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Research article

Existence of positive solutions for nonlinear boundary value problems involving fractional boundary conditions

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Abstract: This paper focuses on investigating the existence of positive solutions to boundary value problems (BVPs) of higher-order nonlinear fractional differential equations (FDES) involving fractional boundary conditions. We examine the characteristics of Green's functions. By utilizing the powerful methodologies of Schauder's fixed-point theorem, upper and lower solutions, and cone theory techniques, we obtain significant existence results for nonsingular boundary value problems. For singular problems, we employ cone theory techniques in conjunction with the Leray-Schauder nonlinear alternative to obtain positive solutions. To further clarify and validate the primary findings, several illustrative examples are provided.

Keywords: Riemann-Liouville fractional derivative; upper and lower solutions; Leray-Schauder nonlinear alternative; Schauder's fixed-point theorem; cone theory techniques; positive solutions
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1. Introduction

Fractional differential equations (FDEs) have garnered significant attention and have become a focal point of growing interest. This surge in attention is largely driven by the swift advancements within the framework of fractional calculus theory and its expanding scope of applications; such equations are capable of modeling a wide variety of mathematical phenomena. Beyond the realm of mathematics, fractional-order differential equations have found extensive applications across a wide variety of disciplines, including, but not restricted to, chemistry, physics, and biological sciences. For details, see [1–5]. Fractional calculus serves as an advanced theoretical framework that extends traditional calculus to incorporate noninteger order and allows for greater precision in modeling natural phenomena and their underlying mathematical structures [1,4–6].

Given the profound theoretical and practical significance of nonlinear FDEs, the investigation of their solutions has evolved into a prominent research hotspot. A substantial body of literature has emerged, specifically dedicated to establishing the existence and uniqueness of solutions for boundary value problems (BVPs) via nonlinear analysis methods. Key theoretical tools employed in these studies include fixed-point theorems in cones [7–10], the method of upper and lower solutions [11], iterative schemes [12, 13], the mixed monotone method [9, 14], the Leray–Schauder nonlinear alternative [9, 15, 16], Schauder's fixed-point theorem [15, 17, 18], Krasnoselskii's fixed-point theorem [19, 20], and the Banach contraction principle [21–24].

In contrast, research on BVPs of integer-order differential equations is arguably more mature and extensive. This field has a longer history and has accumulated a vast literature covering a wide range of equations. Fourth-order differential equations serve as a prime example, having been extensively investigated due to their critical role in modeling elastic beam deflection; we refer the reader to the literature [25–27]. In the literature, the above questions are typically formulated as

$$\begin{cases} w^{(4)}(x) = H(x, w(x), w''(x)), \quad 0 < x < 1, \\ w(0) = w(1) = w''(0) = w''(1) = 0. \end{cases} \tag{1.1}$$

In [25], the existence for BVP (1.1) is established by solving the fourth-order equation using the method of upper and lower solutions under certain constraints on H.

In [26], Dang et al. explored the following problem:

$$\begin{cases} w^{(4)}(x) = H(x, w(x), w''(x), w'''(x)), \quad 0 < x < 1, \\ w(0) = w'(0) = w''(1) = w'''(1) = 0. \end{cases}$$

They reformulated the problem as operator equations involving functions on the right-hand side and established the contraction properties of the operator under certain suitable conditions. They also analyzed the regularity of the solution and the singularity associated with the iterative process.

In [27], Almuthaybiri et al. studied the nonlinear fourth-order BVP

$$w^{(4)}(x) = \varphi(x, w(x), w'(x), w''(x), w'''(x)), \quad x \in (0, 1), \tag{1.2}$$

where $\varphi : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. Additionally, problem (1.2) is governed by one of two possible two-point boundary conditions:

$$\begin{aligned} &w(0) = p, \ w'(0) = q, \ w(1) = a, \ w'(1) = b, \\ &w(0) = p, \ w'(0) = q, \ w''(1) = c, \ w'''(1) = d. \end{aligned}$$

The existence of solutions was established using Rus's contraction mapping theorem, which creatively applies fixed-point theory in a flexible and optional manner and relies on the use of two distinct metrics within a metric space, with one pair being complete.

The extension of traditional integer-order derivatives to arbitrary orders has led to the recognition of various challenges arising from the applications of FDEs (see [8, 9]). Unlike integer-order derivatives, fractional derivatives are nonlocal operators, meaning they account for the history of the system. This property not only enriches the physical modeling capabilities but also introduces substantial complexities in the mathematical analysis, particularly regarding the characteristics of Green's functions and the construction of appropriate function spaces [28–30]. Consequently, the existence and uniqueness criteria in the fractional setting often differ markedly from those in the integer-order case [8,9,23].

In [9], the authors studied the following BVP:

$$\begin{cases} D_{0+}^{\beta} w(x) = \varphi(x, w(x)), \ \ 0 < x < 1, \\ w(0) = w(1) = w'(0) = w'(1) = 0, \end{cases}$$

where $\beta \in (3,4]$, and D_{0+}^{β} denotes the standard Riemann–Liouville derivative. The properties of positive solutions were examined via cone theory techniques and the Leray–Schauder nonlinear alternative. More recently, the authors in [23] established the existence and uniqueness of positive solutions for a BVP involving fractional boundary conditions

$$\begin{cases} D_{0+}^{\beta} w(x) = \varphi(x, w(x)), \ \ 0 < x < 1, \ \beta \in (3,4], \\ D_{0+}^{\beta-3} w(0) = w(0) = w(1) = w'(1) = 0, \end{cases}$$

where D_{0+}^{β} denotes the Riemann-Liouville fractional derivative and φ is continuous, satisfying a mild Lipschitz assumption.

Motivated by prior studies, this paper aims to rigorously determine the existence of multiple positive solutions for a specific class of nonlinear FDEs. Specifically, we address the following BVP:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = \varphi(t, u(t)), \quad t \in (0,1), \ \alpha \in (3,4], \\ u(0) = D_{0^+}^{\alpha-3} u(0) = D_{0^+}^{\alpha-2} u(1) = D_{0^+}^{\alpha-1} u(1) = 0, \end{cases} \tag{1.3}$$

where $\varphi \in C((0,1) \times \mathbb{R}, \mathbb{R})$, and $D_{0^+}^{\alpha}$ denotes the standard Riemann–Liouville fractional derivative. To the best of our knowledge, the boundary value problem under consideration has not been previously addressed in the literature. The primary objective of this work is to establish the existence of positive solutions, an endeavor that fundamentally relies on the properties of the associated Green's function. Drawing inspiration from [10, 11, 18], we adopt a differentiated methodological approach: For nonsingular problems, we utilize the method of upper and lower solutions alongside Schauder's fixed point theorem and cone theoretic techniques; conversely, for singular problems, we employ cone theory in conjunction with the Leray–Schauder nonlinear alternative.

The remainder of this paper is structured as follows. Section 2 introduces some definitions from fractional calculus and auxiliary lemmas, thereby transforming the BVP (1.3) into an equivalent Fredholm integral equation. In Section 3, we derive crucial inequalities related to the Green's function and establish existence results for the nonsingular case using the aforementioned methods. Section 4 extends the analysis to singular problems via cone theory and the Leray–Schauder alternative. Finally, we provide several examples to demonstrate the validity of our results.

2. Preliminaries

We begin by presenting the fundamental definitions of fractional calculus essential for our analysis.

Definition 2.1. *[5, 31] For a function $u : (0,\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is given by*

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

whenever the integral exists pointwise on $(0,\infty)$.

Definition 2.2. *[5, 31] For a continuous function* $u:(0,\infty)\to\mathbb{R}$*, the Riemann-Liouville fractional derivative of order* $\alpha>0$ *is given by*

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{u(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n=[\alpha]+1$*, and the integral is assumed to exist pointwise on* $(0,\infty)$.

Below, we give the Green's function associated with the BVP for fractional-order differential equations.

Lemma 2.1. *Let* $3<\alpha\leq 4$ *and* $l\in C[0,1]$*. Then, the following BVP:*

$$\begin{cases} D_{0^+}^{\alpha}u(t)=l(t), \ \ 0<t<1, \\ u(0)=D_{0^+}^{\alpha-3}u(0)=D_{0^+}^{\alpha-2}u(1)=D_{0^+}^{\alpha-1}u(1)=0, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)l(s)ds,\ t\in[0,1], \tag{2.2}$$

where

$$G(t,s)=\begin{cases} \dfrac{-t^{\alpha-1}+(t-s)^{\alpha-1}+(\alpha-1)t^{\alpha-2}s}{\Gamma(\alpha)}, & 0\leq s\leq t\leq 1, \\ \dfrac{-t^{\alpha-1}+(\alpha-1)t^{\alpha-2}s}{\Gamma(\alpha)}, & 0\leq t\leq s\leq 1. \end{cases} \tag{2.3}$$

The function $G(t,s)$ *is defined as the Green's function of the BVP (2.1).*

Proof. According to [5,31], BVP (2.1) can be transformed into an equivalent integral equation given by

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+c_4t^{\alpha-4}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}l(s)ds. \tag{2.4}$$

With the condition $u(0)=0$, we obtain $c_4=0$.

Applying the operator $D_{0+}^{\alpha-3}$ to the above equation, we obtain

$$\begin{aligned} D_{0^+}^{\alpha-3}u(t) &= I_{0^+}^3 l(t)+c_1D_{0^+}^{\alpha-3}t^{\alpha-1}+c_2D_{0^+}^{\alpha-3}t^{\alpha-2}+c_3D_{0^+}^{\alpha-3}t^{\alpha-3} \\ &= I_{0^+}^3 l(t)+\frac{\Gamma(\alpha)c_1t^2}{2}+\Gamma(\alpha-1)c_2t+c_3\Gamma(\alpha-2). \end{aligned}$$

The condition $D_{0^+}^{\alpha-3}u(0)=0$ implies $c_3=0$. Moreover, taking the $\alpha-1$ and $\alpha-2$ order fractional derivative of (2.4) yields

$$D_{0^+}^{\alpha-1}u(t)=I_{0^+}^1 l(t)+c_1\Gamma(\alpha),$$

$$D_{0^+}^{\alpha-2}u(t)=I_{0^+}^2 l(t)+c_1\Gamma(\alpha)t+c_2\Gamma(\alpha-1).$$

Using the conditions $D_{0^+}^{\alpha-2}u(1) = D_{0^+}^{\alpha-1}u(1) = 0$, we obtain

$$c_1 = \frac{-\int_0^1 l(s)ds}{\Gamma(\alpha)},$$

$$c_2 = \frac{\int_0^1 sl(s)ds}{\Gamma(\alpha-1)}.$$

Therefore, the solution to problem (2.1) is uniquely determined by

$$\begin{aligned} u(t) &= \frac{\displaystyle\int_0^t l(s)(t-s)^{\alpha-1}ds}{\Gamma(\alpha)} - \frac{t^{\alpha-1}\displaystyle\int_0^1 l(s)ds}{\Gamma(\alpha)} + \frac{t^{\alpha-2}\displaystyle\int_0^1 sl(s)ds}{\Gamma(\alpha-1)} \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t l(s)(t-s)^{\alpha-1}ds - \int_0^1 l(s)t^{\alpha-1}ds + \int_0^1 l(s)(\alpha-1)t^{\alpha-2}sds\right]. \end{aligned}$$

□

The following lemma presents the key properties of the Green's function.

Lemma 2.2. *The Green's function $G(t,s)$ defined by (2.3) satisfies:*

(i) $G(t,s) \geq 0, \ \forall\, t, s \in [0,1]$;

(ii) $\frac{(\alpha-3)w(t)s^2}{2\Gamma(\alpha)} \leq G(t,s) \leq \frac{(\alpha-1)(\alpha-2)s^2}{\Gamma(\alpha)}, \ w(t) = t^{\alpha-2}, \ t, s \in [0,1]$;

(iii) $G(t,s) \leq \frac{(\alpha-1)(\alpha-2)t^{\alpha-3}s^2}{\Gamma(\alpha)}, \ t, s \in [0,1]$;

(iv) $G(t,s) \leq \frac{(\alpha-1)(\alpha-2)w(t)s}{\Gamma(\alpha)}, w(t) = t^{\alpha-2}, \ t, s \in [0,1]$;

(v) $\max_{t\in[0,1]} G(t,s) = G(1,s), \ \ \forall\, 0 \leq s \leq 1$.

Proof. Property (i) follows as a consequence from Property (ii).

Observing the expression of $G(t,s)$, it is evident that when $0 \leq t \leq s \leq 1$,

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-2}[(\alpha-1)s-t]}{\Gamma(\alpha)} \leq \frac{t^{\alpha-2}(\alpha-1)s}{\Gamma(\alpha)} \leq \frac{t^{\alpha-2}s(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \\ &\leq \frac{t^{\alpha-3}s^2(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \leq \frac{(\alpha-1)(\alpha-2)s^2}{\Gamma(\alpha)}. \end{aligned}$$

Therefore, (iii) and (iv) hold for $t \leq s$. On the other hand, for $\alpha \in (3,4]$ and $s \geq t$, we obtain

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-2}((\alpha-1)s-t)}{\Gamma(\alpha)} = \frac{t^{\alpha-2}((\alpha-2)s-t+s)}{\Gamma(\alpha)} \\ &\geq \frac{(\alpha-2)t^{\alpha-2}s}{\Gamma(\alpha)} \geq \frac{(\alpha-3)t^{\alpha-2}s^2}{2\Gamma(\alpha)}. \end{aligned}$$

Assuming $a > 0$, $b > 0$, and $\gamma > 1$, we can acquire

$$\gamma b^{\gamma-1}(a-b) \le a^{\gamma} - b^{\gamma} \le \gamma a^{\gamma-1}(a-b). \tag{2.5}$$

When $s \le t$, using the inequality (2.5) twice, we have

$$\begin{aligned} G(t,s) &= \frac{(t-s)^{\alpha-1}-t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\alpha-1)st^{\alpha-2}}{\Gamma(\alpha)} \le \frac{(\alpha-1)(t-s)^{\alpha-2}((t-s)-t)}{\Gamma(\alpha)} + \frac{(\alpha-1)st^{\alpha-2}}{\Gamma(\alpha)} \\ &= \frac{(\alpha-1)(t-s)^{\alpha-2}(-s)}{\Gamma(\alpha)} + \frac{(\alpha-1)st^{\alpha-2}}{\Gamma(\alpha)} = (\alpha-1)s\frac{\left(t^{\alpha-2}-(t-s)^{\alpha-2}\right)}{\Gamma(\alpha)} \\ &\le (\alpha-1)s\frac{(\alpha-2)t^{\alpha-3}s}{\Gamma(\alpha)} \le (\alpha-1)s\frac{(\alpha-2)t^{\alpha-3}t}{\Gamma(\alpha)} = \frac{(\alpha-1)(\alpha-2)t^{\alpha-2}s}{\Gamma(\alpha)}. \end{aligned}$$

Therefore, (iii) and (iv) hold for $s \le t$. To complete the proof, we proceed by analyzing two separate cases.

Case 1: When $\frac{t}{2} \le s \le t$,

$$\begin{aligned} G(t,s) &= \frac{(t-s)^{\alpha-1}-t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\alpha-1)st^{\alpha-2}}{\Gamma(\alpha)} \ge \frac{(t-s)^{\alpha-1}-t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\alpha-1)t^{\alpha-1}}{2\Gamma(\alpha)} \\ &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\alpha-3)t^{\alpha-1}}{2\Gamma(\alpha)} \ge \frac{(\alpha-3)t^{\alpha-1}}{2\Gamma(\alpha)} \ge \frac{(\alpha-3)t^{\alpha-2}s}{2\Gamma(\alpha)} \ge \frac{(\alpha-3)t^{\alpha-2}s^2}{2\Gamma(\alpha)}. \end{aligned}$$

Case 2: When $0 \le s < \frac{t}{2}$,

$$\begin{aligned} G(t,s) &= \frac{(t-s)^{\alpha-1}-t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\alpha-1)st^{\alpha-2}}{\Gamma(\alpha)} \\ &= -\frac{1}{\Gamma(\alpha-1)}\int_{t-s}^{t} u^{\alpha-2}du + \frac{1}{\Gamma(\alpha-1)}\int_{t-s}^{t} t^{\alpha-2}du \\ &= \frac{1}{\Gamma(\alpha-1)}\int_{t-s}^{t}(t^{\alpha-2}-u^{\alpha-2})du \ge \frac{(\alpha-2)}{\Gamma(\alpha-1)}\int_{t-s}^{t} u^{\alpha-3}(t-u)du \ \ \text{(applying (2.5))} \\ &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_{t-s}^{t}(tu^{\alpha-3}-u^{\alpha-2})du \\ &= \frac{(\alpha-1)}{\Gamma(\alpha)}(t^{\alpha-1}-t(t-s)^{\alpha-2}) - \frac{(\alpha-2)}{\Gamma(\alpha)}(t^{\alpha-1}-(t-s)^{\alpha-1}) \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\alpha-1)}{\Gamma(\alpha)}s(t-s)^{\alpha-2} = \frac{(\alpha-1)}{\Gamma(\alpha)}\int_{t-s}^{t}u^{\alpha-2}du - \frac{(\alpha-1)}{\Gamma(\alpha)}\int_{t-s}^{t}(t-s)^{\alpha-2}du \\ &= \frac{(\alpha-1)}{\Gamma(\alpha)}\int_{t-s}^{t}(u^{\alpha-2}-(t-s)^{\alpha-2})du \\ &\ge \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}(t-s)^{\alpha-3}\int_{t-s}^{t}(u-t+s)du \ \ \text{(applying (2.5))} \\ &= \frac{(\alpha-1)(\alpha-2)}{2\Gamma(\alpha)}(t-s)^{\alpha-3}s^2 \ge \frac{(\alpha-1)(\alpha-2)}{2^{\alpha-2}\Gamma(\alpha)}t^{\alpha-3}s^2 = \frac{1}{2\Gamma(\alpha)}\frac{(\alpha-1)}{2^{\alpha-3}}(\alpha-2)t^{\alpha-3}s^2 \\ &\ge \frac{1}{2\Gamma(\alpha)}\frac{(\alpha-1)}{2}(\alpha-2)t^{\alpha-3}s^2 \ge \frac{(\alpha-2)}{2\Gamma(\alpha)}t^{\alpha-3}s^2 \ge \frac{(\alpha-3)}{2\Gamma(\alpha)}t^{\alpha-2}s^2. \end{aligned}$$

To rigorously prove that Property (iv) holds, let us define the following:

$$
\begin{aligned}
G_0(t,s) &= \frac{(t-s)^{\alpha-1}+(\alpha-1)t^{\alpha-2}s-t^{\alpha-1}}{\Gamma(\alpha)}, \quad && t\in[s,1],\\
G_1(t,s) &= \frac{(\alpha-1)t^{\alpha-2}s-t^{\alpha-1}}{\Gamma(\alpha)}, && t\in[0,s].
\end{aligned}
$$

Additionally, it is important to note that $\dfrac{\partial G_0}{\partial t}=\dfrac{(t-s)^{\alpha-2}-t^{\alpha-2}+(\alpha-2)t^{\alpha-3}s}{\Gamma(\alpha-1)}$. Let $g=\dfrac{\partial G_0}{\partial t}$ for each admissible (t,s). Evidently, $g>0$ on its domain precisely when $(t-s)^{\alpha-2}+(\alpha-2)t^{\alpha-3}s-t^{\alpha-2}>0$ on the same domain. This inequality holds because of the observation that

$$
\int_{t-s}^{t}(\alpha-2)(t^{\alpha-3}-\xi^{\alpha-3})d\xi\geq 0.
$$

Thus, $g>0$ and $G_0(t,s)\leq G_0(1,s)$. On the other hand, for each fixed admissible $s\in[0,1]$, note that $\dfrac{\partial G_1}{\partial t}=\dfrac{(\alpha-1)\left[(\alpha-2)s-t\right]t^{\alpha-3}}{\Gamma(\alpha)}\geq 0$; accordingly, the behavior of $G_1(t,s)$ is such that it increases with t. In fact, $G_1(t,s)\leq G_1(s,s)=G_0(s,s)\leq G_0(1,s)=G(1,s)$. Consequently, (v) holds and one can clearly observe that

$$
G(1,s)=\frac{(\alpha-1)s+(1-s)^{\alpha-1}-1}{\Gamma(\alpha)}. \tag{2.6}
$$

Thus, (ii)–(v) are proved. □

Remark 2.1. *Lemma 2.2 implies that* $u(t)\geq 0$ *whenever* $l(t)\geq 0$ *for* $t\in[0,1]$.

We now define the lower and upper solutions for the BVP (1.3).

Definition 2.3. *A function* $\beta\in C[0,1]$ *is called a lower solution of the BVP (1.3) if it satisfies*

$$
\begin{aligned}
&D_{0^+}^{\alpha}\beta(t)\leq\varphi(t,\beta(t)),\ 0<t<1,\ 3<\alpha\leq 4,\\
&\beta(0)\leq 0,\ D_{0^+}^{\alpha-3}\beta(0)\leq 0,\ D_{0^+}^{\alpha-2}\beta(1)\leq 0,\ D_{0^+}^{\alpha-1}\beta(1)\geq 0.
\end{aligned}
$$

Definition 2.4. *A function* $\gamma\in C[0,1]$ *is called an upper solution of the BVP (1.3) if it satisfies*

$$
\begin{aligned}
&D_{0^+}^{\alpha}\gamma(t)\geq\varphi(t,\gamma(t)),\ 0<t<1,\ 3<\alpha\leq 4,\\
&\gamma(0)\geq 0,\ D_{0^+}^{\alpha-3}\gamma(0)\geq 0,\ D_{0^+}^{\alpha-2}\gamma(1)\geq 0,\ D_{0^+}^{\alpha-1}\gamma(1)\leq 0.
\end{aligned}
$$

Theorem 2.1. *Let* $G(t,s)$ *be defined by (2.3). There exists a constant* $\tau>0$ *such that*

$$
\frac{\min_{t\in[\frac{1}{4},1]}G(t,s)}{\max_{t\in[0,1]}G(t,s)}=\frac{G(\frac{1}{4},s)}{G(1,s)}\geq\tau.
$$

Proof. Observe that expression (2.3) implies that

$$\min_{t\in[\frac{1}{4},1]} G(t,s) = \begin{cases} G_0\left(\dfrac{1}{4},s\right), & s\in\left(0,\frac{1}{4}\right] \\ G_1\left(\dfrac{1}{4},s\right), & s\in\left[\frac{1}{4},1\right) \end{cases}$$

$$= \begin{cases} \dfrac{\left(\frac{1}{4}-s\right)^{\alpha-1}-\left(\frac{1}{4}\right)^{\alpha-1}+(\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2}s}{\Gamma(\alpha)}, & s\in\left(0,\frac{1}{4}\right] \\ \dfrac{-\left(\frac{1}{4}\right)^{\alpha-1}+(\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2}s}{\Gamma(\alpha)}, & s\in\left[\frac{1}{4},1\right). \end{cases}$$

To facilitate the following analysis, we introduce the ratio function

$$h(s)=\frac{G(\frac{1}{4},s)}{G(1,s)},\quad s\in[0,1],$$

and proceed to find its lower bound by dividing the interval into two cases.

Case 1: $s\in\left[0,\frac{1}{4}\right]$.

In this interval, the function $h(s)$ takes the form

$$h(s)=\frac{-\left(\frac{1}{4}\right)^{\alpha-1}+(\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2}s+\left(\frac{1}{4}-s\right)^{\alpha-1}}{(1-s)^{\alpha-1}-1+(\alpha-1)s}.$$

To analyze the behavior of the numerator, we define the auxiliary function as

$$u(s)=-\left(\frac{1}{4}\right)^{\alpha-1}+(\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2}s+\left(\frac{1}{4}-s\right)^{\alpha-1},\ s\in\left[0,\frac{1}{4}\right].$$

Instead of studying $h(s)$ directly, we first investigate the simplified ratio:

$$h_1(s)=\frac{u(s)}{s^2},\ s\in\left(0,\frac{1}{4}\right].$$

Differentiating $h_1(s)$ yields

$$h_1'(s)=\frac{su'(s)-2u(s)}{s^3}:=\frac{h_2(s)}{s^3},$$

where

$$u'(s)=(\alpha-1)\left[\left(\frac{1}{4}\right)^{\alpha-2}-\left(\frac{1}{4}-s\right)^{\alpha-2}\right].$$

To determine the sign of $h_2(s)$, we compute its derivative and rearrange the terms:

$$h_2'(s)=(\alpha-1)\left(\frac{1}{4}-s\right)^{\alpha-3}\left[-\left(\frac{1}{4}\right)^{\alpha-2}\left(\frac{1}{4}-s\right)^{3-\alpha}+\frac{1}{4}+(\alpha-3)s\right]:=(\alpha-1)\left(\frac{1}{4}-s\right)^{\alpha-3}h_3(s).$$

Now, we analyze the core function $h_3(s)$. Its derivative is given by

$$h_3'(s)=(\alpha-3)\left[1-\left(\frac{1}{1-4s}\right)^{\alpha-2}\right].$$

Since $\alpha \in (3,4]$ and $s \in \left[0, \frac{1}{4}\right)$, we have $\frac{1}{1-4s} > 1$, which implies $\left(\frac{1}{1-4s}\right)^{\alpha-2} > 1$. Consequently, $h_3'(s) < 0$ for all $s \in \left[0, \frac{1}{4}\right)$. Because $h_3(s)$ is strictly decreasing and $h_3(0) = 0$, it follows that $h_3(s) < 0$ for $s \in \left(0, \frac{1}{4}\right]$. Since $(\alpha-1)\left(\frac{1}{1-4s}\right)^{\alpha-3} > 0$ for $\alpha \in (3,4]$, we immediately obtain $h_2'(s) < 0$. This strict monotonicity, combined with $h_2(0) = 0$, guarantees that $h_2(s) < 0$. As the denominator $s^3 > 0$, we conclude that $h_1'(s) < 0$ on $\left(0, \frac{1}{4}\right]$. Therefore, $h_1(s)$ is strictly decreasing, and its minimum on this interval is attained at the right endpoint:

$$h_1(s) \geq h_1(\frac{1}{4}) = (\alpha-2)\left(\frac{1}{4}\right)^{\alpha-3}, \ s \in \left[0, \frac{1}{4}\right]. \tag{2.7}$$

Assuming that $\gamma > 2$, we can obtain that

$$(1+a)^{\gamma} - 1 - \gamma a \leq \frac{\gamma(\gamma-1)}{2}a^2, \ \ a \in (-1,0).$$

Since $\alpha - 1 > 2$ for $\alpha \in (3,4]$, applying the above inequality with $a = -s$ yields

$$(1-s)^{\alpha-1} - 1 + (\alpha-1)\,s \leq \frac{(\alpha-1)(\alpha-2)}{2}s^2, \ \ s \in \left[0, \frac{1}{4}\right].$$

Consequently, combining this upper bound for the denominator with the lower bound for $h_1(s)$ obtained via (2.7), we deduce that

$$\begin{aligned} h(s) =& \frac{-\left(\frac{1}{4}\right)^{\alpha-1} + (\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2} s + \left(\frac{1}{4}-s\right)^{\alpha-1}}{(1-s)^{\alpha-1} - 1 + (\alpha-1)s} \\ \geq& \frac{-\left(\frac{1}{4}\right)^{\alpha-1} + (\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2} s + \left(\frac{1}{4}-s\right)^{\alpha-1}}{\frac{(\alpha-1)(\alpha-2)}{2}s^2} \\ =& \frac{2}{(\alpha-1)(\alpha-2)}h_1(s) \geq \frac{2}{\alpha-1}\left(\frac{1}{4}\right)^{\alpha-3}, \ s \in \left[0, \frac{1}{4}\right]. \end{aligned}$$

Case 2: $s \in \left[\frac{1}{4}, 1\right]$. In this interval, the function $h(s)$ simplifies to

$$h(s) = \frac{-\left(\frac{1}{4}\right)^{\alpha-1} + (\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2} s}{(1-s)^{\alpha-1} - 1 + (\alpha-1)s}.$$

Next, we aim to prove that

$$h(s) \geq \left(\frac{1}{4}\right)^{\alpha-1}. \tag{2.8}$$

First, we note that the denominator of $h(s)$, denoted as $v(s) = (1-s)^{\alpha-1} - 1 + (\alpha-1)s$, is strictly positive for $s \in (0,1]$. Indeed, we have $v(0) = 0$ and $v'(s) = (\alpha-1)\left(1-(1-s)^{\alpha-2}\right) > 0$ for $s \in (0,1]$ and $\alpha \in (3,4]$. Because the denominator is positive, inequality (2.8) is equivalent to showing that the numerator minus $\left(\frac{1}{4}\right)^{\alpha-1}$ multiplied by the denominator is nonnegative. To this end, we define an auxiliary function:

$$g(s) = -\left(\frac{1}{4}\right)^{\alpha-1} + (\alpha-1)\left(\frac{1}{4}\right)^{\alpha-2} s - \left(\frac{1}{4}\right)^{\alpha-1}((1-s)^{\alpha-1} - 1 + (\alpha-1)s), \ s \in \left[\frac{1}{4}, 1\right].$$

Simplifying the expression of $g(s)$, we obtain

$$g(s) = \left(\frac{1}{4}\right)^{\alpha-1} (3(\alpha - 1)s - (1 - s)^{\alpha-1}).$$

Since $\alpha \in (3, 4]$ and $s \in \left[\frac{1}{4}, 1\right]$, we have

$$g(s) \geq \left(\frac{1}{4}\right)^{\alpha-1} (\frac{3(\alpha - 1)}{4} - (1 - s)^{\alpha-1}) \geq \left(\frac{1}{4}\right)^{\alpha-1} (\frac{3 \cdot 2}{4} - (1 - s)^{\alpha-1}) > 0.$$

Thus, inequality (2.8) holds true. Combining the lower bounds established in Cases 1 and 2, we finally obtain that for all $s \in [0, 1]$,

$$h(s) = \frac{\min_{t\in[\frac{1}{4},1]} G(t, s)}{\max_{t\in[0,1]} G(t, s)} = \frac{G(\frac{1}{4}, s)}{G(1, s)} \geq \min\left\{\frac{2}{(\alpha - 2)}\left(\frac{1}{4}\right)^{\alpha-3}, \left(\frac{1}{4}\right)^{\alpha-1}\right\} = \left(\frac{1}{4}\right)^{\alpha-1} := \tau. \tag{2.9}$$

□

3. Positive solution of the nonsingular problem

Herein, we employ Krasnoselskii's fixed point theorem, Schauder's fixed-point theorem, and cone theory techniques to establish the existence of positive solutions for BVP (1.3) associated with nonsingular fractional differential equations.

Let $T : X \to X$ be defined by

$$(Tu)(t) = \int_0^1 G(t, s)\varphi(s, u(s))ds, \tag{3.1}$$

where $X = C[0, 1]$ denotes the Banach space endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. It is clear that u is a solution of the BVP (1.3) precisely when u is a fixed point of the operator T.

Moreover, for clarity and to facilitate the forthcoming analysis, we present the following assumptions:

$$\delta_1 = \left[\int_0^1 G(1, s)ds\right]^{-1} = \frac{2\Gamma(\alpha + 1)}{(\alpha - 2)(\alpha - 1)}, \tag{3.2}$$

$$\delta_2 = \left[\int_{\frac{1}{4}}^1 G(1, s)ds\right]^{-1} = \frac{32\Gamma(\alpha + 1)}{15\alpha(\alpha - 1) - 24\alpha + 32\left(\frac{3}{4}\right)^{\alpha-1}}. \tag{3.3}$$

Moreover, for clarity and facilitating the forthcoming analysis, we present the following three assumptions: For $t, s \in [0, 1]$, we consider

(H_1) The function $\varphi : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous;

(H_2) There exists a constant $r_1 > 0$ such that $\varphi(t, u) \leq \delta_1 r_1$ holds for all $(t, u) \in [0, 1] \times [0, r_1]$, where δ_1 is given by (3.2);

(H_3) There exists a constant $r_2 > 0$ such that $\varphi(t,u) \geq \delta_2 r_2$ holds for all $(t,u) \in [\frac{1}{4},1] \times [\tau r_2, r_2]$, where the parameters δ_2 and τ are given by (3.3) and (2.9), respectively.

Hereafter, we state and prove our principal result.

Theorem 3.1. *If $r_2 > r_1 > 0$ and conditions (H_1)–(H_3) hold, then the BVP (1.3) admits at least one positive solution.*

Proof. Define the cone $K = \{u \in X : \min_{t\in[\frac{1}{4},1]} u(t) \geq \tau\|u\|\}$. For the operator T defined by (3.1), we easily observe that

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)\varphi(s,u(s))ds \leq \int_0^1 G(1,s)\varphi(s,u(s))ds\\ &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi(s,u(s))ds = \|Tu\|,\end{aligned}$$

and

$$\begin{aligned}\min_{t\in[\frac{1}{4},1]} (Tu)(t) &\geq \tau \int_0^1 G(1,s)\varphi(s,u(s))ds\\ &= \tau \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi(s,u(s))ds = \tau\|Tu\|.\end{aligned}$$

Hence, $Tu \in K$. Furthermore, we establish the complete continuity of T explicitly by applying the Arzela–Ascoli theorem. Now, let $\Omega_1 = \{u \in K : \|u\| < r_1\}$ and $\Omega_2 = \{u \in K : \|u\| < r_2\}$. Note that for $u \in K \cap \partial\Omega_1$ and $t \in [0,1]$, we have $u(t) \leq r_1$. Consequently,

$$\|Tu\| = \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi(s,u(s))ds \leq r_1\delta_1 \int_0^1 G(1,s)ds = r_1,$$

and it follows that $\|Tu\| \leq \|u\|$ when $u \in K \cap \partial\Omega_1$. Similarly, for $u \in K \cap \partial\Omega_2$, we have $u(t) \geq \tau r_2$ for $t \in [\frac{1}{4},1]$. Then,

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)\varphi(s,u(s))\,ds \geq \int_{\frac{1}{4}}^1 G(1,s)\varphi(s,u(s))ds\\ &\geq r_2\delta_2 \int_{\frac{1}{4}}^1 G(1,s)ds = r_2.\end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$ when $u \in K \cap \partial\Omega_2$. Finally, by Krasnoselskii's fixed point theorem [32–34], the operator T has a fixed point in $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$, which implies that the BVP (1.3) has a solution u_0 with $r_1 \leq \|u_0\| \leq r_2$. □

Remark 3.1. *In Theorem 3.1, we used the cone*

$$K = \left\{u \in X : \min_{t\in[\frac{1}{4},1]} u(t) \geq \tau\,\|u\|\right\}$$

instead of the cone $K_1 = \left\{u \in X : \min_{t\in[\frac{1}{4},1]} u(t) \ge \frac{(\alpha-3)\left(\frac{1}{4}\right)^{\alpha-2}}{2(\alpha-1)(\alpha-2)}\|u\|\right\}$*, which can be derived from Lemma 2.2 (ii). The reason is that* τ *in front of the norm* $\|u\|$ *in the cone K is larger. A larger* τ *enables us to derive more refined a priori estimates. Subsequently, these refined estimates afford a more stringent control over the upper and lower bounds of the solutions. Such a stringent control is beneficial for establishing the stability and convergence of solutions. Moreover, it holds substantial importance in areas such as numerical computations.*

Example 3.1.

$$\begin{cases} D^{\alpha}_{0^+}u(t) = \varphi(t,u(t)), \quad t \in (0,1), \\ u(0) = D^{\alpha-3}_{0^+}u(0) = D^{\alpha-2}_{0^+}u(1) = D^{\alpha-1}_{0^+}u(1) = 0. \end{cases} \tag{3.4}$$

Here, $\varphi(t,u) = \dfrac{1}{1+t^2}\left(3u^2 + \dfrac{\Gamma(\alpha)}{2}u\sin^2(\pi u)\right)$*. It is straightforward to verify that condition (*H_1*) is satisfied. Let* $r_1 = \min\left\{1, \dfrac{\delta_1}{6}\right\}$*, where* δ_1 *is given by (3.2). For any* $u \in [0, r_1]$*, since* $r_1 \le \dfrac{\delta_1}{6}$ *implies* $3u \le \dfrac{\delta_1}{2}$*, we have*

$$\varphi(t,u) \le 3u^2 + \frac{\Gamma(\alpha)}{2}u \le \frac{\delta_1}{2}u + \frac{2}{\alpha-2}\cdot\frac{\alpha}{\alpha-1}\cdot\frac{\Gamma(\alpha)}{2}u = \frac{\delta_1}{2}u + \frac{\delta_1}{2}u = \delta_1 u \le \delta_1 r_1.$$

Thus, condition (H_2*) holds. Next, let* $r_2 = \max\left\{2, \dfrac{2\delta_2}{3\tau^2}\right\}$*, where* δ_2 *is given by (3.3). Then, for any* $u \in [\tau r_2, r_2]$ *and* $t \in \left[\dfrac{1}{4}, 1\right]$*, we obtain*

$$\varphi(t,u) \ge \frac{3}{2}u^2 \ge \frac{3}{2}\tau^2 r_2^2.$$

Since $r_2 \ge \dfrac{2\delta_2}{3\tau^2}$*, it follows that*

$$\frac{3}{2}\tau^2 r_2^2 \ge \frac{3}{2}\tau^2\cdot\frac{2\delta_2}{3\tau^2}r_2 = \delta_2 r_2.$$

Hence, condition (H_3*) is satisfied. Therefore, by Theorem 3.1, we conclude that the BVP (1.3) has at least one positive solution.*

To establish the existence of positive solutions, we employ the method of upper and lower solutions. If u is a positive solution of (2.1) with $l(t) \ge 0$ for $t \in [0,1]$, then it follows from Lemma 2.2 that there exist $R > r > 0$ such that

$$rw(t) \le u(t) \le Rw(t), \ t \in [0,1], \tag{3.5}$$

where

$$r = \frac{\alpha-3}{2\Gamma(\alpha)}\int_0^1 s^2 l(s)ds,$$

$$R = \frac{\alpha-2}{\Gamma(\alpha-1)}\int_0^1 sl(s)ds,$$

and $w(t)$ is defined as in Lemma 2.2 with $w(t) = t^{\alpha-2}$.

Theorem 3.2. *Assume that* $\varphi : [0,1]\times[0,+\infty)\to[0,+\infty)$ *satisfies:*

(H_4) φ *is continuous and nondecreasing in the second variable;*
(H_5) $\varphi(t,w(t))\neq 0$ *for all* $t\in[0,1]$*;*
(H_6) There exists $\mu\in(0,1)$ *such that*

$$k^{\mu}\varphi(t,u)\leq\varphi(t,ku),\ \forall\ 0\leq k\leq 1.$$

Then, the BVP (1.3) has a positive solution.

Proof. First, we define the function

$$u(t)=\int_0^1 G(t,s)\varphi(s,w(s))ds.$$

In view of Lemma 2.1, $u(t)$ is a positive solution of the linear BVP:

$$\begin{cases} D^{\alpha}_{0^+}u(t)=\varphi(t,w(t)), & 0<t<1,\\ u(0)=D^{\alpha-3}_{0^+}u(0)=D^{\alpha-2}_{0^+}u(1)=D^{\alpha-1}_{0^+}u(1)=0. \end{cases}$$

Define the positive constants d_1 and d_2 as

$$d_1=\min\left\{1,\frac{\alpha-3}{2\Gamma(\alpha)}\int_0^1 s^2\varphi(s,w(s))ds\right\}>0,\ d_2=\max\left\{1,\frac{\alpha-2}{\Gamma(\alpha-1)}\int_0^1 s\varphi(s,w(s))ds\right\}. \tag{3.6}$$

From inequalities (3.5) and (3.6), we obtain the following estimate for $u(t)$:

$$d_1w(t)\leq u(t)\leq d_2w(t),\ t\in[0,1].$$

Now, let $\beta(t)=k_1u(t)$ and $\gamma(t)=k_2u(t)$, where the constants k_1 and k_2 are chosen to satisfy

$$0<k_1\leq\min\{\frac{1}{d_2},(d_1)^{\frac{\mu}{1-\mu}}\},\ k_2\geq\max\{\frac{1}{d_1},(d_2)^{\frac{\mu}{1-\mu}}\}.$$

Consequently, the following inequalities hold:

$$\begin{gathered} k_1d_1\leq\frac{\beta(t)}{w(t)}\leq k_1d_2\leq 1,\\ \frac{1}{k_2d_2}\leq\frac{w(t)}{\gamma(t)}\leq\frac{1}{k_2d_1}\leq 1,\\ (k_1d_1)^{\mu}\geq k_1,(k_2d_2)^{\mu}\leq k_2. \end{gathered}$$

These inequalities, combined with the properties of φ, imply that

$$\varphi(t,\beta(t))=\varphi\left(t,\frac{\beta(t)}{w(t)}w(t)\right)\geq\varphi(t,k_1d_1w(t))\geq(k_1d_1)^{\mu}\varphi(t,w(t))\geq k_1\varphi(t,w(t))=D^{\alpha}_{0^+}\beta(t),$$

and similarly,

$$\varphi(t,\gamma(t))=\varphi\left(t,\frac{\gamma(t)}{w(t)}w(t)\right)\leq\varphi(t,k_2d_2w(t))\leq(k_2d_2)^{\mu}\varphi(t,w(t))\leq k_2\varphi(t,w(t))=D^{\alpha}_{0^+}\gamma(t).$$

Furthermore, both $\beta(t) = k_1u(t)$ and $\gamma(t) = k_2u(t)$ satisfy the boundary conditions (1.3). Therefore, $\beta(t)$ and $\gamma(t)$ are lower and upper solutions of (1.3), respectively.

We consider the modified fractional boundary value problem:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = h(t, u(t)), \quad 0 < t < 1, \\ u(0) = D^{\alpha-3}_{0^+}u(0) = D^{\alpha-2}_{0^+}u(1) = D^{\alpha-1}_{0^+}u(1) = 0, \end{cases} \tag{3.7}$$

where the auxiliary function h is defined by

$$h(t, u) = \begin{cases} \varphi(t, \beta(t)), & \text{if } u \le \beta(t), \\ \varphi(t, u), & \text{if } \beta(t) \le u \le \gamma(t), \\ \varphi(t, \gamma(t)), & \text{if } u \ge \gamma(t). \end{cases}$$

Since $\varphi(t, u)$ is continuous and nondecreasing with respect to u, we conclude that

$$h(\cdot, u(\cdot)) \in C[0, 1], \tag{3.8}$$

and

$$\varphi(t, \beta(t)) \le h(t, u(t)) \le \varphi(t, \gamma(t))$$

for any $u \in X$. Thus, there exists a constant $M > 0$ such that

$$|h(t, u(t))| \le M, \quad \forall t \in [0, 1],\ u \in X. \tag{3.9}$$

Define the integral operator $S : X \to X$ by

$$(S u)(t) = \int_0^1 G(t, s)h(s, u(s))\, ds,$$

where $G(t, s)$ is the Green's function given by (2.3). Next, we prove that the operator $S : X \to X$ is completely continuous. For $u \in X$, by (3.9) and Lemma 2.2 (ii), we have

$$|(S u)(t)| \le \int_0^1 G(t, s)|h(s, u(s))|\, ds \le \frac{M(\alpha - 1)(\alpha - 2)}{\Gamma(\alpha)} \int_0^1 s^2\, ds = \frac{M(\alpha - 1)(\alpha - 2)}{3\Gamma(\alpha)},$$

implying that $\|S u\| \le \frac{M(\alpha-1)(\alpha-2)}{3\Gamma(\alpha)}$. This proves the boundedness of the set $S(X)$. The continuity of the function $G(t, s)$ on $[0, 1] \times [0, 1]$ guarantees its uniform continuity. Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|G(t_2, s) - G(t_1, s)| < \epsilon, \quad \text{whenever} |t_2 - t_1| < \delta,\ s \in [0, 1].$$

Hence, for any $u \in X$ and $t_1, t_2 \in [0, 1]$ with $|t_2 - t_1| < \delta$, we have

$$|(S u)(t_1) - (S u)(t_2)| \le \int_0^1 |G(t_1, s) - G(t_2, s)||h(s, u(s))|\, ds \le M\epsilon.$$

This proves the equicontinuity of the set $S(X)$. Furthermore, due to the continuity of $G(t, s)$ and $h(t, u)$, the operator S is continuous. By the Arzela-Ascoli theorem, we conclude that $S : X \to X$ is completely continuous. Consequently, Schauder's fixed-point theorem [34] implies that the operator S has a fixed point $u^* \in X$, which is a solution of problem (3.7).

It remains to verify that u^* lies between β and γ. Let $w(t) = \gamma(t) - u^*(t)$. Since γ is an upper solution, we obtain

$$\begin{cases} D_{0^+}^{\alpha} w(t) \geq \varphi(t, \gamma(t)) - h(t, u^*(t)) \geq 0, \\ w(0) = D_{0^+}^{\alpha-3} w(0) = D_{0^+}^{\alpha-2} w(1) = D_{0^+}^{\alpha-1} w(1) = 0. \end{cases}$$

Noting that $w(t)$ satisfies the boundary conditions, Remark 2.1 implies $w(t) \geq 0$ on $[0, 1]$, that is, $u^*(t) \leq \gamma(t)$. A similar argument yields $\beta(t) \leq u^*(t)$. Therefore, $u^*(t)$ satisfies $\beta(t) \leq u^*(t) \leq \gamma(t)$, which implies $h(t, u^*(t)) = \varphi(t, u^*(t))$. Thus, $u^*(t)$ is a positive solution of the original BVP (1.3). □

Example 3.2. *Consider the BVP:*

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) = \varphi(t, u(t)), \quad 0 < t < 1, \\ u(0) = D_{0^+}^{\frac{1}{2}} u(0) = D_{0^+}^{\frac{3}{2}} u(1) = D_{0^+}^{\frac{5}{2}} u(1) = 0, \end{cases} \tag{3.10}$$

where the nonlinear term is given by $\varphi(t, u) = t + u^{\mu}$ *with* $0 < \mu < 1$.

For any constant k *such that* $0 \leq k \leq 1$, *we observe that* $k^{\mu} < 1$. *This implies the following inequality:*

$$K^{\mu} \varphi(t, u) = k^{\mu} t + k^{\mu} u^{\mu} \leq t + (ku)^{\mu} = \varphi(t, ku).$$

Consequently, all the hypotheses of Theorem 3.2 are fulfilled. Therefore, problem (3.10) admits a positive solution.

Next, we present some new upper and lower estimates for $G(t, s)$, different from those given in Lemma 2.2. Then, we apply these new estimates to obtain criteria for the existence of a positive solution to the BVP (1.3).

Theorem 3.3. *For* $t, s \in [0, 1]$, *we have*

$$t^{\alpha-2} G(1, s) \leq G(t, s) \leq 2(\alpha - 2) t^{\alpha-3} G(1, s).$$

Proof. From Lemma 2.2 (iii), $G(t, s) \leq \dfrac{(\alpha - 1)(\alpha - 2) t^{\alpha-3} s^2}{\Gamma(\alpha)}$. First, we analyze the relationship between $G(t, s)$ and s^2. By Eq (2.6), $G(1, s)$ can be expressed as

$$\begin{aligned} G(1, s) = \frac{(\alpha - 1)s + (1 - s)^{\alpha-1} - 1}{\Gamma(\alpha)} &= \frac{(\alpha - 1) \displaystyle\int_0^s -(1 - t)^{\alpha-2} dt + (\alpha - 1)s}{\Gamma(\alpha)} \\ &= \frac{(\alpha - 1) \displaystyle\int_0^s \left[1 - (1 - t)^{\alpha-2}\right] dt}{\Gamma(\alpha)}. \end{aligned}$$

Define $h(t) = 1 - (1 - t)^{\alpha-2}$ for $t \in [0, 1]$. Clearly, $h(0) = 0$ and $h(1) = 1$. Calculating the first and second derivatives of $h(t)$, we obtain

$$h'(t) = (1 - t)^{\alpha-3} (\alpha - 2) \geq 0,$$

$$h''(t) = (1-t)^{\alpha-4}(\alpha-2)(\alpha-3) \leq 0.$$

Thus, $h(t)$ is a monotonically increasing and concave function. The concavity of the function implies that

$$h(t) = h(t \cdot 1 + (1-t) \cdot 0) \geq th(1) + (1-t)h(0) = t.$$

Substituting this inequality back, we can obtain

$$G(1,s) = \frac{(\alpha-1)\displaystyle\int_0^s \left[1-(1-t)^{\alpha-2}\right]dt}{\Gamma(\alpha)} \geq \frac{(\alpha-1)\displaystyle\int_0^s tdt}{\Gamma(\alpha)} = \frac{(\alpha-1)s^2}{2\Gamma(\alpha)}.$$

This establishes the relationship between $G(1,s)$ and s^2:

$$s^2 \leq \frac{2\Gamma(\alpha)}{\alpha-1}G(1,s). \tag{3.11}$$

Combining (*iii*) in Lemma 2.2 with (3.11), it follows that

$$G(t,s) \leq 2(\alpha-2)t^{\alpha-3}G(1,s).$$

Next, we analyze the left-hand side of the inequality. When $s \leq t$,

$$G(t,s) = \frac{-t^{\alpha-1} + (t-s)^{\alpha-1} + (\alpha-1)t^{\alpha-2}s}{\Gamma(\alpha)} = \frac{t^{\alpha-1}\left[-1 + (\alpha-1)\frac{s}{t} + (1-\frac{s}{t})^{\alpha-1}\right]}{\Gamma(\alpha)}.$$

Let $f(u) = -1 + (\alpha-1)u + (1-u)^{\alpha-1}$ for $u \in [0,1]$. Obviously, $f(0) = 0$ and $f(1) = (\alpha-2) > 0$. Differentiating $f(u)$, we obtain

$$f'(u) = (\alpha-1)\left[1-(1-u)^{\alpha-2}\right] \geq 0,$$

$$f''(u) = (\alpha-1)(\alpha-2)(1-u)^{\alpha-3} \geq 0.$$

Hence, f is a strictly increasing convex function. By convexity and the fact that $f(0) = 0$, we have

$$f(s) = f(t \cdot \frac{s}{t} + (1-t)0) \leq tf(\frac{s}{t}).$$

Noting that $f(s) = \Gamma(\alpha)G(1,s)$, the inequality above implies $G(1,s) \leq \frac{t}{t^{\alpha-1}}G(t,s)$. Consequently, for $s \leq t$, we obtain $G(t,s) \geq t^{\alpha-2}G(1,s)$.

On the other hand, when $t \leq s$,

$$\begin{aligned} G(t,s) - t^{\alpha-2}G(1,s) &= \frac{-t^{\alpha-1} + (\alpha-1)t^{\alpha-2}s}{\Gamma(\alpha)} - \frac{t^{\alpha-2}\left[-1 + (\alpha-1)s + (1-s)^{\alpha-1}\right]}{\Gamma(\alpha)} \\ &= \frac{t^{\alpha-2}\left[-t + (\alpha-1)s + 1 - (\alpha-1)s - (1-s)^{\alpha-1}\right]}{\Gamma(\alpha)} \end{aligned}$$

$$= \frac{t^{\alpha-2}\left[1-t-(1-s)^{\alpha-1}\right]}{\Gamma(\alpha)}$$
$$\geq \frac{t^{\alpha-2}\left[1-s-(1-s)^{\alpha-1}\right]}{\Gamma(\alpha)} \geq 0.$$

Therefore, the inequality $G(t,s) \geq t^{\alpha-2}G(1,s)$ also holds for $t \leq s$. □

We now turn our attention to the existence of positive solutions for the BVP (1.3). We impose the following assumption:

(H_7) There exist constants $\mu_2 > \mu_1 > 0$ such that

$$\inf_{u\in F}\int_0^1 G(1,s)\varphi(s,u(s))ds \geq \mu_1 \text{ and } \sup_{u\in F}\int_0^1 G(1,s)\varphi(s,u(s))ds \leq \mu_2,$$

where the set F is defined by

$$F = \left\{u \in X : \mu_1 t^{\alpha-2} \leq u(t) \leq 2(\alpha-2)\mu_2 t^{\alpha-3},\ \forall\, t \in [0,1]\right\}.$$

Theorem 3.4. *If conditions (H_1) and (H_7) are satisfied, then the BVP (1.3) admits a positive solution in F.*

Proof. By Lemma 2.1, the existence of a solution to the BVP (1.3) is equivalent to finding a fixed point of a completely continuous operator T.

We first demonstrate that the set F is closed, bounded, and convex. The convexity and closedness of F follow directly from its definition. To establish boundedness, let $u \in F$. Then, for all $t \in [0,1]$, we have

$$0 \leq u(t) \leq 2(\alpha-2)\mu_2 t^{\alpha-3} \leq 2(\alpha-2)\mu_2,$$

implying that $\|u\| \leq 2(\alpha-2)\mu_2$.

Next, we prove that $T(F) \subset F$. For any $u \in F$, applying Theorem 3.3 and assumption (H_7) yields

$$Tu(t) \geq t^{\alpha-2}\int_0^1 \varphi(s,u(s))G(1,s)ds \geq \mu_1 t^{\alpha-2},$$

and

$$Tu(t) \leq 2(\alpha-2)t^{\alpha-3}\int_0^1 \varphi(s,u(s))G(1,s)ds \leq 2(\alpha-2)\mu_2 t^{\alpha-3}.$$

Combining these inequalities confirms that $Tu \in F$, and hence $T(F) \subset F$. Consequently, Schauder's fixed-point theorem [34] guarantees the existence of a fixed point of T, which is the desired positive solution of (1.3). □

Now, we present the following two corollaries.

Corollary 3.1. *Suppose there exist constants* $\mu_2 > \mu_1 > 0$ *such that* $\varphi(t,\cdot)$ *is nondecreasing on* $[0,\mu_2]$ *for any* $t \in [0,1]$. *Furthermore, assume that*

$$\int_0^1 \varphi(s,\mu_1 s^{\alpha-2})G(1,s)ds \geq \mu_1,$$
$$\int_0^1 \varphi(s,2(\alpha-2)\mu_2 s^{\alpha-3})G(1,s)ds \leq \mu_2.$$

Consequently, BVP (1.3) has a positive solution within the set F.

Corollary 3.2. *Suppose there exist constants* $\mu_2 > \mu_1 > 0$ *such that* $\varphi(t,\cdot)$ *is nonincreasing on* $[0,2(\alpha-2)\mu_2]$ *for any* $t \in [0,1]$. *Furthermore, assume that*

$$\int_0^1 \varphi(s,2(\alpha-2)\mu_2 s^{\alpha-3})G(1,s)ds \geq \mu_1,$$
$$\int_0^1 \varphi(s,\mu_1 s^{\alpha-2})G(1,s)ds \leq \mu_2.$$

Consequently, BVP (1.3) has a positive solution within the set F.

Example 3.3. *Consider the BVP (1.3) with* $\varphi(t,u) = t^3 u^{\frac{1}{2}}$. *It can be easily verified that the conditions of Corollary 3.1 are satisfied. Specifically, the constants* μ_1 *and* μ_2 *can be chosen to satisfy the following inequalities:*

$$\mu_1 \leq \left[\frac{2(\alpha^2+4\alpha-14)}{\Gamma(\alpha)(8+\alpha)(6+\alpha)} + \frac{\Gamma(\frac{6+\alpha}{2})}{\Gamma(\frac{6+3\alpha}{2})}\right]^2,$$

$$\mu_2 \geq 2(\alpha-2)\left[\frac{2(\alpha-1)}{\Gamma(\alpha)(\alpha+7)} + \frac{\Gamma(\frac{\alpha+5}{2})}{\Gamma(\frac{3\alpha+5}{2})} - \frac{2}{\Gamma(\alpha)(\alpha+5)}\right]^2.$$

On the other hand, setting $\varphi(t,u) = \arcsin\frac{1}{1+u}$, *it can be readily verified that Corollary 3.2 holds. In this case,* μ_1 *and* μ_2 *can be chosen to satisfy the following inequalities:*

$$\mu_2 \geq \frac{\pi(\alpha-2)}{4\alpha\Gamma(\alpha-2)},$$

$$\mu_1 \leq \frac{\mu_2}{1+2\mu_2(\alpha-2)}.$$

4. Positive solutions of the singular problem

In this section, we establish the existence of solutions for the singular BVP (1.3). For $b \in C[0,1]$, we use the notation $b \succ 0$ to indicate that $b(t) \geq 0$ on $[0,1]$ and is not identically zero.

Theorem 4.1. *Suppose that the following assumptions are satisfied:*

(A_1) For any constant $N > 0$, *there exists a function* $\phi_N \succ 0$ *such that the inequality* $\varphi(t,u) \geq \phi_N(t)$ *holds for all* $t \in [0,1]$ *and* $u \in (0,N]$.

(A_2) There exist nonnegative continuous functions $p(u)$ and $q(u)$ such that $\varphi(t,u) = p(u) + q(u) \geq 0$ for $(t,u) \in [0,1] \times (0,+\infty)$. Moreover, $p(u)$ is nonincreasing and $\dfrac{q(u)}{p(u)}$ is nondecreasing on $(0,+\infty)$.

(A_3) A constant $k_0 > 0$ exists such that $p(ab) \leq k_0 p(a)p(b)$ for all $a, b > 0$.

(A_4) $\int_0^1 p(\bar{w}(s))ds < +\infty$, where $\bar{w}(t) = \frac{\alpha-3}{2(\alpha-1)(\alpha-2)}w(t)$, and $w(t) = t^{\alpha-2}$ is defined in Lemma 2.2.

(A_5) There exists a positive number $\bar{r}$ satisfying

$$\frac{(\alpha-2)}{\Gamma(\alpha-1)}(p(\bar{r}) + q(\bar{r}))k_0 \int_0^1 s^2 p(\bar{w}(s))ds < \bar{r}.$$

Then, the BVP (1.3) admits a positive solution u that satisfies $0 < \|u\| < \bar{r}$.

Proof. According to (A_5), we take $n_0 \in \{1,2,3,\cdots\}$ satisfying

$$\frac{(\alpha-2)}{\Gamma(\alpha-1)}(p(\bar{r}) + q(\bar{r}))k_0 \int_0^1 s^2 p(\bar{w}(s))ds + \frac{1}{n_0} < \bar{r}.$$

Let $N_0 = \{n_0, n_0+1, \ldots\}$ denote the set of integers greater than or equal to n_0. Fix $n \in N_0$. For $\lambda \in [0,1]$, we now investigate the integral equations:

$$u(t) = \frac{1}{n} + \lambda \int_0^1 G(t,s)\varphi_n(s,u(s))ds, \tag{4.1}$$

where the truncated function φ_n is defined by $\varphi_n(t,u) = \varphi(t, \max\{u, \frac{1}{n}\})$.

We now prove that for any $\lambda \in [0,1]$, every solution u of the integral equations (4.1) satisfies $\|u\| \neq \bar{r}$. Suppose, on the contrary, that $\|u\| = \bar{r}$. Clearly, $u(t) \geq \frac{1}{n}$ for $t \in [0,1]$. By Lemma 2.2 (ii), we have

$$\|u\| \leq \frac{1}{n} + \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\lambda \int_0^1 s^2 \varphi_n(s,u(s))ds.$$

Furthermore, applying Lemma 2.2 (ii) yields

$$\begin{aligned} u(t) &\geq \frac{\alpha-3}{2\Gamma(\alpha)}\lambda w(t) \int_0^1 s^2\varphi_n(s,u(s))ds + \frac{1}{n} \geq \frac{1}{n} + w(t)\left(\|u\| - \frac{1}{n}\right)\frac{\alpha-3}{2(\alpha-1)(\alpha-2)} \\ &\geq \frac{\alpha-3}{2(\alpha-1)(\alpha-2)}w(t)\|u\| = \bar{w}(t)\bar{r}, \end{aligned}$$

where $\bar{w}(t)$ is as defined in (A_4).

Using conditions (A_1) and (A_2), we deduce that for every $t \in [0,1]$,

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)\varphi_n(s,u(s))ds + \frac{1}{n} = \lambda \int_0^1 G(t,s)\varphi(s,u(s))ds + \frac{1}{n} \\ &\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_0^1 s^2\varphi(s,u(s))ds + \frac{1}{n} \\ &\leq \frac{(\alpha-2)}{\Gamma(\alpha-1)} \int_0^1 s^2 p(u(s))\left(1 + \frac{q(u(s))}{p(u(s))}\right)ds + \frac{1}{n} \end{aligned}$$

$$\begin{aligned}
&\le \frac{(\alpha-2)}{\Gamma(\alpha-1)}\int_0^1 s^2 p(\bar{w}(s)\bar{r})\left(1+\frac{q(\bar{r})}{p(\bar{r})}\right)ds+\frac{1}{n}\\
&\le \frac{(\alpha-2)}{\Gamma(\alpha-1)}\left(1+\frac{q(\bar{r})}{p(\bar{r})}\right)k_0 p(\bar{r})\int_0^1 s^2 p(\bar{w}(s)))ds+\frac{1}{n}\\
&= \frac{(\alpha-2)}{\Gamma(\alpha-1)}(p(\bar{r})+q(\bar{r}))k_0\int_0^1 s^2 p(\bar{w}(s)))ds+\frac{1}{n}.
\end{aligned}$$

Thus, taking the norm yields

$$\bar{r}=\|u\|\le \frac{(\alpha-2)}{\Gamma(\alpha-1)}(p(\bar{r})+q(\bar{r}))k_0\int_0^1 s^2 p(\bar{w}(s))ds+\frac{1}{n_0}.$$

This contradicts the choice of n_0 based on (A_5). Therefore, $\|u\|\ne\bar{r}$.

By the Leray-Schauder nonlinear alternative [35], the integral equation

$$u(t)=\frac{1}{n}+\int_0^1 G(t,s)\varphi_n(s,u(s))ds$$

admits at least one solution u_n in $B_{\bar{r}}=\{u\in C[0,1]:\|u\|\le\bar{r}\}$.

Under the assumption (A_1), there is a function $\phi_{\bar{r}}(t)$ such that $\varphi(t,u(t))\ge\phi_{\bar{r}}(t)$ for $t\in[0,1]$. Let $\zeta=\frac{\alpha-3}{2\Gamma(\alpha)}\int_0^1 s^2\phi_{\bar{r}}(s)ds>0$. Since $u_n(t)\ge\frac{1}{n}$, we have

$$\begin{aligned}
u_n(t)&=\frac{1}{n}+\int_0^1 G(t,s)\varphi_n(s,u_n(s))ds=\int_0^1 G(t,s)\varphi(s,u_n(s))ds+\frac{1}{n}\\
&\ge\int_0^1 G(t,s)\phi_{\bar{r}}(s)ds\ge\frac{\alpha-3}{2\Gamma(\alpha)}w(t)\int_0^1 s^2\phi_{\bar{r}}(s)ds=\zeta w(t)>0,\ t\in(0,1].
\end{aligned}$$

To obtain a solution of (1.3), we need only show that the sequence $\{u_n\}$ is equicontinuous on $[0,1]$. For any $t_2,t_1\in[0,1]$, we estimate

$$\begin{aligned}
&|u_n(t_2)-u_n(t_1)|\\
&=\left|\int_0^1 (G(t_2,s)-G(t_1,s))\,\varphi(s,u_n(s))ds\right|\le\int_0^1|G(t_2,s)-G(t_1,s)|\,\varphi(s,u_n(s))ds\\
&\le\left(1+\frac{q(\bar{r})}{p(\bar{r})}\right)\int_0^1|G(t_2,s)-G(t_1,s)|\,p(\zeta w(s))ds\\
&\le\left(1+\frac{q(\bar{r})}{p(\bar{r})}\right)\int_0^1|G(t_2,s)-G(t_1,s)|\,k_0 p\,(w(s))\,p(\zeta)ds\\
&=\left(1+\frac{q(\bar{r})}{p(\bar{r})}\right)k_0 p(\zeta)\int_0^1|G(t_2,s)-G(t_1,s)|\,p\,(w(s))\,ds.
\end{aligned}$$

Now, the continuity of function $G(t,s)$ on $[0,1]\times[0,1]$ guarantees its uniform continuity. Therefore, for $\epsilon>0$, there exists $\delta>0$ such that

$$|G(t_2,\xi)-G(t_1,\xi)|<\epsilon,\ \text{whenever } |t_2-t_1|<\delta,\ s\in[0,1].$$

Then, we have

$$\begin{aligned}
&|u_n(t_2) - u_n(t_1)|\\
&\leq \left(1 + \frac{q(\bar{r})}{p(\bar{r})}\right) k_0 p(\zeta) \int_0^1 |G(t_2, s) - G(t_1, s)|\, p\,(w(s))\, ds\\
&\leq \epsilon \left(1 + \frac{q(\bar{r})}{p(\bar{r})}\right) k_0 p(\zeta) \int_0^1 p\,(w(s))\, ds.
\end{aligned}$$

By assumption (A_4), the integral converges, confirming that $\{u_n\}$ is equicontinuous. According to the Arzela-Ascoli theorem [32], there exists a subsequence $\{u_{n_k}\}$ and a function $u \in C[0,1]$ such that $u_{n_k} \to u$ uniformly on $[0,1]$. Furthermore, u satisfies the inequality

$$\zeta w(t) \leq u(t) < \bar{r}.$$

Finally, Lebesgue's dominated convergence theorem guarantees that the limit function u is a positive solution of BVP (1.3) satisfying $0 < \|u\| < \bar{r}$. □

Theorem 4.2. *Assume that conditions* $(A_2) - (A_5)$ *hold. Furthermore, suppose the following condition is satisfied:*

(A_6) *There exists a constant* $\bar{R} > \bar{r}$ *such that*

$$p(\bar{R}) \int_0^1 G(1, s) \left(1 + \frac{q(\bar{w}(s)\bar{R})}{p(\bar{w}(s)\bar{R})}\right) ds \geq \bar{R}.$$

Then, the BVP (1.3) admits a positive solution $\bar{u}$ *satisfying* $\bar{r} < \|\bar{u}\| \leq \bar{R}$.

Proof. Let $P = \{u \in X : u(t) \geq \bar{w}(t)\|u\|, t \in [0,1]\}$ be a cone in $C[0,1]$, where $\bar{w}(t)$ is defined in (A_4). Define two sets $\Omega_1 = \{u \in C[0,1] : \|u\| < \bar{r}\}$ and $\Omega_2 = \{u \in C[0,1] : \|u\| < \bar{R}\}$. We aim to apply Krasnoselskii's fixed point theorem to the operator T defined by (3.1) on the set $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

First, for $u \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$, we have $\bar{w}(t)\bar{r} \leq u(t) \leq \bar{R}$ for $t \in [0,1]$. Then, by $(A_2) - (A_4)$, we obtain

$$\begin{aligned}
Tu(t) = \int_0^1 G(t, s)\varphi(s, u(s))ds &\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_0^1 s^2 p(u(s)) \left(1 + \frac{q(u(s))}{p(u(s))}\right) ds\\
&\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \left(1 + \frac{q(\bar{R})}{p(\bar{R})}\right) \int_0^1 s^2 p(\bar{w}(s)\bar{r}) ds\\
&\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \left(1 + \frac{q(\bar{R})}{p(\bar{R})}\right) k_0 p(\bar{r}) \int_0^1 s^2 p(\bar{w}(s)) ds\\
&\leq \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \left(1 + \frac{q(\bar{R})}{p(\bar{R})}\right) k_0 p(\bar{r}) \int_0^1 s^2 p(\bar{w}(s)) ds < +\infty, \quad t \in [0,1],
\end{aligned}$$

which implies that the operator T is well-defined on $P \cap (\Omega_2 \setminus \Omega_1)$. Furthermore, standard arguments show that the operator T is completely continuous. It remains to verify the boundary conditions.

For $u \in P \cap \partial\Omega_1$, we have $\|u\| = \bar{r}$, which implies $\bar{w}(t)\bar{r} \leq u(t) \leq \bar{r}$ for $t \in [0,1]$. By assumptions (A_2) and (A_4), we obtain

$$
\begin{aligned}
Tu(t) = \int_0^1 G(t,s)\varphi(s,u(s))ds &\le \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^1 s^2 p(u(s))\left(1+\frac{q(u(s))}{p(u(s))}\right)ds\\
&\le \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(1+\frac{q(\bar{r})}{p(\bar{r})}\right)k_0\int_0^1 s^2 p(\bar{w}(s)\bar{r})ds\\
&\le \frac{(\alpha-2)}{\Gamma(\alpha-1)}(p(\bar{r})+q(\bar{r}))k_0\int_0^1 s^2 p(\bar{w}(s))ds < \bar{r} = \|u\|.
\end{aligned}
$$

From condition (A_5), it follows that $\|Tu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$.

On the other hand, for $u \in P \cap \partial\Omega_2$, we have $\|u\| = \bar{R}$, so $\bar{w}(t)\bar{R} \le u(t) \le \bar{R}$. Consequently,

$$
\begin{aligned}
(Tu)(1) = \int_0^1 G(1,s)\varphi(s,u(s))ds &= \int_0^1 G(1,s)\left(1+\frac{q(u(s))}{p(u(s))}\right)p(u(s))ds\\
&\ge \int_0^1 p(\bar{R})G(1,s)\left(1+\frac{q(\bar{w}(s)\bar{R})}{p(\bar{w}(s)\bar{R})}\right)ds \ge \bar{R} = \|u\|.
\end{aligned}
$$

By assumptions (A_6), we have $\|Tu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_2$. Therefore, by Krasnoselskii's fixed point theorem, the operator T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. This implies that the BVP (1.3) admits a positive solution $\bar{u}$ satisfying $\bar{r} < \|\bar{u}\| \le \bar{R}$. □

While Theorems 4.1 and 4.2 independently establish the existence of a single positive solution within specific norm bounds ($0 < \|u\| < \bar{r}$ and $\bar{r} < \|\bar{u}\| \le \bar{R}$, respectively), their combination yields a more profound insight. Because these two norm intervals do not overlap, the corresponding solutions must be distinct. This non-overlapping property provides the exact mechanism to guarantee multiple solutions. Merging these two independent findings, we obtain the following multiplicity result:

Theorem 4.3. *Assume that conditions* (A_1)–(A_6) *hold. Then, the BVP (1.3) admits two distinct solutions, denoted* $\bar{u}$ *and* u*, which satisfy the inequality* $0 < \|u\| < \bar{r} < \|\bar{u}\| \le \bar{R}$.

Example 4.1. *Consider the BVP*

$$
\begin{aligned}
&D_{0^+}^{\alpha}u(t) = u^{-c}(t) + m_0 u^{d}(t), \quad 0 < t < 1,\\
&u(0) = D_{0^+}^{\alpha-3}u(0) = D_{0^+}^{\alpha-2}u(1) = D_{0^+}^{\alpha-1}u(1) = 0,
\end{aligned}
\tag{4.2}
$$

where c and d are positive constants, and $m_0 > 0$ *represents a fixed parameter.*

Corollary 4.1. *Suppose that* $0 < c < \frac{1}{2}$, $d > 0$.

(i) If $d < 1$*, then the BVP (4.2) possesses a solution that is nonnegative for each* $m_0 > 0$.

(ii) If $d \ge 1$*, then the BVP (4.2) possesses a solution that is nonnegative for each* $0 < m_0 < m_1$*, where* m_1 *denotes a strictly positive constant.*

(iii) If $d > 1$*, then the BVP (4.2) possesses at least two solutions that are nonnegative for each* $0 < m_0 < m_1$.

Proof. Let $p(u) = u^{-c}$, $q(u) = m_0 u^d$, $k_0 = 1$, and $\phi_N = N^{-c}$; it is easy to verify that (A_1)–(A_3) are satisfied. Since $\bar{w}(t) = \frac{\alpha-3}{2(\alpha-1)(\alpha-2)}t^{\alpha-2}$ and $0 < c < \frac{1}{2}$, (A_4) is also satisfied. If condition (A_5) is satisfied, it follows that $m_0 < \dfrac{M\bar{r}^{1+c}-1}{\bar{r}^{c+d}}$, where $M = \frac{2^c\Gamma(\alpha-1)(3-c\alpha+2c)(\alpha-1)^c}{(\alpha-2)^{1-c}(\alpha-3)^c} > 0$. Let $g(r) = \frac{Mr^{1+c}-1}{r^{c+d}}$. Then,

$$\sup_{r>0} g(r) = \begin{cases} +\infty, & 0<d<1,\\ M, & d=1,\\ g\left(\left(\dfrac{M(d-1)}{c+d}\right)^{\frac{1}{1-c}}\right), & d>1. \end{cases}$$

Thus, the BVP (4.2) has a nonnegative solution if either $d < 1$ and $m_0 > 0$, or $d = 1$ and $m_0 < M$.

If $d > 1$, condition (A_6) becomes

$$m_0 \geq \frac{\bar{R}^{1+c}-B}{C\bar{R}^{c+d}}, \tag{4.3}$$

for some $\bar{R} > 0$, where

$$B = \int_0^1 G(1,s)ds = \frac{\alpha^2-3\alpha+2}{2\Gamma(\alpha+1)},$$

and

$$\begin{aligned} C &= \int_0^1 G(1,s)(w(s))^{c+d}ds = \left(\frac{\alpha-3}{2(\alpha-1)(\alpha-2)}\right)^{c+d}\int_0^1 G(1,s)s^{(\alpha-2)(c+d)}ds\\ &= \left(\frac{\alpha-3}{2(\alpha-1)(\alpha-2)}\right)^{c+d}\left(\frac{1}{\Gamma(\alpha-1)[(\alpha-2)(c+d)+2]}+\frac{\Gamma((\alpha-2)(c+d)+1)}{\Gamma((\alpha-2)(c+d)+1+\alpha)}\right.\\ &\quad\left.-\frac{1}{\Gamma(\alpha)[(\alpha-2)(c+d)+1]}\right). \end{aligned}$$

Note that $d > 1$, so when $\bar{R} \to +\infty$, the right-hand side of (4.3) tends to zero. Therefore, for any specified value $m_0 \in \left(0, g\left(\left(\frac{M(d-1)}{c+d}\right)^{\frac{1}{1-c}}\right)\right)$, one can always choose $\bar{R} > \bar{r}$ such that (4.3) holds. Consequently, BVP (4.2) admits an additional nonnegative solution $\bar{u}$. This indicates that (iii) holds. □

5. Conclusions

This paper systematically investigates the existence of multiple positive solutions for a class of higher-order nonlinear fractional boundary value problems with fractional boundary conditions, specifically represented by the BVP (1.3). The cornerstone of our analysis lies in the rigorous construction and comprehensive examination of the associated Green's function, which yields key inequalities essential for subsequent derivations. Capitalizing on these properties, we employ a diverse array of fixed-point techniques tailored to different scenarios. For the nonsingular case, we combine the method of upper and lower solutions, Schauder's fixed-point theorem, and cone theory to establish distinct criteria guaranteeing positive solutions. Extending the framework to the singular case, we successfully derive further existence criteria by integrating cone theory with the Leray-Schauder nonlinear alternative. Ultimately, several illustrative examples are provided to corroborate the applicability and validity of our theoretical findings. In the future, we will continue to study multi-term fractional differential equations to obtain new results.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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