S., S.K.Y., and C.M.; Project administration, S.S., and S.K.Y. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. A. H. Alkhaldi, M. D. Siddiqi, M. A. Khan, L. S. Alqahtani, Imperfect fluid generalized Robertson Walker spacetime admitting Ricci–Yamabe metric, *Adv. Math. Phys.*, **2021** (2021), 2485804, 10. https://doi.org/10.1155/2021/2485804
2. S. Azami, Some results on h-almost Ricci–Bourguignon solitons, *Afr. Mat.*, **33** (2022), Article 8. https://doi.org/10.1007/s13370-021-00953-3
3. N. Basu, A. Bhattacharyya, Conformal Ricci soliton in Kenmotsu manifolds, *Glob. J. Adv. Res. Class. Mod. Geom.*, **4** (2015), 15–21.
4. A. M. Blaga, Harmonic aspects in an η-Ricci soliton, *Int. Electron. J. Geom.*, **13** (2020), 41–49. https://doi.org/10.36890/iejg.573919
5. A. M. Blaga, Solitons and geometrical structures in a perfect fluid spacetime, *Rocky Mountain J. Math.*, **50** (2020), 41–53. https://doi.org/10.1216/rmj.2020.50.41
6. J. P. Bourguignon, Ricci curvature and Einstein metrics, in: D. Ferus et al. (eds.), *Global Differential Geometry and Global Analysis* (Proc. Conf., Berlin, 1979), Lecture Notes in Math., 838, Springer, Berlin, 1981, 42–63. https://doi.org/10.1007/BFb0089767
7. B. Y. Chen, Classification of torqued vector fields and its applications to Ricci solitons, *Kragujevac J. Math.*, **41** (2017), 239–250.
8. B. Chow, S. C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, et al., *The Ricci Flow: Techniques and Applications. Part I: Geometric Aspects*, Math. Surveys Monogr., 135, Amer. Math. Soc., Providence, RI, 2007. https://doi.org/10.1090/surv/163
9. U. C. De, G. C. Ghosh, On generalized quasi Einstein manifolds, *Kyungpook Math. J.*, **44** (2004), 607–615.
10. S. Dwivedi, Some results on Ricci–Bourguignon solitons and almost solitons, *Canad. Math. Bull.*, **64** (2021), 591–604.
11. J. Eells Jr, J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86** (1964), 109–160. https://doi.org/10.2307/2373155.
12. A. E. Fischer, An introduction to conformal Ricci flow, *Class. Quantum Grav.*, **21** (2004), 171–218. https://doi.org/10.1088/0264-9381/21/1/010

13. S. Guler, S. A. Demirbag, Study of generalized quasi-Einstein spacetimes with applications in general relativity, *Int. J. Theor. Phys.*, **55** (2016), 548–562. https://doi.org/10.1007/s10773-015-2716-6

14. R. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geom.*, **17** (1982), 255–306.

15. S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Univ. Press, Cambridge, 1973.

16. S. H. Henry, Brane inflation: string theory viewed from the cosmos, *Lect. Notes Phys.*, **737** (2008), 949–974. https://doi.org/10.1007/978-3-540-74353-8_33

17. R. Jackiw, V. P. Nair, S. Y. Pi, A. P. Polychronakos, Perfect fluid theory and its extensions, *J. Phys. A: Math. Gen.*, **37** (2004), R327–R432. https://doi.org/10.1088/0305-4470/37/42/R01

18. Y. Li, M. D. Siddiqi, M. A. Khan, I. Al-Dayel, M. Youssef, Solitonic effect on relativistic string cloud spacetime attached with strange quark matter, *AIMS Math.*, **9** (2024), 14487–14503. https://doi.org/10.3934/math.2024710

19. Y. Li, S. K. Yadav, S. Shenawy, N. B. Turki, Ricci–Yamabe metric in $f(\mathcal{R},\mathcal{T})$-gravity model coupled with magnetized quark matter, *Int. J. Geom. Methods Mod. Phys.*, **22** (2025), 2550232. https://doi.org/10.1142/S0219887825502329

20. L. Mandel, E. Wolf, *Optical Coherence and Quantum Optics*, Cambridge Univ. Press, Cambridge, 1995.

21. M. Novello, M. J. Rebouças, The stability of a rotating universe, *Astrophys. J.*, **225** (1978), 719–724. https://doi.org/10.1086/156531

22. B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, *Pure Appl. Math.*, 103, Academic Press, New York, 1983.

23. J. M. Overduin, P. S. Wesson, Dark matter and background light, *Phys. Rep.*, **402** (2004), 267–406. https://doi.org/10.1016/j.physrep.2004.07.006

24. P. J. E. Peebles, B. Ratra, The cosmological constant and dark energy, *Rev. Mod. Phys.*, **75** (2003), 559–606. https://doi.org/10.1103/RevModPhys.75.559

25. A. G. Popov, Exact formula for constructing solutions of the Liouville equation $\Delta_2 x = e^x$ from solutions of the Laplace equation $\Delta_2 y = 0$, *Dokl. Akad. Nauk*, **333** (1993), 440–441.

26. M. M. Praveena, C. S. Bagewadi, M. R. Krishnamurthy, Solitons of Kählerian space-time manifolds, *Int. J. Geom. Methods Mod. Phys.*, **18** (2021), 2150021. https://doi.org/10.1142/S0219887821500211

27. R. K. Sachs, W. Hu, *General Relativity for Mathematicians*, Springer, New York, 1997.

28. V. Sahni, A. Starobinsky, The case for a positive cosmological Λ-term, *Int. J. Mod. Phys. D*, **9** (2000), 373–444. https://doi.org/10.1142/S0218271800000542

29. J. Satish, R. Venkateswarlu, Bulk viscous fluid cosmological models in $f(R,T)$ gravity, *Chin. J. Phys.*, **54** (2016), 830–838. https://doi.org/10.1016/j.cjph.2016.09.004

30. S. Shenawy, U. C. De, N. B. Turki, Mixed quasi-Einstein $M(QE)_n$ relativistic spacetimes with applications, *Rep. Math. Phys.*, **96** (2025), 313–326. https://doi.org/10.1016/S0034-4877(25)00076-X

31. M. D. Siddiqi, I. Al-Dayel, Geometric perspective of relativistic bulk viscous fluid string spacetime, *Axioms*, **14** (2025), 674. https://doi.org/10.3390/axioms14090674

32. M. D. Siddiqi, S. K. Chaubey, M. N. I. Khan, $f(R,T)$-gravity model with perfect fluid admitting Einstein solitons, *Mathematics*, **10** (2022), 82. https://doi.org/10.3390/math10010082

33. M. D. Siddiqi, U. C. De, Relativistic magneto-fluid spacetimes, *J. Geom. Phys.*, **170** (2021), 104370. https://doi.org/10.1016/j.geomphys.2021.104370

34. M. D. Siddiqi, F. Mofarreh, Soliton geometry of modified gravity models engaged with strange quark matter fluid and Penrose singularity theorem, *Symmetry*, **17** (2025), 1767. https://doi.org/10.3390/sym17101767

35. M. D. Siddiqi, F. Mofarreh, A. N. Siddiqui, S. A. Siddiqui, Geometrical structure in a relativistic thermodynamical fluid spacetime, *Axioms*, **12** (2023), 138. https://doi.org/10.3390/axioms12020138

36. M. D. Siddiqi, S. A. Siddiqui, Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime, *Int. J. Geom. Methods Mod. Phys.*, **17** (2020), 2050083. https://doi.org/10.1142/S0219887820500838

37. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge Monogr. Math. Phys., Cambridge Univ. Press, Cambridge, 2003.

38. A. Vilenkin, A. C. Wall, Cosmological singularity theorems and black holes, *Phys. Rev. D*, **89** (2014), 064035. https://doi.org/10.1103/PhysRevD.89.064035

39. S. K. Yadav, S. Shenawy, H. Alohali, C. Mantica, h-Almost conformal η-Ricci–Bourguignon solitons and spacetime symmetry in barotropic fluids within $f(R,T)$ gravity, *Symmetry*, **17** (2025), 1794. https://doi.org/10.3390/sym17111794

40. S. K. Yadav, S. Shenawy, B. Turki, Y. Li, Investigating string cloud spacetime with energy-momentum tensor constraints in general relativity, *AIMS Math.*, **10** (2025), 9.

41. S. K. Yadav, S. Shenawy, B. Turki, Ricci–Yamabe metric in $f(R,T)$-gravity model coupled with magnetized quark matter, *Int. J. Geom. Methods Mod. Phys.*, **22** (2025), 2550232. https://doi.org/10.1142/S021988782550232X