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Research article

Exploring Gould–Hopper Sheffer–based Appell polynomials via operational approach and Riordan arrays

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Abstract: The operational and algebraic framework offers a powerful and systematic approach for investigating the structural properties of hybrid special polynomial families. In this work, we introduce a new class of Gould–Hopper Sheffer-based Appell polynomials (GHSbAP) by combining the Gould–Hopper polynomial structure with the general theory of Sheffer and Appell sequences. The offered construction is implemented by means of exponential generating functions and operational methods based upon the principle of monomiality. Basic properties of the GHSbAP family are constructed including generating functions, series representations, operational identities, quasi-monomial behavior, and differential equations. Further, the computationally efficient characterization of determinant representation is derived through the relation between Sheffer sequences and generalized Riordan arrays. A number of illustrative cases such as Gould–Hopper-Sheffer based Bernoulli and Euler polynomials are demonstrated.

Keywords: Sheffer sequences; Gould–Hopper–Sheffer sequences; monomiality principle; operational rules; determinant approach; Riordan arrays
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1. Introduction

The theory of special functions is a mathematical physics theory, which forms a backbone to the analytical modeling of various physical phenomena. These functions provide highly effective means of building simplified yet physically significant models, which allow the derivation of closed-form analytic solutions, and help to better perceive the underlying mechanics of the problem under study.

In many cases, a specific physical system will naturally encourage the investigation of new features of known families of special functions, or give rise to the construction of new classes of special polynomials. These generalizations often display complex structural features and discover interrelationships that occur in various fields of mathematics and theoretical physics. As a result, a reformulation of a physical model using special functions not only results in a more compact and elegant representation in a mathematical form, but also reduces the numerical treatment of governing equations. In addition, such a method can reveal some unforeseen connections between seemingly unrelated fields of physical science.

Gould–Hopper polynomials, also known as higher-order Hermite polynomials or Kampé de Fériet polynomials, constitute an important generalization of the classical Hermite family. These polynomials were introduced to extend the scope of Hermite polynomials to higher orders and play a fundamental role in operational calculus, combinatorial analysis, and the theory of special functions. They are defined as follows (see [1]):

$$g_n^m(\Theta_1,\Theta_2)=\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{\Theta_2^k\Theta_1^{n-mk}}{k!(n-mk)!}. \tag{1.1}$$

Two-variable Hermite polynomials, also known as the Gould–Hopper polynomials (2VGHP) denoted by $\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)$, constitute a natural two-dimensional extension of the classical Hermite family. These polynomials play a significant role in operational methods, multivariate analysis, and the study of higher-order differential equations. They are characterized by the following generating function (see [2]):

$$e^{\Theta_1\varkappa+\Theta_2\varkappa^m}=\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}. \tag{1.2}$$

Operational methods provide a natural and efficient framework for deriving the core properties of classical and generalized special functions, while also facilitating the systematic construction of hybrid polynomial families. In particular, quasi-monomial polynomials can be analyzed effectively through multiplicative and derivative operators within an operational setting. When combined with the monomiality principle, this approach yields a coherent structure for studying multivariable systems and for treating partial differential equations arising in mathematical physics. Numerous hybrid families related to Gould–Hopper, Hermite, Legendre, Laguerre, Appell, and Sheffer polynomials have been developed and examined in recent studies (see, for example, [3–5]). These extensions unify classical families within a broader analytic framework and generate new operational identities [6–8] and differential relations [9, 10]. In particular, the operational structure of the Gould–Hopper polynomials $\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)$ endows them with a strong algebraic foundation. Such identities play a central role in deriving recurrence relations, differential equations, and explicit representations. Specifically, the Gould–Hopper polynomials $\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)$ satisfy the following operational identities:

$$\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)=\exp\left(\Theta_2\frac{\partial^m}{\partial\Theta_1^m}\right)\{\Theta_1^n\}. \tag{1.3}$$

An effective solution to this problem is to adopt a synergistic approach where both operational techniques and the monomiality principle are used, which offers potent analytical means of solving a broad range of partial differential equations typically arising in the physical world. This concept may be traced to J. F. Steffensen [11], who established concepts of poweroids and monomiality. Dattoli [12] subsequently reformulated and broadened the monomiality principle in significant ways, broadening its domain and analytical flexibility and its analytical and interpretative variety. The fulfillment of the following fundamental axioms determines the algebraic and operational structure of the monomiality principle:

$$\mathcal{R}_{n+1}(\Theta_1) = \widehat{\mathcal{M}}\{\mathcal{R}_n(\Theta_1)\}, \tag{1.4}$$

$$n\,\mathcal{R}_{n-1}(\Theta_1) = \widehat{\mathcal{P}}\{\mathcal{R}_n(\Theta_1)\}, \tag{1.5}$$

and the operators that correspond are subject to the commutation relations of the Weyl algebra, which guarantee consistency of the monomiality structure as follows:

$$[\widehat{\mathcal{P}}, \widehat{\mathcal{M}}] = \widehat{1}. \tag{1.6}$$

The quasi-monomial property in turn induces a system of natural operational identities, which is central to the derivation and manipulation of the associated polynomial families:

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\{\mathcal{R}_n(\Theta_1)\} = n\,\mathcal{R}_n(\Theta_1), \tag{1.7}$$

$$\mathcal{R}_n(\Theta_1) = \widehat{\mathcal{M}}^n\{1\}, \tag{1.8}$$

$$\exp(\varkappa\widehat{\mathcal{M}})\{1\} = \sum_{n=0}^{\infty} \mathcal{R}_n(\Theta_1)\frac{\varkappa^n}{n!}. \tag{1.9}$$

In particular, for the polynomials of the form $\mathcal{Y}_n^{[m]}(\Theta_1, \Theta_2)$, the following quasi-monomial identities, are obeyed by these polynomials [13, 14]:

$$\widehat{M}_{\mathcal{Y}_n^{[m]}} := \Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}, \tag{1.10}$$

$$\widehat{P}_{\mathcal{Y}_n^{[m]}} := \frac{\partial}{\partial\Theta_1}. \tag{1.11}$$

One of the most important and far-reaching classes of the special families of polynomials in modern mathematics is the Sheffer sequences [15]. They naturally occur in many fields, including applied mathematics, theoretical and mathematical physics, approximation theory, combinatorics and operator calculus. Their broad applicability is due to the many orthogonal and nonorthogonal polynomials of classical and modern families, such as Hermite, Laguerre, Bernoulli, and Appell polynomials, which can be regarded as either specific cases or limiting cases of the general Sheffer framework. There are a number of equivalent definitions of Sheffer sequences, the most widespread being in terms of their exponential generating function or an apposite recurrence equation. Such formulations give a unified point of view that makes it easy to derive analytically and to algebraically manipulate a wide range of different systems of polynomials.

Theoretically, a polynomial series that takes the form of a sequence of polynomials $\{\mathcal{U}_n(\Theta_1)\}_{n=0}^{\infty}$, where each $\mathcal{U}_n(\Theta_1)$ is a polynomial of exact degree n, is known as a Sheffer A-type zero sequence [15], hereafter referred to simply as a Sheffer-type sequence if it admits an exponential generating function of the form

$$\mathcal{R}(\varkappa)\exp(\Theta_1\mathcal{Y}(\varkappa)) = \sum_{n=0}^{\infty}\mathcal{U}_n(\Theta_1)\frac{\varkappa^n}{n!}, \tag{1.12}$$

where $\mathcal{R}(\varkappa)$ and $\mathcal{Y}(\varkappa)$ are analytic functions in a neighborhood of $\varkappa = 0$, admitting the power series expansions of the form:

$$\mathcal{R}(\varkappa) = \sum_{n=0}^{\infty}\mathcal{R}_n\frac{\varkappa^n}{n!}, \quad \mathcal{R}_0 \neq 0 \tag{1.13}$$

and

$$\mathcal{Y}(\varkappa) = \sum_{n=1}^{\infty}\mathcal{Y}_n\frac{\varkappa^n}{n!}, \quad \mathcal{Y}_1 \neq 0, \tag{1.14}$$

respectively.

The importance of Sheffer-type sequences is that they are structurally general and operationally diverse. They give a connector between combinatorial identities, operational approaches and analytic forms of physical models. Sheffer sequences allow an elegant derivation of recurrence equations, differential equations, and integral equations of a system of polynomials which occur in quantum mechanics, signal processing, probability theory, and statistical mechanics through their generating functions. Therefore, they both generalize classical results and give rise to new, problem-specific families of special functions.

According to Roman [16], the sequence of polynomials $\mathcal{U}_n(\Theta_1)$ is uniquely determined by two (formal) power series,

$$\sigma(\varkappa) = \sum_{n=0}^{\infty}\sigma_n\frac{\varkappa^n}{n!}, \quad \sigma_0 = 0, \sigma_1 \neq 0 \tag{1.15}$$

and

$$\mu(\varkappa) = \sum_{n=0}^{\infty}\mu_n\frac{\varkappa^n}{n!}, \quad \mu_0 \neq 0, \tag{1.16}$$

respectively. The exponential generating function of $\mathcal{U}_n(\Theta_1)$ is then given by

$$\frac{1}{\mu(\sigma^{-1}(\varkappa))}\exp(\Theta_1(\sigma^{-1}(\varkappa))) = \sum_{n=0}^{\infty}\mathcal{U}_n(\Theta_1)\frac{\varkappa^n}{n!} \tag{1.17}$$

for all Θ_1 in $\mathbb{C}$, where $\sigma^{-1}(\varkappa)$ is the compositional inverse of $\sigma(\varkappa)$.

In view of Eqs (1.8), (1.9), (1.12), and (1.13), it follows that

$$\mathcal{R}(\varkappa) = \frac{1}{\mu(\sigma^{-1}(\varkappa))} \tag{1.18}$$

and

$$\mathcal{Y}(\varkappa) = \sigma^{-1}(\varkappa). \tag{1.19}$$

The sequence $\mathcal{U}_n(\Theta_1)$ in Eq (1.17) is the Sheffer sequence. For the pair $(\mu(\varkappa), \sigma(\varkappa))$, the sequence $\mathcal{U}_n(\Theta_1)$ defined by (1.17) is the Sheffer, and the pair $(1, \sigma(\varkappa))$ is called the associated Sheffer specified by

$$\exp(\Theta_1 \mathcal{Y}(\varkappa)) = \sum_{n=0}^{\infty} \mathbb{P}_n(\Theta_1) \frac{\varkappa^n}{n!}. \tag{1.20}$$

Further, the Sheffer sequence for $(\mu(\varkappa), \varkappa)$ becomes the Appell sequence for $\mu(\varkappa) = \varkappa$ given by the relation [15]

$$\mathcal{R}(\varkappa) \exp(\Theta_1 \varkappa) = \sum_{n=0}^{\infty} \mathcal{R}_n(\Theta_1) \frac{\varkappa^n}{n!}. \tag{1.21}$$

Following the framework developed by Roman and Rota [17], a polynomial family $\mathcal{U}_n(\Theta_1)$ is said to form a Sheffer sequence corresponding to a binomial-type polynomial sequence $\mathbb{P}_n(\Theta_1)$ if and only if it fulfills the functional relation given below:

$$\mathcal{U}_n(\Theta_1 + \Theta_2) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{U}_k(\Theta_1) \mathbb{P}_{n-k}(\Theta_2), \tag{1.22}$$

for all $n \geqslant 0$ and for every $\Theta_2 \in \mathbb{R}$, where n denotes the degree of the polynomial, and $\mathbb{R}$ is a field of characteristic zero.

Determinant methods play an important role in several areas of mathematical physics, including wave propagation, nonlinear evolution equations, and relativistic models. The determinant characterization of Sheffer sequences developed by Wang and Costabile [18–21] offers a systematic basis for representing structured polynomial families. This framework is compatible with operational techniques and is especially useful for interpolation and symbolic computation.

Let $(\mathcal{U}_n(\Theta_1))_{n \in \mathbb{N}}$ denote the Sheffer sequence corresponding to the pair $(\mu(\varkappa), \sigma(\varkappa))$, satisfying the relation

$$\Theta_1^n = \sum_{k=0}^{n} \mathcal{R}_{n,k} \, \mathcal{U}_k(\Theta_1), \tag{1.23}$$

then, according to Costabile et al. [18–20] and Wang [21, 22], the determinant form of $\mathcal{U}_n(\Theta_1)$ polynomials given as $\mathcal{U}_n(\Theta_1)$ can be represented in the following determinant form:

$$\begin{aligned} \mathcal{U}_0(\Theta_1) &= \tfrac{1}{\mathcal{R}_{0,0}}, \\ \mathcal{U}_n(\Theta_1) &= \tfrac{(-1)^n}{\mathcal{R}_{0,0}\, \mathcal{R}_{1,1}\, \ldots\, \mathcal{R}_{n,n}} \begin{vmatrix} 1 & \Theta_1 & \Theta_1^2 & \cdots & \Theta_1^{n-1} & \Theta_1^n \\ \mathcal{R}_{0,0} & \mathcal{R}_{1,0} & \mathcal{R}_{2,0} & \cdots & \mathcal{R}_{n-1,0} & \mathcal{R}_{n,0} \\ 0 & \mathcal{R}_{1,1} & \mathcal{R}_{2,1} & \cdots & \mathcal{R}_{n-1,1} & \mathcal{R}_{n,1} \\ 0 & 0 & \mathcal{R}_{2,2} & \cdots & \mathcal{R}_{n-1,2} & \mathcal{R}_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \mathcal{R}_{n-1,n-1} & \mathcal{R}_{n,n-1} \end{vmatrix}, \end{aligned} \tag{1.24}$$

where $\mathcal{R}_{n,k}$ denotes the (n, k)-th element of the Riordan array associated with the pair $(\mu(\varkappa), \sigma(\varkappa))$ [22].

A generalized Riordan array corresponding to a given sequence $(\mathscr{C}_n)_{n \in \mathbb{N}_0}$ is defined as an ordered pair $(\mu(\varkappa), \sigma(\varkappa))$. It constitutes an infinite lower triangular matrix $(\mathcal{R}_{n,k})_{0 \leq k \leq n < \infty}$, whose entries are constructed according to a specific recurrence or generating rule. This rule systematically links the coefficients of the array to the generating functions $\mu(\varkappa)$ and $\sigma(\varkappa)$, thereby establishing a structured framework for representing and manipulating Sheffer-type polynomial sequences. Such arrays serve as a powerful combinatorial and operational tool, allowing explicit formulation of coefficients, transformation properties, and structural identities within polynomial families and related operator calculus formulations. It satisfies the following axiom:

$$\mathcal{R}_{n,k} = \left[\frac{\varkappa^n}{\mathscr{C}_n}\right]\mu(\varkappa)\frac{(\sigma(\varkappa))^k}{\mathscr{C}_k},$$

where expressions such as $\mu(\varkappa)\,(\sigma(\varkappa))^k/\mathscr{C}_k$ are referred to as the column generating functions of the corresponding Riordan array. These generating functions play a central role in characterizing the structure of the array, as each column is generated by one such function. They encapsulate the entire combinatorial and algebraic behavior of the Riordan array, allowing compact representation, transformation, and computation of polynomial coefficients and related sequences derived from $(\mu(\varkappa), \sigma(\varkappa))$.

Note. Although Riordan arrays provide an elegant combinatorial and matrix-theoretic interpretation of the polynomial coefficients, it is important to emphasize that the core development of the present work is fundamentally operational in nature. The quasi-monomial structure, differential identities, and multiplicative–derivative operator formulations are derived independently of the Riordan framework. The Riordan representation is therefore presented as a complementary structural tool rather than as the primary source of the results.

For, $\mathcal{Y}(\varkappa) = \varkappa$ in (1.12), the significant class of polynomials in mathematical physics, namely the Appell polynomials $\mathcal{R}_n(\Theta_1)$, are obtained, which are specified by

$$\mathcal{R}(\varkappa)e^{\Theta_1\varkappa} = \sum_{n=0}^{\infty}\mathcal{R}_n(\Theta_1)\frac{\varkappa^n}{n!}, \tag{1.25}$$

where

$$\mathcal{R}(\varkappa) = \sum_{n=0}^{\infty}\mathcal{R}_n\frac{\varkappa^n}{n!}. \tag{1.26}$$

The primary motivation behind introducing Gould–Hopper Sheffer-based Appell polynomials (GHSbAP) lies in creating a unified operational framework that simultaneously preserves the higher-order structure of Gould–Hopper polynomials and the lowering property of Appell systems within the flexibility of the Sheffer class. Although classical Gould–Hopper polynomials already possess rich operational significance, embedding them into a Sheffer–Appell structure allows one to systematically control their algebraic, differential, and combinatorial properties through analytic generating pairs. This extension is therefore not merely formal; it provides a structurally coherent platform from which new identities, operator relations, determinant representations, and hybrid families naturally emerge.

The paper is organized as follows. Section 2 introduces the GHSbAP, where their generating function is established, and the corresponding series and determinant representations are developed within the Sheffer and Riordan array framework. Section 3 presents the operational formalism based on the monomiality principle, deriving the multiplicative and derivative operators, quasi-monomial structure, recurrence relations, explicit formulas, and the associated differential equation.

Section 4 provides illustrative examples, including the Gould–Hopper Sheffer-based Bernoulli and Euler polynomials as special cases, highlighting their generating functions and operational properties. Finally, Section 5 concludes the paper with remarks on the mathematical significance of the results and outlines possible directions for future research, including further hybrid extensions and applications in applied mathematics and theoretical physics.

2. Gould–Hopper Sheffer-based Appell polynomials

To introduce the GHSbAP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$, we first establish the exponential generating function involving analytic functions $\mathcal{R}(\varkappa)$ and $\mathcal{Y}(\varkappa)$ together with an Appell-type lowering property with respect to the variable Θ_1. This hybrid formulation yields a well-defined multivariable class whose operational identities, recurrence relations, and explicit representations follow in a unified and systematic manner.

Theorem 2.1. *For the GHSbAP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$, the succeeding generating function is demonstrated:*

$$\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right)=\sum_{n=0}^{\infty}{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!},\tag{2.1}$$

or, equivalently

$$\frac{1}{\mu(\sigma^{-1}(\varkappa))}\frac{1}{\mu(\varkappa)}\exp\left(\Theta_1\sigma^{-1}(\varkappa)+\Theta_2(\sigma^{-1}(\varkappa))^m\right)=\sum_{n=0}^{\infty}{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}.\tag{2.2}$$

Proof. We take the Gould–Hopper polynomials $\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)$ as the base sequence in the Sheffer generating function (1.12) for $\mathcal{U}_n(\Theta_1)$. To this end, we replace Θ_1 in (1.12) with the multiplicative operator $\widehat{M}_{\mathcal{Y}_n^{[m]}}$, that is, the raising (multiplication) operator associated with the polynomials $\mathcal{Y}_n^{[m]}(\Theta_1,\Theta_2)$, and obtain

$$\mathcal{R}(\varkappa)\exp\left(\widehat{M}_{\mathcal{Y}_n^{[m]}}\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right)=\sum_{n=0}^{\infty}\mathcal{U}_n\left(\widehat{M}_{\mathcal{Y}_n^{[m]}}\right)\frac{\varkappa^n}{n!},\tag{2.3}$$

which, on using expression (1.10) of $\widehat{M}_{\mathcal{Y}_n^{[m]}}$ on the left-hand side (l.h.s.) becomes

$$\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\left(\Theta_1+m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}(\varkappa)\right)=\sum_{n=0}^{\infty}\mathcal{U}_n\left(\widehat{M}_{\mathcal{Y}_n^{[m]}}\right)\frac{\varkappa^n}{n!}.\tag{2.4}$$

Now, by employing the explicit form of the multiplicative operator $\widehat{M}_{\mathcal{Y}_n^{[m]}}$ given in (1.10), we can rewrite the above relation in an operational form suitable for further manipulation. Next, we decouple the exponential operator appearing on the l.h.s. of the resulting equation by applying the Crofton identity:

$$f\left(z+m\mu\frac{d^m}{dz^{m-1}}\right)\{1\}=\exp\left(\mu\frac{d^m}{dz^m}\right)\{f(z)\},\tag{2.5}$$

which allows the exponential of a composite differential operator to be factorized into a product of simpler exponential operators. Hence, to decouple the exponential operator on the l.h.s. of (2.4), we find

$$\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_2\frac{\partial^m}{\partial\Theta_2^m}\right)\exp\left(\Theta_1\mathcal{Y}(\varkappa)\right)=\sum_{n=0}^{\infty}\mathcal{U}_n\left(\Theta_1+m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\frac{\varkappa^n}{n!}.\tag{2.6}$$

Also, by expanding the first exponential term on the left-hand side of Eq (2.6) via the binomial expansion:

$$(1-\varkappa)^{-\lambda}=\sum_{l=0}^{\infty}(\lambda)_l\frac{\varkappa^l}{l!}, \tag{2.7}$$

we arrive at the assertion (2.1) by denoting

$$\mathcal{U}_n\left(\Theta_1+m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)={}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2).$$

Moreover, in view of Eqs (1.15) and (1.16), the generating function (2.1) can be rewritten in an equivalent form, namely as Eq (2.2). □

Remark 2.1. *For, $\mathcal{R}(\varkappa)=1$, GHSbAP ${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$ reduce to the Gould–Hopper Sheffer polynomials (GHSP), denoted by ${}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)$, represented by*

$$\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right)=\sum_{n=0}^{\infty}{}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}. \tag{2.8}$$

Theorem 2.2. *For the GHSbAP ${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$, the following identity holds:*

$${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{R}_k\,{}_{y}\mathcal{U}_{n-k}^{[m]}(\Theta_1,\Theta_2). \tag{2.9}$$

Proof. Equation (2.1) can be written as

$$(\mathcal{R}(\varkappa))(\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right))=\sum_{n=0}^{\infty}{}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}. \tag{2.10}$$

Using the expansion (1.26) and (2.8) on the l.h.s. of preceding expression, we have

$$\sum_{k=0}^{\infty}\mathcal{R}_k\frac{\varkappa^k}{k!}\sum_{n=0}^{\infty}{}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}=\sum_{n=0}^{\infty}{}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}. \tag{2.11}$$

Replacing n by $n-k$ on the l.h.s. of Eq (2.11) and subsequently equating the coefficients of like powers of $\varkappa$ on both sides of the resulting equation, we obtain the assertion (2.9). □

Determinants have independently emerged as a powerful tool for addressing problems arising in the reflectionless transmission of electromagnetic waves through dielectric media, the analysis of the Korteweg–de Vries equation, relativistic gravitational fields, and several other areas of theoretical physics. The determinant definition of Sheffer sequences proposed by W. Wang [21] in 2014 provides a systematic and effective framework that motivates the development of determinant representations for newly introduced hybrid special polynomial families.

Equivalent in strength to the operational approach, the determinant method proves particularly useful in the solution of general linear interpolation problems and in diverse computational applications. Inspired by this framework, we introduce the determinant definition of the Gould–Hopper Sheffer-based Appell polynomials (GHSbAP) ${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$ by establishing the following result.

Theorem 2.3. *The generalized GHSbAP* ${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$ *has the following determinant representation:*

$$
\begin{aligned}
&{}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\\
&=\frac{(-1)^n}{(\beta_0)^{n+1}}\times\begin{vmatrix}
1 & {}_{y}\mathcal{U}_1^{[m]}(\Theta_1,\Theta_2) & {}_{y}\mathcal{U}_2^{[m]}(\Theta_1,\Theta_2) & \cdots & {}_{y}\mathcal{U}_{n-1}^{[m]}(\Theta_1,\Theta_2) & {}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)\\
\chi_0 & \chi_1 & \chi_2 & \cdots & \chi_{n-1} & \chi_n\\
0 & \chi_0 & \binom{2}{1}\chi_1 & \cdots & \binom{n-1}{1}\chi_{n-2} & \binom{n}{1}\chi_{n-1}\\
0 & 0 & \chi_0 & \cdots & \binom{n-1}{1}\chi_{n-3} & \binom{n}{2}\chi_{n-2}\\
\vdots & \vdots & \vdots & \cdots & \vdots & \\
0 & 0 & 0 & \cdots & \chi_0 & \binom{n}{n-1}\chi_1
\end{vmatrix},
\end{aligned}
\tag{2.12}
$$

where

$$\sum_{n=0}^{\infty}{}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}=\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right),\ \frac{1}{\mathcal{R}(\varkappa)}=\sum_{k=0}^{\infty}\chi_k\frac{\varkappa^k}{k!}.$$

Proof. Using the generation function (2.1), we get

$$\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right)=\left(\sum_{k=0}^{\infty}\chi_k\frac{\varkappa^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}\right).$$

Hence,

$$\sum_{n=0}^{\infty}{}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}=\left(\sum_{k=0}^{\infty}\chi_k\frac{\varkappa^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}\right).$$

Applying the Cauchy product formula to the above series, we obtain

$$\sum_{n=0}^{\infty}{}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\chi_k\,{}_{y}\mathcal{U}\mathcal{R}_{n-k}^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}.$$

On comparing the coefficients of $\varkappa^n/n!$ on both sides of the resulting polynomial identity, we obtain

$${}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)=\sum_{k=0}^{n}\binom{n}{k}\chi_k\,{}_{y}\mathcal{U}\mathcal{R}_{n-k}^{[m]}(\Theta_1,\Theta_2),\quad n\in\mathbb{N}_0.$$

Thus, we arrive at a coupled system of equations, explicitly written as

$$
\begin{aligned}
{}_{y}\mathcal{U}_0^{[m]}(\Theta_1,\Theta_2)&=\chi_0\,{}_{y}\mathcal{U}\mathcal{R}_0^{[m]}(\Theta_1,\Theta_2),\\
{}_{y}\mathcal{U}_1^{[m]}(\Theta_1,\Theta_2)&=\chi_0\,{}_{y}\mathcal{U}\mathcal{R}_1^{[m]}(\Theta_1,\Theta_2)+\chi_1\,{}_{y}\mathcal{U}\mathcal{R}_0^{[m]}(\Theta_1,\Theta_2),\\
{}_{y}\mathcal{U}_2^{[m]}(\Theta_1,\Theta_2)&=\chi_0\,{}_{y}\mathcal{U}\mathcal{R}_2^{[m]}(\Theta_1,\Theta_2)+\binom{2}{1}\chi_1\,{}_{y}\mathcal{U}\mathcal{R}_1^{[m]}(\Theta_1,\Theta_2)+\chi_2\,{}_{y}\mathcal{U}\mathcal{R}_0^{[m]}(\Theta_1,\Theta_2),\\
&\vdots\\
{}_{y}\mathcal{U}_{n-1}^{[m]}(\Theta_1,\Theta_2)&=\chi_0\,{}_{y}\mathcal{U}\mathcal{R}_{n-1}^{[m]}(\Theta_1,\Theta_2)+\binom{n-1}{1}\chi_1\,{}_{y}\mathcal{U}\mathcal{R}_{n-2}^{[m]}(\Theta_1,\Theta_2)+\cdots+\chi_{n-1}\,{}_{y}\mathcal{U}\mathcal{R}_0^{[m]}(\Theta_1,\Theta_2),\\
{}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)&=\chi_0\,{}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)+\binom{n}{1}\chi_1\,{}_{y}\mathcal{U}\mathcal{R}_{n-1}^{[m]}(\Theta_1,\Theta_2)+\cdots+\chi_n\,{}_{y}\mathcal{U}\mathcal{R}_0^{[m]}(\Theta_1,\Theta_2).
\end{aligned}
$$

Applying Cramer's rule, we get

$$
{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)=\frac{\begin{vmatrix}
\mathcal{X}_0 & 0 & \cdots & 0 & {}_{\mathcal{Y}}\mathcal{U}_0^{[m]}(\Theta_1,\Theta_2)\\
\mathcal{X}_1 & \mathcal{X}_0 & \cdots & 0 & {}_{\mathcal{Y}}\mathcal{U}_1^{[m]}(\Theta_1,\Theta_2)\\
\mathcal{X}_2 & \binom{2}{1}\mathcal{X}_1 & \cdots & 0 & {}_{\mathcal{Y}}\mathcal{U}_2^{[m]}(\Theta_1,\Theta_2)\\
\mathcal{X}_3 & \binom{3}{2}\mathcal{X}_2 & \cdots & 0 & {}_{\mathcal{Y}}\mathcal{U}_3^{[m]}(\Theta_1,\Theta_2)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\mathcal{X}_{n-1} & \binom{n-1}{1}\mathcal{X}_{n-2} & \cdots & \mathcal{X}_0 & {}_{\mathcal{Y}}\mathcal{U}_{n-1}^{[m]}(\Theta_1,\Theta_2)\\
\mathcal{X}_n & \binom{n}{1}\mathcal{X}_{n-1} & \cdots & \binom{n}{n-1}\mathcal{X}_1 & {}_{\mathcal{Y}}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)
\end{vmatrix}}{\begin{vmatrix}
\mathcal{X}_0 & 0 & \cdots & 0 & 0\\
\mathcal{X}_1 & \mathcal{X}_0 & \cdots & 0 & 0\\
\mathcal{X}_2 & \binom{2}{1}\mathcal{X}_1 & \cdots & 0 & 0\\
\mathcal{X}_3 & \binom{3}{2}\mathcal{X}_2 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\mathcal{X}_{n-1} & \binom{n-1}{1}\mathcal{X}_{n-2} & \cdots & \mathcal{X}_0 & 0\\
\mathcal{X}_n & \binom{n}{1}\mathcal{X}_{n-1} & \cdots & \binom{n}{n-1}\mathcal{X}_1 & \mathcal{X}_0
\end{vmatrix}}.
$$

Taking the transpose of the preceding equation, we obtain

$$
{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)=\frac{1}{(\mathcal{X}_0)^{n+1}}\times\begin{vmatrix}
\mathcal{X}_0 & \mathcal{X}_1 & \cdots & \mathcal{X}_{n-1} & \mathcal{X}_n\\
0 & \mathcal{X}_0 & \cdots & \binom{n-1}{1}\mathcal{X}_{n-2} & \binom{n}{1}\mathcal{X}_{n-1}\\
0 & 0 & \cdots & \binom{n-1}{1}\mathcal{X}_{n-3} & \binom{n}{2}\mathcal{X}_{n-2}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & \mathcal{X}_0 & \binom{n}{n-1}\mathcal{X}_1\\
{}_{\mathcal{Y}}\mathcal{U}_0^{[m]}(\Theta_1,\Theta_2) & {}_{\mathcal{Y}}\mathcal{U}_1^{[m]}(\Theta_1,\Theta_2) & \cdots & {}_{\mathcal{Y}}\mathcal{U}_{n-1}^{[m]}(\Theta_1,\Theta_2) & {}_{\mathcal{Y}}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2)
\end{vmatrix}.
$$

Thus, the proof is completed by applying elementary row operations. □

Remark 2.2. *For $\mathcal{Y}(\varkappa)=\varkappa$, the Sheffer polynomials $\mathcal{U}_n(\Theta_1)$ reduce to the Appell polynomials $\mathcal{R}_n(\Theta_1)$. Consequently, by choosing $\sigma(\varkappa)=\varkappa$ (equivalently, $\sigma^{-1}(\varkappa)=\varkappa$) on the l.h.s. of Eq* (2.1)*, we recover the corresponding results for the two-iterated Gould–Hopper-based Appell polynomials ${}_{y^{[m]}}\mathcal{R}_n^{[2]}(\Theta_1,\Theta_2)$; see [6] for related developments.*

3. Quasi-monomial properties

In order to frame the GHSbAP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$, within the framework of the monomiality principle, we establish the following result. This formulation enables the identification of appropriate raising and lowering operators that govern the quasi-monomial behavior of the sequence. Moreover, it provides a systematic mechanism for deriving recurrence relations and operational identities. The result below thus clarifies the algebraic structure underlying the GHSbAP family and sets the stage for further analytical developments.

Theorem 3.1. *The GHSbAP* ${}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)$ *are quasi-monomial under the action of the following multiplicative and derivative operators:*

$$\widehat{M}_{{}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_n^{[m]}} := \left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}'(\mathcal{Y}^{-1}(\partial_{\Theta_1})) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} + \frac{\mathcal{U}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{U}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}, \tag{3.1}$$

or, in an equivalent form,

$$\widehat{M}_{{}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_n^{[m]}} := \left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}} - \frac{\mu'(\partial_{\Theta_1})}{\mu(\partial_{\Theta_1})}\right)\frac{1}{\sigma'(\partial_{\Theta_1})} - \frac{\mu'(\sigma(\partial_{\Theta_1}))}{\mu(\sigma(\partial_{\Theta_1}))}, \tag{3.2}$$

and

$$\widehat{P}_{{}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_n^{[m]}} := \mathcal{Y}^{-1}(\partial_{\Theta_1}), \tag{3.3}$$

or, in an equivalent form,

$$\widehat{P}_{{}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_n^{[m]}} := \sigma(\partial_{\Theta_1}), \tag{3.4}$$

respectively.

Proof. Let us consider

$$\partial_{\Theta_1}\left(\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right)\right) = \mathcal{Y}(\varkappa)\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right). \tag{3.5}$$

Because $\mathcal{Y}(\varkappa)$ admits the power series expansion given in Eq (1.14), the above identity can be rewritten in the following equivalent form:

$$\mathcal{Y}^{-1}(\partial_{\Theta_1})\left(\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right)\right) = \varkappa\left(\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right)\right). \tag{3.6}$$

Differentiating Eq (2.1) partially with respect to $\varkappa$ and then invoking Eq (2.6), we obtain

$$\begin{aligned}&\left(\widehat{M}_{\mathcal{Y}_n^{[m]}}\mathcal{Y}'(\varkappa) + \frac{\mathcal{R}'(\varkappa)}{\mathcal{R}(\varkappa)} + \frac{\mathcal{U}'(\varkappa)}{\mathcal{U}(\varkappa)}\right)\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right)\\ &= \sum_{n=0}^{\infty}{}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_{n+1}^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}.\end{aligned} \tag{3.7}$$

Beacause $\mathcal{R}(\varkappa)$ and $\mathcal{U}(\varkappa)$ are invertible formal power series and because the logarithmic derivatives $\mathcal{R}'(\varkappa)/\mathcal{R}(\varkappa)$ and $\mathcal{U}'(\varkappa)/\mathcal{U}(\varkappa)$ admit Taylor expansions in powers of $\varkappa$, it follows, in view of identity (3.6), that Eq (3.7) can be rewritten as

$$\begin{aligned}&\left(\widehat{M}_{\mathcal{Y}_n^{[m]}}\mathcal{Y}'\left(\mathcal{Y}^{-1}(\partial_{\Theta_1})\right) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} + \frac{\mathcal{U}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{U}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}\right)\left\{\mathcal{R}(\varkappa)\mathcal{U}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right)\right\}\\ &= \sum_{n=0}^{\infty}{}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_{n+1}^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}.\end{aligned} \tag{3.8}$$

Again, by invoking the expression (2.1) on the l.h.s. of Eq (3.8) and performing a legitimate rearrangement of the resulting series, we arrive at

$$\sum_{n=0}^{\infty}\left(\left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial_{\Theta_1}^{m-1}}\right)\mathcal{Y}'\left(\mathcal{Y}^{-1}(\partial_{\Theta_1})\right) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} + \frac{\mathcal{U}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{U}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}\right){}_{\mathcal{Y}\mathcal{U}}\mathcal{R}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}$$

$$= \sum_{n=0}^{\infty} {}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n+1}^{[m]}(\Theta_1, \Theta_2)\frac{\varkappa^n}{n!}. \tag{3.9}$$

By comparing the coefficients of like powers of $\varkappa$ and invoking the monomiality principle in Eq (1.4), we obtain assertion (3.1). Moreover, in view of relations (1.15) and (1.16), assertions (3.2) and (3.4) can be expressed equivalently as Eqs (3.1) and (3.3), respectively.

To establish assertion (3.4), we substitute the generating function (2.1) into both sides of identity (3.7), which yields

$$\mathcal{Y}^{-1}(\partial_{\Theta_1})\left\{\sum_{n=0}^{\infty} {}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2)\frac{\varkappa^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n-1}^{[m]}(\Theta_1, \Theta_2)\frac{\varkappa^n}{(n-1)!}. \tag{3.10}$$

Rearranging the order of summation on the l.h.s. of Eq (3.10) and subsequently equating the coefficients of identical powers of $\varkappa$ on both sides of the resulting identity, we obtain

$$\mathcal{Y}^{-1}(\partial_{\Theta_1})\left\{{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2)\right\} = n\,{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n-1}^{[m]}(\Theta_1, \Theta_2), \quad n \geqslant 1, \tag{3.11}$$

which, in view of the monomiality principle given by Eq (1.5), yields the assertion (3.3). Furthermore, using relation (1.15), Eq (3.3) can be expressed equivalently in the form of Eq (3.4). □

Remark 3.1. *In view of Eq* (1.8) *and using Eqs* (3.1) *and* (3.2)*, we deduce the following consequence of Theorem 3.1.*

Corollary 3.1. *The GHSbAP* ${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2)$ *have the following explicit representations:*

$${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2) = \widehat{M}^{n}_{{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}}\{1\},$$

$${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2) = \left(\left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}'(\mathcal{Y}^{-1}(\partial_{\Theta_1})) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} + \frac{\mathcal{U}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{U}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}\right)^{n}\{1\}, \tag{3.12}$$

or, equivalently

$${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2) = \left(\left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}} - \frac{\mu'(\partial_{\Theta_1})}{\mu(\partial_{\Theta_1})}\right)\frac{1}{\sigma'(\partial_{\Theta_1})} - \frac{\mu'(\sigma(\partial_{\Theta_1}))}{\mu(\sigma(\partial_{\Theta_1}))}\right)^{n}\{1\}. \tag{3.13}$$

It is well-known that the fundamental properties of a quasi-monomial polynomial sequence can be systematically derived from the explicit forms of its multiplicative and derivative operators. Accordingly, to obtain the differential equation satisfied by the GHSbAP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2)$, we employ the previously established expressions of the corresponding raising and lowering operators and combine them within the framework of the monomiality principle. This approach leads naturally to a linear differential equation characterizing the GHSbAP family.

Theorem 3.2. *The GHSbAP* ${}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2)$ *satisfy the following differential equation:*

$$\Bigg(\Bigg(\Bigg(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}} - \frac{\mu'(\partial_{\Theta_1})}{\mu(\partial_{\Theta_1})}\Bigg)\frac{1}{\sigma'(\partial_{\Theta_1})} - \frac{\mu'(\sigma(\partial_{\Theta_1}))}{\mu(\sigma(\partial_{\Theta_1}))}\Bigg)\frac{\sigma(\partial_{\Theta_1})}{\sigma'(\partial_{\Theta_1})} - n\Bigg)\,{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2) = 0, \tag{3.14}$$

or, equivalently,

$$\Bigg(\Bigg(\Bigg(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\Bigg)\mathcal{Y}'(\mathcal{Y}^{-1}(\partial_{\Theta_1})) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} + \frac{\mathcal{U}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{U}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}\Bigg)\mathcal{Y}^{-1}(\partial_{\Theta_2}) - n\Bigg)\,{}_{\mathcal{Y}}\mathcal{U}\mathcal{R}_{n}^{[m]}(\Theta_1, \Theta_2) = 0. \tag{3.15}$$

Proof. Using Eqs (3.1) and (3.3) in (1.7), we get assertion (3.14), and similarly, using Eqs (3.2) and (3.4) in Eq (1.7), we get assertion (3.15). □

Further, we establish an operational rule that connects the GHSbAP ${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1, \Theta_2)$ with the Sheffer–Appell polynomials ${}_{\mathcal{U}}\mathcal{R}_n(\Theta_1)$. This connection highlights the structural compatibility between the hybrid Gould–Hopper framework and the classical Sheffer–Appell theory. To this end, we prove the following result.

Theorem 3.3. *The following operational representation connecting the GHSAbP ${}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1, \Theta_2)$ with the Sheffer–Appell polynomials ${}_{\mathcal{U}}\mathcal{R}_n(\Theta_1)$ holds true:*

$$ {}_{y}\mathcal{U}\mathcal{R}_n^{[m]}(\Theta_1, \Theta_2) = \exp\left(\Theta_2 \frac{\partial^m}{\partial^m_{\Theta_1}}\right)\{{}_{\mathcal{U}}\mathcal{R}_n(\Theta_1)\}. \tag{3.16} $$

Proof. In view of Eq (2.6), the result follows directly from identity (2.5). □

4. Examples

The Sheffer framework establishes a deep connection between combinatorial identities, operator theory, and special functions via the monomiality principle. Because of its flexible analytic structure, it allows the systematic construction of hybrid and generalized polynomial systems with controllable properties. Beyond its theoretical significance, the Sheffer class plays an important role in solving differential and integral equations appearing in mathematical physics. Consequently, the Sheffer framework continues to provide a coherent foundation for extending classical polynomial families into broader analytic contexts. The following specializations (Table 1) are not presented merely as substitutions of parameters, but as natural consequences of the general operational structure developed in earlier sections. These examples illustrate how classical families emerge within the unified Gould–Hopper–Sheffer framework while inheriting the associated multiplicative and derivative operators.

Table 1. Some known Sheffer polynomials.

S.No.	$\mathcal{R}(\varkappa); \mathcal{Y}(\varkappa)$	$\mu(\varkappa); \sigma(\varkappa)$	Generating functions	Polynomials
I.	$e^{-\varkappa^m}; \nu\varkappa$	$e^{(\frac{\varkappa}{\nu})^m}; \frac{\varkappa}{\nu}$	$\exp(\nu\Theta_1\varkappa - \varkappa^m) = \sum_{n=0}^{\infty} \mathcal{Y}_{n,m,\nu}(\Theta_1)\frac{\varkappa^n}{n!}$	Generalized Hermite polynomials $\mathcal{Y}_{n,m,\nu}(\Theta_1)$ [1]
II.	$(1-\varkappa)^{-\alpha-1}; \frac{\varkappa}{\varkappa-1}$	$(1-\varkappa)^{-\alpha-1}; \frac{\varkappa}{\varkappa-1}$	$\frac{1}{(1-\varkappa)^{\alpha+1}} \exp\left(\frac{\Theta_1\varkappa}{\varkappa-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(\Theta_1)\varkappa^n$	Generalized Laguerre polynomials $n!L_n^{(\alpha)}(\Theta_1)$ [2]
III.	$\frac{\varkappa}{1-\varkappa};\ \ln\left(\frac{1+\varkappa}{1-\varkappa}\right)$	$\frac{2}{e^{\varkappa}-1};\ \frac{e^{\varkappa}-1}{e^{\varkappa}+1}$	$\frac{\varkappa}{1-\varkappa}\left(\frac{1+\varkappa}{1-\varkappa}\right)^{\Theta_1} = \sum_{n=0}^{\infty} \mathcal{P}_n(\Theta_1)\frac{\varkappa^n}{n!}$	Pidduck polynomials $\mathcal{P}_n(\Theta_1)$ [2]
IV.	$e^{\beta\varkappa}; 1 - e^{\varkappa}$	$(1-\varkappa)^{-\beta}; \ln(1-\varkappa)$	$\exp\left(\beta\varkappa + \Theta_1(1-e^{\varkappa})\right) = \sum_{n=0}^{\infty} a_n^{(\beta)}(\Theta_1)\frac{\varkappa^n}{n!}$	Actuarial polynomials $a_n^{(\beta)}(\Theta_1)$ [6]
V.	$e^{-\varkappa}; \ln\left(1 + \frac{\varkappa}{a}\right)$	$\exp\left(a(e^{\varkappa}-1)\right); a(e^{\varkappa}-1)$	$e^{-\varkappa}\left(1 + \frac{\varkappa}{a}\right)^{\Theta_1} = \sum_{n=0}^{\infty} c_n(\Theta_1; a)\frac{\varkappa^n}{n!}$	Poisson-Charlier polynomials $c_n(\Theta_1; a)$ [2]
VI.	$\left(1 + (1+\varkappa)^{\lambda}\right)^{-\mu}; \ln(1+\varkappa)$	$\left(1 + e^{\lambda\varkappa}\right)^{\mu}; e^{\varkappa} - 1$	$\left(1 + (1+\varkappa)^{\lambda}\right)^{-\mu}(1+\varkappa)^{\Theta_1} = \sum_{n=0}^{\infty} s_n(\Theta_1; \lambda, \mu)\frac{\varkappa^n}{n!}$	Peters polynomials $s_n(\Theta_1; \lambda, \mu)$ [8]
VII.	$\frac{\varkappa}{\ln(1+\varkappa)}; \ln(1+\varkappa)$	$\frac{\varkappa}{e^{\varkappa}-1}; e^{\varkappa} - 1$	$\frac{\varkappa}{\ln(1+\varkappa)}(1+\varkappa)^{\Theta_1} = \sum_{n=0}^{\infty} b_n(\Theta_1)\frac{\varkappa^n}{n!}$	Bernoulli polynomials of the second kind $b_n(\Theta_1)$ [9]
VIII.	$\frac{2}{2+\varkappa}; \ln(1+\varkappa)$	$\frac{1}{2}(1+e^{\varkappa}); e^{\varkappa} - 1$	$\frac{2}{2+\varkappa}(1+\varkappa)^{\Theta_1} = \sum_{n=0}^{\infty} \mathcal{R}_n(\Theta_1)\frac{\varkappa^n}{n!}$	Related polynomials $\mathcal{R}_n(\Theta_1)$ [9]
IX.	$\frac{1}{\sqrt{1+\varkappa^2}}; \arctan(\varkappa)$	$\sec\varkappa; \tan\varkappa$	$\frac{1}{\sqrt{1+\varkappa^2}} \exp(\Theta_1 \arctan(\varkappa)) = \sum_{n=0}^{\infty} \mathcal{R}_n(\Theta_1)\frac{\varkappa^n}{n!}$	Hahn polynomials $\mathcal{R}_n(\Theta_1)$ [6]
X.	$(1-4\varkappa)^{-\frac{1}{2}}$	$\frac{1+\varkappa}{(1-\varkappa)^a}; \frac{1}{4} - \frac{1}{4}\left(\frac{1+\varkappa}{1-\varkappa}\right)^2$	$(1-4\varkappa)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4\varkappa}}\right)^{a-1} \exp\left(\frac{-4\Theta_1\varkappa}{(1+\sqrt{1-4\varkappa})^2}\right) = \sum_{n=0}^{\infty} \mathcal{R}_n(a, \Theta_1)\varkappa^n$	Shively's pseudo-Laguerre polynomials $\mathcal{R}_n(a, \Theta_1)$ [9]

Example 4.1. Taking $\mathcal{U}(\varkappa)\ (or\ 1/\mu(\varkappa)) = \varkappa/(e^{\varkappa} - 1)$ of the Bernoulli polynomials $\mathcal{B}_n(\Theta_1)$ on the l.h.s. of generating function (2.1) and (2.2), we find the following.

The following generating function characterizes the Gould-Hopper Sheffer-based Bernoulli polynomials (GHSbBP), denoted by ${}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}(\Theta_1, \Theta_2)$:

$$\frac{\varkappa}{e^{\varkappa} - 1}\mathcal{R}(\varkappa) \exp\left(\Theta_1\mathcal{Y}(\varkappa) + \Theta_2(\mathcal{Y}(\varkappa))^m\right) = \sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}(\Theta_1, \Theta_2)\frac{\varkappa^n}{n!}, \tag{4.1}$$

or, equivalently,

$$\frac{\varkappa}{e^{\varkappa} - 1}\frac{1}{\mu(\sigma^{-1}(\varkappa))} \exp\left(\Theta_1\sigma^{-1}(\varkappa) + \Theta_2(\sigma^{-1}(\varkappa))^m\right) = \sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}(\Theta_1, \Theta_2)\frac{\varkappa^n}{n!}. \tag{4.2}$$

The GHSbBP, denoted by ${}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}(\Theta_1, \Theta_2)$, admit the following series representation, which follows directly from their generating function and provides an explicit construction in terms of the underlying Gould–Hopper and Sheffer polynomials:

$${}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}(\Theta_1, \Theta_2) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k\ {}_{\mathcal{Y}\mathcal{U}}{}_{n-k}^{[m]}(\Theta_1, \Theta_2). \tag{4.3}$$

The GHSbBP ${}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}(\Theta_1, \Theta_2)$ form a quasi-monomial sequence with respect to the following multiplicative and derivative operators:

$$\widehat{M}_{{}_{\mathcal{Y}\mathcal{U}}\mathcal{B}_n^{[m]}} = \left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}^{'}(\mathcal{Y}^{-1}(\partial_{\Theta_1})) + \frac{\mathcal{R}^{'}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} + \frac{e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}(1 - (\mathcal{Y}^{-1}(\partial_{\Theta_1}))) - 1}{\mathcal{Y}^{-1}(\partial_{\Theta_1})(e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))} - 1)}, \tag{4.4}$$

or, equivalently,

$$\widehat{M}_{{}_{y}\mathcal{U}\mathcal{B}_n^{[m]}} = \left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}} - \frac{\mu'(\partial_{\Theta_1})}{\mu(\partial_{\Theta_1})}\right)\frac{1}{\sigma'(\partial_{\Theta_1})} + \frac{e^{\sigma(\partial_{\Theta_1})}(1-\sigma(\partial_{\Theta_1}))-1}{\sigma(\partial_{\Theta_1})(e^{\sigma(\partial_{\Theta_1})}-1)}, \tag{4.5}$$

and

$$\widehat{P}_{{}_{y}\mathcal{U}\mathcal{B}_n^{[m]}} = \mathcal{Y}^{-1}(\partial_{\Theta_1}), \tag{4.6}$$

or, equivalently,

$$\widehat{P}_{{}_{y}\mathcal{U}\mathcal{B}_n^{[m]}} = \sigma(\partial_{\Theta_1}), \tag{4.7}$$

respectively.

The GHSbBP ${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2)$ satisfy the following differential equation:

$$\Bigg(\left(\Theta_1 + m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))\mathcal{Y}^{-1}(\partial_{\Theta_1}) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}\mathcal{Y}^{-1}(\partial_{\Theta_1})$$
$$+ \frac{e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}(1-(\mathcal{Y}^{-1}(\partial_{\Theta_1})))-1}{\mathcal{Y}^{-1}(\partial_{\Theta_1})(e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}-1)}\mathcal{Y}^{-1}(\partial_{\Theta_1}) - n\Bigg){}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2) = 0.$$

The following relation establishes an explicit connection between the classical Bernoulli polynomials $\mathcal{B}_n(\Theta_1)$ and the GHSbBP ${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2)$, and is given by

$${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2) = \exp\left(\Theta_2\frac{\partial^m}{\partial^m_{\Theta_1}}\right)\{\mathcal{U}\mathcal{B}_n(\Theta_1)\}. \tag{4.8}$$

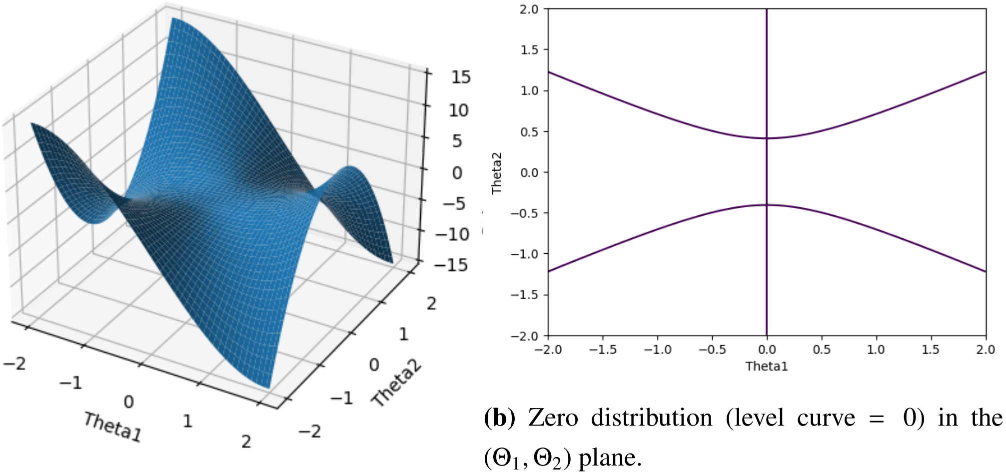
Taking $\chi_0 = 1$ and $\chi_i = 1/(i+1)$, for $i = 1, 2, \ldots, n$, in Eq (2.12), we obtain the determinantal definition of the Gould–Hopper Sheffer-based Bernoulli polynomials ${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2)$.

Consequently, the generalized Gould–Hopper Sheffer-based Bernoulli polynomials ${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2)$ admit the following determinant representation:

$${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2) = (-1)^n$$
$$\times\begin{vmatrix} 1 & {}_{y}\mathcal{U}_1^{[m]}(\Theta_1,\Theta_2) & {}_{y}\mathcal{U}_2^{[m]}(\Theta_1,\Theta_2) & \ldots & {}_{y}\mathcal{U}_{n-1}^{[m]}(\Theta_1,\Theta_2) & {}_{y}\mathcal{U}_n^{[m]}(\Theta_1,\Theta_2) \\ 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \ldots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n} \\ 0 & 0 & 1 & \ldots & \binom{n-1}{1}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}, \tag{4.9}$$

where ${}_{y}\mathcal{U}_1^{[m]}(\Theta_1,\Theta_2)$, $\quad (n = 0, 1, 2, \cdots, n)$ are called the Gould–Hopper Sheffer polynomials of degree n.

The graphical behavior of the Gould–Hopper Sheffer-based Bernoulli polynomials ${}_{y}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2)$ is illustrated through surface and contour plots (see Figure 1). The surface plot shows the interaction between the variables Θ_1 and Θ_2, and the zero contour highlights the geometric distribution of roots in the plane.



(a) Surface plot of ${}_{\mathcal{Y}}\mathcal{U}\mathcal{B}_n^{[m]}(\Theta_1,\Theta_2)$.

(b) Zero distribution (level curve = 0) in the (Θ_1,Θ_2) plane.

Figure 1. Surface plot and zero distribution.

Example 4.2. Taking $\mathcal{U}(\varkappa)$ $(or\ 1/\mu(\varkappa)) = 2/(e^{\varkappa}+1)$ of the Euler polynomials $\mathcal{E}_n(\Theta_1)$ in the l.h.s. of generating function (2.1) and (2.2), we find the following.

The following generating function holds for the Gould–Hopper Sheffer-based Euler polynomials (GHSbEP), denoted by ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)$:

$$\frac{2}{e^{\varkappa}+1}\mathcal{R}(\varkappa)\exp\left(\Theta_1\mathcal{Y}(\varkappa)+\Theta_2(\mathcal{Y}(\varkappa))^m\right)=\sum_{n=0}^{\infty}{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!},\tag{4.10}$$

or, equivalently,

$$\frac{2}{e^{\varkappa}+1}\frac{1}{\mu(\sigma^{-1}(\varkappa))}\exp\left(\Theta_1\sigma^{-1}(\varkappa)+\Theta_2(\sigma^{-1}(\varkappa))^m\right)=\sum_{n=0}^{\infty}{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)\frac{\varkappa^n}{n!}.\tag{4.11}$$

The GHSbEP, denoted by ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)$, admit the following series representation, which follows directly from their generating function and provides an explicit construction in terms of the underlying Gould–Hopper and Sheffer polynomials: ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)$ as

$${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\,{}_{\mathcal{Y}}\mathcal{U}_{n-k}^{[m]}(\Theta_1,\Theta_2).\tag{4.12}$$

The GHSbEP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)$, form a quasi-monomial sequence in the sense of the monomiality principle. In particular, there exist well-defined multiplicative (raising) and derivative (lowering) operators under which this polynomial family behaves analogously to the ordinary monomial basis. These operators generate the fundamental operational structure of the GHSbEP and enable the derivation of their recurrence relations, differential equations, and other structural properties. Specifically, the polynomials ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1,\Theta_2)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\widehat{M}_{{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}}=\left(\Theta_1+m\Theta_2\frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))+\frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}+\frac{e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}}{(e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}+1)},\tag{4.13}$$

or, equivalently,

$$\widehat{M}_{{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}} = \left(\Theta_1 + m\Theta_2 \frac{\partial^{m-1}}{\partial\Theta_1^{m-1}} - \frac{\mu'(\partial_{\Theta_1})}{\mu(\partial_{\Theta_1})}\right) \frac{1}{f'(\partial_{\Theta_1})} + \frac{e^{\sigma(\partial_{\Theta_1})}}{(e^{\sigma(\partial_{\Theta_1})}+1)}, \tag{4.14}$$

and

$$\widehat{P}_{{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}} = \mathcal{Y}^{-1}(\partial_{\Theta_1}), \tag{4.15}$$

or, equivalently,

$$\widehat{P}_{{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}} = \sigma(\partial_{\Theta_1}), \tag{4.16}$$

respectively.

The GHSbEP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2)$ satisfy the following differential equation:

$$\begin{aligned}&\left(\left(\Theta_1 + m\Theta_2 \frac{\partial^{m-1}}{\partial\Theta_1^{m-1}}\right)\mathcal{Y}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))\mathcal{Y}^{-1}(\partial_{\Theta_1}) + \frac{\mathcal{R}'(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}{\mathcal{R}(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}\mathcal{Y}^{-1}(\partial_{\Theta_1}) + \frac{e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}}{(e^{(\mathcal{Y}^{-1}(\partial_{\Theta_1}))}+1)}\mathcal{Y}^{-1}(\partial_{\Theta_1}) - n\right)\\ &\times {}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2) = 0.\end{aligned} \tag{4.17}$$

The following connection formula establishes an explicit relationship between the classical Euler polynomials $\mathcal{E}_n(\Theta_1)$ and the GHSbEP ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2)$. This relation highlights how the GHSEP family naturally extends the Euler polynomials through the incorporation of the Gould–Hopper structure and the underlying Sheffer framework, thereby preserving the essential algebraic and operational properties of the classical case:

$${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2) = \exp\left(\Theta_2 \frac{\partial^m}{\partial^m_{\Theta_1}}\right)\{{}_{\mathcal{U}}\mathcal{E}_n(\Theta_1)\}. \tag{4.18}$$

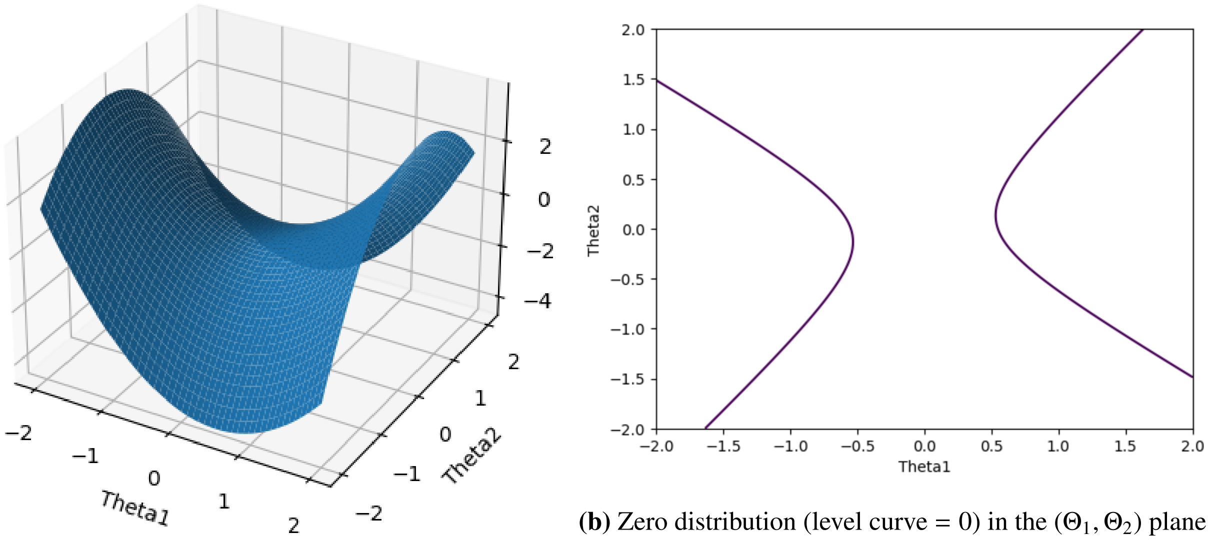
Taking $\chi_0 = 1$ and $\chi_i = 1/(i+1)$, for $i = 1, 2, \ldots, n$, in Eq (2.12), we obtain the determinantal definition of the Gould–Hopper Sheffer-based Euler polynomials ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2)$. This determinant formulation provides an explicit algebraic representation of the polynomial sequence and highlights its intrinsic Sheffer-type structure within the Gould–Hopper framework.

Consequently, the generalized Gould–Hopper Sheffer-based Euler polynomials ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2)$ admit the following determinant representation:

$$\begin{aligned}&{}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2) =(-1)^n\\ &\times \begin{vmatrix} 1 & {}_{\mathcal{Y}}\mathcal{U}_1^{[m]}(\Theta_1, \Theta_2) & {}_{\mathcal{Y}}\mathcal{U}_2^{[m]}(\Theta_1, \Theta_2) & \ldots & {}_{\mathcal{Y}}\mathcal{U}_{n-1}^{[m]}(\Theta_1, \Theta_2) & {}_{\mathcal{Y}}\mathcal{U}_n^{[m]}(\Theta_1, \Theta_2) \\ 1 & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \ldots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{1}\frac{1}{2} \\ 0 & 0 & 1 & \ldots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{2}\frac{1}{2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix},\end{aligned} \tag{4.19}$$

where ${}_{\mathcal{Y}}\mathcal{U}_1^{[m]}(\Theta_1, \Theta_2)$, $\quad (n = 0, 1, 2, \cdots, n)$ are called the Gould–Hopper Sheffer polynomials of degree n.

The graphical behavior of the Gould–Hopper Sheffer-based Euler polynomials ${}_{\mathcal{Y}}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2)$ is illustrated through surface and contour plots (see Figure 2). The surface plot demonstrates the combined influence of the variables Θ_1 and Θ_2 on the polynomial structure, revealing smooth variations and interaction patterns induced by the hybrid formulation. The corresponding zero contour represents the level set ${}_{y}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2) = 0$, highlighting the geometric distribution of roots in the (Θ_1, Θ_2) plane and providing insight into the qualitative behavior of the polynomial family.



(b) Zero distribution (level curve = 0) in the (Θ_1, Θ_2) plane.

(a) Surface plot of ${}_{y}\mathcal{U}\mathcal{E}_n^{[m]}(\Theta_1, \Theta_2)$.

Figure 2. Graphical visualization of the GHSbEP, illustrating the surface behavior and corresponding zero distribution.

On the novelty of the present results. Several of the identities obtained in this paper arise directly from the operational construction and cannot be derived solely from classical Riordan-array properties. In particular, the quasi-monomial operators, the derived differential equations, and the hybrid convolution structures stem from the analytic interplay between the Gould–Hopper operator and the Sheffer generating pair. This demonstrates that the introduced family possesses an intrinsic structural depth that extends beyond matrix-based interpretations.

5. Conclusions and future perspectives

In this paper, we have developed a unified operational and determinantal framework for the study of GHSbAP. By embedding the Gould–Hopper structure within Sheffer-type formalism and applying the monomiality principle, we obtained a coherent collection of analytical results, including generating functions, operational identities, quasi-monomial relations, differential equations, and determinant representations, which together highlight the rich algebraic structure of the proposed family. The determinantal approach complements the operational method and proves effective for symbolic computation and interpolation problems, and the connections established with classical Appell, Bernoulli, and Euler polynomials emphasize the unifying nature of the construction.

The framework also suggests several directions for future research, such as multivariate and matrix extensions, orthogonality and asymptotic analysis, and applications in fractional calculus, integral transforms, and boundary-value problems. Because Hermite and Gould–Hopper polynomials naturally appear in models of quantum oscillators, heat transfer, and wave propagation, the present extension provides a flexible symbolic tool for studying higher-order and perturbed systems. Although the current work is primarily theoretical, the structural insights developed here may support future applications in mathematical physics, computational methods, and operator-based approximation techniques.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Mohamed Rhaima: Conceptualization, methodology, operational approach, Riordan array formulation, writing—original draft preparation; Waseem Ahmad Khan: Conceptualization, formal analysis, investigation, operational approach and Riordan array formulation, validation, writing—original draft preparation, writing—review and editing; Shahid Ahmad Wani: Conceptualization, methodology, investigation, operational approach and Riordan array formulation, validation, writing—original draft preparation, writing—review and editing, supervision, project administration; Georgia Irina Oros: Conceptualization, formal analysis, operational approach and Riordan array formulation, validation, writing—original draft preparation, writing—review and editing, supervision, project administration. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no competing interests.

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