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Research article

Impulsive nonlocal boundary value problems for (k,ψ)-Hilfer proportional fractional differential equations: Existence, stability, and application to pantograph equations

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Abstract: This paper examines a class of impulsive boundary value problems related to fractional pantograph differential equations governed by the (k,ψ)-Hilfer proportional fractional derivative. The problem is reformulated into an equivalent integral equation, which provides a convenient framework for further analysis. The existence and uniqueness of solutions are derived using Banach's fixed-point theorem. Moreover, several forms of Ulam stability are examined. Illustrative examples and graphical computations are included to support the applicability of the theoretical results.

Keywords: (k,ψ)-Hilfer proportional fractional operator; impulsive conditions; nonlocal integral boundary conditions; existence and uniqueness; Ulam-Hyers stability
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1. Introduction

Nowadays, fractional differential equations ($\mathcal{FDE}$s) have become a primary mathematical tool for modeling hereditary effects across various fields, including engineering, biology, fluid systems, and physics; we refer the reader to see [1, 2] and references therein. Researchers employ these equations to address a broad range of studies, ranging from the theoretical analysis of solution existence to the development of analytical and numerical solution techniques. In particular, impulsive differential equations ($\mathcal{DE}$s) are fundamental tools for modeling dynamical systems that experience sudden changes in state at specific moments. Addtionally, impulsive $\mathcal{FDE}$s play an important role in describing phenomena involving abrupt changes, such as synaptic jumps in biological neural networks, mechanical shock processes, and trajectory correction systems [3, 4]. On the other hand, implicit $\mathcal{FDE}$s naturally emerge in various applications, including optimal control theory and heat transfer models with nonlinear feedback effects [5–7]. The integration of fractional dynamics, impulsive effects, and implicit nonlinearities gives rise to a mathematically challenging yet practically significant class of systems, which constitutes the main focus of the present study. Their rich theoretical foundation and broad applicability have led to considerable research interest.

Moreover, the pantograph equation ($\mathcal{PGE}$), introduced in [8], is a well-known $\mathcal{DE}$ incorporating a proportional time-delay term. It has been widely employed to model a variety of processes and phenomena whose evolution depends on their past states. There has been substantial research activity devoted to the study and development of impulsive $\mathcal{FDE}$s with time-delay effects. Many of them have investigated the existence and uniqueness of solutions, qualitative behavior, and various stability notions for impulsive delay $\mathcal{DE}$s. For example, Ahmed et al. [9] proved the existence and uniqueness of solutions in 2020 for an impulsive fractional pantograph differential equation ($\mathcal{FPGDE}$) with generalized anti-periodic boundary conditions involving the Caputo fractional operator. In 2021, Khaminsou et al. [10] examined a fractional pantograph boundary value problem with instantaneous impulses under the Caputo proportional fractional derivative with respect to another function. In 2024, Kaewsuwan et al. [11] investigated a class of nonlinear impulsive fractional integro-differential equations under the $\mathcal{PGE}$. Their analysis was carried out under multi-point integral boundary conditions within the framework of the (ρ_k, ψ_k)-Hilfer fractional derivative operator ($\mathcal{HFDO}$). For related works, we refer the reader to [12–14] and the references therein. Additionally, Ulam-type stability has become an essential tool in the qualitative analysis of $\mathcal{PDE}$s, particularly when exact solutions are difficult to obtain. It provides a rigorous framework to ensure that approximate solutions remain close to exact solutions under small perturbations. In particular, Ulam-Hyers (UH) stability and its generalizations, namely generalized Ulam-Hyers (GUH), Ulam-Hyers-Rassias (UHR), and generalized Ulam-Hyers-Rassias (GUHR) stability, have been widely applied to impulsive and non-impulsive fractional models. These concepts have demonstrated their effectiveness in studying the robustness of solutions. For related results, we refer the reader to [15–18] and the references cited therein.

Compared with Kaewsuwan et al. [11], the present work is developed in a different operator setting and problem structure. Specifically, we study an impulsive $\mathcal{FPGDE}$ of order $\alpha_\ell \in (0, 1]$ governed by the (k_ℓ, ψ_ℓ)-Hilfer proportional fractional operator, whereas Ref. [11] considered a higher-order impulsive pantograph integro-differential equation under the (ρ_k, ψ_k)-$\mathcal{HFDO}$. In addition, our impulsive condition is formulated through the (k_ℓ, ψ_ℓ)-Riemann-Liouville proportional fractional integral operator, which leads to a new equivalent integral equation and corresponding Ulam-type stability results. Our research seeks to address this gap in the literature on nonlinear impulsive fractional boundary value problems, as the qualitative analysis and stability theory for impulsive $\mathcal{FPGDE}$ with mixed boundary conditions have not yet been systematically developed within the framework of the (k_ℓ, ψ_ℓ)-Hilfer proportional fractional derivative operator ((k_ℓ, ψ_ℓ)-$\mathcal{HPFDO}$). Motivated by this gap and the previous research background review, the present work aims to establish a comprehensive qualitative framework for the impulsive (k_ℓ, ψ_ℓ)-Hilfer proportional fractional differential equation ($\mathcal{HPFDE}$) subject to mixed boundary conditions of the form

$$\begin{cases} {}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}u(t) = f(t,u(t),u(\lambda t)), \quad t \in \mathcal{J}_\ell, \quad t \neq t_\ell, \quad \ell = 0,1,\ldots,m, \\ {}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell}u(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}}u(t_\ell^-) = \phi_\ell(u(t_\ell)), \quad \ell = 1,2,\ldots,m, \\ u(T) = \displaystyle\sum_{i=0}^{m}\mu_i\, {}_{t_i,k_i}\mathcal{I}^{\sigma_i,\rho_i;\psi_i}u(\xi_i) + \mathcal{A}, \qquad \xi_i \in (t_i,t_{i+1}], \end{cases} \tag{1.1}$$

where ${}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}$ denotes the (k_ℓ,ψ_ℓ)-$\mathcal{HPFDO}$ of order $\alpha_\ell \in (0,1]$ and type $\beta_\ell \in [0,1]$, $\rho_\ell \in (0,1]$, $k_\ell > 0$, $\mathcal{J}_\ell := (t_\ell,t_{\ell+1}] \subset (a,b]$ for $\ell = 0,1,2,\ldots,m$, with $0 \le a = t_0 < t_1 < \ldots < t_m < t_{m+1} \le b \le T$; $\mathcal{J}_0 := [a,t_1]$, $\mathcal{J} := [a,b]$, ${}_{t_\ell^+,k_\ell}\mathcal{I}^{q,\rho_\ell;\psi_\ell}$ is the (k_ℓ,ψ_ℓ)-Riemann-Liouville proportional fractional integral operator (RL-$\mathcal{PFIO}$) with order $q \in \{(k_\ell-\gamma_\ell),(k_{\ell-1}-\gamma_{\ell-1}),\sigma_\ell\}$ with $q > 0$, $\ell = 1,2,\ldots,m$, $\xi_i \in (t_i,t_{i+1}]$, $\mathcal{A} \in \mathbb{R}$, and $\lambda \in [0,1]$. The main contributions of this study are the derivation of an equivalent integral formulation for the proposed impulsive fractional $\mathcal{PGDE}$, the establishment of sufficient conditions for the existence and uniqueness of solutions via Banach's fixed-point theorem, and the analysis of several Ulam-type stability results, including UH, GUH, UHR, and GUHR stability results. Illustrative examples and graphical computations are also provided to illustrate the theoretical results and to show the influence of the main fractional and operator parameters.

The structure of the paper is as follows: Section 2 provides the necessary operator definitions and preliminary lemmas, which are essential for converting the problem under consideration into an equivalent integral equation. Section 3 establishes the existence and uniqueness of solutions using Banach's fixed-point theorem, while Section 4 addresses the Ulam stability of the solutions. Illustrative examples with graphical computations are presented in Section 5 to validate and illustrate the main results. The conclusions of this study are presented in the final section.

2. Preliminaries

This section presents the fundamental definitions, lemmas, and auxiliary tools used throughout the study; further details can be found in [19–27]. We next introduce the notation employed for simplifying computations in this paper.

$$ {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s) = (\psi(\tau)-\psi(s))^{\frac{\alpha}{k}-1}\exp\left(\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(s))\right). \tag{2.1}$$

Definition 2.1. *([27]). Let $\alpha > 0$, $k > 0$, $\rho \in (0,1]$, and $u \in L^1([a,b],\mathbb{R})$. Hence, the (k,ψ)-RL-$\mathcal{PFIO}$ of α of u can be written as*

$$ {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}u(\tau) = \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)u(s)ds, \tag{2.2}$$

where $\Gamma_k(u) = \int_0^\infty s^{u-1}\exp\left(-\frac{s^k}{k}\right)ds$, u is an element in complex number $\mathbb{C}$, that is $Re(u) > 0$, and $\Gamma(u) = \Gamma_k(u)$ as $k \to 1$, $\Gamma_k(z+k) = z\Gamma_k(z)$, $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$, $\Gamma_k(k) = 1$.

Definition 2.2. *([27]). Let $\alpha > 0$, $k > 0$, $\rho \in (0,1]$, $u \in C([a,b],\mathbb{R})$, $\psi(\tau) \in C^n([a,b],\mathbb{R})$ with $\psi'(\tau) \neq 0$, and $n = 1,2,\ldots$ so that $n = \lfloor\alpha/k\rfloor + 1$. Hence, the (k,ψ)-RL-$\mathcal{PFIO}$ of α of u can be written as*

$$ {}^{\mathbb{RL}}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau) \;=\; \frac{{}_{k}\mathfrak{D}^{n,\rho;\psi}}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(\tau,s)\psi'(\tau)u(s)ds$$

$$= {}_{k}\mathfrak{D}^{n,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}u(\tau)\Big),$$

where ${}_{k}\mathfrak{D}^{1,\rho;\psi}u(\tau) = {}_{k}\mathfrak{D}^{\rho;\psi}u(\tau) = (1-\rho)u(\tau) + k\rho\frac{u'(\tau)}{\psi'(\tau)}$ *and* ${}_{k}\mathfrak{D}^{n,\rho;\psi} = \underbrace{{}_{k}\mathfrak{D}^{\rho;\psi}{}_{k}\mathfrak{D}^{\rho;\psi}\cdots{}_{k}\mathfrak{D}^{\rho;\psi}}_{n\ times}$.

Definition 2.3. *([27]). Let* $\alpha > 0$, $k > 0$, $\rho \in (0,1]$, $\beta \in [0,1]$, $u \in C^n([a,b],\mathbb{R})$, $\psi(\tau) \in C^n([a,b],\mathbb{R})$ *with* $\psi'(\tau) \neq 0$, *and* $n \in \mathbb{N}$ *so that* $n = \lfloor \alpha/k \rfloor + 1$. *Hence, the* (k,ψ)*-*$\mathcal{HPFDO}$ *of* α *and* β *of* u *can be written as*

$${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau) = {}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}u(\tau)\Big)\Big).$$

Next, we list several important properties required for the analysis.

Lemma 2.4. *([27]). Assume* α, $\delta \in [0,\infty)$, k, $\gamma \in (0,\infty)$, $\rho \in (0,1]$, $\omega \in \mathbb{R}$, *and* $\omega/k > -1$. *Hence,*

(i) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(\tau,a)\Big] = \frac{\Gamma_k(\omega)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega+\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega+\alpha}{k}-1}(\tau,a)$.

(ii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\Big] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a)$.

(iii) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{\delta,\rho;\psi}u(\tau)\Big) = {}_{a,k}\mathcal{I}^{\delta,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}u(\tau)\Big) = {}_{a,k}\mathcal{I}^{\delta+\alpha,\rho;\psi}u(\tau)$.

(iv) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau)\Big) = u(\tau) - \sum\limits_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-i}(\tau,a)}{\rho^{\frac{\gamma-ki}{k}}\Gamma_k(\gamma+k-ki)}\Big[{}_{k}\mathfrak{D}^{n-i,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\gamma,\rho;\psi}u(a^+)\Big)\Big]$,
where $\gamma = \alpha + \beta(nk-\alpha)$.

Next, we employ the following auxiliary result:

Lemma 2.5. *([27]). Let* ω, $\eta \in \mathbb{C}$, $Re(\eta) > 0$, $Re(\omega) > 0$, $k > 0$, $\rho \in (0,1]$, $m \in \mathbb{N}$, $Re(\eta) > Re(\omega)$, *and* $n = \lfloor Re(\omega)/k \rfloor + 1$. *Then we obtain the following relations:*

$${}^{H}_{a,k}\mathfrak{D}^{\omega,\beta,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\Big) = {}_{a,k}\mathcal{I}^{\eta-\omega,\rho;\psi}f(t). \tag{2.3}$$

Let the weighted space be denoted by

$$C_{\psi}^{1-\frac{\gamma}{k}}(\mathcal{J},\mathbb{R}) = \Big\{u : (a,b] \to \mathbb{R} | u(a^+) \text{ exists, and } {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma}{k}}(t,a)u(t) \in C(\mathcal{J},\mathbb{R})\Big\},\ \gamma \in (0,1],$$

where $C_{\psi}^{1-\frac{\gamma}{k}} = C_{\psi}^{1-\frac{\gamma}{k}}(\mathcal{J},\mathbb{R})$. The weighted space of piece-wise continuous functions is defined as

$$\begin{aligned}\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J},\mathbb{R}) \ &= \ \Big\{u : (a,b] \to \mathbb{R} |\ u \in C_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}},\ \ell = 0,1,2,\ldots,m, \\ &\qquad {}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell}u(t_\ell^+),\ {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}}u(t_\ell^-) \text{ exist, and} \\ &\qquad {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}}u(t_\ell^-) = {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}}u(t_\ell),\ \ell = 1,\ldots,m\Big\}.\end{aligned}$$

Observe that $\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}} = \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J},\mathbb{R})$ is a Banach space equipped with

$$\|u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} = \sup_{t\in\mathcal{J}}\Big|{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)u(t)\Big|.$$

For ease of notation, we set the following symbols:

$$\mathcal{F}_u(t) \ := \ f(t,u(t),u(\lambda t)), \tag{2.4}$$

$$\mathcal{G}_\ell(u) \ := \ {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell+(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell}\mathcal{F}_u(t_{\ell+1}) + \phi_{\ell+1}(u(t_{\ell+1})), \tag{2.5}$$

for $\ell = 0,1,\ldots,m-1$.

Lemma 2.6. *Let $\alpha_\ell \in (0,1)$, $\beta_\ell \in [0,1]$, $\rho_\ell \in (0,1]$, $k_\ell > 0$, $\gamma_\ell = \alpha_\ell + \beta_\ell(k_\ell - \alpha_\ell)$, $\psi_\ell \in C(\mathcal{J},\mathbb{R})$ with $\psi_\ell' > 0$, $\ell = 0,1,2,\ldots,m$, $\mathcal{F}_u \in C_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}$, and $\Delta_1 \neq 0$. Then the linear impulsive (k_ℓ,ψ_ℓ)-$\mathcal{HPFDE}$ with mixed boundary conditions*

$$\begin{cases} {}_{t_\ell^+,k_\ell}^{\;H}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}u(t) = \mathcal{F}_u(t), \quad t \in \mathcal{J}_\ell, \quad t \neq t_\ell, \quad \ell = 0,1,\ldots,m, \\ {}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell}u(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}}u(t_\ell^-) = \phi_\ell(u(t_\ell)), \quad \ell = 1,2,\ldots,m, \\ u(T) = \displaystyle\sum_{i=0}^{m}\mu_i \, {}_{t_i,k_i}\mathcal{I}^{\sigma_i,\rho_i;\psi_i}u(\xi_i) + \mathcal{A}, \qquad \xi_i \in (t_i,t_{i+1}], \end{cases} \tag{2.6}$$

is satisfying to the following integral equation, $u \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}$, as

$$\begin{aligned} u(t) &= \frac{{}_{k_\ell}^{\rho_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m}\frac{\mu_i\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\mathcal{G}_j(u) - \frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\mathcal{G}_j(u) \\ &\quad + \sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i) + \mathcal{A} - {}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\Bigg\}\Bigg] \\ &\quad + \frac{{}_{k_\ell}^{\rho_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\sum_{j=0}^{\ell-1}\mathcal{G}_j(u) + {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_u(t), \end{aligned} \tag{2.7}$$

where

$$\Delta_1 := \frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} - \sum_{i=0}^{m}\frac{\mu_i\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}. \tag{2.8}$$

Proof. Let $u \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}$ be a solution of the problem (2.6). For $t \in [t_0,t_1]$, we have

$$u(t) = \frac{{}_{k_0}^{\rho_0}\Psi_{\psi_0}^{\frac{\gamma_0}{k_0}-1}(t,t_0)}{\rho_0^{\frac{\gamma_0}{k_0}-1}\Gamma_{k_0}(\gamma_0)}c_1 + {}_{t_0,k_0}\mathcal{I}^{\alpha_0,\rho_0;\psi_0}\mathcal{F}_u(t),$$

where $c_1 = {}_{t_0,k_0}\mathcal{I}^{(k_0-\gamma_0),\rho_0;\psi_0}u(t_0^+)$. Using Lemma 2.4, we obtain

$${}_{t_0,k_0}\mathcal{I}^{(k_0-\gamma_0),\rho_0;\psi_0}u(t) = c_1 + {}_{t_0,k_0}\mathcal{I}^{\alpha_0+(k_0-\gamma_0),\rho_0;\psi_0}\mathcal{F}_u(t). \tag{2.9}$$

Substituting $t = t_1$ into (2.9), we have

$${}_{t_0,k_0}\mathcal{I}^{(k_0-\gamma_0),\rho_0;\psi_0}u(t_1) = c_1 + {}_{t_0,k_0}\mathcal{I}^{\alpha_0+(k_0-\gamma_0),\rho_0;\psi_0}\mathcal{F}_u(t_1). \tag{2.10}$$

For $t \in (t_1,t_2]$, we obtain

$$u(t) = \frac{{}_{k_1}^{\rho_1}\Psi_{\psi_1}^{\frac{\gamma_1}{k_1}-1}(t,t_1)}{\rho_1^{\frac{\gamma_1}{k_1}-1}\Gamma_{k_1}(\gamma_1)}\,{}_{t_1,k_1}\mathcal{I}^{(k_1-\gamma_1),\rho_1;\psi_1}u(t_1^+) + {}_{t_1,k_1}\mathcal{I}^{\alpha_1,\rho_1;\psi_1}\mathcal{F}_u(t). \tag{2.11}$$

From the impulsive condition, ${}_{t_1^+,k_1}\mathcal{I}^{(k_1-\gamma_1),\rho_1;\psi_1}u(t_1^+) = {}_{t_0^+,k_0}\mathcal{I}^{(k_0-\gamma_0),\rho_0;\psi_0}u(t_1^-) + \phi_1(u(t_1))$, implies that

$$u(t) \quad = \quad \frac{{}^{\rho_1}_{k_1}\Psi_{\psi_1}^{\frac{\gamma_1}{k_1}-1}(t,t_1)}{\rho_1^{\frac{\gamma_1}{k_1}-1}\Gamma_{k_1}(\gamma_1)}\left(c_1 + \mathcal{G}_0(u)\right) + {}_{t_1,k_1}\mathcal{I}^{\alpha_1,\rho_1;\psi_1}\mathcal{F}_u(t).$$

Applying Lemma 2.4, we get

$${}_{t_1,k_1}\mathcal{I}^{(k_1-\gamma_1),\rho_1;\psi_1}u(t) \quad = \quad c_1 + {}_{t_0,k_0}\mathcal{I}^{\alpha_0+(k_0-\gamma_0),\rho_0;\psi_0}\mathcal{F}_u(t_1) + \phi_1(u(t_1)) + {}_{t_1,k_1}\mathcal{I}^{\alpha_1+(k_1-\gamma_1),\rho_1;\psi_1}\mathcal{F}_u(t).$$

In particular $t = t_2$, we have

$${}_{t_1,k_1}\mathcal{I}^{(k_1-\gamma_1),\rho_1;\psi_1}u(t_2) \quad = \quad c_1 + {}_{t_0,k_0}\mathcal{I}^{\alpha_0+(k_0-\gamma_0),\rho_0;\psi_0}\mathcal{F}_u(t_1) + \phi_1(u(t_1)) + {}_{t_1,k_1}\mathcal{I}^{\alpha_1+(k_1-\gamma_1),\rho_1;\psi_1}\mathcal{F}_u(t_2).$$

Under the impulsive condition, ${}_{t_2^+,k_2}\mathcal{I}^{(k_2-\gamma_2),\rho_2;\psi_2}u(t_2^+) = {}_{t_1^+,k_1}\mathcal{I}^{(k_1-\gamma_1),\rho_1;\psi_1}u(t_2^-) + \phi_2(u(t_2))$, for $t \in (t_2, t_3]$, we get

$$u(t) = \frac{{}^{\rho_2}_{k_2}\Psi_{\psi_2}^{\frac{\gamma_2}{k_2}-1}(t,t_2)}{\rho_2^{\frac{\gamma_2}{k_2}-1}\Gamma_{k_2}(\gamma_2)}\left(c_1 + \sum_{j=0}^{1}\mathcal{G}_j(u)\right) + {}_{t_2,k_2}\mathcal{I}^{\alpha_2,\rho_2;\psi_2}\mathcal{F}_u(t).$$

Then, for $t \in (t_3, t_4]$, we have

$$u(t) = \frac{{}^{\rho_3}_{k_3}\Psi_{\psi_3}^{\frac{\gamma_3}{k_3}-1}(t,t_3)}{\rho_3^{\frac{\gamma_3}{k_3}-1}\Gamma_{k_3}(\gamma_3)}\left(c_1 + \sum_{j=0}^{2}\mathcal{G}_j(u)\right) + {}_{t_3,k_3}\mathcal{I}^{\alpha_3,\rho_3;\psi_3}\mathcal{F}_u(t).$$

Repeating the above process, for any $t \in (t_\ell, t_{\ell+1}]$, with $\ell = 0, 1, , \ldots, m$, one has

$$u(t) = \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\left(c_1 + \sum_{j=0}^{\ell-1}\mathcal{G}_j(u)\right) + {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_u(t). \tag{2.12}$$

By Lemma 2.4, we obtain

$${}_{t_\ell,k_\ell}\mathcal{I}^{\sigma_\ell,\rho_\ell;\psi_\ell}u(t) \quad = \quad \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell+\sigma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell+\sigma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell+\sigma_\ell)}\left(c_1 + \sum_{j=0}^{\ell-1}\mathcal{G}_j(u)\right) + {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell+\sigma_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_u(t).$$

Applying the mixed boundary condition, $u(T) = \sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\sigma_i,\rho_i;\psi_i}u(\xi_i) + \mathcal{A}$, we have

$$\begin{aligned}
&\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\left(c_1 + \sum_{j=0}^{m-1}\mathcal{G}_j(u)\right) + {}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\\
= \quad &\sum_{i=0}^{m}\frac{\mu_i\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\left(c_1 + \sum_{j=0}^{i-1}\mathcal{G}_j(u)\right) + \sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i) + \mathcal{A}.
\end{aligned} \tag{2.13}$$

Solving Eq (2.13), we obtain that

$$\begin{aligned} c_1 &= \frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m}\frac{\mu_i\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\mathcal{G}_j(u)-\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\mathcal{G}_j(u) \\ &\quad+\sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i)+\mathcal{A}-{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\Bigg\}, \end{aligned} \tag{2.14}$$

where Δ_1 is given by (2.8). Inserting the value c_1 in (2.12), we obtain the solution (2.7).

Conversely, suppose that u satisfies (2.7). Taking ${}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}$ into both sides of (2.7) and using Lemmas 2.4 and 2.5, we get that

$$\begin{aligned} &{}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}u(t) \\ &= {}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}\Bigg(\Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m}\frac{\mu_i\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\mathcal{G}_j(u)-\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\mathcal{G}_j(u) \\ &\quad+\sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i)+\mathcal{A}-{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\Bigg\}\Bigg]\frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg) \\ &\quad+{}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}\Bigg(\frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\sum_{j=0}^{\ell-1}\mathcal{G}_j(u)\Bigg)+{}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}\Big({}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_u(t)\Big) \\ &= \mathcal{F}_u(t). \end{aligned}$$

Next, we verify that u satisfies mixed boundary conditions. By directly evaluating (2.7), we obtain

$$\begin{aligned} u(T) &= \frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m}\frac{\mu_i\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\mathcal{G}_j(u)-\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\mathcal{G}_j(u) \\ &\quad+\sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i)+\mathcal{A}-{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\Bigg\}\Bigg] \\ &\quad+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\mathcal{G}_j(u)+{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\sigma_i,\rho_i;\psi_i}u(\xi_i) &= \sum_{i=0}^{m}\frac{\mu_i\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m}\frac{\mu_i\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\mathcal{G}_j(u) \\ &\quad-\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\mathcal{G}_j(u)+\sum_{i=0}^{m}\mu_i\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i)+\mathcal{A} \end{aligned}$$

$$-{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\Bigg\}\Bigg] + \frac{\mu_i\, {}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i, t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} \sum_{j=0}^{\ell-1} \mathcal{G}_j(u)$$
$$+\sum_{i=0}^{m} \mu_i\, {}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i).$$

Therefore, $u(T) = \sum_{i=0}^{m} \mu_i\, {}_{t_i,k_i}\mathcal{I}^{\sigma_i,\rho_i;\psi_i} u(\xi_i) + \mathcal{A}$. The proof is finished. □

3. Existence and uniqueness result via Banach's fixed-point theorem

By Lemma 2.6, we define an operator $Q : \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R}) \to \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ by

$$\begin{aligned}(Qu)(t) \;=\;& \Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m} \frac{\mu_i\, {}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i, t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} \sum_{j=0}^{i-1} \mathcal{G}_j(u) - \frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T, t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} \sum_{j=0}^{m-1} \mathcal{G}_j(u) \\ &+ \sum_{i=0}^{m} \mu_i\, {}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_u(\xi_i) + \mathcal{A} - {}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_u(T)\Bigg\}\Bigg] \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t, t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)} \\ &+ \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t, t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)} \sum_{j=0}^{\ell-1} \mathcal{G}_j(u) + {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_u(t). \end{aligned} \tag{3.1}$$

It follows that the problem (1.1) has a solution if and only if the operator Q has fixed points. In addition, we set

$$\Lambda_1 := \frac{1}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}, \tag{3.2}$$

$$\begin{aligned}\Lambda_2 :=\;& \sum_{i=0}^{m} \frac{|\mu_i|\, {}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i, t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} \sum_{j=0}^{i-1} \frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1}, t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\ &+ \frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T, t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} \sum_{j=0}^{m-1} \frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1}, t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\ &+ \sum_{i=1}^{m} \frac{|\mu_i|\, {}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i, t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)} + \frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T, t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}, \end{aligned} \tag{3.3}$$

$$\Lambda_3 := \sum_{j=0}^{m-1} \frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1}, t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}, \tag{3.4}$$

$$\Lambda_4 := \frac{{}^{\rho_m}_{k_m}\Psi^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}_{\psi_m}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}, \tag{3.5}$$

$$\Lambda_5 := \sum_{i=0}^{m}\frac{i\ |\mu_i|\,{}^{\rho_i}_{k_i}\Psi^{\frac{\gamma_i+\sigma_i}{k_i}-1}_{\psi_i}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} + \frac{m\,{}^{\rho_m}_{k_m}\Psi^{\frac{\gamma_m}{k_m}-1}_{\psi_m}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}. \tag{3.6}$$

Lemma 3.1. (*Banach's fixed-point theorem [28]*) *Let $\mathcal{B}$ be a non-empty closed subset of a Banach space $\mathcal{X}$. Then, any contraction mapping Q from $\mathcal{B}$ into itself has a unique fixed point.*

Theorem 3.2. *Suppose that $\psi_\ell \in C(\mathcal{J})$, where $\psi'_\ell(t) > 0$, $\ell = 0, 1, 2, \ldots, m$, $t \in \mathcal{J}$ and $f \in C(\mathcal{J}\times\mathbb{R}^2, \mathbb{R})$, $\phi_\ell \in C(\mathbb{R}, \mathbb{R})$ for $\ell = 1, 2, \ldots, m$, which satisfy the following assumptions:*

($\mathbb{H}_1$) *There exists a real positive constant $\mathcal{L}$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le {}^{\rho_\ell}_{k_\ell}\Psi^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}(t,t_\ell)\mathcal{L}\Big(|u_1-u_2| + |v_1-v_2|\Big),$$

for all $t \in \mathcal{J}$ and u_i, $v_i \in \mathbb{R}$ for $i = 1, 2$.

($\mathbb{H}_2$) *There exists a real positive constant Φ such that*

$$|\phi_\ell(u) - \phi_\ell(v)| \le \Phi\,{}^{\rho_\ell}_{k_\ell}\Psi^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}(t,t_\ell)\,|u-v|,$$

for all $t \in \mathcal{J}$ and u, $v \in \mathbb{R}$ for $\ell = 1, 2, \ldots, m$.

Then, the problem (1.1) *has a unique solution if*

$$\Theta_1 2\mathcal{L} + \Theta_2\Phi < 1, \tag{3.7}$$

where Θ_1, Θ_2 are provided by

$$\Theta_1 := \Lambda_1(\Lambda_2 + |\Delta_1|\Lambda_3) + \Lambda_4, \tag{3.8}$$

$$\Theta_2 := \Lambda_1(\Lambda_5 + |\Delta_1|m). \tag{3.9}$$

Proof. We change the problem (1.1) into a fixed-point problem, that is, $u = Qu$, where the operator Q is defined by (3.1). The proof will be separated into two steps as follows:

Step 1: It is shown that $Q\mathcal{B}_\mathcal{R} \subset \mathcal{B}_\mathcal{R}$.

Assume that $\sup_{t\in\mathcal{J}}|f(0,0,0)| := \mathbb{F} < \infty$ and $\mathbb{I} := \max\{|\phi_\ell(0)| : \ell = 1, 2, \ldots, m\}$. Let $\mathcal{B}_\mathcal{R} := \{u \in \mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell} : \|u\|_{\mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}} \le \mathcal{R}\}$ be a bounded, closed, and convex subset of $\mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}$ with the constant radius

$$\mathcal{R} \ge \frac{\Theta_1\mathbb{F} + \Theta_2\mathbb{I} + \Lambda_1|\Delta_1||\mathcal{A}|}{1 - (\Theta_1 2\mathcal{L} + \Theta_2\Phi)}.$$

For any $u \in \mathcal{B}_\mathcal{R}$, we have

$$\left|{}^{\rho_\ell}_{k_\ell}\Psi^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}(t,t_\ell)(Qu)(t)\right|$$

$$
\begin{aligned}
\leq\ & \frac{1}{|\Delta_1|\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\left|\mathcal{G}_j(u)\right|+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left|\mathcal{G}_j(u)\right| \\
&+\sum_{i=0}^{m}|\mu_i|\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\,|\mathcal{F}_u(\xi_i)|+|\mathcal{A}|+{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\,|\mathcal{F}_u(T)|\Bigg] \\
&+\frac{1}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\sum_{j=0}^{\ell-1}\left|\mathcal{G}_j(u)\right|+{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)\,{}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\,|\mathcal{F}_u(t)|. \qquad (3.10)
\end{aligned}
$$

Then, we get the following results under the assumptions $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$ as

$$
\begin{aligned}
|\mathcal{F}_u(t)| &\leq |f(t,u(t),u(\lambda t))-f(0,0,0)|+|f(0,0,0)| \\
&\leq {}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)\mathcal{L}\Big(|u(t)|+|u(\lambda t)|\Big)+|f(0,0,0)| \\
&\leq 2\mathcal{L}\,\|u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\mathbb{F} \\
&\leq 2\mathcal{L}\mathcal{R}+\mathbb{F}, \qquad (3.11) \\
|\phi_\ell(u(t_\ell))| &\leq |\phi_\ell(u(t_\ell))-\phi_\ell(0)|+|\phi_\ell(0)|\leq\Phi\mathcal{R}+\mathbb{I}. \qquad (3.12)
\end{aligned}
$$

Applying (3.11) and (3.12), we have

$$
\begin{aligned}
\left|\mathcal{G}_j(u)\right| &\leq {}_{t_j,k_j}\mathcal{I}^{\alpha_j+(k_j-\gamma_j),\rho_j;\psi_j}\left|\mathcal{F}_u(t_{j+1})\right|+\left|\phi_{j+1}(u(t_{j+1}))\right| \\
&\leq (2\mathcal{L}\mathcal{R}+\mathbb{F})_{t_j,k_j}\mathcal{I}^{\alpha_j+(k_j-\gamma_j),\rho_j;\psi_j}(1)(t_{j+1})+\Phi\mathcal{R}+\mathbb{I} \\
&= \left(\frac{2\mathcal{L}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\Phi\right)\mathcal{R}+\frac{\mathbb{F}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\mathbb{I}. \qquad (3.13)
\end{aligned}
$$

Then, taking (3.11)–(3.13) into (3.10), we have

$$
\begin{aligned}
&\left|{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)(Qu)(t)\right| \\
\leq\ & \frac{1}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\left\{\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\right. \\
&\times\sum_{j=0}^{i-1}\left[\left(\frac{2\mathcal{L}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\Phi\right)\mathcal{R}+\frac{\mathbb{F}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\mathbb{I}\right] \\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left[\left(\frac{2\mathcal{L}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\Phi\right)\mathcal{R}+\frac{\mathbb{F}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\mathbb{I}\right] \\
&\left.+\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)}(2\mathcal{L}\mathcal{R}+\mathbb{F})+|\mathcal{A}|+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}(2\mathcal{L}\mathcal{R}+\mathbb{F})\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left[\left(\frac{2\mathcal{L}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\Phi\right)\mathcal{R}+\frac{\mathbb{F}\,{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\mathbb{I}\right]\\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}(2\mathcal{L}\mathcal{R}+\mathbb{F})\\
\leq\;&\frac{(2\mathcal{L}\mathcal{R}+\mathbb{F})}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}\\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)}\\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\Bigg)+\frac{(2\mathcal{L}\mathcal{R}+\mathbb{F})}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}\\
&+\frac{(2\mathcal{L}\mathcal{R}+\mathbb{F}){}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}+\frac{(\Phi\mathcal{R}+\mathbb{I})}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{i\;|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\\
&+\frac{m\,{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg)+\frac{(\Phi\mathcal{R}+\mathbb{I})m}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}+\frac{|\mathcal{A}|}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\\
\leq\;&(\Lambda_1(\Lambda_2+|\Delta_1|\,\Lambda_3)+\Lambda_4)(2\mathcal{L}\mathcal{R}+\mathbb{F})+\Lambda_1(\Lambda_5+|\Delta_1|\,m)(\Phi\mathcal{R}+\mathbb{I})+\Lambda_1\,|\Delta_1|\,|\mathcal{A}|\\
=\;&\Theta_1(2\mathcal{L}\mathcal{R}+\mathbb{F})+\Theta_2(\Phi\mathcal{R}+\mathbb{I})+\Lambda_1\,|\Delta_1|\,|\mathcal{A}|\leq\mathcal{R},
\end{aligned}
\tag{3.14}
$$

which implies that $\mathcal{Q}\mathcal{B}_{\mathcal{R}}\subset\mathcal{B}_{\mathcal{R}}$.

Step 2: We show that $\mathcal{Q}$ is a contraction.

Let $u,v\in\mathcal{B}_{\mathcal{R}}$. Then, for each $t\in\mathcal{J}$, we consider

$$
\begin{aligned}
&\left|{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)((\mathcal{Q}u)(t)-(\mathcal{Q}v)(t))\right|\\
\leq\;&\frac{1}{|\Delta_1|\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\left|\mathcal{G}_j(u)-\mathcal{G}_j(v)\right|\\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left|\mathcal{G}_j(u)-\mathcal{G}_j(v)\right|+\sum_{i=0}^{m}|\mu_i|\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\left|\mathcal{F}_u(s)-\mathcal{F}_v(s)\right|(\xi_i)\\
&+{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_u(s)-\mathcal{F}_v(s)\right|(T)\Bigg]+\frac{1}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left|\mathcal{G}_j(u)-\mathcal{G}_j(v)\right|\\
&+{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{1-\frac{\gamma_m}{k_m}}(T,t_m)\,{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_u(s)-\mathcal{F}_v(s)\right|(T)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{2\mathcal{L}}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{|\mu_i|\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\frac{{}_{k_j}^{\rho_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\
&\quad+\frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}_{k_j}^{\rho_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\sum_{i=1}^{m}\frac{|\mu_i|\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)} \\
&\quad+\frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\Bigg)\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\frac{2\mathcal{L}}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}_{k_j}^{\rho_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\
&\quad+\frac{2\mathcal{L}\,{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\frac{\Phi m}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} \\
&\quad+\frac{\Phi}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{i\ |\mu_i|\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}+\frac{m\ {}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg)\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} \\
&\leq 2\mathcal{L}(\Lambda_1(\Lambda_2+|\Delta_1|\,\Lambda_3)+\Lambda_4)\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\Lambda_1(\Lambda_5+|\Delta_1|\,m)\Phi\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} \\
&= (\Theta_1 2\mathcal{L}+\Theta_2\Phi)\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}},
\end{aligned}$$

which implies that

$$\|\mathcal{Q}u-\mathcal{Q}v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}\leq(\Theta_1 2\mathcal{L}+\Theta_2\Phi)\,\|u-v\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}.$$

Since $\Theta_1 2\mathcal{L}+\Theta_2\Phi<1$, it follows from Banach's fixed-point theorem (Lemma 3.1) that $\mathcal{Q}$ is a contraction. Hence, $\mathcal{Q}$ has a unique fixed point that is the unique solution of the problem (1.1) on $\mathcal{J}$. The proof is done. ◻

4. Ulam stability results

In this section, we study UH, GUH, UHR, and GUHR stability of problem (1.1).

Definition 4.1. *The problem* (1.1) *is said to be UH stable if, for every* $\varepsilon>0$, *there exists a constant* $C_f>0$ *such that, for any solution* $z\in\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J},\mathbb{R})$ *of*

$$\begin{cases}\left|{}_{t_\ell^+,k_\ell}^{H}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell}z(t)-\mathcal{F}_z(t)\right|\leq\epsilon,\\[2mm] \left|{}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell}z(t_\ell^+)-{}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}}z(t_\ell^-)-\phi_\ell(z(t_\ell))\right|\leq\epsilon,\end{cases}\tag{4.1}$$

there exists a unique solution $u\in\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J},\mathbb{R})$ *of problem* (1.1) *with*

$$|z(t)-u(t)|\leq C_f\epsilon,\quad t\in\mathcal{J}.$$

Definition 4.2. *The problem* (1.1) *is said to be GUH stable if, for* $\varepsilon > 0$ *and set of positive real numbers* $\mathbb{R}^+$*, there exists* $\chi \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\chi(0) = 0$ *such that, for any solution* $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *of*

$$\begin{cases} \left| {}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell} z(t) - \mathcal{F}_z(t) \right| \leq \chi(t), \\ \left| {}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell} z(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}} z(t_\ell^-) - \phi_\ell(z(t_\ell)) \right| \leq \upsilon, \end{cases} \tag{4.2}$$

there exists $\epsilon > 0$ *and a unique solution* $u \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *of problem* (1.1) *with*

$$|z(t) - u(t)| \leq \chi(\epsilon), \quad t \in \mathcal{J}.$$

Definition 4.3. *The problem* (1.1) *is said to be UHR stable with respect to* (χ, υ) *if, for* $\varepsilon > 0$*, there exists a real constant* $C_{f,\chi_f} > 0$ *such that for any solution* $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *of*

$$\begin{cases} \left| {}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell} z(t) - \mathcal{F}_z(t) \right| \leq \epsilon\chi(t), \\ \left| {}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell} z(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}} z(t_\ell^-) - \phi_\ell(z(t_\ell)) \right| \leq \epsilon\upsilon, \end{cases} \tag{4.3}$$

there exists a unique solution $u \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *of problem* (1.1) *with*

$$|z(t) - u(t)| \leq C_{f,\chi_f}\epsilon(\upsilon + \chi(t)), \quad t \in \mathcal{J}.$$

Definition 4.4. *The problem* (1.1) *is said to be GUHR stable with respect to* (χ, υ) *if there exists a real constant* $C_{f,\chi_f} > 0$*, such that for any solution* $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *of* (4.2)*, there exists a unique solution* $u \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *of problem* (1.1) *with*

$$|z(t) - u(t)| \leq C_{f,\chi_f}(\upsilon + \chi(t)), \quad t \in \mathcal{J}.$$

Remark 4.5. *By Definitions 4.1–4.4, we can find out that:* (*i*) *Definition 4.1 → Definition 4.2;* (*ii*) *Definition 4.3 → Definition 4.4; and (iii) Definition* 4.3 *→ Definition 4.1.*

Remark 4.6. *The function* $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *is a solution of the inequality* (4.1)*. There exists a function* $w \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *with a sequence* w_ℓ *for* $\ell = 1, 2, ..., m$*, which depends on a function* z*, such that*

(*i*) $|w(t)| \leq \epsilon$, $|w_\ell| \leq \epsilon$, $t \in \mathcal{J}$;
(*ii*) ${}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell} z(t) = \mathcal{F}_z(t) + w(t)$, $t \in \mathcal{J}$;
(*iii*) ${}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell} z(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}} z(t_\ell^-) = \phi_\ell(z(t_\ell)) + w_\ell$, $t \in \mathcal{J}$.

Remark 4.7. *The function* $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *is a solution of the inequality* (4.2)*. There exists a function* $w \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J}, \mathbb{R})$ *with a sequence* w_ℓ *for* $\ell = 1, 2, ..., m$*, which depends on a function* z*, such that*

(*i*) $|w(t)| \leq \chi(t)$, $|w_\ell| \leq \upsilon$, $t \in \mathcal{J}$;
(*ii*) ${}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell} z(t) = \mathcal{F}_z(t) + w(t)$, $t \in \mathcal{J}$;

(*iii*) ${}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell} z(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}} z(t_\ell^-) = \phi_\ell(z(t_\ell)) + w_\ell,\ t \in \mathcal{J}.$

Remark 4.8. *The function* $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J},\mathbb{R})$ *is a solution of the inequality* (4.3). *There exists a function* $w \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(\mathcal{J},\mathbb{R})$ *with a sequence* w_ℓ *for* $\ell = 1, 2, ..., m$, *which depends on a function* z, *such that*

(*i*) $|w(t)| \leq \epsilon\chi(t)$, $|w_\ell| \leq \epsilon\upsilon$, $t \in \mathcal{J}$;

(*ii*) ${}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell} z(t) = \mathcal{F}_z(t) + w(t)$, $t \in \mathcal{J}$;

(*iii*) ${}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell} z(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}} z(t_\ell^-) = \phi_\ell(z(t_\ell)) + w_\ell,\ t \in \mathcal{J}.$

4.1. Ulam-Hyers stability results

We now proceed to state the UH stability result.

Theorem 4.9. *Let* $\alpha_\ell \in (0,1)$, $\beta_\ell \in [0,1]$, $\rho_\ell \in (0,1]$, $k_\ell > 0$, $\gamma_\ell = \alpha_\ell + \beta_\ell(k_\ell - \alpha_\ell)$, $\psi_\ell \in C(\mathcal{J},\mathbb{R})$ *with* $\psi_\ell' > 0$, $\ell = 0, 1, 2, \ldots, m$, $\mathcal{F}_u \in C_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}$, *and* $\Delta_1 \neq 0$. *If* $(\mathbb{H}_1)$, $(\mathbb{H}_2)$ *and* (3.7) *are fulfilled, then the problem* (1.1) *is UH stable on* $\mathcal{J}$.

Proof. Suppose that $z \in \mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}$ is the solution of inequality (4.1). Applying conditions (*ii*) and (*iii*) from Remark 4.6 together with Lemma 2.6, we obtain

$$\begin{cases} {}^{H}_{t_\ell^+,k_\ell}\mathfrak{D}^{\alpha_\ell,\beta_\ell,\rho_\ell;\psi_\ell} z(t) = \mathcal{F}_z(t) + w(t), \quad t \in \mathcal{J}_\ell, \quad t \neq t_\ell, \quad \ell = 0, 1, \ldots, m, \\ {}_{t_\ell^+,k_\ell}\mathcal{I}^{(k_\ell-\gamma_\ell),\rho_\ell;\psi_\ell} z(t_\ell^+) - {}_{t_{\ell-1}^+,k_{\ell-1}}\mathcal{I}^{(k_{\ell-1}-\gamma_{\ell-1}),\rho_{\ell-1};\psi_{\ell-1}} z(t_\ell^-) = \phi_\ell(z(t_\ell)) + w_\ell,\ \ \ell = 1, 2, \ldots, m, \\ z(T) = \displaystyle\sum_{i=0}^{m} \mu_i \, {}_{t_i,k_i}\mathcal{I}^{\sigma_i,\rho_i;\psi_i} z(\xi_i) + \mathcal{A}, \quad \xi_i \in (t_i, t_{i+1}]. \end{cases} \tag{4.4}$$

Then, the solution of (4.4) is given by

$$\begin{aligned} z(t) \;=\;& \frac{{}_{k_\ell}^{\rho_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)} \Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m} \frac{\mu_i \, {}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} \sum_{j=0}^{i-1}\mathcal{G}_j(z) - \frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} \sum_{j=0}^{m-1}\mathcal{G}_j(z) \\ & + \sum_{i=0}^{m}\mu_i \, {}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_z(\xi_i) + \mathcal{A} - {}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_z(T)\Bigg\}\Bigg] + \frac{{}_{k_\ell}^{\rho_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)} \sum_{j=0}^{\ell-1}\mathcal{G}_j(z) \\ & + {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_z(t) + \frac{{}_{k_\ell}^{\rho_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)} \sum_{j=0}^{\ell-1}\mathcal{G}_j(w) + {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\mathcal{F}_w(t) \\ & + \frac{{}_{k_\ell}^{\rho_\ell}\Psi_{\psi_\ell}^{\frac{\gamma_\ell}{k_\ell}-1}(t,t_\ell)}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)} \Bigg[\frac{1}{\Delta_1}\Bigg\{\sum_{i=0}^{m} \frac{\mu_i \, {}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} \sum_{j=0}^{i-1}\mathcal{G}_j(w) - \frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} \sum_{j=0}^{m-1}\mathcal{G}_j(w) \\ & + \sum_{i=0}^{m}\mu_i \, {}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\mathcal{F}_w(\xi_i) - {}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\mathcal{F}_w(T)\Bigg\}\Bigg]. \end{aligned} \tag{4.5}$$

Using (i) in Remark 4.6 via ($\mathbb{H}_1$) and ($\mathbb{H}_2$), we have

$$
\begin{aligned}
&\left|{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)(z(t)-u(t))\right| \\
\leq\ & \frac{1}{|\Delta_1|\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\left|\mathcal{G}_j(z)-\mathcal{G}_j(u)\right| \\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left|\mathcal{G}_j(z)-\mathcal{G}_j(u)\right|+\sum_{i=0}^{m}|\mu_i|\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\left|\mathcal{F}_z(s)-\mathcal{F}_u(s)\right|(\xi_i) \\
&+{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_z(s)-\mathcal{F}_u(s)\right|(T)\Bigg]+\frac{1}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left|\mathcal{G}_j(z)-\mathcal{G}_j(u)\right| \\
&+{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{1-\frac{\gamma_m}{k_m}}(T,t_m)\,{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_z(s)-\mathcal{F}_u(s)\right|(T) \\
&+\frac{1}{|\Delta_1|\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\left|\mathcal{G}_j(w)\right|+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\left|\mathcal{G}_j(w)\right| \\
&+\sum_{i=0}^{m}|\mu_i|\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\left|\mathcal{F}_w(\xi_i)\right|+{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_w(T)\right|\Bigg] \\
&+\frac{1}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\sum_{j=0}^{\ell-1}\left|\mathcal{G}_j(w)\right|+{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)\,{}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\left|\mathcal{F}_w(t)\right| \\
\leq\ & \frac{2\mathcal{L}}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)} \\
&+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\Bigg)\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\frac{2\mathcal{L}}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)\,\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\
&+\frac{2\mathcal{L}\,{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\frac{\Phi m}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} \\
&+\frac{\Phi}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{i\ |\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}+\frac{m\,{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg)\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}
\end{aligned}
$$

$$\begin{aligned}
&+\frac{\epsilon}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi^{\frac{\gamma_i+\sigma_i}{k_i}-1}_{\psi_i}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\frac{{}^{\rho_j}_{k_j}\Psi^{\frac{\alpha_j-\gamma_j}{k_j}+1}_{\psi_j}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}\\
&+\frac{{}^{\rho_m}_{k_m}\Psi^{\frac{\gamma_m}{k_m}-1}_{\psi_m}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi^{\frac{\alpha_j-\gamma_j}{k_j}+1}_{\psi_j}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi^{\frac{\alpha_i+\sigma_i}{k_i}}_{\psi_i}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)}\\
&+\frac{{}^{\rho_m}_{k_m}\Psi^{\frac{\alpha_m}{k_m}}_{\psi_m}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\Bigg)+\frac{\epsilon}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi^{\frac{\alpha_j-\gamma_j}{k_j}+1}_{\psi_j}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\frac{\epsilon\,{}^{\rho_m}_{k_m}\Psi^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}_{\psi_m}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\\
&+\frac{\epsilon}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{i\;|\mu_i|\,{}^{\rho_i}_{k_i}\Psi^{\frac{\gamma_i+\sigma_i}{k_i}-1}_{\psi_i}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}+\frac{m\,{}^{\rho_m}_{k_m}\Psi^{\frac{\gamma_m}{k_m}-1}_{\psi_m}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg)+\frac{\epsilon\, m}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\\
\leq\;& 2\mathcal{L}(\Lambda_1(\Lambda_2+|\Delta_1|\,\Lambda_3)+\Lambda_4)\,\|z-u\|_{\mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}}+\Lambda_1(\Lambda_5+|\Delta_1|\,m)\Phi\,\|z-u\|_{\mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}}\\
&+(\Lambda_1(\Lambda_2+|\Delta_1|\,\Lambda_3)+\Lambda_4)\epsilon+\Lambda_1(\Lambda_5+|\Delta_1|\,m)\epsilon\\
=\;& (\Theta_1 2\mathcal{L}+\Theta_2\Phi)\,\|z-u\|_{\mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}}+(\Theta_1+\Theta_2)\epsilon. \qquad (4.6)
\end{aligned}$$

Consequently, $\|z-u\|_{\mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}} \leq C_f\epsilon$, where C_f is defined as

$$C_f := \frac{\Theta_1+\Theta_2}{1-(\Theta_1 2\mathcal{L}+\Theta_2\Phi)}. \qquad (4.7)$$

Therefore, the problem (1.1) is UH stable. □

Corollary 4.10. *In Theorem 4.9, if we set* $\chi(\epsilon) = C_f\epsilon$ *such that* $\chi(0) = 0$*, then the problem* (1.1) *is GUH stable.*

4.2. Ulam-Hyers-Rassias stability results

Prior to establishing the UHR stability result, we state the following assumption:

($\mathbb{H}_3$) There exists a non-decreasing function $\chi \in C(\mathcal{J},\mathbb{R})$, and there is a positive real number $C_\chi > 0$, for any $\epsilon > 0$, so that

$$ {}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\chi(t) \leq C_\chi\chi(t).$$

Theorem 4.11. *Suppose that* $\alpha_\ell \in (0,1)$, $\beta_\ell \in [0,1]$, $\rho_\ell \in (0,1]$, $k_\ell > 0$, $\gamma_\ell = \alpha_\ell + \beta_\ell(k_\ell - \alpha_\ell)$, $\psi_\ell \in C(\mathcal{J},\mathbb{R})$ *with* $\psi'_\ell > 0$, $\ell = 0,1,2,\ldots,m$, $\mathcal{F}_u \in C^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}$, *and* $\Delta_1 \neq 0$. *If* ($\mathbb{H}_1$), ($\mathbb{H}_2$), *and* (3.7) *are fulfilled, then the problem* (1.1) *is UHR stable with respect to* (χ, υ) *on* $\mathcal{J}$.

Proof. Suppose that $z \in \mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}$ is the solution of inequality (4.3) and $u \in \mathcal{PC}^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}$ is the solution of the problem (1.1). Using an argument similar to that in the proof of Theorem 4.9 and (i) in Remark 4.8, together with ($\mathbb{H}_1$)–($\mathbb{H}_3$), we obtain

$$\left|{}^{\rho_\ell}_{k_\ell}\Psi^{1-\frac{\gamma_\ell}{k_\ell}}_{\psi_\ell}(t,t_\ell)(z(t)-u(t))\right|$$

$$
\begin{aligned}
\leq\ & \frac{1}{|\Delta_1|\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\Big|\mathcal{G}_j(z)-\mathcal{G}_j(u)\Big| \\
& +\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\Big|\mathcal{G}_j(z)-\mathcal{G}_j(u)\Big|+\sum_{i=0}^{m}|\mu_i|\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\left|\mathcal{F}_z(s)-\mathcal{F}_u(s)\right|(\xi_i) \\
& +{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_z(s)-\mathcal{F}_u(s)\right|(T)\Bigg]+\frac{1}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\Big|\mathcal{G}_j(z)-\mathcal{G}_j(u)\Big| \\
& +{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{1-\frac{\gamma_m}{k_m}}(T,t_m)\,{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_z(s)-\mathcal{F}_u(s)\right|(T) \\
& +\frac{1}{|\Delta_1|\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\Bigg[\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\Big|\mathcal{G}_j(w)\Big|+\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\Big|\mathcal{G}_j(w)\Big| \\
& +\sum_{i=0}^{m}|\mu_i|\,{}_{t_i,k_i}\mathcal{I}^{\alpha_i+\sigma_i,\rho_i;\psi_i}\left|\mathcal{F}_w(\xi_i)\right|+{}_{t_m,k_m}\mathcal{I}^{\alpha_m,\rho_m;\psi_m}\left|\mathcal{F}_w(T)\right|\Bigg] \\
& +\frac{1}{\rho_\ell^{\frac{\gamma_\ell}{k_\ell}-1}\Gamma_{k_\ell}(\gamma_\ell)}\sum_{j=0}^{\ell-1}\Big|\mathcal{G}_j(w)\Big|+{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t,t_\ell)\,{}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\left|\mathcal{F}_w(t)\right| \\
\leq\ & \frac{2\mathcal{L}}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\
& +\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)} \\
& +\frac{{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\Bigg)\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\frac{2\mathcal{L}}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)\,\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)} \\
& +\frac{2\mathcal{L}\,{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\frac{\Phi m}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} \\
& +\frac{\Phi}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{i\ \,|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}+\frac{m\,{}^{\rho_m}_{k_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg)\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}} \\
& +\frac{\epsilon C_\chi\chi(t)}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\Bigg(\sum_{i=0}^{m}\frac{|\mu_i|\,{}^{\rho_i}_{k_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\sum_{j=0}^{i-1}\frac{{}^{\rho_j}_{k_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}_{k_j}^{\rho_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}+\sum_{i=1}^{m}\frac{|\mu_i|\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\alpha_i+\sigma_i}{k_i}}(\xi_i,t_i)}{\rho_i^{\frac{\alpha_i+\sigma_i}{k_i}}\Gamma_{k_i}(\alpha_i+\sigma_i+k_i)}\\
&+\left.\frac{{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}\right)+\frac{\epsilon C_\chi\chi(t)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\sum_{j=0}^{m-1}\frac{{}_{k_j}^{\rho_j}\Psi_{\psi_j}^{\frac{\alpha_j-\gamma_j}{k_j}+1}(t_{j+1},t_j)}{\rho_j^{\frac{\alpha_j-\gamma_j}{k_j}+1}\Gamma_{k_j}(\alpha_j-\gamma_j+2k_j)}\\
&+\frac{(2\mathcal{L}\mathcal{R}+\mathbb{F})\,{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\alpha_m}{k_m}+1-\frac{\gamma_m}{k_m}}(T,t_m)}{\rho_m^{\frac{\alpha_m}{k_m}}\Gamma_{k_m}(\alpha_m+k_m)}+\frac{\epsilon\upsilon}{|\Delta_1|\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\left(\sum_{i=0}^{m}\frac{i\ |\mu_i|\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)}\right.\\
&+\left.\frac{m\,{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\right)+\frac{\epsilon\upsilon m}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)}\\
\leq\ & 2\mathcal{L}(\Lambda_1(\Lambda_2+|\Delta_1|\,\Lambda_3)+\Lambda_4)\,\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+\Lambda_1(\Lambda_5+|\Delta_1|\,m)\Phi\,\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}\\
&+\Big(C_\chi\chi(t)(\Lambda_1(\Lambda_2+|\Delta_1|\,\Lambda_3)+\Lambda_4)+\upsilon\Lambda_1(\Lambda_5+|\Delta_1|\,m)\Big)\epsilon\\
=\ & (\Theta_1 2\mathcal{L}+\Theta_2\Phi)\,\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}+(C_\chi\Theta_1+\Theta_2)\epsilon(\chi(t)+\upsilon).
\end{aligned}
\tag{4.8}
$$

This implies that $\|z-u\|_{\mathcal{PC}_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}}\leq C_{f,\chi_f}\epsilon(\chi(t)+\upsilon)$, where

$$
C_{f,\chi_f}:=\frac{C_\chi\Theta_1+\Theta_2}{1-(\Theta_1 2\mathcal{L}+\Theta_2\Phi)}. \tag{4.9}
$$

Hence, the problem (1.1) is UHR stable with respect to (χ,υ). □

Corollary 4.12. *If $\epsilon=1$ and $\chi(0)=0$ are chosen in Theorem 4.11, then the problem* (1.1) *becomes GUHR stable with respect to* (χ,υ).

5. Examples

In this section, we provide two illustrative examples with graphical computations to demonstrate the applicability of our theoretical results.

Example 5.1. *Consider the nonlinear impulsive (k_ℓ,ψ_ℓ)-$\mathcal{HPFDE}$ with mixed boundary conditions of the form*

$$
\begin{cases}
{}^{H}_{t_\ell^+,\frac{\ell+5}{4}}\mathfrak{D}^{\frac{\ell+6}{8},\frac{\ell+2}{5},\frac{\ell+8}{10};\psi_\ell}u(t)=f(t,u(t),u((0.1)t)),\ t\neq t_\ell,\ \ell=0,1,2,\\
{}_{t_\ell^+,\frac{\ell+5}{4}}\mathcal{I}^{(\frac{\ell+5}{4}-\gamma_\ell),\frac{\ell+8}{10};\psi_\ell}u(t_\ell^+)-{}_{t_{\ell-1}^+,\frac{\ell+4}{4}}\mathcal{I}^{(\frac{\ell+4}{4}-\gamma_{\ell-1}),\frac{\ell+7}{10};\psi_{\ell-1}}u(t_\ell^-)=\phi_\ell(u(t_\ell)),\quad \ell=1,2,\\
u(\frac{12}{7})=\sum_{i=0}^{2}(\frac{3i+2}{7})_{t_i,\frac{i+5}{4}}\mathcal{I}^{\frac{i+2}{5-i},\frac{i+8}{10};\psi_i}u(\frac{4i+1}{7})+1,\quad \xi_i\in(t_i,t_{i+1}].
\end{cases}
\tag{5.1}
$$

Here, $\alpha_\ell=(\ell+6)/8,\beta_\ell=(\ell+2)/5,\rho_\ell=(\ell+8)/10,k_\ell=(\ell+5)/4,\psi_\ell(t)=(\ell+1-e^{-(\ell+2)t})/(\ell+3)$, $t_\ell=4\ell/7$ for $\ell=0,1,2$ with $T=12/7$, and $\lambda=0.1$. The parameters of mixed boundary conditions are given by $\mu_i = (3i+2)/7$, $\sigma_i = (i+2)/(5-i)$, $\xi_i = (4i+1)/7$ for $i = 0, 1, 2$, and $\mathcal{A} = 1$. Utilizing all parameters, we obtain $\Delta_1 \approx 2.129370$, $\Lambda_1 \approx 0.465964$, $\Lambda_2 \approx 0.718382$, $\Lambda_3 \approx 0.364747$, $\Lambda_4 \approx 0.013087$, $\Lambda_5 \approx 3.597920$, $\Theta_1 \approx 0.709731$, and $\Theta_2 \approx 3.660920$. The following functions are used to validate the theoretical results:

$$f(t, u(t), u(0.1t)) = e^{-3t} + \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t, t_\ell)}{(t+2)^2 + 12}\left(\frac{|u(t)|}{|u(t)|+1} + \frac{|u(0.1t)|}{|u(0.1t)|+2}\right), \tag{5.2}$$

$$\phi_\ell(u(t_\ell)) = \sin(3\pi t_\ell) + \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t, t_\ell)e^{-5t_\ell}}{2\cos^2(4\pi t_\ell) + 25}. \tag{5.3}$$

For any $u_i, v_i \in \mathbb{R}$, $i = 1, 2$, and $t \in [0, 12/7]$, it follows that

$$\begin{aligned} |f(t, u_1, v_1) - f(t, u_2, v_2)| &\leq \frac{{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t, t_\ell)}{16}\Big(|u_1 - u_2| + |v_1 - v_2|\Big), \\ |\phi_\ell(u) - \phi_\ell(v)| &\leq \frac{1}{25}{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{1-\frac{\gamma_\ell}{k_\ell}}(t, t_\ell)\,|u - v|\,. \end{aligned}$$

$(\mathbb{H}_1)$ and $(\mathbb{H}_2)$ in Theorem 3.2 are satisfied via, $\mathcal{L} = 1/16$ and $\Phi = 1/25$, which implies that $\Theta_1 2\mathcal{L} + \Theta_2\Phi \approx 0.235153 < 1$. Since all conditions in Theorem 3.2 are satisfied, the problem (5.1) has a unique solution on $[0, T]$. Furthermore, from (4.7), we get

$$C_f := \frac{\Theta_1 + \Theta_2}{1 - (\Theta_1 2\mathcal{L} + \Theta_2\Phi)} \approx 5.714410 > 0.$$

Then, the problem (5.1) is UH stable on $[0, T]$. Taking $\chi(\epsilon) = C_f\epsilon$ via $\chi(0) = 0$, the problem (5.1) is GUH stable on $[0, T]$.

Moreover, by taking $\chi(t) = {}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{2-k_\ell}{k_\ell}}(t, t_\ell)$ into $(\mathbb{H}_3)$, we have

$${}_{t_\ell,k_\ell}\mathcal{I}^{\alpha_\ell,\rho_\ell;\psi_\ell}\chi(t) = \frac{\Gamma_{k_\ell}(2)\,{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\alpha_\ell}{k_\ell}}(t, t_\ell)}{\rho_\ell^{\frac{\alpha_\ell}{k_\ell}}\Gamma_{k_\ell}(2 + \alpha_\ell)}\,{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{2-k_\ell}{k_\ell}}(t, t_\ell) \leq \frac{\Gamma_{k_\ell}(2)\,{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\alpha_\ell}{k_\ell}}(t, t_\ell)}{\rho_\ell^{\frac{\alpha_\ell}{k_\ell}}\Gamma_{k_\ell}(2 + \alpha_\ell)}\chi(t).$$

Then, we have

$$\mathfrak{C}_\chi = \max_{k\in\{0,1,2\}}\left\{\frac{\Gamma_{k_\ell}(2)\,{}^{\rho_\ell}_{k_\ell}\Psi_{\psi_\ell}^{\frac{\alpha_\ell}{k_\ell}}(t, t_\ell)}{\rho_\ell^{\frac{\alpha_\ell}{k_\ell}}\Gamma_{k_\ell}(2 + \alpha_\ell)}\right\} \approx 0.318368.$$

Applying (4.9), one has

$$C_{f,\chi_f} := \frac{C_\chi\Theta_1 + \Theta_2}{1 - (\Theta_1 2\mathcal{L} + \Theta_2\Phi)} \approx 5.081910.$$

Therefore, by all conclusions in Theorem 4.11, the problem (5.1) is UHR stable on $[0, T]$. Finally, if we take $\epsilon = 1$, the problem (5.1) is GUHR stable on $[0, T]$.

In addition, we represent the relationship between the constant values Θ_1, Θ_2 and the quantity $\mathcal{Y} := \Theta_1 2\mathcal{L} + \Theta_2$. The corresponding computed values are reported in Tables 1–3 for various choices of the parameters α_ℓ, β_ℓ, ρ_ℓ, and k_ℓ. The computational illustration is divided into three representative cases. Moreover, Figures 1–3 illustrate the graphical relationships between $\mathcal{Y}$ and the considered parameters.

Case I. In this case, we display the computed values of Θ_1, Θ_2 and $\mathcal{Y}$ with respect to the parameters $\alpha_\ell \in (0, 1]$ and $\beta_\ell \in [0, 1]$, as shown in Table 1 and Figure 1.

Case II. In this case, we present the computed values of Θ_1, Θ_2 and $\mathcal{Y}$ with respect to the parameters $\alpha_\ell \in (0, 1]$ and $\rho_\ell \in (0, 1]$, as reported in Table 2 and Figure 2.

Case III. Finally, we study the effect of the proportional order $k_\ell > 0$ together with the fractional order $\alpha_\ell \in (0, 1]$. The corresponding computed values are provided in Table 3 and Figure 3.

Table 1. Computed values of Θ_1, Θ_2 and the condition $\mathcal{Y} < 1$ for varying α_ℓ and β_ℓ.

α_ℓ	β_ℓ	Θ_1	Θ_2	$\mathcal{Y} < 1$
0.10	0.00	0.052076	0.389936	0.022107
0.18	0.10	0.198590	1.297320	0.076717
0.26	0.20	0.366766	2.057810	0.128158
0.34	0.30	0.540441	2.655410	0.173771
0.42	0.40	0.703978	3.097780	0.211908
0.50	0.50	0.844096	3.404390	0.241688
0.58	0.60	0.950926	3.599030	0.262827
0.66	0.70	1.018360	3.705520	0.275516
0.74	0.80	1.044240	3.745660	0.280356
0.82	0.90	1.030090	3.738450	0.278299
0.90	1.00	0.980596	3.699810	0.270566

Table 2. Computed values of $\mathcal{Y}$ under the condition $\mathcal{Y} < 1$ for varying $\alpha_\ell \in (0, 1]$ and $\rho_\ell \in (0, 1]$.

α_ℓ	ρ_ℓ	Θ_1	Θ_2	$\mathcal{Y} < 1$
0.10	0.05	1.064560	2.050020	0.215071
0.18	0.14	1.471070	2.511840	0.284358
0.26	0.23	1.505490	2.782400	0.299482
0.34	0.32	1.410460	2.981120	0.295553
0.42	0.41	1.279160	3.138880	0.285450
0.50	0.50	1.143080	3.269130	0.273650
0.58	0.59	1.013730	3.378980	0.261875
0.66	0.68	0.895144	3.472690	0.250801
0.74	0.77	0.788401	3.553130	0.240675
0.82	0.86	0.693248	3.622150	0.231542
0.90	0.95	0.608882	3.681170	0.223357

Table 3. Numerical values of Θ_1, Θ_2 and the condition $\mathcal{Y} < 1$ for varying $\alpha_\ell \in (0, 1]$ and $k_\ell \in [1.1, 1.5]$.

α_ℓ	k_ℓ	Θ_1	Θ_2	$\mathcal{Y}$
0.10	1.10	2.847433	3.520883	0.496764
0.18	1.14	2.314145	3.527536	0.430370
0.26	1.18	1.929431	3.533945	0.382537
0.34	1.22	1.643127	3.540111	0.346995
0.42	1.26	1.423802	3.546036	0.319817
0.50	1.30	1.251331	3.551727	0.298485
0.58	1.34	1.112528	3.557192	0.281354
0.66	1.38	0.998544	3.562439	0.267316
0.74	1.42	0.903305	3.567478	0.255612
0.82	1.46	0.822538	3.572319	0.245710
0.90	1.50	0.753173	3.576972	0.237226

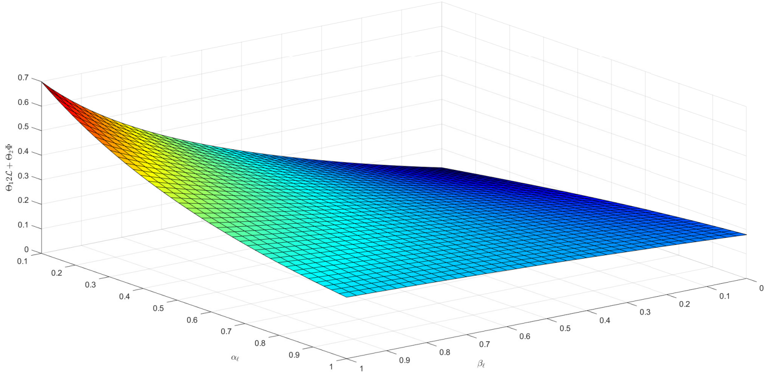

Figure 1. Graphical representation of $\mathcal{Y}$ with respect to α_ℓ and β_ℓ in Case I.

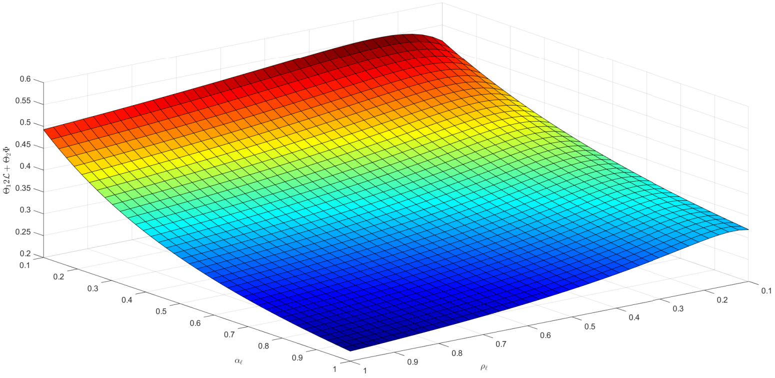

Figure 2. Graphical representation of $\mathcal{Y}$ with respect to α_ℓ and ρ_ℓ in Case II.

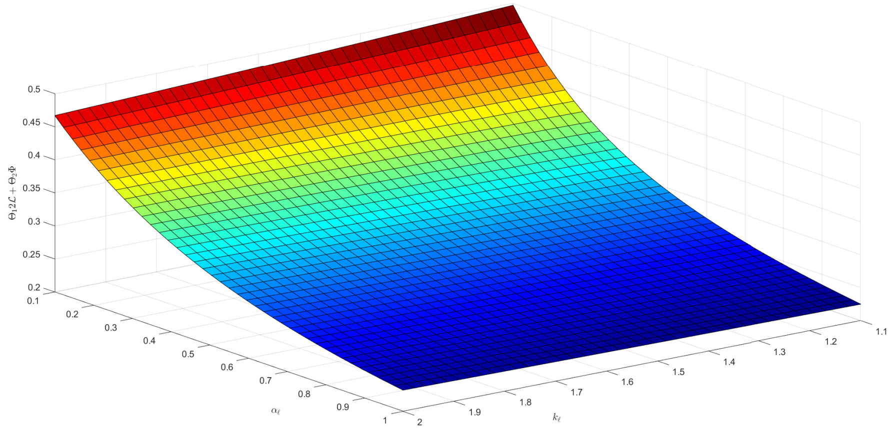

Figure 3. Computed values of Θ_1, Θ_2 and the condition $\mathscr{Y} < 1$ for varying $\alpha_\ell \in (0, 1]$ and $k_\ell \in [1.1, 1.5]$.

Example 5.2. *Consider the linear impulsive* (k_ℓ, ψ_ℓ)*-*$\mathcal{HPFDE}$ *with mixed boundary conditions of the form*

$$\begin{cases} {}_{t_\ell^+,\frac{\ell+1}{2}}^{H}\mathfrak{D}^{\frac{\ell+1}{6},\frac{\ell+5}{8},\frac{\ell+3}{6};\psi_\ell}u(t) = 0, \;\; t \neq t_\ell, \;\; \ell = 0,1,2, \\ {}_{t_\ell^+,\frac{\ell+1}{2}}\mathcal{I}^{(\frac{\ell+1}{2}-\gamma_\ell),\frac{\ell+3}{6};\psi_\ell}u(t_\ell^+) - {}_{t_{\ell-1}^+,\frac{\ell}{2}}\mathcal{I}^{(\frac{\ell}{2}-\gamma_{\ell-1}),\frac{\ell+2}{6};\psi_{\ell-1}}u(t_\ell^-) = 1, \quad \ell = 1,2, \\ u(1) = \sum_{i=0}^{2}(\frac{i+1}{8})\,{}_{t_i,\frac{i+1}{2}}\mathcal{I}^{\frac{i+1}{i+3},\frac{i+3}{6};\psi_i}u(\frac{2i+1}{6}) + 2, \quad \xi_i \in (t_i, t_{i+1}]. \end{cases} \tag{5.4}$$

Here, $t_\ell = \ell/3$ for $\ell = 0,1,2$ with $f(t,u(t),u(\lambda t)) = 0$, and $T = 1$. The parameters of mixed boundary conditions are given by $\mu_i = (i+1)/8$, $\sigma_i = (i+1)/(i+3)$, and $\xi_i = (2i+1)/6$, for $i = 0,1,2$, and $\mathscr{A} = 2$. From Lemma 2.6, we can derive the solution of (5.4) on [0, 1] as

$$u(t) = \begin{cases} \dfrac{{}_{k_0}^{\rho_0}\Psi_{\psi_0}^{\frac{\gamma_0}{k_0}-1}(t,t_0)}{\rho_0^{\frac{\gamma_0}{k_0}-1}\Gamma_{k_0}(\gamma_0)}\left[\dfrac{1}{\Delta_1}\left\{\displaystyle\sum_{i=0}^{m}\dfrac{i\mu_i\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} - \dfrac{2\,{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} + 2\right\}\right], & t \in [t_0,t_1], \\ \dfrac{{}_{k_1}^{\rho_1}\Psi_{\psi_1}^{\frac{\gamma_1}{k_1}-1}(t,t_1)}{\rho_1^{\frac{\gamma_1}{k_1}-1}\Gamma_{k_1}(\gamma_1)}\left[\dfrac{1}{\Delta_1}\left\{\displaystyle\sum_{i=0}^{m}\dfrac{i\mu_i\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} - \dfrac{2\,{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} + 2\right\} + 1\right], & t \in (t_1,t_2], \\ \dfrac{{}_{k_2}^{\rho_2}\Psi_{\psi_2}^{\frac{\gamma_2}{k_2}-1}(t,t_2)}{\rho_2^{\frac{\gamma_2}{k_2}-1}\Gamma_{k_2}(\gamma_2)}\left[\dfrac{1}{\Delta_1}\left\{\displaystyle\sum_{i=0}^{m}\dfrac{i\mu_i\,{}_{k_i}^{\rho_i}\Psi_{\psi_i}^{\frac{\gamma_i+\sigma_i}{k_i}-1}(\xi_i,t_i)}{\rho_i^{\frac{\gamma_i+\sigma_i}{k_i}-1}\Gamma_{k_i}(\gamma_i+\sigma_i)} - \dfrac{2\,{}_{k_m}^{\rho_m}\Psi_{\psi_m}^{\frac{\gamma_m}{k_m}-1}(T,t_m)}{\rho_m^{\frac{\gamma_m}{k_m}-1}\Gamma_{k_m}(\gamma_m)} + 2\right\} + 2\right], & t \in (t_2,t_3]. \end{cases} \tag{5.5}$$

To further demonstrate the qualitative behavior of the analytical solution obtained in (5.5), we present numerical illustrations based on different choices of the kernel function $\psi(t)$. In particular, four representative cases are considered, corresponding to polynomial, exponential, logarithmic, and trigonometric kernels. For each case, the fractional parameters α, β, ρ, and k are varied to examine their influence on the solution profile across the impulsive points. The resulting solution is plotted on [0, 1], and the corresponding graphical behaviors are discussed below.

Case I. $\psi_\ell(t) = t^3 + (2\ell + 1)t$ with $\alpha_\ell = (\ell + 1)/6$, $\beta_\ell = (\ell + 5)/8$, $\rho_\ell = (\ell + 3)/6$, and $k_\ell = (\ell + 1)/2$. Under this choice, the explicit solution (5.5) reduces to

$$u(t) = \begin{cases} \dfrac{(0.096467)\exp\left(-\frac{2t}{1+t}\right)}{\left(\frac{t}{1+t}\right)^{\frac{1}{4}}}, & t \in [0, 1/3], \\ \dfrac{(0.689587)\exp\left(-\frac{t}{2(1+2t)} + 0.1\right)}{\left(\frac{t}{1+2t} - 0.2\right)^{\frac{1}{6}}}, & t \in (1/3, 2/3], \\ \dfrac{(1.768950)\exp\left(-\frac{2t}{15(1+3t)} + 0.029630\right)}{\left(\frac{t}{1+3t} - 0.222222\right)^{\frac{1}{12}}}, & t \in (2/3, 1]. \end{cases} \tag{5.6}$$

The graphical representation of the solution for this case is shown in Figure 4.

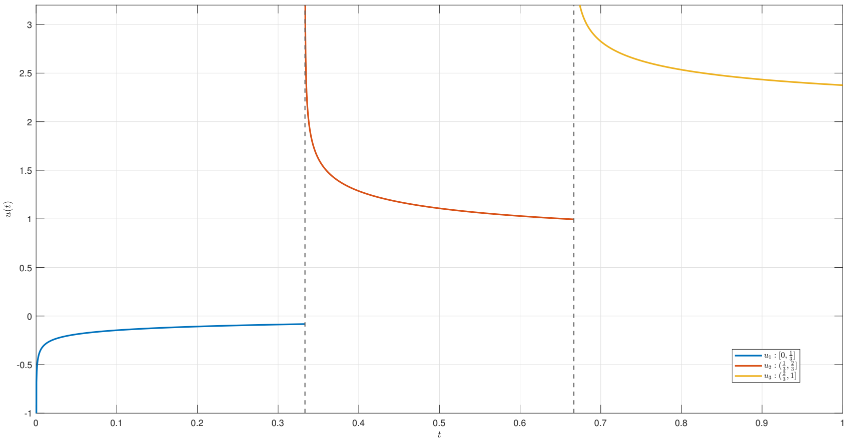

Figure 4. Graphical representation of the solution $u(t)$ corresponding to $\psi(t) = t^3 + (2\ell + 1)t$, $\alpha_\ell = \frac{\ell+1}{6}$, $\beta_\ell = \frac{\ell+5}{8}$, $\rho_\ell = \frac{\ell+3}{6}$, and $k_\ell = \frac{\ell+1}{2}$ in Example 5.2, Case I.

Case II. $\psi_\ell(t) = e^{(\ell+1)t} - 1$ with $\alpha_\ell = \ln(\ell + 2)/\ln(\ell + 3)$, $\beta_\ell = \ell/(\ell + 1)$, $\rho_\ell = \exp(-1/(\ell + 1))$, and $k_\ell = \ell + 1 + \sin(\pi/(2\ell + 2))$. In this case, the solution (5.5) can be written as

$$u(t) = \begin{cases} \dfrac{1.02240\,\exp\left(-0.859145e^{t} + 0.859145\right)}{(e^{t} - 1)^{0.684534}}, & t \in [0, 1/3], \\ \dfrac{3.92130\exp\left(-0.239637e^{2t} + 0.466748\right)}{(e^{2t} - 1.94773)^{0.353629}}, & t \in (1/3, 2/3], \\ \dfrac{5.72864\exp\left(-0.113032e^{3t} + 0.835196\right)}{(e^{3t} - 7.38906)^{0.251300}}, & t \in (2/3, 1]. \end{cases} \tag{5.7}$$

The graphical representation of the solution for this case is shown in Figure 5.

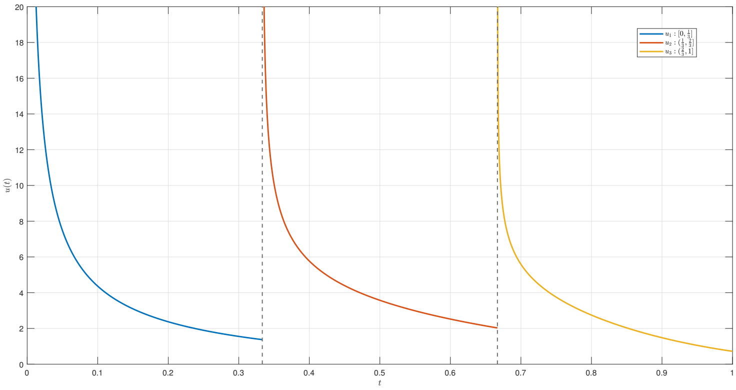

Figure 5. Graphical representation of the solution $u(t)$ corresponding to $\psi_\ell(t) = e^{(\ell+1)t} - 1$, $\alpha_\ell = \frac{\ln(\ell+2)}{\ln(\ell+3)}, \beta_\ell = \frac{\ell}{\ell+1}, \rho_\ell = e^{-\frac{1}{\ell+1}}$, and $k_\ell = \ell + 1 + \sin\left(\frac{\pi}{2\ell+2}\right)$ in Example 5.2, Case II.

Case III. $\psi_\ell(t) = t + \ln(1 + \ell t)$ with $\alpha_\ell = \exp(1/(1 + \ell)), \beta_\ell = (7 - \ell)/(\ell + 8), \rho_\ell = 1/(3\ell + 6)$, and $k_\ell = (\ell + 1)/(2 + \ell)$. In this case, the solution (5.5) can be written as

$$u(t) = \begin{cases} 62.4251 t^{0.554570} e^{-10t}, & t \in [0, 1/3], \\ 57.3257(t + \ln(t + 1) - 0.621013)^{0.491027} e^{-10.5t - 10.5\ln(t+1) + 6.52064}, & t \in (1/3, 2/3], \\ 55.0352(t + \ln(2t + 1) - 1.51396)^{0.430407} e^{-12t - 12\ln(2t+1) + 18.1676}, & t \in (2/3, 1]. \end{cases} \tag{5.8}$$

The graphical representation of the solution for this case is shown in Figure 6.

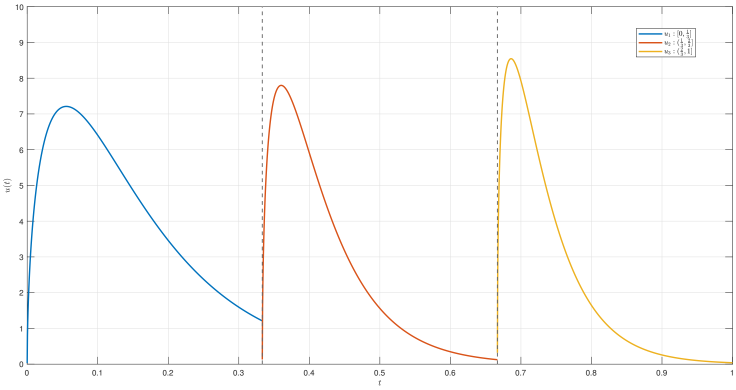

Figure 6. Graphical representation of the solution $u(t)$ corresponding to $\psi_\ell(t) = t + \ln(1 + \ell t)$, $\alpha_\ell = \exp\left(\frac{1}{1+\ell}\right), \beta_\ell = \frac{7-\ell}{\ell+8}, \rho_\ell = \frac{1}{3\ell+6}$, and $k_\ell = \frac{\ell+1}{\ell+2}$ in Example 5.2, Case III.

Case IV. $\psi_\ell(t) = \arcsin(((\ell+1)t)/(1+(\ell+1)t))$ with $\alpha_\ell = \sin(\pi/(4(\ell+2))), \beta_\ell = \sin((\pi(\ell+1))/(2(\ell+2)))$, $\rho_\ell = \cos(\pi/(3(\ell+2)))$, and $k_\ell = 1 + \ell + \sin(\pi/(2\ell+2))$. In this case, the solution (5.5) can be written as

$$u(t)=\begin{cases}\dfrac{0.154593\exp\left(-0.0906220\arcsin\left(\frac{t}{1+t}\right)\right)}{\left(\arcsin\left(\frac{t}{1+t}\right)\right)^{0.227233}}, & t\in[0,1/3],\\ \dfrac{1.19359\exp\left(-0.0256709\arcsin\left(\frac{2t}{1+2t}\right)+0.0105640\right)}{\left(\arcsin\left(\frac{2t}{1+2t}\right)-0.411517\right)^{0.120104}}, & t\in(1/3,2/3],\\ \dfrac{2.25579\exp\left(-0.0104284\arcsin\left(\frac{3t}{1+3t}\right)+0.00760990\right)}{\left(\arcsin\left(\frac{3t}{1+3t}\right)-0.729728\right)^{0.071730}}, & t\in(2/3,1].\end{cases} \tag{5.9}$$

The corresponding numerical profile of the solution is presented in Figure 7.

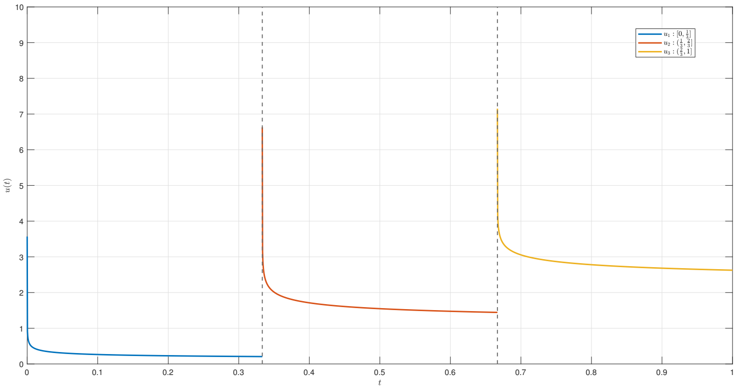

Figure 7. Graphical representation of the solution $u(t)$ corresponding to $\psi_\ell(t) = \arcsin\left(\frac{(\ell+1)t}{1+(\ell+1)t}\right)$, $\alpha_\ell = \sin\left(\frac{\pi}{4(\ell+2)}\right)$, $\beta_\ell = \sin\left(\frac{\pi(\ell+1)}{2(\ell+2)}\right)$, $\rho_\ell = \cos\left(\frac{\pi}{3(\ell+2)}\right)$, and $k_\ell = 1+\ell+\sin\left(\frac{\pi}{2(\ell+2)}\right)$ in Example 5.2, Case IV.

Table 4. Specification of the functions $\psi_\ell(t)$ and the corresponding parameters $\alpha_\ell, \beta_\ell, \rho_\ell$, and k_ℓ for the four considered cases.

Case	$\boldsymbol{\alpha_\ell}$	$\boldsymbol{\beta_\ell}$	$\boldsymbol{\rho_\ell}$	$\boldsymbol{k_\ell}$	$\boldsymbol{\psi_\ell(t)}$
I	$\dfrac{\ell+1}{6}$	$\dfrac{\ell+5}{8}$	$\dfrac{\ell+3}{6}$	$\dfrac{\ell+1}{2}$	$t^3+(2\ell+1)t$
II	$\dfrac{\ln(\ell+2)}{\ln(\ell+3)}$	$\dfrac{\ell}{\ell+1}$	$e^{-\frac{1}{\ell+1}}$	$\ell+1+\sin(\frac{\pi}{2\ell+2})$	$e^{(\ell+1)t}-1$
III	$\exp(\frac{1}{1+\ell})$	$\dfrac{7-\ell}{\ell+8}$	$\dfrac{1}{3\ell+6}$	$\dfrac{\ell+1}{\ell+2}$	$t+\ln(1+\ell t)$
IV	$\sin(\frac{\pi}{4(\ell+2)})$	$\sin(\frac{\pi(\ell+1)}{2(\ell+2)})$	$\cos(\frac{\pi}{3(\ell+2)})$	$1+\ell+\sin(\frac{\pi}{2\ell+2})$	$\arcsin(\frac{(\ell+1)t}{1+(\ell+1)t})$

6. Conclusions

The results obtained in this study are novel and provide a significant contribution to the existing literature on (k_ℓ, ψ_ℓ)-Hilfer proportional fractional pantograph differential impulsive boundary value problems. In this work, we presented significant results for the (k_ℓ, ψ_ℓ)-Hilfer proportional fractional derivative operator by establishing an equivalent fractional integral formulation of the associated nonlinear fractional differential equation. We relied on Banach's fixed-point theorem to prove the desired existence and uniqueness results. Moreover, several types of Ulam stability results were established to ensure the existence of solutions to the considered problem. To further substantiate the validity of the obtained theoretical results, two numerical examples were presented to demonstrate their effectiveness and confirm their correctness. In Example 5.1, a nonlinear impulsive (k_ℓ, ψ_ℓ)-Hilfer proportional fractional pantograph boundary value problem was examined. By explicitly computing the associated constants and verifying the contractive conditions satisfied by Theorem 3.2, we confirmed the existence and uniqueness of the solution. Moreover, by Theorems 4.9 and 4.11, the problem was shown to satisfy Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability. The accompanying computational and graphical illustrations further demonstrated how the parameters influence the stability condition, thereby highlighting the effectiveness of the theoretical criteria. In Example 5.2, a linear impulsive problem was considered, allowing the explicit construction of analytical solutions under four different choices of kernel functions $\psi(t)$, as shown in Table 4. The computed solutions were demonstrated through various representative kernel functions. These results highlighted the crucial influence of the kernel and fractional parameters on the qualitative behavior of solutions across impulsive points. Consequently, these graphical and analytical results further confirmed the applicability of the proposed framework.

This research contributes to the qualitative theory of nonlinear impulsive fractional boundary value problems involving a general kernel function. Future research may extend these results to higher-order impulsive systems or to problems subject to different boundary conditions.

Author contributions

Weerawat Sudsutad: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing, formal analysis, supervision; Jutarat Kongson: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing; Chatthai Thaiprayoon: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing; Aphirak Aphithana: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing, formal analysis, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. I. Podlubny, *Fractional Differential Equations*, New York: Academic Press, 1999.
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and application of fractional differential equations*, Amsterdam: North-Holland Mathematics Studies, 2006.
3. I. Stamova, G. Stamov, *Applied impulsive mathematical models*, Cham: Springer, 2016.
4. M. Benchohra, J. Henderson, S. K. Ntouyas, *Impulsive differential equations and inclusions*, New York: Hindawi Publishing Corporation, 2006.
5. S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in fractional differential equations*, New York: Springer, 2012.
6. Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2014.
7. K. Diethelm, *The analysis of fractional differential equations: An Application-oriented exposition using differential operators of Caputo type*, Berlin: Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2
8. J. R. Ockendon, A. B. Tayler, The dynamics of a current collection system for an electric locomotive, *Proc. A*, **322** (1971), 447–468. http://doi.org/10.1098/rspa.1971.0078
9. I. Ahmed, P. Kumam, J. Abubakar, P. Borisut, K. Sitthithakerngkiet, Solutions for impulsive fractional pantograph differential equation via generalized anti-periodic boundary value condition, *Adv. Differ. Equ.,* **2020** (2020), 477. http://doi.org/10.1186/s13662-020-02887-4
10. B. Khaminsou, W. Sudsutad, C. Thaiprayoon, J. Alzabut, S. Pleumpreedaporn, Analysis of impulsive boundary value pantograph problems via Caputo proportional fractional derivative under Mittag–Leffler functions, *Fractal Fract.*, **5** (2021), 251. http://doi.org/10.3390/fractalfract5040251
11. M. Kaewsuwan, C. Thaiprayoon, A. Aphithana, J. Kongson, W. Sae-dan, W. Sudsutad, Nonlinear impulsive (ρ_k, ψ_k)-Hilfer fractional pantograph integro-differential equations under nonlocal integral boundary conditions, *J. Nonlinear Funct. Anal.*, **2024** (2024), 10. http://doi.org/10.23952/jnfa.2024.10
12. J. R. Wang, Z. Lin, On the impulsive fractional anti-periodic BVP modelling with constant coefficients, *J. Appl. Math. Comput.*, **46** (2014), 107–121. http://doi.org/10.1007/s12190-013-0740-7
13. M. Zuo, X. Hao, L. Liu, Y. Cui, Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions, *Bound. Value Probl.*, **2017** (2017), 161. http://doi.org/10.1186/s13661-017-0892-8
14. M. I. Abbas, On the initial value problems for the Caputo–Fabrizio impulsive fractional differential equations, *Asian-Eur. J. Math.*, **14** (2020), 2150073. http://doi.org/10.1142/S179355712150073X

15. S. M. Ulam, *A collection of mathematical problems*, New York: Interscience, 1960.
16. D. H. Hyers, On the stability of the linear functional equations, *Proc. Natl. Acad. Sci.,* **27** (1941), 222–224. http://doi.org/10.1073/pnas.27.4.222
17. T. M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300. http://doi.org/10.2307/2042795
18. S. M. Jung, *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, New York: Springer, 2011.
19. Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, S. M. Kang, Generalized Riemann–Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities, *IEEE Access*, **6** (2018), 64946–64953. http://doi.org/10.1109/ACCESS.2018.2878266
20. R. Diaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, arXiv: math/0405596, 2007. http://doi.org/10.48550/arXiv.math/0405596
21. K. D. Kucche, A. D. Mali, On the nonlinear (k,ψ)-Hilfer proportional fractional differential equations, *Chaos Soliton. Fract.*, **152** (2021), 111335. http://doi.org/10.1016/j.chaos.2021.111335
22. A. Salim, M. Benchohra, J. E. Lazreg, J. Henderson, On k-generalized ψ-Hilfer boundary value problems with retardation and anticipation, *Adv. Theory Nonlinear Anal. Appl.*, **6** (2022), 173–190. http://doi.org/10.31197/atnaa.973992
23. W. Sudsutad, W. Lewkeeratiyutkul, C. Thaiprayoon, J. Kongson, Existence and stability results for impulsive (k,ψ)-Hilfer fractional double integro-differential equation with mixed nonlocal conditions, *AIMS Mathematics*, **8** (2023), 20437–20476. http://doi.org/10.3934/math.20231042
24. J. Wang, M. Feckan, Y. Zhou, Presentation of solutions of impulsive fractional Langevin equations and existence results, *Eur. Phys. J. Spec. Top.*, **222** (2013), 1857–1874. http://doi.org/10.1140/epjst/e2013-01969-9
25. M. A. Almalahi, S. K. Panchal, Some existence and stability results for ψ-Hilfer fractional implicit differential equation with periodic conditions, *J. Math. Anal. Model.*, **1** (2020), 1–19. http://doi.org/10.48185/jmam.v1i1.4
26. M. Kaewsuwan, R. Phuwapathanapun, W. Sudsutad, J. Alzabut, C. Thaiprayoon, J. Kongson, Nonlocal impulsive fractional integral boundary value problem for (ρ_k,ϕ_k)-Hilfer fractional integro-differential equations, *Mathematics*, **10** (2022), 3874. http://doi.org/10.3390/math10203874
27. W. Sudsutad, J. Kongson, C. Thaiprayoon, On generalized (k,ψ)-Hilfer proportional fractional operator and its application to the higher-order Cauchy problem, *Bound. Value Probl.*, **2024** (2024), 83. http://doi.org/10.1186/s13661-024-01891-x
28. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003.

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