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Research article

Multiplicity results for some higher order iterative systems

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Abstract: Our study was concerned with the higher order iterative system where the nonlinearities exhibited dependence on the first-order derivatives

$$\begin{cases} u_i^{(n)}(x) + \lambda_i f_i(x, u_{i+1}(x), u'_{i+1}(x)) = 0, & 1 \le i \le m, x \in [0,1]; \\ u_{m+1}(x) = u_1(x), & x \in [0,1], \end{cases}$$

with two-point boundary conditions

$$u_i(0) = u_i'(0) = \cdots = u_i^{(n-2)}(0) = u_i^{(r)}(1) = 0,$$

where $m \ge 2, n \ge 3, i \in \{1, 2, \cdots, m\}$, $r \in \{1, 2, \cdots, n-2\}$ but fixed and $\lambda_i \in (0, \infty)$ were constants. By giving the corresponding Green's function and its related properties, the form of the solution of the system could be obtained. Combined with the fixed-point theorem of functional type on cones due to Bai and Ge, we established novel conclusions on the existence of multiple positive solutions for higher order iterative systems. A notable feature was the involvement of first derivatives in the nonlinear terms.

Keywords: iterative system; multiplicity; positive solutions; Green function; fixed-point theorem
Mathematics Subject Classification: 34B18

1. Introduction

Iterative systems, such as the core research object of dynamical system theory, provide a basic framework for understanding complex dynamic behaviors emerging from simple rules. The research not only deepens our mathematical understanding of the nature of nonlinear phenomena such as bifurcation and chaos [1], but also provides an indispensable tool for dynamic modeling in physics [2], engineering, and life sciences [3]. It is the core mathematical tool for describing the mechanical behavior of one-dimensional space structures. Numerous studies concerning two-point boundary conditions have appeared in the literature over the past several decades. We also mention the recent papers [4–7], where some existence results were investigated for certain systems subject to two-point boundary conditions.

In 2013, Prasad, et al. [8] demonstrated that a minimum of one positive solution exists for the second order iterative system

$$\begin{cases} u_i''(x) + \lambda_i a_i(x) f_i(u_{i+1}(x)) = 0, & 1 \le i \le m, x \in [x_1, x_3]; \\ u_{m+1}(x) = u_1(x), & x \in [x_1, x_3], \end{cases}$$

for each $1 \le i \le m$, satisfying the three-point boundary conditions

$$\alpha_i u_i'(x_1) - \beta_i u_i'(x_1) = 0, \ \ \gamma_i u_i(x_3) + \delta_i u_i'(x_3) = u_i'(x_2).$$

Via the Guo-Krasnosel'skiĭ fixed point theorem on cones, criteria ensuring the presence of a positive solution was derived.

As far as the authors know, the complexity of constructing the Green's function is limited when using the Guo-Krasnosel'skiĭ fixed point theorem to study the solvability of the equation. At present, most of the research on higher order iterative systems focus on multi-point boundary value conditions. A breakthrough came in 2022, when researchers initially provided existence results for higher-order iterative systems subject to integral boundary conditions. For example, in 2022, Sreedhar, Kanakayya, and Rajendra [9] addressed the question of positive solution existence in the iterative system of higher order differential equations

$$\begin{cases} u_i^{(n)}(x) + \lambda_i a_i(x) f_i(u_{i+1}(x)) = 0, & 1 \le i \le m, x \in [0, 1]; \\ u_{m+1}(x) = u_1(x), & x \in [0, 1], \end{cases} \tag{1}$$

for each $1 \le i \le m$, obeying nonhomogeneous integral boundary conditions

$$u_i(0) = u_i'(0) = \cdots = u_i^{(n-2)}(0) = 0, \ \ u_i^{(r)}(1) - \eta_i \int_0^1 g_i(x) u_i^{(r)}(x) dx = \beta_i, \tag{2}$$

where $r \in \{1, 2, \cdots, n-2\}$ but fixed. Through an analysis of the eigenvalue λ_i precise range, the existence of results was derived via the fixed point theorem. Furthermore, in 2024, employing analogous techniques, they [10] investigated the presence of positive solutions for the higher order iterative system (1) and (2) subject to integral boundary conditions and devoid of eigenvalue λ_i.

While the above problems considered the case that the nonlinear term is just dependent on the unknown function, there are other works that consider the nonlinear term with lower order derivation. In 2004, Bai and Ge [11] established that triple positive solutions exist for the second-order two-point boundary value problem

$$u''(x) + f(x, u(x), u'(x)) = 0, \quad x \in (0, 1),$$
$$u(0) = u(1) = 0,$$

where $f : [0, 1] \times [0, \infty) \times R \to [0, \infty)$ is continuous. A new fixed-point theorem of functional type in a cone is established, and some new multiplicity results are obtained. Then, Bai and Ge applied this theorem to the four-point boundary value problem [12].

In 2021, Webb [13] investigated the solvability and multiplicity of positive solutions for nonlinear second-order nonlocal boundary value problems where the nonlinear term depends on the derivative

$$u''(x) + f(x, u(x), u'(x)) = 0,$$
$$a_0 u(0) - b_0 u'(0) = \beta_0[u], \ \ a_1 u(1) + b_1 u'(1) = \beta_1[u],$$

where a_0, a_1, b_0, b_1 are nonnegative constants and the terms $\beta_i[u] = \int_0^1 u(x)dB_i(x)$ are positive linear functionals defined via Riemann–Stieltjes integrals where B_i are nonnegative, nondecreasing functions. By using the degree theory, the existence of multiple positive for nonlinear second-order nonlocal boundary value problems are obtained.

Beyond ordinary differential equations, the study of quasilinear Partial Differential Equations systems involving chemotaxis and fluid interactions has also attracted considerable attention; for instance, Zheng and Ke [14] recently established eventual smoothness and stabilization for a three-dimensional Keller-Segel-Navier-Stokes system modeling coral fertilization, providing insights into the existence and boundedness of solutions for such quasilinear problems.

Based on the above discussions, the objective of this work is to determine the existence of multiple positive solutions for higher order iterative systems involving the first-order derivative

$$\begin{cases} u_i^{(n)}(x) + \lambda_i f_i(x, u_{i+1}(x), u_{i+1}'(x)) = 0, & 1 \le i \le m, x \in [0,1]; \\ u_{m+1}(x) = u_1(x), & x \in [0,1], \end{cases} \tag{3}$$

with two-point boundary conditions

$$u_i(0) = u_i'(0) = \cdots = u_i^{(n-2)}(0) = u_i^{(r)}(1) = 0, \tag{4}$$

where $m \ge 2, n \ge 3, i \in \{1, 2, \cdots, m\}$, $f_i : [0,1] \times [0,\infty) \times R \to [0,\infty)$ are continuous, $r \in \{1, 2, \cdots, n-2\}$ but fixed and $\lambda_i \in (0,\infty)$ are constants.

The above iterative model can dynamically integrate the interactions among multi-species, and can describe the complex changes of population growth rate more comprehensively and accurately. This characteristic serves the key application field of food chain population dynamics analysis in ecology. For example, in environmental protection, the model can provide a more scientific population assessment basis for the formulation of species conservation strategies; in ecological restoration projects, its accurate dynamic simulation ability helps to set restoration goals, evaluate restoration effectiveness, and optimize restoration paths, thus improving the pertinence and effectiveness of the project.

The interesting point of this work is the higher order iterative system with the nonlinear terms involved the first order derivative. Compared with previous studies on the higher order iterative systems, we use the fixed-point theorem of functional type on cones due to Bai and Ge [11], but not from Guo-Krasnosel'skiĭ.

The article is organized as follows. In Section 2, we first outline the requisite background on cone theory in ordered Banach spaces. After which, the solution of iterative system (3) and (4) is expressed in an equivalent manner through the determination of its corresponding Green's function, and some relevant properties are obtained. In Section 3, we establish new multiplicity for the iterative system (3) and (4) through a geometric analysis of the nonlinear term by Lemma 2.1. Finally, an example is presented.

2. Preliminariess

To establish our main results, we first review the necessary definitions and preliminaries on cone theory in ordered Banach spaces. The solution to the iterative system (3) and (4) is then reformulated as an equivalent integral equation involving specific kernel functions. Furthermore, several key inequalities for these kernels are derived in this section.

Definition 1. *[15] Let E be a real Banach space. A nonempty convex closed set $Q \subset E$ is called a cone if it satisfies*
(i) $su \in Q$ for every $u \in Q$ and $s \geq 0$,
(ii) $u = 0$ whenever both u and $-u$ belong to Q.

Definition 2. *[15] A map $\psi : Q \to [0, \infty)$ is termed a nonnegative continuous concave functional on Q if it is continuous and satisfies the inequality*

$$\psi(su + (1-s)v) \geq s\psi(u) + (1-s)\psi(v)$$

for any $u, v \in Q$ and $s \in [0, 1]$. Correspondingly, a map $\zeta : Q \to [0, \infty)$ is called a nonnegative continuous convex functional on Q if it is continuous and fulfills

$$\zeta(su + (1-s)v) \leq s\zeta(u) + (1-s)\zeta(v)$$

for any $u, v \in Q$ and $s \in [0, 1]$.

Definition 3. *[11] Let $r > b > 0$, $K > 0$ be constants, ψ a nonnegative continuous concave functional and ζ, ξ nonnegative continuous convex functionals on the cone Q. Define convex sets*

$$Q(\zeta, r; \xi, K) = \{u \in Q \mid \zeta(u) < r;\ \xi(u) < K\},$$
$$\overline{Q}(\zeta, r; \xi, K) = \{u \in Q \mid \zeta(u) \leq r; \xi(u) \leq K\},$$
$$Q(\zeta, r; \xi, K; \psi, b) = \{u \in Q \mid \zeta(u) < r;\ \xi(u) < K;\ \psi(u) > b\},$$
$$\overline{Q}(\zeta, r; \xi, K; \psi, b) = \{u \in Q \mid \zeta(u) \leq r; \xi(u) \leq K;\ \psi(u) \geq b\}.$$

The conditions on the nonnegative continuous convex functionals ζ, ξ will be imposed as follows:
(B1) There exists $M > 0$ with the property that $\|u\| \leq M \max\{\zeta(u), \xi(u)\}$, for any $u \in Q$.
(B2) For all positive numbers r and L, one has $Q(\zeta, r; \xi, K) \neq \emptyset$.

Lemma 1. *(Bai and Ge [11]) Let E be a Banach space, and $Q \subset E$ a cone. For constants satisfying $r_2 \geq d > b > r_1 > 0$ and $K_2 \geq K_1 > 0$, suppose the nonnegative continuous convex functionals ζ, ξ fulfill (B1) and (B2). Let ψ be a nonnegative continuous concave functional on Q satisfying $\psi(u) \leq \zeta(u)$ for every $u \in \overline{Q}(\zeta, r_2; \xi, K_2)$. $T : \overline{Q}(\zeta, r_2; \xi, K_2) \to \overline{Q}(\zeta, r_2; \xi, K_2)$ is a completely continuous operator. Assume*
(D1) $\{u \in \overline{Q}(\zeta, r; \xi, K_2; \psi, b) \mid \psi(u) > b\} \neq \emptyset, \psi(Tu) > b$ for $u \in \overline{Q}(\zeta, r; \xi, K_2; \psi, b)$,
(D2) $\zeta(Tu) < r_1, \xi(Tu) < K_1$ for all $y \in \overline{Q}(\zeta, r_1; \xi, K_1)$,
(D3) $\psi(Tu) > b$ for all $u \in \overline{Q}(\zeta, r_2; \xi, K_2; \psi, b)$ and $\zeta(Tu) > d$.
Then, T possesses at least three distinct fixed points u_1, u_2, and u_3 in $\overline{Q}(\zeta, r_2; \xi, K_2)$, which satisfy

$$u_1 \in Q(\zeta, r_1; \xi, K_1), \quad u_2 \in \{\overline{Q}(\zeta, r_2; \xi, K_2; \psi, b) \mid \psi(u) > b\}$$

and

$$u_3 \in \overline{Q}(\zeta, r_2; \xi, K_2) \setminus (\overline{Q}(\zeta, r_2; \xi, K_2; \psi, b) \cup \overline{Q}(\zeta, r_1; \xi, K_1)).$$

The following lemmas can be proved with standard way, and we omit the details here.

Lemma 2. *If* $g(x) \in C([0,1])$*, then*

$$u_i^{(n)}(x) + g(x) = 0, \quad 1 \le i \le m, \quad x \in [0,1], \tag{5}$$

where the boundary condition (4) has a unique solution

$$u_i(x) = \lambda_i \int_0^1 H(x,s)g(s)ds, \tag{6}$$

where

$$H(x,s) = \frac{1}{(n-1)!} \begin{cases} x^{n-1}(1-s)^{n-r-1} - (x-s)^{n-1}, & 0 \le s \le x \le 1; \\ x^{n-1}(1-s)^{n-r-1}, & 0 \le x \le s \le 1. \end{cases} \tag{7}$$

Lemma 3. *The Green's function satisfies* $H(x,s) \ge 0$ *for* $0 \le x, s \le 1$*, and*

$$H(x,s) \ge \frac{1}{4^{n-1}} \max_{x \in [0,1]} H(x,s), \qquad x \in \left[\frac{1}{4}, \frac{3}{4}\right], s \in [0,1].$$

By Lemma 2.2, the solution of iterative system (3) and (4) can be written as

$$u_i(x) = \lambda_i \int_0^1 H(x,s) f_i(s, u_{i+1}(s), u'_{i+1}(s))ds, \quad 1 \le i \le m, x \in [0,1].$$

If we solve for u_1, according to

$$u_{m+1}(x) = u_1(x), \ x \in [0,1],$$

we can get $u_2, u_3, \cdots, u_m$ in order. Hence, $(u_1(x), u_2(x), \cdots, u_m(x))$ constitutes a solution to the higher order iterative system (3) and (4).

3. Multiplicity results of positive solution

Findings regarding multiple positive solutions to iterative system (3) and (4) are detailed in this section. For construction, let $B = \{u : u \in C^1([0,1], \mathbb{R})\}$ be the Banach space with the ordering $u \le v$ if $u(x) \le y(x)$ for each $x \in [0,1]$. The norm

$$\|u\| = \max\left\{ \max_{x \in [0,1]} |u(x)|, \max_{x \in [0,1]} |u'(x)| \right\}.$$

Define the cone Q in B as

$$Q = \left\{ u \in B : u(x) \ge 0, \min_{x \in [\frac{1}{4}, \frac{3}{4}]} u(x) \ge \frac{1}{4^{n-1}} \max_{x \in [0,1]} u(x) \right\},$$

and functionals

$$\zeta(u) = \max_{x \in [0,1]} |u(x)|, \ \ \xi(u) = \max_{x \in [0,1]} |u'(x)|, \ \ \psi(u) = \min_{x \in [\frac{1}{4}, \frac{3}{4}]} |u(x)|, \quad u \in B.$$

Thus, $\zeta, \xi : B \to [0, \infty)$ are nonnegative continuous convex functionals satisfying (B1) and (B2). Meanwhile, ψ constitutes a nonnegative continuous concave functional, and it holds that $\psi(u) \le \zeta(u)$ for every $u \in B$.

We now define operators $T_i : Q \to B$ as

$$T_1 u_2(x) = \lambda_1 \int_0^1 H(x,s) f_1(s, u_2(s), u_2'(s)) ds,$$

$$T_2 u_3(x) = \lambda_2 \int_0^1 H(x,s) f_2(s, u_3(s), u_3'(s)) ds,$$

$$\cdots$$

$$T_m u_{m+1}(x) = T_m u_1(x) = \lambda_m \int_0^1 H(x,s) f_m(s, u_1(s), u_1'(s)) ds.$$

Thus, if $u_1 \in Q$ satisfies

$$u_1 = (T_1 T_2 \cdots T_m) u_1 \triangleq T u_1,$$

then the solution of the higher order iterative system (3) and (4) can be written as

$$(u_1, T_2 T_3 \cdots T_m u_1, \cdots, T_{m-1} T_m u_1, T_m u_1).$$

Lemma 4. *The operators $T, T_i, 1 \le i \le m : Q \to Q$ are all completely continuous.*

Proof. By applying the Arzelà-Ascoli theorem, using a method analogous to the proof in [16], one can complete the proof with a standard way. □

Define constants

$$M_i = \max_{0 \le x \le 1} \lambda_i \int_0^1 H(x,s) ds, \quad N_i = \max_{x \in [0,1]} \left\{ \lambda_i \int_0^1 \left| \frac{\partial H(x,s)}{\partial x} \right| ds \right\},$$
$$C_i = \frac{\lambda_i}{4^{n-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} \max_{x \in [\frac{1}{4}, \frac{3}{4}]} H(x,s) ds, \qquad 1 \le i \le m,$$

and

$$M = \max\{M_1, M_2, \cdots, M_m\}, \quad N = \max\{N_1, N_1, \cdots, N_m\}, \quad C = \min\{C_1, C_2, \cdots, C_m\}.$$

Theorem 1. *Suppose there exist constants $r_2 \ge 4^{n-1} b > b > r_1 > 0, K_2 \ge K_1 > 0$ such that $\frac{b}{C} \le \min\{\frac{r_2}{M}, \frac{K_2}{N}\}$. If:*
(A1) $f_i(x,u,v) < \min\{\frac{r_1}{M}, \frac{K_1}{N}\}$, for $(x,u,v) \in [0,1] \times [0,r_1] \times [-K_1, K_1]$,
(A2) $f_i(x,u,v) > \frac{b}{C}$, for $(x,u,v) \in [\frac{1}{4}, \frac{3}{4}] \times [b, 4^{n-1} b] \times [-K_2, K_2]$,
(A3) $f_i(x,u,v) \le \min\{\frac{r_2}{M}, \frac{K_2}{N}\}$, for $(x,u,v) \in [0,1] \times [0,r_2] \times [-K_2, K_2]$.
Then, the higher order iterative system (3) and (4) possesses at least three positive solutions:

$$(\hat{u}_1, T_2 T_3 \cdots T_m \hat{u}_1, \cdots, T_{m-1} T_m \hat{u}_1, T_m \hat{u}_1),$$

$$(\hat{u}_2, T_2 T_3 \cdots T_m \hat{u}_2, \cdots, T_{m-1} T_m \hat{u}_2, T_m \hat{u}_2),$$

and

$$(\hat{u}_3, T_2T_3\cdots T_m\hat{u}_3, \cdots, T_{m-1}T_m\hat{u}_3, T_m\hat{u}_3),$$

such that

$$\max_{0\le x\le 1}\hat{u}_1(x) < r_1, \quad \max_{0\le x\le 1}|\hat{u}_1'(x)| < K_1,$$

$$b \le \min_{\frac{1}{4}\le x\le\frac{3}{4}}\hat{u}_2(x) \le \max_{0\le x\le 1}\hat{u}_2(x) \le r_2, \quad \max_{0\le x\le 1}|\hat{u}_2'(x)| \le K_2,$$

$$\max_{0\le x\le 1}\hat{u}_3(x) \le 4^{n-1}b, \quad \min_{\frac{1}{4}\le x\le\frac{3}{4}}\hat{u}_3(x) \le b, \quad \max_{0\le x\le 1}|\hat{u}_3'(x)| \le K_2.$$

Proof. If $u_1 \in \overline{Q}(\zeta, r_2; \xi, K_2)$, then $\zeta(u_1) \le r_2, \xi(u_1) \le K_2$, and assumption (A3) implies $f_m(x, u_1(x), u_1'(x)) \le \min\{\frac{r_2}{M}, \frac{K_2}{N}\}$. Consequently,

$$\begin{aligned}\zeta(u_m) &= \zeta(T_m u_{m+1}) = \zeta(T_m u_1)\\ &= \max_{x\in[0,1]}\left|\lambda_m\int_0^1 H(x,s)f_m(s,u_1(s),u_1'(s))ds\right|\\ &\le \frac{r_2}{M}\max_{x\in[0,1]}\left|\lambda_m\int_0^1 H(x,s)ds\right|\\ &\le r_2,\end{aligned}$$

and

$$\begin{aligned}\xi(u_m) &= \max_{x\in[0,1]}|u_m'(x)|\\ &= \max_{x\in[0,1]}\left\{\left|\lambda_m\int_0^1\frac{\partial H(x,s)}{\partial x}f_m(s,u_1(s),u_1'(s))ds\right|\right\}\\ &\le \frac{K_2}{N}\cdot N_m\\ &\le K_2,\end{aligned}$$

i.e., $u_m \in \overline{Q}(\zeta, r_2; \xi, K_2)$. Then, the assumption (A3) implies $f_{m-1}(x, u_m(x), u_m'(x)) \le \min\{\frac{r_2}{M}, \frac{K_2}{N}\}$. Consequently,

$$\begin{aligned}\zeta(u_{m-1}) &= \zeta(T_{m-1}u_m)\\ &= \max_{x\in[0,1]}\left|\lambda_{m-1}\int_0^1 H(x,s)f_{m-1}(s,u_m(s),u_m'(s))ds\right|\\ &\le \frac{r_2}{M}\max_{x\in[0,1]}\left|\lambda_{m-1}\int_0^1 H(x,s)ds\right|\\ &\le r_2,\end{aligned}$$

and

$$\begin{aligned}\xi(u_{m-1}) &= \max_{x\in[0,1]}|u_{m-1}'(x)|\\ &= \max_{x\in[0,1]}\left\{\left|\lambda_{m-1}\int_0^1\frac{\partial H(x,s)}{\partial x}f_{m-1}(s,u_m(s),u_m'(s))ds\right|\right\}\end{aligned}$$

$$\leq \frac{K_2}{N} \cdot N_{m-1}$$
$$\leq K_2,$$

i.e., $u_{m-1} \in \overline{Q}(\zeta, r_2; \xi, K_2)$.

Proceeding in a similar way, we obtain

$$\zeta(T_1 T_2 \cdots T_m u_1) = \zeta(T u_1) \leq r_2, \ \ \xi(T_1 T_2 \cdots T_m u_1) = \xi(T u_1) \leq K_2.$$

Consequently, we have $T : \overline{Q}(\zeta, r_2; \xi, K_2) \to \overline{Q}(\zeta, r_2; \xi, K_2)$. Proceeding analogously, for any $u_1 \in \overline{Q}(\zeta, r_1; \xi, K_1)$, it follows that $0 \leq u_1(x) \leq r_1, -K_1 \leq u_1'(x) \leq K_1$. By assumption (A1), this implies $f_i(x, u(x), v) < \min\{\frac{r_1}{M}, \frac{K_1}{N}\}$, for $0 \leq x \leq 1$. Following a similar reasoning as before, we deduce $T : \overline{Q}(\zeta, r_1; \xi, K_1) \to Q(\zeta, r_1; \xi, K_1)$. Hence, (D2) is fulfilled.

For (D1), we consider the constant function $u_0(x) = 4^{n-1}b$, $0 \leq x \leq 1$. One can readily check that $u_0(x) = 4^{n-1}b \in \overline{Q}(\zeta, 4^{n-1}b; \xi, K_2; \psi, b)$, $\psi(u_0) = \psi(4^{n-1}b) > b$, and, consequently, $\{u \in \overline{Q}(\zeta, 4^{n-1}b; \xi, K_2; \psi, b) \mid \psi(u) > b\} \neq \emptyset$. For $u_1 \in \overline{Q}(\zeta, 4^{n-1}b; \xi, K_2; \psi, b)$, there is $|u_1(x)| \leq 4^{n-1}b, |u_1'(x)| \leq K_2$ for $0 \leq x \leq 1$ and $u_1(x) \geq b$ for $\frac{1}{4} \leq x \leq \frac{3}{4}$. The first step showed that for $x \in [0, 1]$, $|(T_i T_{i+1} \cdots T_m u_1)(x)| \leq 4^{n-1}b$, $|(T_i T_{i+1} \cdots T_m u_1)'(x)| \leq K_2 (1 \leq i \leq m)$. In addition, from assumption (A2), it yields that $f(x, u_1(x), u_1'(x)) > \frac{b}{C}$ for $\frac{1}{4} \leq x \leq \frac{3}{4}$. In the following, we estimate $\psi(T u_1)$.

By the use of the assumption (A2), we have $f_m(x, u_1(x), u_1'(x)) > \frac{b}{C}$ for $\frac{1}{4} \leq x \leq \frac{3}{4}$, and one gets

$$\begin{aligned}
\psi(u_m) &= \psi(T_m u_1) \\
&= \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \left| \lambda_m \int_0^1 H(x, s) f_m(s, u_1(s), u_1'(s)) ds \right| \\
&\geq \left| \frac{\lambda_m}{4^{n-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} \max_{x \in [\frac{1}{4}, \frac{3}{4}]} H(x, s) f_m(s, u_1(s), u_1'(s)) ds \right| \\
&> \frac{b}{C} \cdot \left| \frac{\lambda_m}{4^{n-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} \max_{x \in [\frac{1}{4}, \frac{3}{4}]} H(x, s) ds \right| \\
&\geq b.
\end{aligned}$$

By the use of the assumption (A2) again, we have $f_{m-1}(x, u_m(x), u'{}_m(x)) > \frac{b}{C}$ for $\frac{1}{4} \leq x \leq \frac{3}{4}$, and one has

$$\begin{aligned}
\psi(u_{m-1}) &= \psi(T_{m-1} u_m) \\
&= \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \left| \lambda_{m-1} \int_0^1 H(x, s) f_{m-1}(s, u_m(s), u_m'(s)) ds \right| \\
&\geq \left| \frac{\lambda_{m-1}}{4^{n-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} \max_{x \in [\frac{1}{4}, \frac{3}{4}]} H(x, s) f_{m-1}(s, u_1(s), u_1'(s)) ds \right| \\
&> \frac{b}{C} \cdot \left| \frac{\lambda_m}{4^{n-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} \max_{x \in [\frac{1}{4}, \frac{3}{4}]} H(x, s) ds \right| \\
&\geq b.
\end{aligned}$$

By the iterative method, we get $\psi(u_i) \geq b$, for $1 \leq i \leq m$. Therefore,

$$\psi(Tu_1) = \psi(T_1 T_2 \ldots T_m u_1) \geq b, \qquad \text{for every } \; u_1 \in \overline{Q}(\zeta, 4^{n-1}b; \xi, K_2; \psi, b).$$

The foregoing argument demonstrates that condition (D1) is held.

For (D3), to proceed with the proof, we suppose that $u_1 \in \overline{Q}(\zeta, r_2; \xi, K_2; \psi, b)$ with $\zeta(Tu_1) > 4^{n-1}b$. According to ψ and $Tu_1 \in Q$, it obtains

$$\begin{aligned}
\psi(Tu_1) &= \min_{\frac{1}{4} \leq x \leq \frac{3}{4}} |(Tu_1)(x)| \\
&\geq \frac{1}{4^{n-1}} \cdot \max_{0 \leq x \leq 1} |(Tu_1)(x)| \\
&= \frac{1}{4^{n-1}} \cdot \zeta(Tu_1) \\
&> \frac{1}{4^{n-1}} \cdot 4^{n-1} b \\
&= b.
\end{aligned}$$

(D3) is obtained. Thus, Lemma 2.1 directly gets that the operator T possesses positive fixed-points $\hat{u}_1, \hat{u}_2$, and $\hat{u}_3$ in $\overline{Q}(\zeta, r_2; \xi, K_2)$ with

$$\max_{0 \leq x \leq 1} \hat{u}_1(x) < r_1, \quad \max_{0 \leq x \leq 1} |\hat{u}'{}_1(x)| < K_1,$$

$$b \leq \min_{\frac{1}{4} \leq x \leq \frac{3}{4}} \hat{u}_2(x) \leq \max_{0 \leq x \leq 1} \hat{u}_2(x) \leq r_2, \quad \max_{0 \leq x \leq 1} |\hat{u}_2'(x)| \leq K_2,$$

$$\max_{0 \leq x \leq 1} \hat{u}_3(x) \leq 4^{n-1}b, \quad \min_{\frac{1}{4} \leq x \leq \frac{3}{4}} \hat{u}_3(x) \leq b, \quad \max_{0 \leq x \leq 1} |\hat{u}'{}_3(x)| \leq K_2.$$

After the above discussion, the higher order iterative system (3)-(4) has three positive solutions

$$(\hat{u}_1, T_2 T_3 \cdots T_m \hat{u}_1, \cdots, T_{m-1} T_m \hat{u}_1, T_m \hat{u}_1),$$

$$(\hat{u}_2, T_2 T_3 \cdots T_m \hat{u}_2, \cdots, T_{m-1} T_m \hat{u}_2, T_m \hat{u}_2),$$

and

$$(\hat{u}_3, T_2 T_3 \cdots T_m \hat{u}_3, \cdots, T_{m-1} T_m \hat{u}_3, T_m \hat{u}_3).$$

The proof is complete. □

An inspection of the proof of Theorem 3.1 reveals that, by suitably combining conditions of the type (A1)–(A3), problem (3) and (4) can be shown to admit an arbitrary number of positive solutions.

Remark 1. *The method used in Theorem 3.1 can be similarly applied to iterative systems where the nonlinearities involve higher order derivatives. One would need to employ fixed point theorems with more functionals, leading to more involved conditions and discussions.*

Corollary 1. *Assume the existence of constants* $0 < r_1 < b_1 < 4^{n-1}b_1 \leq r_2 < b_2 < 4^{n-1}b_2 \leq \cdots \leq r_p$ *and* $0 < K_1 \leq K_2 \leq \cdots \leq L_{p-1}$*(with* $p \in N$*) satisfying* $\frac{b_j}{C} \leq \min\{\frac{r_{j+1}}{M}, \frac{L_{j+1}}{N}\}$*. If, for all* $1 \leq i \leq m$ *and* $1 \leq j \leq p$*, the* f_i *meets the following requirements:*
(E1) $f_i(x, u, v) < \min\{\frac{r_j}{M}, \frac{L_j}{N}\}$*, for* $(x, u, v) \in [0, 1] \times [0, r_j] \times [-L_j, L_j]$*;*
(E2) $f_i(x, u, v) > \frac{b_j}{C}$*, for* $(x, u, v) \in [\frac{1}{4}, \frac{3}{4}] \times [b_j, 4^{n-1}b_j] \times [-L_{j+1}, L_{j+1}]$*,*
then the boundary value problem (3) and (4) possesses at least $2p - 1$ *positive solutions.*

Proof. For the base case $p = 1$, condition (E1) implies that the operator satisfies $T : \overline{Q}(\zeta, r_1; \xi, K_1) \to Q(\zeta, r_1; \xi, K_1) \subset \overline{Q}(\zeta, r_1; \xi, K_1)$. By the Schauder's fixed-point theorem, T admits at least one fixed point $\hat{u}_1 \in Q(\zeta, r_1; \xi, K_1)$. When $p = 2$, Theorem 3.1 (taking $c_1 = r_2$) directly applies, yielding at least three fixed points $\hat{u}_2, \hat{u}_3$, and $\hat{u}_4$. Hence, the proof is completed via induction. □

Remark 2. *The proposed method can be extended to more general boundary conditions. Specifically, when the boundary condition (4) is substituted with the nonlocal condition described in [17], the corresponding Green's function can be explicitly constructed. Applying Theorem 2.1, a parallel argument to that of Theorem 3.1 establishes that, under assumptions (A1)–(A3), there exist at least three solutions.*

4. Example

In this section, we provide specific examples to illustrate that hypotheses (A1)–(A3) are common growth conditions, and their rationality and feasibility are well-grounded.

Example 1. *Consider the case with $n = 3$, $m = 3$, and $r = 1$, and address the problem*

$$\begin{cases} u_1'''(x) + \lambda_1 f_1(x, u_2(x), u'_2(x)) = 0, & x \in [0,1]; \\ u_2'''(x) + \lambda_2 f_2(x, u_3(x), u'_3(x)) = 0, & x \in [0,1]; \\ u_3'''(x) + \lambda_3 f_3(x, u_1(x), u'_1(x)) = 0, & x \in [0,1], \end{cases}$$

with the boundary conditions

$$u_i(0) = u_i'(0) = u_i'(1) = 0, \ \textit{for} \ \ i = 1,2,3,$$

where

$$f_1(x,u,v) = g(u) + \sin x + \left(\frac{|v|}{3000}\right)^3,$$
$$f_2(x,u,v) = g(u) + \cos x + \left(\frac{|v|}{4000}\right)^3,$$
$$f_3(x,u,v) = g(u) + \sin(2x) + \left(\frac{|v|}{5000}\right)^3,$$

and

$$g(u) = \begin{cases} 20u, & 0 \le u \le 1; \\ 20 + 1980(u-1), & 1 \le u \le 2; \\ 2000, & u \ge 2. \end{cases}$$

Select $\lambda_1 = 0.5, \lambda_2 = 1, \lambda_3 = 1.5, r_1 = 1, b = 2, r_2 = 1000, K_1 = 10, K_2 = 3000$. *With these values, there are*

$$M = \frac{1}{128},\ N = \frac{1}{4},\ C = \frac{13}{6144},$$

and

$$\min\left\{\frac{r_1}{M}, \frac{K_1}{N}\right\} = 40,\ \frac{b}{C} \approx 945.23,\ \min\left\{\frac{r_2}{M}, \frac{K_2}{N}\right\} = 12000.$$

Now, we check that $f_i(x,u,v)(i=1,2,3)$ *satisfy all conditions of Theorem 3.1.*

For (A1), when $0 \le x \le 1, 0 \le u \le 1, |v| \le 10,$ *we have*

$$f_1(x,u,v) = g(u) + \sin x + \left(\frac{|v|}{3000}\right)^3 \le 1 + \left(\frac{10}{3000}\right)^3 < 40.$$

Using the same verification method, one obtains f_2 *and* $f_3 < 40$.

For (A2), when $1/4 \le x \le 3/4, 2 \le u \le 32, |v| \le 3000,$ *we have*

$$f_1(x,u,v) = g(u) + \sin x + \left(\frac{|v|}{3000}\right)^3 \ge 1999 + \left(\frac{3000}{3000}\right)^3 > 945.23.$$

Following the same verification procedure, we get f_2 *and* $f_3 > 945.23$.

For (A3), when $0 \le x \le 1, 0 \le u \le 1000, |v| \le 3000,$ *we have*

$$f_1(x,u,v) = g(u) + \sin x + \left(\frac{|v|}{3000}\right)^3 \le 2000 + 1 + 1 \le 12000.$$

By the same verification approach, we obtain f_2 *and* $f_3 < 12000$.

Consequently, each conditions in Theorem 3.1 can be fulfilled, and employing this theorem allows the solution to the aforementioned problem to be expressed as $(\hat{u}_1, T_2T_3\hat{u}_1, T_3\hat{u}_1)$, $(\hat{u}_2, T_2T_3\hat{u}_2, T_3\hat{u}_2)$, *and* $(\hat{u}_3, T_2T_3\hat{u}_3, T_3\hat{u}_3)$ *such that*

$$\max_{0\le x\le 1} \hat{u}_1(x) < 1, \quad \max_{0\le x\le 1} |\hat{u}_1'(x)| < 10,$$

$$2 \le \min_{\frac{1}{4}\le x\le \frac{3}{4}} \hat{u}_2(x) \le \max_{0\le x\le 1} \hat{u}_2(x) \le 1000, \quad \max_{0\le x\le 1} |\hat{u}_2'(x)| \le 3000,$$

$$\max_{0\le x\le 1} \hat{u}_3(x) \le 32, \quad \min_{\frac{1}{4}\le x\le \frac{3}{4}} \hat{u}_3(x) \le 2, \quad \max_{0\le x\le 1} |\hat{u}_3'(x)| \le 3000.$$

5. Conclusions

This paper studies the existence of multiple positive solutions for a class of higher order iterative systems with two-point boundary conditions, where the nonlinear terms depend on the first-order derivatives. By constructing the Green's function and applying the fixed-point theorem of functional type on cones due to Bai and Ge, we establish the existence of at least three distinct positive solutions under suitable growth conditions. A concrete example is provided to verify the theoretical results. The work extends existing results on iterative systems to the case with derivative-dependent nonlinearities.

Author contributions

All authors write and review this manuscript equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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