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Research article

A parametric logarithmic improvement of the critical Hardy inequality and stability of the deficit

Ghaliah Alhamzi[1], **Wael Mahmoud Mohammad Salameh**[2], **Prakash Jadhav**[3] **and Mdi Begum Jeelani**[1,*]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

[2] Faculty of Information Technology, Abu Dhabi University, Abu Dhabi, United Arab Emirates

[3] Department of Mechanical Engineering, SRM University AP, Andhra Pradesh, India

* **Correspondence:** Email: mbshaikh@imamu.edu.sa.

Abstract: We address a classification problem for critical one-dimensional Hardy forms perturbed by a logarithmic remainder. On $(0,1)$, with the gauge $\log\frac{e}{x} = \log(e/x)$, we construct an explicit one-parameter family of Riccati weights that yields an identity-level ground-state representation. This produces a continuum of logarithmic improvements of the critical Hardy inequality with a computable remainder coefficient and an explicit positive ground state. We then derive a quantitative interior stability estimate: The Hardy deficit controls the distance to the associated ground state on every interior subinterval. We further classify the constant-coefficient logarithmic remainder class by reducing the associated ground-state ordinary differential equation (ODE) to a Euler equation in the logarithmic variable, and we obtain an interior compactness statement for sequences with vanishing deficit. As an application, we prove positivity and a priori bounds for a class of Dirichlet Schrödinger problems with critical singular potentials.

Keywords: Hardy inequality; logarithmic remainder; ground-state representation; Riccati equation; deficit stability; critical Schrödinger operator
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1. Introduction

Hardy-type inequalities occupy a central role in real and functional analysis because they control singular weights through derivatives and enable coercivity for variational problems with critical potentials. The classical one-dimensional inequality of Hardy provides a sharp bound of the form

$$\int_0^1 \frac{|f(x)|^2}{x^2}\,\mathrm{d}x \le 4\int_0^1 |f'(x)|^2\,\mathrm{d}x, \qquad f \in C_c^\infty(0,1),$$

and its multidimensional analogues yield the critical inverse-square potential in elliptic theory and spectral analysis; see Hardy [1], the monograph of Hardy, Littlewood, Pólya, Leibloss [2,3], Kufner [4–6], and the standard references [7–9]. Improvements of critical Hardy inequalities with lower-order positive remainders form a large literature, [10–12]. For logarithmic and optimal remainder structures in the Hilbert setting, see for instance Brezis and Marcus [13], Filippas and Tertikas [14], Barbatis, Filippas, and Tertikas [15], and Tertikas and Zographopoulos [18]. For ground-state and factorization methods, see Frank and Seiringer [16], Pinchover and Tintarev [22], and Bobkov and Götze [23]. Recent stability-oriented developments include [24–26], while Kutev and Rangelov [27] give a distinct logarithmic improvement in a higher-dimensional ball with double singular structure.

A guiding principle behind several sharp inequalities is a ground-state representation: One factors the quadratic form into a nonnegative square by solving a Riccati equation associated with a positive solution of a second-order ordinary differential equation (ODE). This paper develops a parametric version of that principle for a specific classification problem: Determine, within a constant-coefficient logarithmic remainder class, which potentials admit an exact factorization and quantify stability of the induced deficit.

We use the gauge $\log\frac{e}{x}$ rather than $\log(1/x)$ for a structural reason. The shift by e guarantees

$$\log\frac{e}{x} \ge 1 \qquad \text{for all } x \in (0,1),$$

so every logarithmic denominator remains strictly positive up to the endpoint $x = 1$, and powers of the gauge stay uniformly normalized there. The same choice makes the logarithmic variable

$$t = \log\frac{e}{x}$$

range over $(1,\infty)$ instead of $(0,\infty)$, which simplifies the Euler reduction in Section 4 and the cutoff construction used in the sharpness proof.

The contributions of the paper are as follows. We introduce a one-parameter Riccati family that produces an identity-level factorization for the critical Hardy form augmented by a logarithmic remainder. This yields a continuum of improved Hardy inequalities with an explicit remainder coefficient and an explicit positive ground state. We then prove an interior stability estimate showing that the resulting deficit controls the distance (in a concrete L^2 sense on interior subintervals) from the ground state, and we derive an interior compactness statement for sequences with vanishing deficit. We also provide a direct classification of constant-coefficient logarithmic remainders by an explicit reduction of the ground-state ODE to a Euler equation in the logarithmic variable. Finally, we apply the resulting positivity and stability to a class of Dirichlet Schrödinger problems with critical singular potentials, obtaining existence, uniqueness, and an a priori bound in the natural energy space associated with the factorized form.

Applied significance

The analytic mechanism developed here has two concrete uses. First, the exact factorization identifies the correct coercive energy for a borderline singular Schrödinger operator, which leads directly to a well-posed Dirichlet theory in the natural energy space of the problem. Second, the stability estimate shows that a small Hardy deficit forces proximity to a distinguished profile on every interior subinterval. This type of control is useful in spectral approximation, in singular perturbation arguments, and in variational problems where one needs a quantitative description of near-extremizing sequences for critical operators; compare the stability perspective in [24] and the operator-theoretic viewpoint in [11, 12].

The paper is organized as follows. Section 2 fixes notation and proves an algebraic identity underlying ground-state representations. Section 3 proves the parametric logarithmic improvement and the exact factorization identity, and it identifies the corresponding ground state. Section 4 establishes a sharp threshold for constant-coefficient logarithmic remainders and classifies the associated ground-state ODE by a logarithmic change of variables. Section 5 proves quantitative interior stability and an interior compactness statement for near-minimizers. Section 6 gives an application to a Dirichlet Schrödinger boundary value problem. Section 7 closes with concluding remarks and future directions.

2. Preliminaries and notation

Throughout, we write

$$L(x) := \log\frac{e}{x}, \qquad x \in (0,1).$$

Then, $L(x) \geq 1$ on $(0,1)$ and $L'(x) = -1/x$. We write $C_c^\infty(0,1)$ for smooth compactly supported functions on $(0,1)$. For $f \in C_c^\infty(0,1)$, all boundary terms at 0 and 1 vanish in integrations by parts.

We begin with a basic identity that we use repeatedly.

Lemma 2.1 (A one-dimensional factorization identity)**.** *Let $\psi \in C^1(0,1)$ be real-valued and let $f \in C_c^\infty(0,1)$ be real-valued. Then,*

$$\int_0^1 |f'(x)|^2\,\mathrm{d}x = \int_0^1 |f'(x) + \psi(x)f(x)|^2\,\mathrm{d}x + \int_0^1 \left(\psi'(x) - \psi(x)^2\right)|f(x)|^2\,\mathrm{d}x. \tag{2.1}$$

Proof. Expand the square:

$$|f' + \psi f|^2 = |f'|^2 + 2\psi f f' + \psi^2 f^2.$$

Integrate over $(0,1)$ and integrate by parts in the mixed term:

$$\int_0^1 2\psi f f'\,\mathrm{d}x = \int_0^1 \psi (f^2)'\,\mathrm{d}x = [\psi f^2]_0^1 - \int_0^1 \psi' f^2\,\mathrm{d}x.$$

Since f has compact support in $(0,1)$, the boundary term equals 0, hence,

$$\int_0^1 |f' + \psi f|^2\,\mathrm{d}x = \int_0^1 |f'|^2\,\mathrm{d}x - \int_0^1 \psi' f^2\,\mathrm{d}x + \int_0^1 \psi^2 f^2\,\mathrm{d}x.$$

Rearranging yields (2.1). □

Remark 2.2. Lemma 2.1 admits the same form for complex-valued f when ψ is real-valued. Indeed,

$$|f' + \psi f|^2 = |f'|^2 + 2\Re(\psi f \overline{f'}) + \psi^2|f|^2,$$

and integration by parts applies to $\Re(\psi f \overline{f'}) = \frac{1}{2}\psi(|f|^2)'$.

3. A parametric logarithmic improvement

3.1. Riccati weights and the improved potential

Definition 3.1 (Riccati family and potential)**.** Fix a parameter $a \in \mathbb{R}$. Define, for $x \in (0,1)$,

$$\psi_a(x) := -\frac{1}{2x} + \frac{a}{2x\,L(x)}, \tag{3.1}$$

and define the associated potential

$$V_a(x) := \psi_a'(x) - \psi_a(x)^2. \tag{3.2}$$

Proposition 3.2 (Explicit form of V_a)**.** *For every $a \in \mathbb{R}$ and $x \in (0,1)$,*

$$V_a(x) = \frac{1}{4x^2} + \frac{a(2-a)}{4x^2\,L(x)^2}. \tag{3.3}$$

Proof. From (3.1),

$$\psi_a(x) = -\frac{1}{2}x^{-1} + \frac{a}{2}x^{-1}L(x)^{-1}.$$

Differentiate term-by-term:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(-\tfrac{1}{2}x^{-1}\Big) = \frac{1}{2}x^{-2}.$$

For the second term, set $g(x) := x^{-1}L(x)^{-1}$. Then,

$$\begin{aligned} g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{-1})\,L^{-1} + x^{-1}\,\frac{\mathrm{d}}{\mathrm{d}x}(L^{-1}) \\ &= -x^{-2}L^{-1} + x^{-1}\big(-L^{-2}L'\big). \end{aligned}$$

Using $L' = -x^{-1}$, we obtain

$$g'(x) = -x^{-2}L^{-1} + x^{-2}L^{-2}.$$

Therefore,

$$\psi_a'(x) = \frac{1}{2}x^{-2} + \frac{a}{2}\Big(-x^{-2}L^{-1} + x^{-2}L^{-2}\Big).$$

Next, compute $\psi_a(x)^2$:

$$\psi_a(x)^2 = \left(-\frac{1}{2}x^{-1} + \frac{a}{2}x^{-1}L^{-1}\right)^2 = \frac{1}{4}x^{-2} - \frac{a}{2}x^{-2}L^{-1} + \frac{a^2}{4}x^{-2}L^{-2}.$$

Subtracting yields

$$\begin{aligned} \psi_a'(x) - \psi_a(x)^2 &= \big(\tfrac{1}{2} - \tfrac{1}{4}\big)x^{-2} + \frac{a}{2}\Big(-x^{-2}L^{-1} + x^{-2}L^{-2}\Big) + \frac{a}{2}x^{-2}L^{-1} - \frac{a^2}{4}x^{-2}L^{-2} \\ &= \frac{1}{4}x^{-2} + \Big(\frac{a}{2} - \frac{a^2}{4}\Big)x^{-2}L^{-2} = \frac{1}{4x^2} + \frac{a(2-a)}{4x^2L(x)^2}, \end{aligned}$$

which equals (3.3). $\square$

3.2. Exact factorization and the improved Hardy inequality

Theorem 3.3 (Exact ground-state representation and inequality)**.** *Fix $a \in \mathbb{R}$ and let V_a be given by* (3.3)*. For every real-valued $f \in C_c^\infty(0,1)$,*

$$\int_0^1 |f'(x)|^2\,\mathrm{d}x - \int_0^1 V_a(x)\,|f(x)|^2\,\mathrm{d}x = \int_0^1 \left|f'(x) + \psi_a(x) f(x)\right|^2 \mathrm{d}x, \tag{3.4}$$

with ψ_a given by (3.1). In particular, for $a \in [0,2]$,

$$\int_0^1 \frac{|f(x)|^2}{x^2}\,\mathrm{d}x + \; a(2-a)\int_0^1 \frac{|f(x)|^2}{x^2\,L(x)^2}\,\mathrm{d}x \le 4\int_0^1 |f'(x)|^2\,\mathrm{d}x. \tag{3.5}$$

Proof. Lemma 2.1 with $\psi = \psi_a$ yields

$$\int_0^1 |f'|^2\,\mathrm{d}x = \int_0^1 |f' + \psi_a f|^2\,\mathrm{d}x + \int_0^1 (\psi_a' - \psi_a^2)\,|f|^2\,\mathrm{d}x.$$

By Definition 3.1, $\psi_a' - \psi_a^2 = V_a$, hence (3.4) follows.

Assume now $a \in [0,2]$. Proposition 3.2 yields

$$V_a(x) = \frac{1}{4x^2} + \frac{a(2-a)}{4x^2\,L(x)^2},$$

and (3.4) gives

$$\int_0^1 |f'|^2\,\mathrm{d}x \ge \int_0^1 \left(\frac{1}{4x^2} + \frac{a(2-a)}{4x^2\,L(x)^2}\right)|f|^2\,\mathrm{d}x.$$

Multiplying by 4 and rearranging yields (3.5). □

Definition 3.4 (Ground state)**.** For $a \in \mathbb{R}$, define

$$u_a(x) := x^{1/2}\,L(x)^{a/2}, \qquad x \in (0,1). \tag{3.6}$$

Proposition 3.5 (Ground-state form of the deficit)**.** *Fix $a \in \mathbb{R}$. For every real-valued $f \in C_c^\infty(0,1)$,*

$$\int_0^1 |f'(x)|^2\,\mathrm{d}x - \int_0^1 V_a(x)|f(x)|^2\,\mathrm{d}x = \int_0^1 u_a(x)^2 \left|\left(\frac{f(x)}{u_a(x)}\right)'\right|^2 \mathrm{d}x. \tag{3.7}$$

Proof. From (3.6),

$$\log u_a(x) = \frac{1}{2}\log x + \frac{a}{2}\log L(x).$$

Differentiate and use $L'(x) = -1/x$:

$$\begin{aligned} \frac{u_a'(x)}{u_a(x)} &= \frac{1}{2x} + \frac{a}{2}\,\frac{L'(x)}{L(x)} \\ &= \frac{1}{2x} - \frac{a}{2x\,L(x)}. \end{aligned}$$

Comparing with (3.1), we obtain the exact identity

$$\psi_a(x) = -\frac{u_a'(x)}{u_a(x)}. \tag{3.8}$$

Define $g := f/u_a$, so $f = u_a g$ and $f' = u_a' g + u_a g'$. Using (3.8), compute

$$\begin{aligned} f'(x) + \psi_a(x) f(x) &= u_a'(x) g(x) + u_a(x) g'(x) + \psi_a(x) u_a(x) g(x) \\ &= u_a(x) g'(x) + (u_a'(x) + \psi_a(x) u_a(x)) g(x) \\ &= u_a(x) g'(x). \end{aligned}$$

Insert this identity into (3.4):

$$\int_0^1 |f'|^2 \, \mathrm{d}x - \int_0^1 V_a |f|^2 \, \mathrm{d}x = \int_0^1 |u_a g'|^2 \, \mathrm{d}x = \int_0^1 u_a^2 |g'|^2 \, \mathrm{d}x,$$

which equals (3.7). □

Remark 3.6. The identity (3.7) places the present logarithmic improvement in the same Riccati/ground-state framework used in [16, 23]. In the current borderline one-dimensional setting, the explicit profile u_a makes the factorization completely elementary and yields a direct route to the threshold and stability statements below.

Theorem 3.7 (ODE for the ground state and kernel characterization)**.** *Fix $a \in \mathbb{R}$, let V_a be given by* (3.3)*, and let u_a be given by* (3.6)*. Then $u_a \in C^2(0, 1)$ and*

$$-u_a''(x) = V_a(x)\, u_a(x) \qquad \text{for all } x \in (0, 1). \tag{3.9}$$

Moreover, for every $f \in C_c^\infty(0, 1)$,

$$\int_0^1 u_a(x)^2 \left| \left(\frac{f(x)}{u_a(x)} \right)' \right|^2 \mathrm{d}x = 0 \quad \Longrightarrow \quad f \equiv 0. \tag{3.10}$$

Proof. By (3.8), one has $u_a' = -\psi_a u_a$, hence,

$$u_a'' = -(\psi_a' u_a + \psi_a u_a').$$

Use again $u_a' = -\psi_a u_a$ to obtain

$$u_a'' = -(\psi_a' u_a + \psi_a(-\psi_a u_a)) = -(\psi_a' - \psi_a^2) u_a = -V_a u_a,$$

which yields (3.9).

For (3.10), assume the left-hand side equals 0. Since $u_a > 0$ on $(0, 1)$, the integrand vanishes almost everywhere, hence

$$\left(\frac{f}{u_a} \right)' = 0 \quad \text{a.e. on } (0, 1).$$

Therefore, f/u_a equals a constant C on $(0, 1)$, so $f(x) = C\, u_a(x)$ for all $x \in (0, 1)$. If $C \neq 0$, then f has support $(0, 1)$ because $u_a > 0$ on $(0, 1)$, which contradicts compact support in $(0, 1)$. Hence, $C = 0$ and $f \equiv 0$. □

4. A threshold for constant logarithmic remainders

Theorem 3.3 yields a family of remainder coefficients through the identity $a(2-a) \in [0,1]$ for $a \in [0,2]$. We now establish a sharp threshold for constant-coefficient logarithmic remainders within the quadratic-form framework on $C_c^\infty(0,1)$, and we classify the corresponding ground-state ODE by an explicit logarithmic change of variables. To streamline the notation in this section, for $c \in \mathbb{R}$, we set

$$W_c(x) := \frac{1}{4x^2} + \frac{c}{4x^2\, L(x)^2}, \qquad x \in (0,1). \tag{4.1}$$

Lemma 4.1 (Logarithmic change of variables)**.** *Let $t = L(x) = \log\frac{e}{x}$, so $t \in (1,\infty)$ and $\frac{\mathrm{d}}{\mathrm{d}x} = -\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}t}$. Let $v \in C^2(1,\infty)$ and define $u(x) := x^{1/2}v(L(x))$ for $x \in (0,1)$. Then,*

$$u''(x) = x^{-3/2}\left(v''(t) - \frac{1}{4}v(t)\right)\Big|_{t=L(x)}. \tag{4.2}$$

Proof. Set $t = L(x)$ and write $u(x) = x^{1/2}v(t)$. Differentiate once:

$$u'(x) = \frac{1}{2}x^{-1/2}v(t) + x^{1/2}v'(t)\,\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}x^{-1/2}v(t) - x^{-1/2}v'(t) = x^{-1/2}\left(\frac{1}{2}v(t) - v'(t)\right).$$

Define $w(t) := \frac{1}{2}v(t) - v'(t)$, so $u'(x) = x^{-1/2}w(t)$. Differentiate again:

$$\begin{aligned} u''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{-1/2})\,w(t) + x^{-1/2}\,\frac{\mathrm{d}}{\mathrm{d}x}w(t) \\ &= -\frac{1}{2}x^{-3/2}w(t) + x^{-1/2}w'(t)\,\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= -\frac{1}{2}x^{-3/2}w(t) - x^{-3/2}w'(t) \\ &= x^{-3/2}\left(-\frac{1}{2}w(t) - w'(t)\right). \end{aligned}$$

Compute $w'(t) = \frac{1}{2}v'(t) - v''(t)$. Therefore,

$$-\frac{1}{2}w(t) - w'(t) = -\frac{1}{2}\left(\frac{1}{2}v(t) - v'(t)\right) - \left(\frac{1}{2}v'(t) - v''(t)\right) = -\frac{1}{4}v(t) + v''(t),$$

which yields (4.2). □

Remark 4.2. Because $t = L(x)$ maps $(0,1)$ onto $(1,\infty)$, every integral obtained after the logarithmic substitution starts at 1. We keep this lower limit explicitly in all subsequent estimates, especially in (4.8) and (4.9), so that the domain shift is transparent throughout the sharpness argument.

Theorem 4.3 (Classification of the constant-coefficient ODE)**.** *Let $c \in \mathbb{R}$, and let W_c be given by* (4.1)*. Let $v \in C^2(1,\infty)$ and define $u(x) := x^{1/2}v(L(x))$. Then,*

$$-u''(x) = W_c(x)\,u(x) \qquad \textit{for all } x \in (0,1) \tag{4.3}$$

holds if and only if

$$v''(t) + \frac{c}{4t^2} v(t) = 0 \qquad \text{for all } t \in (1, \infty). \tag{4.4}$$

If $c \leq 1$, set

$$\alpha_\pm := \frac{1 \pm \sqrt{1 - c}}{2}.$$

Then, the general real-valued solution of (4.4) has the form

$$v(t) = A\, t^{\alpha_+} + B\, t^{\alpha_-} \qquad (A, B \in \mathbb{R}),$$

and the general real-valued solution of (4.3) has the form

$$u(x) = x^{1/2}\Big(A\, L(x)^{\alpha_+} + B\, L(x)^{\alpha_-}\Big) \qquad (A, B \in \mathbb{R}).$$

Proof. Assume (4.3). By Lemma 4.1,

$$-u''(x) = -x^{-3/2}\left(v''(t) - \frac{1}{4} v(t)\right)\Big|_{t=L(x)} = x^{-3/2}\left(-v''(t) + \frac{1}{4} v(t)\right)\Big|_{t=L(x)}.$$

On the other hand,

$$W_c(x)\, u(x) = \left(\frac{1}{4x^2} + \frac{c}{4x^2 t^2}\right) x^{1/2} v(t) = x^{-3/2}\left(\frac{1}{4} + \frac{c}{4t^2}\right) v(t).$$

Equating these expressions yields

$$-v''(t) + \frac{1}{4} v(t) = \left(\frac{1}{4} + \frac{c}{4t^2}\right) v(t),$$

hence (4.4). The converse implication follows by reversing the computation.

Assume now $c \leq 1$. Seek power solutions $v(t) = t^\alpha$ with $\alpha \in \mathbb{R}$. Then

$$v'(t) = \alpha t^{\alpha - 1}, \qquad v''(t) = \alpha(\alpha - 1) t^{\alpha - 2}.$$

Insert into (4.4):

$$\alpha(\alpha - 1) t^{\alpha - 2} + \frac{c}{4t^2} t^\alpha = \left(\alpha(\alpha - 1) + \frac{c}{4}\right) t^{\alpha - 2} = 0.$$

Thus, α satisfies the quadratic equation

$$\alpha^2 - \alpha + \frac{c}{4} = 0.$$

Its discriminant equals $1 - c \geq 0$, and the two real roots are precisely $\alpha_\pm = \frac{1 \pm \sqrt{1-c}}{2}$. Since (4.4) is a linear second-order ODE with continuous coefficients on $(1, \infty)$, the general real solution is

$$v(t) = A t^{\alpha_+} + B t^{\alpha_-} \qquad (A, B \in \mathbb{R}).$$

Substituting $t = L(x)$ yields the stated representation for u. □

Theorem 4.4 (Sharp threshold for constant logarithmic remainders)**.** *Let $c \in \mathbb{R}$ and let W_c be given by* (4.1)*. Then the following statements hold.*

(i) If $c \le 1$, then for every real-valued $f \in C_c^\infty(0,1)$,

$$\int_0^1 W_c(x)\,|f(x)|^2\,\mathrm{d}x \le \int_0^1 |f'(x)|^2\,\mathrm{d}x. \tag{4.5}$$

(ii) If $c > 1$, then there exists a real-valued $f \in C_c^\infty(0,1)$ such that

$$\int_0^1 |f'(x)|^2\,\mathrm{d}x - \int_0^1 W_c(x)\,|f(x)|^2\,\mathrm{d}x < 0. \tag{4.6}$$

Proof. Part (i). Assume $c \le 1$.

If $c \le 0$, then $W_c(x) \le W_0(x) = \frac{1}{4x^2}$ for all $x \in (0,1)$. Apply (3.4) with $a = 0$, for which $V_0 = W_0$, to obtain

$$\int_0^1 W_0(x)\,|f(x)|^2\,\mathrm{d}x \le \int_0^1 |f'(x)|^2\,\mathrm{d}x.$$

Pointwise domination $W_c \le W_0$ yields (4.5).

Assume $0 < c \le 1$. Choose $a \in [0,2]$ solving $a(2-a) = c$. This is possible because the quadratic equation $a^2 - 2a + c = 0$ has discriminant $4(1-c) \ge 0$ and solutions

$$a = 1 \pm \sqrt{1-c},$$

both of which lie in $[0,2]$ for $c \in [0,1]$. For this a, Proposition 3.2 gives $V_a = W_c$. Then, (4.5) follows from (3.4) since the right-hand side of (3.4) is an integral of a square.

Part (ii). Assume $c > 1$ and write $c = 1 + \gamma$ with $\gamma > 0$. Use the case $a = 1$ from Theorem 3.3, where

$$V_1(x) = \frac{1}{4x^2} + \frac{1}{4x^2\,L(x)^2} = W_1(x), \qquad u_1(x) = x^{1/2}L(x)^{1/2}.$$

Define the logarithmic variable $t = L(x)$, so $t \in (1,\infty)$ and

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{x}, \qquad \frac{\mathrm{d}x}{x} = -\,\mathrm{d}t.$$

Fix a smooth function $\eta \in C^\infty(\mathbb{R})$ satisfying

$$0 \le \eta \le 1, \qquad \eta(s) = 0 \text{ for } s \le 0, \qquad \eta(s) = 1 \text{ for } s \ge 1, \qquad |\eta'(s)| \le 2 \text{ for all } s.$$

For parameters $R > 1$ and $T > 2R$, define a cutoff on $(1,\infty)$ by

$$\chi_{R,T}(t) := \eta\Big(\frac{t-R}{R}\Big)\Big(1 - \eta\Big(\frac{t-T}{T}\Big)\Big).$$

Then, $\chi_{R,T} \in C_c^\infty(1,\infty)$, and

$$\chi_{R,T}(t) = 0 \text{ for } t \le R, \qquad \chi_{R,T}(t) = 1 \text{ for } t \in [2R,T], \qquad \chi_{R,T}(t) = 0 \text{ for } t \ge 2T.$$

Define

$$g_{R,T}(t) := \chi_{R,T}(t), \qquad f_{R,T}(x) := u_1(x)\, g_{R,T}(L(x))\,.$$

Since $\operatorname{supp}\chi_{R,T} \subset [R, 2T]$, one has

$$\operatorname{supp} f_{R,T} \subset \{x \in (0,1) : R \le L(x) \le 2T\} = [e^{1-2T}, e^{1-R}],$$

which is a compact subinterval of $(0, 1)$. Because u_1 is smooth on $(0, 1)$ and $x \mapsto L(x)$ is smooth there, the composition $x \mapsto g_{R,T}(L(x))$ is smooth on $(0, 1)$, hence, $f_{R,T} \in C_c^\infty(0, 1)$.

Introduce

$$I_{R,T} := \int_0^1 \frac{|f_{R,T}(x)|^2}{x^2\, L(x)^2}\, \mathrm{d}x, \qquad D_{R,T} := \int_0^1 |f'_{R,T}(x)|^2\, \mathrm{d}x - \int_0^1 W_1(x)|f_{R,T}(x)|^2\, \mathrm{d}x.$$

By Proposition 3.5 with $a = 1$,

$$D_{R,T} = \int_0^1 u_1(x)^2 \left|\left(\frac{f_{R,T}(x)}{u_1(x)}\right)'\right|^2 \mathrm{d}x = \int_0^1 u_1(x)^2 \left|\big(g_{R,T}(L(x))\big)'\right|^2 \mathrm{d}x. \tag{4.7}$$

Compute the derivative by the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(g_{R,T}(L(x))\big) = g'_{R,T}(L(x))\, L'(x) = -\frac{1}{x}\, g'_{R,T}(L(x)).$$

Using $u_1(x)^2 = x\, L(x)$, insert into (4.7):

$$D_{R,T} = \int_0^1 x\, L(x) \cdot \frac{1}{x^2}\, |g'_{R,T}(L(x))|^2\, \mathrm{d}x = \int_0^1 \frac{L(x)}{x}\, |g'_{R,T}(L(x))|^2\, \mathrm{d}x.$$

Change variables $t = L(x)$ with $\frac{\mathrm{d}x}{x} = -\,\mathrm{d}t$ to obtain

$$D_{R,T} = \int_1^\infty t\, |g'_{R,T}(t)|^2\, \mathrm{d}t. \tag{4.8}$$

Next compute $I_{R,T}$. Since $f_{R,T} = u_1 g_{R,T}(L(x))$ and $u_1^2 = x\, L(x)$,

$$\frac{|f_{R,T}(x)|^2}{x^2 L(x)^2} = \frac{x\, L(x)\, |g_{R,T}(L(x))|^2}{x^2 L(x)^2} = \frac{1}{x}\, L(x)^{-1}\, |g_{R,T}(L(x))|^2.$$

Changing variables $t = L(x)$ yields

$$I_{R,T} = \int_1^\infty t^{-1}\, |g_{R,T}(t)|^2\, \mathrm{d}t. \tag{4.9}$$

Since $g_{R,T}(t) = 1$ on $[2R, T]$, one has the lower bound

$$I_{R,T} \ge \int_{2R}^T \frac{1}{t}\, \mathrm{d}t = \log\!\left(\frac{T}{2R}\right). \tag{4.10}$$

Differentiate $\chi_{R,T}$:

$$\chi'_{R,T}(t) = \frac{1}{R}\eta'\Big(\frac{t-R}{R}\Big)\Big(1-\eta\Big(\frac{t-T}{T}\Big)\Big) - \frac{1}{T}\eta\Big(\frac{t-R}{R}\Big)\eta'\Big(\frac{t-T}{T}\Big).$$

Use $(\alpha+\beta)^2 \le 2\alpha^2 + 2\beta^2$ in (4.8) and the bounds $|\eta'| \le 2$, $0 \le \eta \le 1$:

$$\begin{aligned} D_{R,T} &= \int_1^\infty t\,|\chi'_{R,T}(t)|^2\,\mathrm{d}t \\ &\le 2\int_1^\infty t\left|\frac{1}{R}\eta'\Big(\frac{t-R}{R}\Big)\Big(1-\eta\Big(\frac{t-T}{T}\Big)\Big)\right|^2 \mathrm{d}t + 2\int_1^\infty t\left|\frac{1}{T}\eta\Big(\frac{t-R}{R}\Big)\eta'\Big(\frac{t-T}{T}\Big)\right|^2 \mathrm{d}t \\ &\le \frac{8}{R^2}\int_R^{2R} t\,\mathrm{d}t + \frac{8}{T^2}\int_T^{2T} t\,\mathrm{d}t = 24. \end{aligned}$$

Hence,

$$D_{R,T} \le 24. \tag{4.11}$$

Now, consider the quadratic form associated with $W_c = W_1 + \frac{\gamma}{4x^2L(x)^2}$:

$$Q_c[f] := \int_0^1 |f'|^2\,\mathrm{d}x - \int_0^1 W_c|f|^2\,\mathrm{d}x = \Big(\int_0^1 |f'|^2\,\mathrm{d}x - \int_0^1 W_1|f|^2\,\mathrm{d}x\Big) - \frac{\gamma}{4}\int_0^1 \frac{|f|^2}{x^2L(x)^2}\,\mathrm{d}x.$$

For $f = f_{R,T}$, this becomes

$$Q_c[f_{R,T}] = D_{R,T} - \frac{\gamma}{4}I_{R,T}.$$

Combine (4.10) and (4.11):

$$Q_c[f_{R,T}] \le 24 - \frac{\gamma}{4}\log\Big(\frac{T}{2R}\Big).$$

Choose $R > 1$ fixed and then choose $T > 2R$ large enough so that

$$\frac{\gamma}{4}\log\Big(\frac{T}{2R}\Big) > 24.$$

This yields $Q_c[f_{R,T}] < 0$, proving (4.6). □

5. Stability of the deficit

We now show that the deficit in (3.7) controls the distance to the ground state u_a on every interior subinterval.

Definition 5.1 (Deficit functional)**.** Fix $a \in [0,2]$. For real-valued $f \in C_c^\infty(0,1)$, define

$$\mathcal{D}_a[f] := \int_0^1 |f'(x)|^2\,\mathrm{d}x - \int_0^1 V_a(x)\,|f(x)|^2\,\mathrm{d}x. \tag{5.1}$$

Theorem 5.2 (Quantitative stability on interior subintervals)**.** *Fix $a \in [0,2]$ and $\delta \in (0,1)$. Let u_a be given by* (3.6)*. Then for every real-valued $f \in C_c^\infty(0,1)$, there exists a constant $c_\delta(f) \in \mathbb{R}$, for example,*

$$c_\delta(f) := \frac{1}{1-\delta}\int_\delta^1 \frac{f(s)}{u_a(s)}\,\mathrm{d}s, \tag{5.2}$$

such that

$$\int_\delta^1 |f(x) - c_\delta(f)\, u_a(x)|^2 \, \mathrm{d}x \le (1-\delta)^2 \, \frac{\sup_{x\in[\delta,1]} u_a(x)^2}{\inf_{x\in[\delta,1]} u_a(x)^2} \, \mathcal{D}_a[f]. \tag{5.3}$$

Moreover, the ratio factor admits the explicit bound

$$\frac{\sup_{x\in[\delta,1]} u_a(x)^2}{\inf_{x\in[\delta,1]} u_a(x)^2} \le \frac{1}{\delta}\, L(\delta)^a. \tag{5.4}$$

Proof. Let $g := f/u_a$, so $f = u_a g$. Proposition 3.5 gives

$$\mathcal{D}_a[f] = \int_0^1 u_a(x)^2 |g'(x)|^2 \, \mathrm{d}x. \tag{5.5}$$

On $[\delta, 1]$, the function u_a is continuous and strictly positive, hence,

$$m_\delta := \inf_{x\in[\delta,1]} u_a(x)^2 > 0, \qquad M_\delta := \sup_{x\in[\delta,1]} u_a(x)^2 < \infty.$$

Set $\bar{g}_\delta := \frac{1}{1-\delta}\int_\delta^1 g(s)\, \mathrm{d}s$, so $c_\delta(f) = \bar{g}_\delta$ as in (5.2). Then,

$$f(x) - c_\delta(f)u_a(x) = u_a(x)(g(x) - \bar{g}_\delta),$$

hence,

$$\int_\delta^1 |f - c_\delta(f)u_a|^2 \, \mathrm{d}x = \int_\delta^1 u_a(x)^2 \, |g(x) - \bar{g}_\delta|^2 \, \mathrm{d}x \le M_\delta \int_\delta^1 |g(x) - \bar{g}_\delta|^2 \, \mathrm{d}x. \tag{5.6}$$

We now use the one-dimensional Poincaré inequality on the interval $[\delta, 1]$ with the explicit length factor $(1-\delta)^2$. For completeness, we record the elementary argument. For any $x \in [\delta, 1]$,

$$g(x) - \bar{g}_\delta = \frac{1}{1-\delta}\int_\delta^1 \big(g(x) - g(y)\big)\, \mathrm{d}y = \frac{1}{1-\delta}\int_\delta^1 \left(\int_y^x g'(s)\, \mathrm{d}s\right) \mathrm{d}y.$$

Therefore,

$$|g(x) - \bar{g}_\delta| \le \int_\delta^1 |g'(s)|\, \mathrm{d}s \le (1-\delta)^{1/2} \left(\int_\delta^1 |g'(s)|^2 \, \mathrm{d}s\right)^{1/2}.$$

Square and integrate over $x \in [\delta, 1]$ to obtain

$$\int_\delta^1 |g(x) - \bar{g}_\delta|^2 \, \mathrm{d}x \le (1-\delta)^2 \int_\delta^1 |g'(s)|^2 \, \mathrm{d}s.$$

Insert this bound into (5.6):

$$\int_\delta^1 |f - c_\delta(f)u_a|^2 \, \mathrm{d}x \le M_\delta (1-\delta)^2 \int_\delta^1 |g'(s)|^2 \, \mathrm{d}s.$$

Use $m_\delta \le u_a^2$ on $[\delta, 1]$ to compare $\int_\delta^1 |g'|^2$ with (5.5):

$$\int_\delta^1 |g'(s)|^2 \, \mathrm{d}s \le \frac{1}{m_\delta}\int_\delta^1 u_a(s)^2 |g'(s)|^2 \, \mathrm{d}s \le \frac{1}{m_\delta}\mathcal{D}_a[f].$$

Combining the last two estimates yields (5.3).

Finally, since $u_a^2 = x\,L(x)^a$, on $x \in [\delta, 1]$, one has

$$\delta \le x \le 1, \qquad 1 \le L(x) \le L(\delta).$$

Because $a \in [0, 2]$, it follows that

$$u_a(x)^2 = x\,L(x)^a \le L(\delta)^a, \qquad u_a(x)^2 = x\,L(x)^a \ge \delta.$$

Therefore, $M_\delta/m_\delta \le \delta^{-1}L(\delta)^a$, which is (5.4). $\square$

Theorem 5.3 (Interior compactness for vanishing deficit)**.** *Fix $a \in [0, 2]$ and $\delta \in (0, 1)$. Let $(f_n)_{n\ge1} \subset C_c^\infty(0, 1)$ satisfy*

$$\mathcal{D}_a[f_n] \to 0, \qquad \int_\delta^1 |f_n(x)|^2\,\mathrm{d}x = 1 \quad \textit{for all } n.$$

Then, there exists a sequence of constants $(c_n)_{n\ge1} \subset \mathbb{R}$ such that

$$\int_\delta^1 |f_n(x) - c_n u_a(x)|^2\,\mathrm{d}x \to 0.$$

Moreover, $(c_n)_{n\ge1}$ is bounded, and every convergent subsequence $c_{n_k} \to c$ satisfies

$$\int_\delta^1 \left|f_{n_k}(x) - c\,u_a(x)\right|^2\,\mathrm{d}x \to 0.$$

Proof. Apply Theorem 5.2 to each f_n and define $c_n := c_\delta(f_n)$, so that

$$\int_\delta^1 |f_n - c_n u_a|^2\,\mathrm{d}x \le C_{\delta,a}\,\mathcal{D}_a[f_n], \qquad C_{\delta,a} := (1-\delta)^2\,\frac{\sup_{[\delta,1]} u_a^2}{\inf_{[\delta,1]} u_a^2}. \tag{5.7}$$

Since $\mathcal{D}_a[f_n] \to 0$, the right-hand side of (5.7) tends to 0, hence

$$\int_\delta^1 |f_n - c_n u_a|^2\,\mathrm{d}x \to 0.$$

To bound (c_n), use the triangle inequality in $L^2(\delta, 1)$:

$$|c_n|\,\|u_a\|_{L^2(\delta,1)} \le \|c_n u_a - f_n\|_{L^2(\delta,1)} + \|f_n\|_{L^2(\delta,1)} = \|c_n u_a - f_n\|_{L^2(\delta,1)} + 1.$$

Since the first term tends to 0, the sequence (c_n) is bounded.

Finally, let $c_{n_k} \to c$. Then,

$$\|f_{n_k} - c u_a\|_{L^2(\delta,1)} \le \|f_{n_k} - c_{n_k} u_a\|_{L^2(\delta,1)} + |c_{n_k} - c|\,\|u_a\|_{L^2(\delta,1)}.$$

The first term tends to 0 by (5.7), and the second tends to 0 since $c_{n_k} \to c$. Hence, $f_{n_k} \to c u_a$ in $L^2(\delta, 1)$. $\square$

6. Application to a critical Dirichlet Schrödinger problem

We illustrate how the factorization and stability yield a well-posedness statement for a Dirichlet boundary value problem with critical singularities.

Definition 6.1 (Energy space)**.** Fix $a \in [0,2]$. Define $\mathcal{H}_a$ as the completion of $C_c^\infty(0,1)$ under the norm

$$\|f\|_{\mathcal{H}_a} := (\mathcal{D}_a[f])^{1/2},$$

where $\mathcal{D}_a$ is given by (5.1).

Remark 6.2*.* The norm of $\mathcal{H}_a$ is adapted to the critical operator rather than to the plain Sobolev energy. On the core space $C_c^\infty(0,1)$, Hardy's inequality implies

$$\|f\|_{\mathcal{H}_a}^2 = \mathcal{D}_a[f] \le \int_0^1 |f'(x)|^2 \,\mathrm{d}x,$$

so the $H_0^1(0,1)$ energy continuously dominates the present norm. In addition, the ground-state representation (3.7) shows that $\mathcal{H}_a$ measures a weighted H^1-norm of f/u_a. Hence, away from the singular endpoint $x=0$, the space behaves like a standard Sobolev space, while globally it retains the critical singular structure needed for the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - V_a$.

Theorem 6.3 (Dirichlet problem with a critical potential)**.** *Fix $a \in [0,2]$ and let V_a be given by* (3.3)*. For each continuous linear functional $\ell \in \mathcal{H}_a^*$, there exists a unique $u \in \mathcal{H}_a$ such that*

$$\int_0^1 u'(x)\varphi'(x)\,\mathrm{d}x - \int_0^1 V_a(x)\,u(x)\varphi(x)\,\mathrm{d}x = \ell(\varphi) \qquad \textit{for all } \varphi \in C_c^\infty(0,1). \tag{6.1}$$

Moreover,

$$\|u\|_{\mathcal{H}_a} \le \|\ell\|_{\mathcal{H}_a^*}. \tag{6.2}$$

Proof. Define the bilinear form on $C_c^\infty(0,1)$,

$$B_a(\phi,\varphi) := \int_0^1 \phi'(x)\varphi'(x)\,\mathrm{d}x - \int_0^1 V_a(x)\,\phi(x)\varphi(x)\,\mathrm{d}x.$$

By Theorem 3.3, $B_a(\varphi,\varphi) = \mathcal{D}_a[\varphi] \ge 0$. Proposition 3.5 yields the representation

$$\mathcal{D}_a[\varphi] = \int_0^1 u_a(x)^2 \left| \left(\frac{\varphi(x)}{u_a(x)} \right)' \right|^2 \mathrm{d}x,$$

which is quadratic in φ and defines an inner product via polarization. In particular, B_a is symmetric.

By Cauchy–Schwarz in the Hilbert space induced by $\mathcal{D}_a$,

$$|B_a(\phi,\varphi)| \le \mathcal{D}_a[\phi]^{1/2}\,\mathcal{D}_a[\varphi]^{1/2} = \|\phi\|_{\mathcal{H}_a}\,\|\varphi\|_{\mathcal{H}_a}.$$

Hence, B_a extends uniquely to a continuous bilinear form on $\mathcal{H}_a \times \mathcal{H}_a$, still denoted B_a. Moreover,

$$B_a(\phi,\phi) = \|\phi\|_{\mathcal{H}_a}^2 \qquad \text{for all } \phi \in \mathcal{H}_a,$$

so the form is coercive with coercivity constant 1 relative to $\|\cdot\|_{\mathcal{H}_a}$. The Lax–Milgram theorem therefore yields, for every $\ell \in \mathcal{H}_a^*$, a unique $u \in \mathcal{H}_a$ such that

$$B_a(u, \varphi) = \ell(\varphi) \qquad \text{for all } \varphi \in \mathcal{H}_a.$$

Restricting test functions to $C_c^\infty(0, 1)$ gives (6.1). Taking $\varphi = u$ yields

$$\|u\|_{\mathcal{H}_a}^2 = B_a(u, u) = \ell(u) \le \|\ell\|_{\mathcal{H}_a^*} \|u\|_{\mathcal{H}_a},$$

which implies (6.2). □

7. Conclusion and future directions

This paper developed a parametric Riccati framework that yields an exact factorization for the critical Hardy form on (0, 1) augmented by a constant-coefficient logarithmic remainder. The resulting family of improved inequalities comes with an explicit ground state and a ground-state representation of the deficit. The stability theorem shows that small deficit forces proximity to the ground state on any interior subinterval, and the compactness theorem provides a structural description of sequences with vanishing deficit. The threshold theorem identifies a sharp boundary in the constant-coefficient logarithmic remainder class, complemented by an explicit reduction of the ground-state ODE to an Euler equation in the logarithmic variable.

From the applied viewpoint, the main output is a coercive and explicitly factorized energy for a critical singular Schrödinger operator together with a quantitative rigidity principle for near-minimizers. The factorized form isolates the correct singular profile, while the interior stability estimate turns small defect into a concrete approximation statement. This combination is useful in spectral problems, in borderline variational models, and in perturbative settings where the singular profile governs the effective behavior.

Several directions invite further development. One direction concerns nonlinear energies with $p \neq 2$, where Picone-type identities and nonlinear ground-state representations play a key role. Another direction concerns extensions to higher-dimensional domains with boundary geometry, where logarithmic gauges interact with distance-to-boundary functions and curvature terms. A third direction concerns fractional and nonlocal operators, where borderline embeddings generate logarithmic remainders and stability questions connect with spectral estimates and heat kernel bounds. A refined stability theory that approaches the singular endpoint suggests distances built from the ground-state variable f/u_a in weighted Sobolev scales, with potential connections to quantitative spectral theory for critical Schrödinger operators. At a methodological level, it is also natural to look for structural reformulations of the present factorization scheme in settings with additional algebraic state variables or finite-dimensional couplings; compare, for a broad survey of algebraic state-space reorganizations in another context, Yan, Cheng, Feng, Li, and Yue [28].

Author contributions

Ghaliah Alhamzi: Conceptualization, Methodology, Formal analysis, Writing–original draft preparation; Wael Mahmoud Mohammad Salameh: Investigation, Methodology, Formal analysis, Writing–review and editing; Prakash Jadhav: Software, Validation, Visualization, Writing–review and editing; Mdi Begum Jeelani: Supervision, Project administration, Funding acquisition, Conceptualization, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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