AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 13913–13936.
DOI: 10.3934/math.2026572
Received: 23 February 2026
Revised: 11 April 2026
Accepted: 28 April 2026
Published: 18 May 2026

Research article

Multiplicity of solutions to (p, q)-Kirchhoff equations with weight functions via local Palais-Smale condition

Jiaqing Hu[1] **and QiangQiang Yang**[2,*]

[1] School of Mathematics and Statistics, Huaibei Normal University, Huaibei 235000, China
[2] School of Computer Science and Technology, Huaibei Normal University, Huaibei 235000, China

* **Correspondence:** Email: yqq2512741373@163.com.

Abstract: In this paper, we study the following (p, q)-Kirchhoff equation with weight functions:

$$\begin{cases} \sum_{k\in\{p,q\}} \left(-\left(1 + \int_{\mathbb{R}^N} |\nabla u|^k dx\right) \Delta_k u + |u|^{k-2}u \right) = \lambda h(x)|u|^{r-2}u + g(x)|u|^{s-2}u, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases}$$

where $1 < q \leq p$, $\{r, s\} \subset (2p, p*)$, and $p < N < 2p$. By applying a critical point theorem following Perera (J. Anal. Math., 2025), there exists $\Lambda_m \geq 0$ such that the equation has at least $m \in \mathbb{N}$ pairs of nontrivial solutions for every $\lambda > \Lambda_m$. Particularly, we need only prove that the corresponding energy functional satisfies the local Palais-Smale condition, and an explicit expression for Λ_m is given, which generalizes some results in the existing literature and provides a new perspective for searching for multiple solutions of Kirchhoff-type equations.

Keywords: (p, q)-Kirchhoff equation; multiple solutions; variational method and critical point theorem; local Palais-Smale condition
Mathematics Subject Classification: 35J60, 35J20, 35B38

---

1. Introduction

In this paper, we study the (p, q)-Kirchhoff equation with weight functions:

$$\begin{cases} \sum_{k\in\{p,q\}} \left(-\left(1 + \int_{\mathbb{R}^N} |\nabla u|^k dx\right) \Delta_k u + |u|^{k-2}u \right) = \lambda h(x)|u|^{r-2}u + g(x)|u|^{s-2}u, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $1 < q \leq p$, $\{r, s\} \subset (2p, p*)$, $p < N < 2p$, $p* = \frac{Np}{N-p}$, $p < N < 2p$, and weight functions $h, g \in C(\mathbb{R}^N, \mathbb{R})$.

When $p = q = 2$ and $N = 3$, problem (1.1) becomes a Kirchhoff equation of the form

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x, u), \tag{1.2}$$

where $a, b > 0$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, which is closely related to the stationary analog of the equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u),$$

which was proposed by Kirchhoff in 1883 as an extension of the classical D'Alembert's wave equations for the free vibration of elastic strings. Compared to the semi-linear equations (i.e., the integral term disappears), it is much more challenging and interesting to study Eq (1.2) because the appearance of nonlocal term $\int_{\Omega} |\nabla u|^2 dx$ implies that the equation is no longer a pointwise identity. After the pioneering work of Lions [24], Kirchhoff-type problems have attracted considerable attention. For example, the existence, uniqueness and multiplicity of solutions were studied in [1, 16, 20], [4], and [18, 19], respectively, while [27, 36] investigated sign-changing solutions for such problems.

When the integral terms disappears, (1.1) becomes a (p, q)-Laplacian equation of the form

$$\begin{cases} \sum\limits_{k\in\{p,q\}} \left(-\Delta_k u + |u|^{k-2}u\right) = f(x, u), \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{1.3}$$

which comes from finding a stationary solution of the general reaction-diffusion equations and has a wide range of applications [8, 37]. Starting with Ball [8], such (p, q)-Laplace equations have attracted great interest in the past few decades, and many important results were established.

To be more explicit, some results on solutions to Eq (1.3) are summarized in Table 1, where the A-R condition represents the Ambroset-Rabinowitz condition. In addition, one may refer to [23, 34, 35] for the existence of nontrivial solutions to equations with potential functions, and [6, 11, 21] for the existence of nontrivial solutions to Sobolev critical (p, q)-Laplacian equations.

It is worth mentioning that, recently, by means of the concentration compactness principle and a critical point theorem proposed by Perera [29, Theorem 2.1], Liu and Perera [26] considered the existence of multiple nontrivial solutions to (p, q)-Laplacian equation (1.3) when $f(x, u) = \lambda h(x)|u|^{r-2}u + g(x)|u|^{s-2}u$, where $r \in (p, p*)$, $s \in (p, p*]$, h, g satisfy appropriate conditions. Ambrosio and Isernia [3, section 3] studied the (p, q)-Laplacian equation with the Kirchhoff term

$$\begin{cases} \sum\limits_{k\in\{p,q\}} \left(-\left(1 + \int_{\mathbb{R}^N} |\nabla u|^k dx\right)\Delta_k u + |u|^{k-2}u\right) = f(x, u), \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $f(x, u) = f(u)$ satisfies Ambrosetti-Rabinowitz condition. By using the Nehari manifold method developed by Szulkin and Weth [33], Ambrosio and Isernia [3, section 3] proved the existence of a nontrivial solution to Eq (1.4).

Table 1. Results on solutions to Eq (1.3).

p, q	$f(x, u)$	Research results	Methods	References
$p = q = 2$	$\|u\|^{r-2}u, r \in (p, p^*)$	the existence of positive radial solutions	mountain pass theorem	[32]
$p = q = 2$	f satisfies A-R condition	the existence of nodal solutions	Nehari manifold and minimax method	[9]
$p = q \neq 2$	$\|u\|^{r-2}u, r \in (p, p^*)$	the uniqueness of positive solutions	using the results of the relevant ordinary differential equations	[12, 30, 31]
$p = q \neq 2$	$f(x, u) = h(x)\|u\|^{r-2}u$, $r \in (p, p*)$ and $1 < p < 2$	the existence of positive solutions	using the uniqueness of the positive solution and the minimax procedure	[13]
$p = q \neq 2$	$f(x, u)$ satisfies A-R condition	the existence of nontrivial solutions	a mountain pass argument and minimization technique	[28]
$1 < q < p < N$	$f(x, u) = f(u)$ or $\frac{f(x,u)}{u^{p-1}}$ tends to a positive constant as $u \to +\infty$	the existence of nontrivial solutions	Ekeland variational principle, mountain pass theorem, and concentration-compactness principle	[17]
$1 < q < p < N$	$f(x, u)$ satisfies A-R condition	the existence of nontrivial solutions	Ekeland variational principle, mountain pass theorem, and concentration-compactness principle	[22]
$1 < q \leq p < r < p*$	$f(x, u) = \lambda h(x)\|u\|^{r-2}u + g(x)\|u\|^{p*-2}u$	multiple solutions	a critical point theorem of Perera [29] and concentration compactness principle	[26]

However, the existence of multiple solutions to Eq (1.4) is still unknown. Inspired by [3, 26], we will study the multiplicity of nontrivial solutions to Eq (1.4) in the case where $f(x, u) = \lambda h(x)|u|^{r-2}u + g(x)|u|^{s-2}u$, that is, (p, q)-Kirchhoff equation (1.1), by applying the critical point theorem proposed by

Perera [29, Theorem 2.1]. In particular, we only need to prove that the energy functional corresponding to Eq (1.1) satisfies the local Palais-Smale condition ($(P.S.)$ condition for short) via concentration compactness principle of Lions, which is different from the well-known mountain pass theorem and symmetric mountain pass theorem [2], which usually requires us to prove that the functional satisfies $(P.S.)_c$ condition for all $c \in \mathbb{R}$ or $\mathbb{R}^+$.

Let the weight functions h and g satisfy
(f_1) $h \geq 0$, $g \geq 0$, $|\Omega| > 0$ for $\Omega = \{x \in \mathbb{R}^N : hg > 0\}$, where $|\Omega|$ denotes the Lebesgue measure in $\mathbb{R}^N$.
(f_2) $h \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p*}{p*-r}}(\mathbb{R}^N)$, $g \in L^\infty(\mathbb{R}^N)$.
For $m \in \mathbb{N}$, we set

$$f_m(t) = \frac{1}{p}t^p + \frac{1}{q}t^q + \frac{1}{2p}t^{2p} + \frac{1}{2q}t^{2q} - C_2(m)t^s, \ t > 0, \tag{1.5}$$

where $1 < q \leq p$, $s \in (2p, p*)$, and $C_2(m) > 0$ is given by (5.1). It is easy to check that f_m has a unique critical point $t_m > 0$, which is a global maximum point. Let

$$d_s := \inf_{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{1,p}^p}{|u|_s^p}, \ c_s := \left(\frac{1}{2p} - \frac{1}{s}\right) d_s^{\frac{s}{s-p}} |g|_\infty^{\frac{p}{p-s}}, \tag{1.6}$$

where $\|\cdot\|_{1,p}$ and $|\cdot|_t$ denote the norm of Sobolev space $W^{1,p}(\mathbb{R}^N)$ and Lebesgue space $L^t(\mathbb{R}^N)$, $t \in [1, \infty]$, respectively.

We first show the existence result by using the concentration compactness principle of Lions [25] and the mountain pass theorem [2].

Theorem 1.1. *Assume that (f_1) and (f_2) hold. Then, there exists $\Lambda_* \geq 0$ such that (1.1) has at least one nontrivial solution for every $\lambda > \Lambda_*$, which is a mountain pass solution. Moreover, if there exists $m \in \mathbb{N}$ such that $f_m(t_m) \leq c_s$, then $\Lambda_* = 0$. In contrast, $\Lambda_* > 0$.*

Remark 1.1. (i) Λ_* is defined by (4.1) in Theorem 1.1, and $\Lambda_* = 0$ if $|g|_\infty$ is large enough.

(ii) In the proof of Theorem 1.1, since the functional only satisfies the local $(P.S.)$ condition by employing the first concentration compactness principle of Lions, we need to constrain the upper bound of the mountain pass level by restricting the range of λ (i.e., setting $\lambda > \Lambda_*$), see Section 3.

As can be seen from Theorem 1.1, it seems that there is no advantage in searching for the existence of nontrivial solutions by applying the concentration compactness principle and mountain pass theorem. But if we combine the first concentration compactness principle with Perera's critical point theorem [29, Theorem 2.1], the multiplicity of nontrivial solutions to problem (1.1) will be solved.

Theorem 1.2. *Assume that (f_1) and (f_2) hold. Then for every $m \in \mathbb{N}$, there exists $\Lambda_m \geq 0$ such that (1.1) has at least m pairs of nontrivial solutions with positive energy for every $\lambda > \Lambda_m$. Moreover, if $f_m(t_m) \leq c_s$, then $\Lambda_m = 0$. In contrast, $\Lambda_m > 0$.*

Remark 1.2. (i) Λ_m is defined by (5.4) in Theorem 1.2, and for any given $m \in \mathbb{N}$, $\Lambda_m = 0$ when $|g|_\infty$ is large enough.

(ii) For condition (f_2), $h \in L^\infty(\mathbb{R}^N)$ can be removed, or $h \in L^{\frac{p*}{p*-r}}(\mathbb{R}^N)$ may be replaced by other conditions. First, the role of $h \in L^\infty(\mathbb{R}^N)$ is to be able to apply Method 1 in the proof of strong convergence (see Section 3), because we need to guarantee that $\langle \psi'(u), u_n - u \rangle \to 0$ when $u_n \to u$ in

$L^r(\mathbb{R}^N)$, $r \in (2p, p*)$ in Method 1; see (3.29) in Lemma 3.1. In other words, if $h \in L^\infty(\mathbb{R}^N)$ is removed, we can also prove strong convergence by Method 2, and then obtain the conclusions of Theorems 1.1 and 1.2. Second, the role of $h \in L^{\frac{p*}{p*-r}}(\mathbb{R}^N)$ is to ensure that the weak continuity of

$$\psi : W^{1,p}(\mathbb{R}^N) \to \mathbb{R},\ \psi(u) = \int_{\mathbb{R}^N} h|u|^r dx$$

and compactness of ψ' hold; see [28, Lemma 1]. Therefore, in Theorems 1.1 and 1.2, we may replace $h \in L^{\frac{p*}{p*-r}}(\mathbb{R}^N)$ by other conditions, ensuring weak continuity of ψ and compactness of ψ', such as $\lim\limits_{|x|\to\infty} h(x) = 0$.

(iii) Theorem 1.2 is a partial extension of the result of Ambrosio and Isernia [3, Section 3], where the existence of nontrivial solutions to Eq (1.1) was proved when h and g are positive constants by using the Nehari manifold method. This also partially extends the result of Liu and Perera [26, Theorem 1.2], where the existence of multiple solutions to the (p, q)-Laplacian equation was considered.

(iv) In this paper, to obtain the existence of multiple solutions, we only need to prove that the functional satisfies the local $(P.S.)$ condition, which is different from [22], where the authors also need to show that the weak limit of the $(P.S.)_c$ sequence produced by the mountain pass theorem is nonzero after proving that the weak limit is the weak solution of the (p, q)-Laplacian equation; see [22, Section 3] (also see [11, 21] for critical cases).

In order to prove Theorems 1.1 and 1.2, the crucial step is to prove that the functional satisfies the local (P.S.) condition, since it is easy to verify that the functional has a mountain pass structure and a geometric structure satisfying Perera [29, Theorem 2.1], and the critical value $c \in (0, c_s)$. There are two difficulties. First, the embedding $X := W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, $s \in (2p, p*)$ is noncompact. We will follow the process of Liu and Perera [26] by using the first concentration compactness principle to prove the strong convergence of $(P.S.)_c$ sequences in $L^s(\mathbb{R}^N)$. Second, we need to prove the strong convergence of sequences in space X. Since when $p \neq 2$, the space $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space, which means that we cannot get

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx + \int_{\mathbb{R}^N} |u_n|^{p-2}u_n v dx \to \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \int_{\mathbb{R}^N} |u|^{p-2}uv dx,$$

only by the weak convergence $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$. By using the fact that

$$\frac{\partial u_n}{\partial x_j}(x) \to \frac{\partial u}{\partial x_j}(x) \text{ a.e. in } \mathbb{R}^N,\ j \in \{1, 2, \cdots, N\},$$

which was proved in the first step by [3, Theorem 3], we present two methods to prove strong convergence; see Step 4 in Section 3.

Regarding the notation, in this paper, $\|\cdot\|_{1,k}$ and $|\cdot|_t$ denote the norms of the Sobolev space $W^{1,k}(\mathbb{R}^N)$, $k \in \{p, q\}$ and the Lebesgue space $L^t(\mathbb{R}^N)$, $t \in [1, \infty]$, respectively. $\to$ and $\rightharpoonup$ denote the strong and weak convergence in the related function space respectively. := and =: denote definitions. $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. $C, C_1, C_2, ...$ denote positive constants.

The rest of this article is organized as follows. In Section 2, we present some preliminary results, which include the mountain pass theorem and a critical point theorem due to Perera and the first concentration compactness principle of Lions. In Section 3, we show that the functional satisfies the local $(P.S.)$ condition. Sections 4 and 5 are devoted to the proof of Theorems 1.1 and 1.2, respectively.

2. Preliminary results

Let $X := W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ be the Banach space equipped with the norm

$$\|u\| = \|u\|_{1,p} + \|u\|_{1,q}, \text{ where } \|u\|_{1,k} = \left(\int_{\mathbb{R}^N} (|\nabla u|^k + |u|^k) dx \right)^{\frac{1}{k}}, \ k \in \{p, q\}.$$

Define the functional $I : X \to \mathbb{R}$ by

$$I(u) = \sum_{k\in\{p,q\}} \left(\frac{1}{k} \int_{\mathbb{R}^N} (|\nabla u|^k + |u|^k) dx + \frac{1}{2k} \left(\int_{\mathbb{R}^N} |\nabla u|^k dx \right)^2 \right) - \frac{\lambda}{r} \int_{\mathbb{R}^N} h|u|^r dx - \frac{1}{s} \int_{\mathbb{R}^N} g|u|^s dx;$$

then, the critical points of functional I are solutions of (1.1).

Lemma 2.1 ([2, Mountain pass theorem]). *Let X be a Banach space and $I \in C^1(X, \mathbb{R})$ satisfy $(P.S.)$ condition. Assume $I(0) = 0$ and*
(i) there exist $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;
(ii) there exists $e \in X \backslash B_\rho(0)$ such that $I(e) \leq 0$.
Then, the functional I possesses a critical value $c \geq \alpha$, and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}$.

Next, we recall a critical point theorem of Perera [29, Theorem 2.1]. Let X be a Banach space and $B_R(0) = \{u \in X : \|u\| < R\}$. For a symmetric subset $A \subset X\backslash\{0\}$, we denote by $i(A)$ the cohomological index of A, which was introduced by Fadell and Rabinowitz [15]. If A is homeomorphic to the unit sphere S^{m-1} in $\mathbb{R}^m$, then $i(A) = m$.

Lemma 2.2 ([29, Theorem 2.1]). *Let X be a Banach space and let $I \in C^1(X, \mathbb{R})$ be an even functional and satisfy $(P.S.)_c$ condition for $c \in (0, c*)$. If 0 is a strictly local minimizer of I, and there are $R > 0$ and a compact symmetric set $A \subset \partial B_R(0)$ such that $i(A) = m$,*

$$\max_A I \leq 0, \ \ \max_B I < c*,$$

where $B = \{tu \in X : t \in [0, 1], u \in A\}$, then I has m pairs of nonzero critical points with positive critical values.

To prove compactness, we also need the first concentration-compactness principle of Lions [25].

Lemma 2.3 ([25, Lemma I.1]). *Let $\{z_n\}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$z_n \geq 0 \ in \ \mathbb{R}^N, \ \ \int_{\mathbb{R}^N} z_n dx = \Lambda,$$

where $\Lambda > 0$ is fixed. Then, up to a subsequence, one of the following three situations hold:
(i) (Compactness) There exists a sequence $\{y_n\} \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $\int_{B_{R_\varepsilon}(y_n)} z_n dx \geq \Lambda - \varepsilon$ for all $n \in \mathbb{N}$ large enough.

(ii) (Vanishing) For all $R > 0$ there holds

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} z_n dx = 0.$$

(iii) (Dichotomy) There exists $l \in (0,\Lambda)$ such that for any $\varepsilon > 0$, there exist $n_0 \geq 1$ and $0 \leq z_n^1, z_n^2 \in L^1(\mathbb{R}^N)$ satisfying the following for $n > n_0$:

$$\int_{\mathbb{R}^N}\left(z_n - (z_n^1 + z_n^2)\right)dx \leq \varepsilon,\ \left|\int_{\mathbb{R}^N} z_n^1 dx - l\right| \leq \varepsilon,\ \left|\int_{\mathbb{R}^N} z_n^2 dx - (\Lambda - l)\right| \leq \varepsilon,$$

and

$$dist\left(\mathrm{supp}\left(z_n^1\right), \mathrm{supp}\left(z_n^2\right)\right) \to \infty, n \to \infty.$$

Lemma 2.4 ([11, Lemma 4])**.** *Let $1 < p < \infty$, $\{u_n\}$ be bounded in $L^p(\mathbb{R}^N)$, and $u_n(x) \to u(x)$ almost everywhere (a.e.) in $\mathbb{R}^N$. Then, $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^N)$.*

Lemma 2.5 ([10, Brézis-Lieb Lemma])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open subset and $\{u_n\} \subset L^p(\Omega)$, $1 \leq p < \infty$. If $\{u_n\}$ is bounded in $L^p(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in Ω, then*

$$\lim_{n\to\infty}|u_n|_p^p = |u|_p^p + \lim_{n\to\infty}|u_n - u|_p^p.$$

3. Local $(P.S.)$ condition

Lemma 3.1. *The functional I satisfies $(P.S.)_c$ condition for $0 < c < c_s$, where c_s is defined by (1.6).*

Proof. Let $\{u_n\} \subset X$ be a $(P.S.)_c$ sequence of I, that is

$$\begin{aligned} I(u_n) = &\sum_{k\in\{p,q\}}\left(\frac{1}{k}\int_{\mathbb{R}^N}(|\nabla u_n|^k + |u_n|^k)dx + \frac{1}{2k}\left(\int_{\mathbb{R}^N}|\nabla u_n|^k dx\right)^2\right)\\ &-\frac{\lambda}{r}\int_{\mathbb{R}^N} h|u_n|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u_n|^s dx\\ &\to c, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \langle I'(u_n), v\rangle = &\sum_{k\in\{p,q\}}\int_{\mathbb{R}^N}(|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla v + |u_n|^{k-2}u_n v)dx\\ &+\sum_{k\in\{p,q\}}\left(\int_{\mathbb{R}^N}|\nabla u_n|^k dx\right)\int_{\mathbb{R}^N}|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla v dx\\ &-\lambda\int_{\mathbb{R}^N} h|u_n|^{r-2}u_n v dx - \int_{\mathbb{R}^N} g|u_n|^{s-2}u_n v dx\\ &= o_n(1)\|v\| \end{aligned} \tag{3.2}$$

for all $v \in X$.

Step 1. We first prove that $\{u_n\}$ is bounded in X. If $2p < r \leq s < p^*$, then

$$
\begin{aligned}
&1 + c + \|u_n\| \\
&\geq I(u_n) - \frac{1}{r}\langle I'(u_n), u_n\rangle \\
&= \sum_{k\in\{p,q\}} \left(\left(\frac{1}{k} - \frac{1}{r}\right)\int_{\mathbb{R}^N} (|\nabla u_n|^k + |u_n|^k)dx + \left(\frac{1}{2k} - \frac{1}{r}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^k dx\right)^2\right) \\
&\quad \left(\frac{1}{r} - \frac{1}{s}\right)\int_{\mathbb{R}^N} g|u_n|^s dx \\
&\geq \sum_{k\in\{p,q\}} \left(\frac{1}{k} - \frac{1}{r}\right)\|u_n\|_{1,k}^k.
\end{aligned}
\tag{3.3}
$$

If $2p < s < r < p^*$, then

$$
\begin{aligned}
&1 + c + \|u_n\| \\
&\geq I(u_n) - \frac{1}{s}\langle I'(u_n), u_n\rangle \\
&= \sum_{k\in\{p,q\}} \left(\left(\frac{1}{k} - \frac{1}{s}\right)\int_{\mathbb{R}^N} (|\nabla u_n|^k + |u_n|^k)dx + \left(\frac{1}{2k} - \frac{1}{s}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^k dx\right)^2\right) \\
&\quad \left(\frac{1}{s} - \frac{1}{r}\right)\lambda\int_{\mathbb{R}^N} h|u_n|^r dx \\
&\geq \sum_{k\in\{p,q\}} \left(\frac{1}{k} - \frac{1}{s}\right)\|u_n\|_{1,k}^k.
\end{aligned}
\tag{3.4}
$$

From (3.3) and (3.4), it is easy to see that $\{u_n\}$ is bounded in X. Thus, we may assume that

$$u_n \rightharpoonup u \text{ in } X,\ u_n \to u \text{ in } L^t_{loc}(\mathbb{R}^N) \text{ for } t \in [1, p*),\ u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N. \tag{3.5}$$

Step 2. We shall apply the first concentration compactness principle of Lions [25] to show that $u_n \to u$ in $L^s(\mathbb{R}^N)$ for all $s \in (2p, p*)$. We will proceed in the spirit of [7,26], where (p, q)-Laplacian equations with critical and subcritical growth were studied, respectively. Let

$$z_n = |\nabla u_n|^p + |\nabla u_n|^q + |u_n|^p + |u_n|^q + |u_n|^s + \lambda h|u_n|^r.$$

Since $\{u_n\}$ is bounded, up to a subsequence, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} z_n dx = \Lambda > 0. \tag{3.6}$$

In fact, if $\Lambda = 0$, then

$$\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q = \int_{\mathbb{R}^N} (|\nabla u_n|^p + |\nabla u_n|^q + |u_n|^p + |u_n|^q)\, dx \to 0.$$

Hence $u_n \to 0$ in X, and $I(u_n) \to 0$, which contradicts $c > 0$. By (3.6) and Lemma 2.3, we deduce that one of the following three possibilities must occur: *Compactness*, *Vanishing*, and *Dichotomy*.

We claim that *Vanishing* cannot occur. By (3.6), we may assume that $\int_{\mathbb{R}^N} z_n dx \geq \frac{\Lambda}{2} > 0$ for all $n \in \mathbb{N}$, and hence there is $R_n > 0$ such that

$$\int_{B_{R_n(0)}} z_n dx \geq \frac{\Lambda}{4}.$$

If *Vanishing* occurs, then for any $R > 0$, there is $n_0 \geq 1$ such that

$$\int_{B_R(0)} z_n dx \leq \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} z_n dx < \frac{\Lambda}{8}$$

for all $n > n_0$; in particular, we have $\int_{B_{R_{n(0)}}} z_n dx < \frac{\Lambda}{8}$, a contradiction.

We next prove that *Dichotomy* cannot occur. If *Dichotomy* occurs, there exists $l \in (0, \Lambda)$ such that for any $\varepsilon > 0$, there exist $R > 0$, $4R < R_n \to \infty$, and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\left|\int_{B_R(y_n)} z_n dx - l\right| < \varepsilon, \ \left|\int_{B^c_{R_n}(y_n)} z_n dx - (\Lambda - l)\right| < \varepsilon, \ \left|\int_{B_{R_n}(y_n)\backslash B_R(y_n)} z_n dx\right| < \varepsilon, \tag{3.7}$$

where A^c denotes the complement of set A in $\mathbb{R}^N$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $|\varphi|_\infty = 1$, and

$$\varphi(x) = 1 \text{ in } B_1(0), \ \varphi(x) = 0 \text{ in } B_2^c(0).$$

Define $u_n^1 = \varphi_n^1 u_n$, $u_n^2 = \left(1 - \varphi_n^2\right) u_n$, where

$$\varphi_n^1 = \varphi\left(\frac{x - y_n}{R}\right), \ \varphi_n^2 = \varphi\left(\frac{x - y_n}{R_n/2}\right).$$

Then,

$$\text{supp}\left(u_n^1\right) = \left\{x \in \mathbb{R}^N : |x - y_n| \leq 2R\right\}, \ \text{supp}\left(u_n^2\right) = \left\{x \in \mathbb{R}^N : |x - y_n| \geq \frac{R_n}{2}\right\}$$

are two disjoint sets, and

$$\text{supp}\left(u_n^1\right) \subset B_{2R}(y_n) \subset B_{R_n}(y_n), \ \text{supp}\left(u_n^2\right) \subset B^c_{R_n/2}(y_n).$$

Moreover,

$$dist\left(\text{supp}\left(u_n^1\right), \ \text{supp}\left(u_n^2\right)\right) \to \infty, \ n \to \infty,$$

and

$$|\nabla\varphi_n^1|_\infty \leq \frac{1}{R}|\nabla\varphi|_\infty, \ |\nabla\varphi_n^2|_\infty \leq \frac{1}{R}|\nabla\varphi|_\infty.$$

For $k \in \{p, q\}$, by (3.7), we have

$$\begin{aligned}
\int_{B_{R_n}(y_n)\backslash B_R(y_n)} |\nabla u_n^1|^k dx &= \int_{B_{R_n}(y_n)\backslash B_R(y_n)} |\varphi_n^1 \nabla u_n + u_n \nabla\varphi_n^1|^k dx \\
&\leq 2^k \int_{B_{R_n}(y_n)\backslash B_R(y_n)} \left(|\varphi_n^1 \nabla u_n|^k + |u_n \nabla\varphi_n^1|^k\right) dx \\
&\leq 2^k \left(1 + \frac{1}{R}|\nabla\varphi|_\infty^k\right) \int_{B_{R_n}(y_n)\backslash B_R(y_n)} \left(|\nabla u_n|^k + |u_n|^k\right) dx \\
&\leq 2^k \left(1 + \frac{1}{R}|\nabla\varphi|_\infty^k\right) \varepsilon,
\end{aligned}$$

and then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n^1|^k dx &= \int_{B_{R_n}(y_n)} |\nabla u_n^1|^k dx \\ &= \int_{B_R(y_n)} |\nabla u_n^1|^k dx + \int_{B_{R_n}(y_n)\setminus B_R(y_n)} |\nabla u_n^1|^k dx \\ &= \int_{B_R(y_n)} |\nabla u_n|^k dx + o_\varepsilon(1).\end{aligned} \tag{3.8}$$

Similarly, since $B_R(y_n) \subset B_{R_n/2}(y_n)$, we have

$$\begin{aligned}\int_{B_{R_n}(y_n)\setminus B_{R_n/2}(y_n)} |\nabla u_n^2|^k dx &\le \int_{B_{R_n}(y_n)\setminus B_R(y_n)} |\nabla u_n^2|^k dx \\ &= \int_{B_{R_n}(y_n)\setminus B_R(y_n)} |(1-\varphi_n^2)\nabla u_n - u_n \nabla \varphi_n^2|^k dx \\ &\le 2^k \int_{B_{R_n}(y_n)\setminus B_R(y_n)} \left(|(1-\varphi_n^2)\nabla u_n|^k + |u_n \nabla \varphi_n^2|^k\right) dx \\ &\le 2^k \left(1 + \frac{1}{R}|\nabla \varphi|_\infty^k\right) \int_{B_{R_n}(y_n)\setminus B_R(y_n)} \left(|\nabla u_n|^k + |u_n|^k\right) dx \\ &\le 2^k \left(1 + \frac{1}{R}|\nabla \varphi|_\infty^k\right) \varepsilon,\end{aligned}$$

and then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n^2|^k dx &= \int_{B^c_{R_n/2}(y_n)} |\nabla u_n^2|^k dx \\ &= \int_{B^c_{R_n}(y_n)} |\nabla u_n^2|^k dx + \int_{B_{R_n}(y_n)\setminus B_{R_n/2}(y_n)} |\nabla u_n^2|^k dx \\ &= \int_{B^c_{R_n}(y_n)} |\nabla u_n|^k dx + o_\varepsilon(1).\end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9), and using (3.7), we have

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla u_n^2|^k + |\nabla u_n^1|^k) dx &= \int_{B^c_{R_n}(y_n)} |\nabla u_n|^k dx + \int_{B_R(y_n)} |\nabla u_n|^k dx + o_\varepsilon(1) \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^k dx - \int_{B_{R_n}(y_n)\setminus B_R(y_n)} |\nabla u_n|^k dx + o_\varepsilon(1) \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^k dx + o_\varepsilon(1).\end{aligned}$$

Similar arguments yield

$$\int_{\mathbb{R}^N} |u_n^1|^k dx = \int_{B_R(y_n)} |u_n|^k dx + o_\varepsilon(1), \quad \int_{\mathbb{R}^N} |u_n^2|^k dx = \int_{B^c_{R_n}(y_n)} |u_n|^k dx + o_\varepsilon(1), k \in \{p, q\},$$

$$\int_{\mathbb{R}^N} h|u_n^1|^r dx = \int_{B_R(y_n)} h|u_n|^r dx + o_\varepsilon(1), \quad \int_{\mathbb{R}^N} h|u_n^2|^r dx = \int_{B^c_{R_n}(y_n)} h|u_n|^r dx + o_\varepsilon(1),$$

$$\int_{\mathbb{R}^N} g|u_n^1|^s dx = \int_{B_R(y_n)} g|u_n|^s dx + o_\varepsilon(1), \quad \int_{\mathbb{R}^N} g|u_n^2|^s dx = \int_{B^c_{R_n}(y_n)} g|u_n|^s dx + o_\varepsilon(1).$$

Using these estimates and (3.7), as well as the boundedness of $\{u_n^i\}$, $i = 1, 2$, we deduce

$$\begin{aligned} c + o_n(1) + o_\varepsilon(1) = &\sum_{i=1}^{2} \left(\frac{1}{p}\|u_n^i\|_{1,p}^p + \frac{1}{q}\|u_n^i\|_{1,q}^q - \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u_n^i|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u_n^i|^s dx \right) \\ &+ \frac{1}{2p}\left(\sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla u_n^i|^p dx \right)^2 + \frac{1}{2q}\left(\sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla u_n^i|^q dx \right)^2 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} o_n(1) + o_\varepsilon(1) = &\|u_n^i\|_{1,p}^p + \|u_n^i\|_{1,q}^q - \lambda \int_{\mathbb{R}^N} h|u_n^i|^r dx - \int_{\mathbb{R}^N} g|u_n^i|^s dx \\ &+ \left(\int_{\mathbb{R}^N} |\nabla u_n^1|^p dx + \int_{\mathbb{R}^N} |\nabla u_n^2|^p dx \right) \int_{\mathbb{R}^N} |\nabla u_n^i|^p dx \\ &+ \left(\int_{\mathbb{R}^N} |\nabla u_n^1|^q dx + \int_{\mathbb{R}^N} |\nabla u_n^2|^q dx \right) \int_{\mathbb{R}^N} |\nabla u_n^i|^q dx, \end{aligned} \tag{3.11}$$

from (3.1) and (3.2) with $v = u_n^i$ and $\operatorname{supp}\left(u_n^1\right) \cap \operatorname{supp}\left(u_n^2\right) = \emptyset$.

Assume that

$$|u_n^i|_s^p \to \gamma_i, \ \lambda \int_{\mathbb{R}^N} h|u_n^i|^r dx \to \alpha_i, \ \int_{\mathbb{R}^N} g|u_n^i|^s dx \to \beta_i.$$

Since $\int_{\mathbb{R}^N} g|u_n^i|^s dx \le |g|_\infty |u_n^i|_s^s$, letting $n \in \infty$, we have

$$\gamma_i \ge \beta_i^{\frac{p}{s}} |g|_\infty^{-\frac{p}{s}}. \tag{3.12}$$

It follows from (3.11) that

$$\begin{aligned} &\alpha_i + \beta_i + o_n(1) + o_\varepsilon(1) \\ &= \|u_n^i\|_{1,p}^p + \|u_n^i\|_{1,q}^q + \left(\int_{\mathbb{R}^N} |\nabla u_n^1|^p dx + \int_{\mathbb{R}^N} |\nabla u_n^2|^p dx \right) \int_{\mathbb{R}^N} |\nabla u_n^i|^p dx \\ &\ + \left(\int_{\mathbb{R}^N} |\nabla u_n^1|^q dx + \int_{\mathbb{R}^N} |\nabla u_n^2|^q dx \right) \int_{\mathbb{R}^N} |\nabla u_n^i|^q dx, \end{aligned} \tag{3.13}$$

and then using (3.10), we have

$$
\begin{aligned}
c + o_n(1) + o_\varepsilon(1) &= \sum_{i=1}^{2}\left(\frac{1}{p}\|u_n^i\|_{1,p}^p + \frac{1}{q}\|u_n^i\|_{1,q}^q - \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u_n^i|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u_n^i|^s dx\right) \\
&\quad + \frac{1}{2p}\left(\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla u_n^i|^p dx\right)^2 + \frac{1}{2q}\left(\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla u_n^i|^q dx\right)^2 \\
&\geq \sum_{i=1}^{2}\left(\frac{1}{2p}\left(\|u_n^i\|_{1,p}^p + \|u_n^i\|_{1,q}^q\right) - \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u_n^i|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u_n^i|^s dx\right) \\
&\quad + \frac{1}{2p}\left(\left(\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla u_n^i|^p dx\right)^2 + \left(\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla u_n^i|^q dx\right)^2\right) \\
&= \sum_{i=1}^{2}\left(\frac{\alpha_i+\beta_i}{2p} - \frac{\alpha_i}{r} - \frac{\beta_i}{s}\right) \\
&\geq \left(\frac{1}{2p} - \frac{1}{s}\right)(\beta_1+\beta_2)\,.
\end{aligned} \tag{3.14}
$$

If $\{y_n\}$ is bounded, then there exists $\rho > 0$ such that $\{y_n\} \subset B_\rho(0)$. Since

$$
\operatorname{supp}\left(u_n^2\right) \subset B^c_{R_n/2}(y_n) \subset B^c_{R_n/2-\rho}(0),
$$

and

$$
\psi : \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R},\ \psi(u) = \int_{\mathbb{R}^N} h|u|^r dx \tag{3.15}
$$

is weakly continuous, see [28, Lemma 1], we deduce

$$
h|u_n^2|^r \cdot \chi_n^c \to 0 \text{ a.e. in } \mathbb{R}^N,\ \left|h|u_n^2|^r \cdot \chi_n^c\right| \leq h|u_n|^r,\ \int_{\mathbb{R}^N} h|u_n|^r dx \to \int_{\mathbb{R}^N} h|u|^r dx,
$$

where χ_n^c is the indicator function of $B^c_{R_n/2-\rho}(0)$. Applying the Lebesgue dominated theorem, we get

$$
\int_{\mathbb{R}^N} h|u_n^2|^r dx = \int_{B^c_{R_n/2-\rho}(0)} h|u_n^2|^r dx = \int_{\mathbb{R}^N} h|u_n^2|^r \cdot \chi_n^c dx \to 0,
$$

and then $\alpha_2 = 0$. Using (3.13) with $i = 2$ and (3.12), we get

$$
\beta_2 \geq \lim_{n\to\infty}\|u_n^2\|_{1,p}^p \geq d_s \lim_{n\to\infty}|u_n^2|_s^p = d_s\gamma_2 \geq \beta_2^{\frac{p}{s}}|g|_\infty^{-\frac{p}{s}}. \tag{3.16}
$$

Since $\beta_2 > 0$ (indeed, if $\beta_2 = 0$ then $u_n^2 \to 0$ in X by (3.13), and hence $\int_{B^c_{R_n}(y_n)} z_n dx = o_\varepsilon(1)$, which contradicts (3.7)), then $\beta_2 \geq d_s^{\frac{s}{s-p}}|g|_\infty^{\frac{p}{p-s}}$. It follows from (3.14) that

$$
c \geq \left(\frac{1}{2p} - \frac{1}{s}\right) d_s^{\frac{s}{s-p}}|g|_\infty^{\frac{p}{p-s}},
$$

a contradiction.

If $\{y_n\}$ is unbounded, up to a subsequence, we may assume $|y_n| \to \infty$ as $n \to \infty$. Denoting by χ_n the indicator function of $B_{2R}(y_n)$, we get

$$h|u_n^1|^r \cdot \chi_n \to 0 \text{ a.e. in } \mathbb{R}^N.$$

By similar argument, we have $\alpha_1 = 0$ and $\beta_1 \geq d_s^{\frac{s}{s-p}} |g|_\infty^{\frac{p}{p-s}}$, and then

$$c \geq \left(\frac{1}{2p} - \frac{1}{s}\right) d_s^{\frac{s}{s-p}} |g|_\infty^{\frac{p}{p-s}}$$

by (3.14), a contradiction. This completes the proof that *Dichotomy* cannot occur.

Therefore, *Compactness* holds, that is, there exists a sequence $\{y_n\} \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{B^c_{R_\varepsilon}(y_n)} z_n dx \leq \varepsilon \tag{3.17}$$

for all $n \in \mathbb{N}$ large enough.

We claim that $\{y_n\}$ is bounded. Otherwise, we may assume that $|y_n| \to \infty$ as $n \to \infty$. Given $\rho > 0$, $B_\rho(0) \subset B^c_{R_\varepsilon}(y_n)$ for large n, and thus

$$\lambda \int_{B_\rho(0)} h|u_n|^r dx \leq \lambda \int_{B^c_{R_\varepsilon}(y_n)} h|u_n|^r dx \leq \int_{B^c_{R_\varepsilon}(y_n)} z_n dx \leq \varepsilon.$$

Letting $\rho \to \infty$ and then $n \to \infty$, by the weak continuity of the functional ψ given in (3.15), we deduce

$$\lim_{n\to\infty} \lambda \int_{\mathbb{R}^N} h|u_n|^r dx = \lambda \int_{\mathbb{R}^N} h|u|^r dx = 0. \tag{3.18}$$

Hence, by (3.2) with $v = u_n$, we get

$$\|u_n\|^p_{1,p} + \|u_n\|^q_{1,q} + |\nabla u_n|^{2p}_p + |\nabla u_n|^{2q}_q = \int_{\mathbb{R}^N} g|u_n|^s dx + o_n(1).$$

Let $n \to \infty$, and we have

$$\lim_{n\to\infty} \left(\|u_n\|^p_{1,p} + \|u_n\|^q_{1,q} + |\nabla u_n|^{2p}_p + |\nabla u_n|^{2q}_q\right) = \lim_{n\to\infty} \int_N g|u_n|^s dx =: \beta. \tag{3.19}$$

It follows from (3.19) that $\beta > 0$. Otherwise, $u_n \to 0$ in X, and then $I(u_n) \to 0$, a contradiction. Similar to the process of estimating (3.16), we have $\beta \geq d_s^{\frac{s}{s-p}} |g|_\infty^{\frac{p}{p-s}}$. Combining (3.18) with (3.19), we get

$$\begin{aligned}
c + o_n(1) = I(u_n) \\
&= \frac{1}{p}\|u_n\|^p_{1,p} + \frac{1}{q}\|u_n\|^q_{1,q} + \frac{1}{2p}|\nabla u_n|^{2p}_p + \frac{1}{2q}|\nabla u_n|^{2q}_q \\
&- \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u_n|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u_n|^s dx \\
&\geq \frac{1}{2p}\left(\|u_n\|^p_{1,p} + \|u_n\|^q_{1,q} + |\nabla u_n|^{2p}_p + |\nabla u_n|^{2q}_q\right) - \frac{1}{s}\beta \\
&= \left(\frac{1}{2p} - \frac{1}{s}\right)\beta \geq \left(\frac{1}{2p} - \frac{1}{s}\right) d_s^{\frac{s}{s-p}} |g|_\infty^{\frac{p}{p-s}},
\end{aligned}$$

a contradiction. Hence, $\{y_n\}$ is bounded, and we may assume that $|y_n| < \rho$. By (3.17),

$$\int_{B^c_{R_\varepsilon+\rho}(0)} |u_n|^s dx \le \int_{B^c_{R_\varepsilon}(y_n)} z_n dx \le \varepsilon.$$

From the compact embedding $W^{1,p}(B_{R_\varepsilon+\rho}(0)) \hookrightarrow L^s(B_{R_\varepsilon+\rho}(0))$, we have

$$\lim_{n\to\infty} \int_{B_{R_\varepsilon+\rho}(0)} |u_n|^s dx = \int_{B_{R_\varepsilon+\rho}(0)} |u|^s dx.$$

Hence,

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |u_n|^s dx &= \overline{\lim_{n\to\infty}} \left(\int_{B_{R_\varepsilon+\rho}(0)} |u_n|^s dx + \int_{B^\varepsilon_{R_\varepsilon+\rho}(0)} |u|^s dx \right) \\ &\le \int_{B_{R_\varepsilon+\rho}(0)} |u|^s dx + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ yields $|u_n|^s_s \to |u|^s_s$. Using Brezis-Lieb Lemma, we have

$$u_n \to u \text{ in } L^s(\mathbb{R}^N),\ \ s \in (2p, p*). \tag{3.20}$$

Step 3. By (3.20), using the Hölder inequality and the Lebesgue dominated theorem, we get

$$\int_{\mathbb{R}^N} g|u_n|^{s-2} u_n v dx \to \int_{\mathbb{R}^N} g|u|^{s-2} uv dx, \tag{3.21}$$

and

$$\int_{\mathbb{R}^N} g|u_n|^{s-2} u_n(u_n - u) dx \to 0. \tag{3.22}$$

Moreover, since ψ' is compact (see [28, Lemma 1]), we have $\psi'(u_n)v \to \psi'(u)v$, that is

$$\int_{\mathbb{R}^N} h|u_n|^{r-2} u_n v dx \to \int_{\mathbb{R}^N} h|u|^{r-2} uv dx, \tag{3.23}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} h|u_n|^{r-2} u_n(u_n - u) dx &= \psi(u_n) - \frac{1}{r}\psi'(u_n)u \\ &\to \psi(u) - \frac{1}{r}\psi'(u)u = 0.\end{aligned} \tag{3.24}$$

In addition, as in the first step of [3, Theorem 3] where the autonomous (p, q)-Kirchhoff equation was considered, we have

$$\frac{\partial u_n}{\partial x_j}(x) \to \frac{\partial u}{\partial x_j}(x) \text{ a.e. in } \mathbb{R}^N,\ \ j \in \{1, 2, \cdots, N\}. \tag{3.25}$$

Indeed, using $u_n \rightharpoonup u$ in X and $I'(u_n) \to 0$, we have

$$\langle I'(u_n) - I'(u), (u_n - u)\varphi\rangle = o_n(1)$$

for the fixed $\varphi \in C_c^\infty(\mathbb{R}^N)$. According to the proof of [3, Theorem 3], if $\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)\varphi dx \to 0$ as $n \to \infty$, then $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, where $f(x,u) = \lambda h(x)|u|^{r-2}u + g(x)|u|^{s-2}u$. Using the boundedness of $\{u_n\}$ in X and (3.20), by the Hölder inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} g|u_n|^{s-2}u_n(u_n-u)\varphi dx\right| &\le |g|_\infty|\varphi|_\infty \int_{\mathbb{R}^N} |u_n|^{s-1}|u_n-u|dx \\ &\le |g|_\infty|\varphi|_\infty\left(\int_{\mathbb{R}^N}|u_n|^s dx\right)^{\frac{s-1}{s}}\left(\int_{\mathbb{R}^N}|u_n-u|^s dx\right)^{\frac{1}{s}} \\ &\le C|u_n-u|_s^s \\ &\to 0\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^N} h|u_n|^{r-2}u_n(u_n-u)\varphi dx\right| \to 0,$$

and hence (3.25) holds.

Step 4. $u_n \to u$ in X. We present two methods.

Method 1. By (3.25), (3.5), and Lemma 2.4, we have

$$|\nabla u_n|^{k-2}\frac{\partial u_n}{\partial x_j} \rightharpoonup |\nabla u|^{k-2}\frac{\partial u}{\partial x_j} \text{ in } L^{\frac{k}{k-1}}(\mathbb{R}^N)$$

for $k \in \{p,q\}$, and

$$|u_n|^{k-2}u_n \rightharpoonup |u|^{k-2}u \text{ in } L^{\frac{k}{k-1}}(\mathbb{R}^N).$$

Hence for any $v \in X$,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^{k-2}\nabla u_n\cdot\nabla v dx &\to \int_{\mathbb{R}^N}|\nabla u|^{k-2}\nabla u\cdot\nabla v dx, \\ \int_{\mathbb{R}^N}|u_n|^{k-2}u_n v dx &\to \int_{\mathbb{R}^N}|u|^{k-2}uv dx.\end{aligned} \tag{3.26}$$

Let $\lim_{n\to\infty}|\nabla u_n|_p^p = A_1$ and $\lim_{n\to\infty}|\nabla u_n|_q^q = A_2$. It follows from (3.21), (3.23), (3.26), and (3.2) that

$$\begin{aligned}&\sum_{k\in\{p,q\}}\int_{\mathbb{R}^N}(|\nabla u|^{k-2}\nabla u\cdot\nabla v + |u|^{k-2}uv)dx \\ &+ A_1\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla v dx + A_2\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\cdot\nabla v dx \\ &= \lambda\int_{\mathbb{R}^N} h|u|^{r-2}uv dx + \int_{\mathbb{R}^N} g|u|^{s-2}uv dx,\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}&\sum_{k\in\{p,q\}}\int_{\mathbb{R}^N}(|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla v + |u_n|^{k-2}u_n v)dx \\ &+ A_1\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla v dx + A_2\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla v dx \\ &= \lambda\int_{\mathbb{R}^N} h|u_n|^{r-2}u_n v dx + \int_{\mathbb{R}^N} g|u_n|^{s-2}u_n v dx + o_n(1)\end{aligned} \tag{3.28}$$

by (3.2) and the boundedness of $\{u_n\}$ in X. In addition, using the Hölder inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} h|u|^{r-2}u(u_n-u)dx\right| &\leq |h|_\infty \int_{\mathbb{R}^N} |u|^{r-1}|u_n-u|dx \\ &\leq |h|_\infty |u|_r^{r-1}|u_n-u|_r \\ &\to 0,\end{aligned} \tag{3.29}$$

and

$$\left|\int_{\mathbb{R}^N} g|u|^{s-2}u(u_n-u)dx\right| \leq |g|_\infty |u|_s^{s-1}|u_n-u|_s \to 0. \tag{3.30}$$

Thus, let $v = u_n - u$ in (3.27) and (3.28); we get

$$\begin{aligned}&\sum_{k\in\{p,q\}}\int_{\mathbb{R}^N}\left(\left(|\nabla u_n|^{k-2}\nabla u_n-|\nabla u|^{k-2}\nabla u\right)\cdot\nabla(u_n-u)+\left(|u_n|^{k-2}u_n-|u|^{k-2}u\right)(u_n-u)\right)dx \\ &+A_1\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u)dx \\ &+A_2\int_{\mathbb{R}^N}\left(|\nabla u_n|^{q-2}\nabla u_n-|\nabla u|^{q-2}\nabla u\right)\cdot\nabla(u_n-u)dx \\ &=\lambda\int_{\mathbb{R}^N} h\left(|u_n|^{r-2}u_n-|u|^{r-2}u\right)(u_n-u)dx+\int_{\mathbb{R}^N} g\left(|u_n|^{s-2}u_n-|u|^{s-2}u\right)(u_n-u)dx+o_n(1) \\ &=o_n(1)\end{aligned} \tag{3.31}$$

by (3.22), (3.24), (3.29), and (3.30). Using the inequality [14]

$$|x-y|^t \leq \begin{cases} C\,(x-y)\left(|x|^{t-2}x-|y|^{t-2}y\right), & t\geq 2, \\ C\,(x-y)\left(|x|^{t-2}x-|y|^{t-2}y\right)^{\frac{t}{2}}(|x|^t+|y|^t)^{\frac{2-t}{2}}, & 1<t<2,\end{cases}$$

we have $u_n \to u$ in both $W^{1,p}(\mathbb{R}^N)$ and $W^{1,q}(\mathbb{R}^N)$. Hence $u_n \to u$ in X.

Method 2. Let $\lim\limits_{n\to\infty}|\nabla u_n|_p^p = A_1$ and $\lim\limits_{n\to\infty}|\nabla u_n|_q^q = A_2$; we have

$$\begin{aligned}&\sum_{k\in\{p,q\}}\int_{\mathbb{R}^N}(|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla(u_n-u)+|u_n|^{k-2}u_n(u_n-u))dx \\ &+A_1\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u)dx+A_2\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(u_n-u)dx \\ &=o_n(1)\end{aligned} \tag{3.32}$$

by (3.2), (3.22), and (3.24). Note that

$$\begin{aligned}
&\sum_{k\in\{p,q\}}\int_{\mathbb{R}^N}(|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla(u_n-u)+|u_n|^{k-2}u_n(u_n-u))dx\\
&=\sum_{k\in\{p,q\}}\left(\|u_n\|_{1,k}^k-\int_{\mathbb{R}^N}\left(|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla u+|u_n|^{k-2}u_nu\right)dx\right)\\
&\geq\sum_{k\in\{p,q\}}\left(\|u_n\|_{1,k}^k-\int_{\mathbb{R}^N}\left(|\nabla u_n|^{k-1}|\nabla u|+|u_n|^{k-1}|u|\right)dx\right)\\
&\geq\sum_{k\in\{p,q\}}\left(\|u_n\|_{1,k}^k-\left(|\nabla u_n|_k^{k-1}|\nabla u|_k+|u_n|_k^{k-1}|u|_k\right)\right)\\
&\geq\sum_{k\in\{p,q\}}\left(\|u_n\|_{1,k}^k-\|u_n\|_{1,k}^{k-1}\|u\|_{1,k}\right)\\
&=\sum_{k\in\{p,q\}}\|u_n\|_{1,k}^{k-1}\left(\|u_n\|_{1,k}-\|u\|_{1,k}\right)
\end{aligned}\tag{3.33}$$

by using the Hölder inequality, and

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla u_n|^{k-2}\nabla u_n\cdot\nabla(u_n-u)dx&\geq|\nabla u_n|_k^k-\int_{\mathbb{R}^N}|\nabla u_n|^{k-1}|\nabla u|dx\\
&\geq|\nabla u_n|_k^{k-1}\left(|\nabla u_n|_k-|\nabla u|_k\right)
\end{aligned}\tag{3.34}$$

for $k\in\{p,q\}$. By $u_n\rightharpoonup u$ in X, we get

$$u_n\rightharpoonup u \text{ in } W^{1,k}(\mathbb{R}^N),\ u_n\rightharpoonup u \text{ in } \mathcal{D}^{1,k}(\mathbb{R}^N) \text{ for } k\in\{p,q\}.$$

Since the norm in $W^{1,k}(\mathbb{R}^N)$ and the norm in $\mathcal{D}^{1,k}(\mathbb{R}^N)$ are weakly lower semicontinuous, then

$$\|u\|_{1,k}\leq\varliminf_{n\to\infty}\|u_n\|_{1,k},\ \|u\|_{\mathcal{D}^{1,k}}\leq\varliminf_{n\to\infty}\|u_n\|_{\mathcal{D}^{1,k}} \text{ for } k\in\{p,q\},\tag{3.35}$$

where $\|u\|_{\mathcal{D}^{1,k}}=|\nabla u|_k$. Combining (3.32)–(3.35), we have

$$\sum_{k\in\{p,q\}}\|u_n\|_{1,k}^{k-1}\left(\|u_n\|_{1,k}-\|u\|_{1,k}\right)=o_n(1).\tag{3.36}$$

Case 1. If $q=p$, then

$$\|u_n\|_{1,p}^{p-1}\left(\|u_n\|_{1,p}-\|u\|_{1,p}\right)=o_n(1).\tag{3.37}$$

We claim that there is $C>0$ such that $\|u_n\|_{1,p}\geq C$. Otherwise, if $u_n\to 0$ in $W^{1,p}(\mathbb{R}^N)$, that is, $u_n\to 0$ in X, then $I(u_n)\to 0$, a contradiction. Hence, by (3.37),

$$\lim_{n\to\infty}\|u_n\|_{1,p}=\|u\|_{1,p}.\tag{3.38}$$

If $p=2$, by $u_n\rightharpoonup u$ in $W^{1,2}(\mathbb{R}^N)$, it is easy to check that $u_n\to u$ in $W^{1,2}(\mathbb{R}^N)$. If $p\neq 2$, then

$$\lim_{n\to\infty}|\nabla u_n|_p=|\nabla u|_p,\ \lim_{n\to\infty}|u_n|_p=|u|_p$$

by (3.38) and the weakly lower semicontinuity of the norm in $W^{1,p}(\mathbb{R}^N)$ and $\mathcal{D}^{1,p}(\mathbb{R}^N)$. By (3.5) and (3.25), using Lemma 2.5 (Brésiz-Lieb Lemma), we have

$$\lim_{n\to\infty}|\nabla u_n-\nabla u|_p^p=\lim_{n\to\infty}|\nabla u_n|_p^p-|\nabla u|_p^p=0,\ \lim_{n\to\infty}|u_n-u|_p^p=\lim_{n\to\infty}|u_n|_p^p-|u|_p^p=0,$$

and hence

$$\lim_{n\to\infty}\|u_n-u\|_{1,p}=0,$$

that is, $u_n\to u$ in X.

Case 2. If $q<p$, then there exists $C>0$ such that $\|u_n\|_{1,k}\geq C$ for $k\in\{p,q\}$. Otherwise, if $u_n\to 0$ in $W^{1,p}(\mathbb{R}^N)$, then $u_n\to 0$ in $L^t(\mathbb{R}^N)$, $t\in(p,p*)$. Thus,

$$\int_{\mathbb{R}^N}g|u_n|^s dx\to 0,\ \text{and}\ \int_{\mathbb{R}^N}h|u_n|^r dx\to 0,$$

since ψ defined by (3.15) is weakly continuous. Hence,

$$o_n(1)=\langle I'(u_n),u_n\rangle=\|u_n\|_{1,q}^q+|\nabla u_n|_q^{2q}+o_n(1),$$

and then $I(u_n)\to 0$, which is a contradiction. If $u_n\to 0$ in $W^{1,q}(\mathbb{R}^N)$, then $u_n\to 0$ in $L^t(\mathbb{R}^N)$, $t\in(q,q*)$. For any α,β satisfying $q<\alpha<q*\leq\beta<p*$, using the Hölder inequality and the boundedness of $\{u_n\}$ in X, we have

$$|u_n|_\beta\leq|u_n|_\alpha^{(1-\lambda)}|u_n|_{p*}^\lambda\leq C|u_n|_\alpha^{(1-\lambda)}\to 0,$$

where $\lambda=\frac{\beta-\alpha}{p*-\alpha}\cdot\frac{p*}{\beta}$. Hence $u_n\to 0$ in $L^t(\mathbb{R}^N)$, $t\in(q,p*)$. Similarly, $I(u_n)\to 0$, which is a contradiction. Therefore, by (3.37),

$$\lim_{n\to\infty}\|u_n\|_{1,k}=\|u\|_{1,k},\ k\in\{p,q\}.$$

By virtue of the weakly lower semicontinuity of the norm,

$$\lim_{n\to\infty}|\nabla u_n|_k=|\nabla u|_k,\ \lim_{n\to\infty}|u_n|_k=|u|_k.$$

By (3.5) and (3.25), using Lemma 2.5 (Brésiz-Lieb Lemma), we have

$$\lim_{n\to\infty}|\nabla u_n-\nabla u|_k^k=\lim_{n\to\infty}|\nabla u_n|_k^k-|\nabla u|_k^k=0,$$

and

$$\lim_{n\to\infty}|u_n-u|_k^k=\lim_{n\to\infty}|u_n|_k^k-|u|_k^k=0$$

for $k\in\{p,q\}$. Hence $u_n\to u$ in X. □

4. Proof of Theorem 1.1

Lemma 4.1. *Assume that* (f_1) *and* (f_2) *hold. Then*
(i) there exist $\rho,\alpha>0$ *such that* $I(u)\geq\alpha>0$ *for any* $u\in X$ *with* $\|u\|=\rho$*;*
(ii) there exists $e\in X$ *with* $\|e\|>\rho$ *such that* $I(e)\leq 0$.

Proof. (i) For any $u \in X$ with $\|u\| < 1$, by Hölder inequality and (1.6),

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q + \frac{1}{2p}|\nabla u|_p^{2p} + \frac{1}{2q}|\nabla u|_q^{2q} \\ &\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u|^s dx \\ &\geq \frac{1}{2p}\left(\|u\|_{1,p}^p + \|u\|_{1,q}^q + |\nabla u|_p^{2p} + |\nabla u|_q^{2q}\right) - \frac{\lambda}{r}|h|_\infty |u|_r^r - \frac{1}{s}|g|_\infty |u|_s^s \\ &\geq \frac{1}{2p}\left(\|u\|_{1,p}^p + \|u\|_{1,q}^q\right) - \frac{\lambda}{r}|h|_\infty d_s^{-\frac{r}{p}}\|u\|_{1,p}^r - \frac{1}{s}|g|_\infty d_s^{-\frac{s}{p}}\|u\|_{1,p}^s \\ &\geq C_1\|u\|^p - \lambda C_2\|u\|^r - C_3\|u\|^s. \end{aligned}$$

Since $\{r, s\} \subset (2p, p*)$, then there is a $\rho \in (0,1)$ small enough such that

$$\alpha := \inf_{\|u\|=\rho} I(u) > 0 = I(0).$$

(ii) For any fixed $u_0 \in X\backslash\{0\}$ and $t > 0$, since

$$\begin{aligned} I(tu_0) &= \frac{t^p}{p}\|u_0\|_{1,p}^p + \frac{t^q}{q}\|u_0\|_{1,q}^q + \frac{t^{2p}}{2p}|\nabla u_0|_p^{2p} + \frac{t^{2q}}{2q}|\nabla u_0|_q^{2q} \\ &\quad - \frac{\lambda t^r}{r}\int_{\mathbb{R}^N} h|u_0|^r dx - \frac{t^s}{s}\int_{\mathbb{R}^N} g|u_0|^s dx, \end{aligned}$$

then $I(t_0u_0) < 0$ for $t = t_0$ large enough, and hence there exists $e := t_0u_0$ with $\|e\| > \rho$ such that $I(e) \leq 0$. □

Proof of Theorem 1.1. By Lemmas 3.1 and 4.1, applying Lemma 2.1 (mountain pass theorem), it is enough to show that

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) < c_s,$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$.

Let $Z := \{u \in X : \text{supp}u \subset \Omega\}$, where Ω is defined by (f_1), then Z is an infinite dimensional subspace of X. Let Z_m be an m-dimensional subspace of Z. On Z_m,

$$[u]_h := \left(\int_{\mathbb{R}^N} h|u|^r\right)^{\frac{1}{r}},\ [u]_g := \left(\int_{\mathbb{R}^N} g|u|^s\right)^{\frac{1}{s}}$$

are norms of $u \in Z_m$. Since $\dim Z_m < \infty$, all norms on Z_m are equivalent. Therefore, for $u \in Z_m$, we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q + \frac{1}{2p}|\nabla u|_p^{2p} + \frac{1}{2q}|\nabla u|_q^{2q} \\ &\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u|^s dx \\ &\leq \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q + \frac{1}{2p}\|u\|^{2p} + \frac{1}{2q}\|u\|^{2q} - \lambda C_1(m)\|u\|^r - C_2(m)\|u\|^s \\ &= f_m(\|u\|) - \lambda C_1(m)\|u\|^r, \end{aligned}$$

where $C_1(m), C_2(m) > 0$ (depending on m) are constants, and f_m is defined by (1.5). Since $s > 2p$, there exists $R = R(m) > 0$ large enough such that $f_m(R) < 0$.

Let

$$A = Z_m \cap \partial B_R(0).$$

Then, for any $\lambda > 0$, taking $e \in A$, we have

$$I(e) \leq f_m(\|e\|) - \lambda C_1(m)\|e\|^r \leq f_m(R) < 0.$$

Since the function f_m is continuous and $f_m(0) = 0$, then there is $\delta \in (0, R)$ such that

$$f_m(t) < c_s, \ \forall t \in [0, \delta].$$

Thus, if $t_1 \in \left[0, \frac{\delta}{R}\right]$, then $\|t_1 e\| \in [0, \delta]$, and hence

$$I(t_1 e) \leq f_m(\|t_1 e\|) < c_s.$$

On the other hand, if $t_1 \in \left[\frac{\delta}{R}, 1\right]$, then $\|t_1 e\| \in [\delta, R]$. Since

$$I(t_1 e) \leq f_m(\|t_1 e\|) - \lambda C_1(m)\|t_1 e\|^r,$$

it is enough to show that $f_m(t) - \lambda C_1(m) t^r < c_s$ for any $t \in [\delta, R]$.

Let

$$\Lambda_m := \begin{cases} \max\limits_{t \in [\delta, R]} \dfrac{f_m(t) - c_s}{C_1(m) t^r}, & \text{if } f_m(t_m) > c_s, \\ 0, & \text{if } f_m(t_m) \leq c_s. \end{cases}$$

Then, when $\lambda > \Lambda_m$, $f_m(t) - \lambda C_1(m)\|t\|^r < c_s$ for any $t \in [\delta, R]$. Hence,

$$\max_{t \in [0,1]} I(te) < c_s$$

when $\lambda > \Lambda_m$, and then $c \leq \max\limits_{t \in [0,1]} I(te) < c_s$.

Let

$$\Lambda_* := \min \{\Lambda_m : m \in \mathbb{N}\}. \tag{4.1}$$

Then, $c < c_s$ when $\lambda > \Lambda_*$. In particular, if there exists a $m \in \mathbb{N}$ such that $\Lambda_m = 0$, that is $\max\limits_{t>0} f_m(t) \leq c_s$, then $\Lambda_* = 0$, where f_m is defined by (1.5).

5. Proof of Theorem 1.2

Proof of Theorem 1.2. For any $u \in X \backslash \{0\}$, when $t > 0$ is small enough,

$$\begin{aligned} I(tu) = t^q \Bigg(& \frac{1}{q}\|u\|_{1,q}^q + \frac{t^{p-q}}{p}\|u\|_{1,p}^p + \frac{t^{2p-q}}{2p}|\nabla u|_p^{2p} + \frac{t^{2q-q}}{2q}|\nabla u|_q^{2q} \\ & - \frac{\lambda t^{r-q}}{r}\int_{\mathbb{R}^N} h|u|^r dx - \frac{t^{s-q}}{s}\int_{\mathbb{R}^N} g|u|^s dx \Bigg) \\ > {} & 0. \end{aligned}$$

Hence, $u = 0$ is a strictly local minimizer of I.

Let $Z := \{u \in X : \text{supp}u \subset \Omega\}$, where Ω is defined by (f_1). Then, Z is an infinite dimensional subspace of X. Let Z_m be an m-dimensional subspace of Z. On Z_m,

$$[u]_h := \left(\int_{\mathbb{R}^N} h|u|^r\right)^{\frac{1}{r}},\ [u]_g := \left(\int_{\mathbb{R}^N} g|u|^s\right)^{\frac{1}{s}}$$

are norms of $u \in Z_m$. Since $\dim Z_m < \infty$, all norms on Z_m are equivalent. Therefore, for $u \in Z_m$, we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q + \frac{1}{2p}|\nabla u|_p^{2p} + \frac{1}{2q}|\nabla u|_q^{2q} \\ &\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N} h|u|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} g|u|^s dx \\ &\leq \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q + \frac{1}{2p}\|u\|^{2p} + \frac{1}{2q}\|u\|^{2q} - \lambda C_1(m)\|u\|^r - C_2(m)\|u\|^s \\ &= f_m(\|u\|) - \lambda C_1(m)\|u\|^r, \end{aligned} \tag{5.1}$$

where $C_1(m), C_2(m) > 0$ (depending on m) are constants, and f_m is defined by (1.5). Since $s > 2p$, there exists $R = R(m) > 0$ large enough such that $f_m(R) < 0$.

Let

$$A := Z_m \cap \partial B_R(0). \tag{5.2}$$

Then $i(A) = m$. For any $\lambda > 0$, if $u \in A$, then

$$I(u) \leq f_m(\|u\|) - \lambda C_1(m)\|u\|^r \leq f_m(R) < 0. \tag{5.3}$$

Hence $\max\limits_{u\in A} I(u) \leq 0$.

Since the function f_m is continuous and $f_m(0) = 0$, then there is a $\delta \in (0, R)$ such that

$$f_m(t) < c_s,\ \forall t \in [0, \delta].$$

Thus, if $t_1 \in \left[0, \frac{\delta}{R}\right]$, then for any $u \in A$, we have $\|t_1 u\| \in [0, \delta]$, and hence

$$I(t_1 u) \leq f_m(\|t_1 u\|) < c_s.$$

On the other hand, if $t_1 \in \left[\frac{\delta}{R}, 1\right]$, then for any $u \in A$, we have $\|t_1 u\| \in [\delta, R]$. Since

$$I(t_1 u) \leq f_m(\|t_1 u\|) - \lambda C_1(m)\|t_1 u\|^r,$$

it is enough to show that $f_m(t) - \lambda C_1(m)t^r < c_s$, that is,

$$\lambda > \frac{f_m(t) - c_s}{C_1(m)t^r}$$

for any $t \in [\delta, R]$.

Let

$$\Lambda_m := \begin{cases} \max\limits_{t\in[\delta,R]} \dfrac{f_m(t) - c_s}{C_1(m)t^r}, & \text{if } f_m(t_m) > c_s, \\ 0, & \text{if } f_m(t_m) \leq c_s, \end{cases} \tag{5.4}$$

and then when $\lambda > \Lambda_m$, $f_m(t) - \lambda C_1(m)\|t\|^r < c_s$ for any $t \in [\delta, R]$. Hence,

$$\max_{u \in B} I(u) < c_s,$$

where $B = \{tu \in X : t \in [0,1], u \in A\}$. Applying Lemma 2.2, I has m pairs of nonzero critical points, which are nontrivial solutions of (1.1).

6. Conclusions

In this paper, we study the multiplicity of solutions to the (p,q)-Kirchhoff equation (1.1) with nonnegative weight functions by using the concentration compactness principle and a critical point theorem of Perera. In particular, we only need to prove that the corresponding energy functional satisfies the local Palais-Smale condition. This work partially generalizes the results of Liu and Perera [26, Theorem 1.2], and provides a new perspective for studying multiple solutions of Kirchhoff-type equations. However, this paper only considers the case of nonnegative and bounded weight functions. As for other cases involving sign-changing or unbounded weight functions, we leave them for further investigation in our future work.

Author contributions

J. Hu: Writing–original draft; Q. Yang: Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This study is supported via funding from Education Department of Anhui Province project number (2025AHGXZK40492).

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. C. Alves, F. Corrêa, T. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.,* **49** (2005), 85–93. http://dx.doi.org/10.1016/j.camwa.2005.01.008
2. A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.,* **14** (1973), 349–381. http://dx.doi.org/10.1016/0022-1236(73)90051-7
3. V. Ambrosio, T. Isernia, A multiplicity result for a (p,q)-Schrodinger-Kirchhoff type equation, *Ann. Mat. Pur. Appl.,* **201** (2022), 943–984. http://dx.doi.org/10.1007/s10231-021-01145-y

4. A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, *T. Am. Math. Soc.,* **348** (1996), 305–330. http://dx.doi.org/10.1090/S0002-9947-96-01532-2
5. A. Bahri, Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, *Rev. Mat. Iberoam.,* **6** (1990), 1–15. http://dx.doi.org/10.4171/RMI/92
6. L. Baldelli, R. Filippucci, Existence of solutions for critical (p, q)-Laplacian equations in $\mathbb{R}^N$, *Commun. Contemp. Math.,* **25** (2023), 2150109. http://dx.doi.org/10.1142/S0219199721501091
7. L. Baldelli, Y. Brizi, R. Filippucci, Multiplicity results for (p, q)-Laplacian equations with critical exponent in $\mathbb{R}^N$ and negative energy, *Calc. Var. Partial Dif.,* **60** (2021), 8–30. http://dx.doi.org/10.1007/s00526-020-01867-6
8. J. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Philos. T. Roy. Soc. A,* **306** (1982), 557–611. http://dx.doi.org/10.1098/rsta.1982.0095
9. T. Bartsch, M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$, *Arch. Rat. Mech. Anal.,* **124** (1993), 261–276. http://dx.doi.org/10.1007/BF00953069
10. H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *P. Am. Math. Soc.,* **88** (1983), 486–490. http://dx.doi.org/10.1090/S0002-9939-1983-0699419-3
11. M. Chaves, G. Ercole, O. Miyagaki, Existence of a nontrivial solution for a (p, q)-Laplacian equation with p-critical exponent in $\mathbb{R}^N$, *Bound. Value Probl.,* **2014** (2014), 1–15. http://dx.doi.org/10.1186/s13661-014-0236-x
12. G. Citti, A uniqueness theorem for radial ground states of the equation $\Delta_p u + f(u) = 0$, *Boll. Un. Mat. Ital.,* **7** (1993), 283–310.
13. G. Citti, F. Uguzzoni, Positive solutions of $-\Delta_p u + u^{p-1} - q(x)u^\alpha = 0$ in $\mathbb{R}^N$, *NoDEA,* **9** (2002), 1–14. http://dx.doi.org/10.1007/s00030-002-8115-z
14. J. Díaz, *Nonlinear partial differential equations and free boundaries*, London: Pitman Advanced publishing Limited, 1985.
15. E. Fadell, P. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, *Invent. Math.,* **45** (1978), 139–174. http://dx.doi.org/10.1007/BF01390270
16. G. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, *J. Math. Anal. Appl.,* **401** (2013), 706–713. http://dx.doi.org/10.1016/j.jmaa.2012.12.053
17. C. He, G. Li, The existence of a nontrivial solution to the p-q-Laplacian problem with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$, *Nonlinear Anal. TMA,* **68** (2008), 1100–1119. http://dx.doi.org/10.1016/j.na.2006.12.008
18. X. He, W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, *Nonlinear Anal. TMA,* **70** (2009), 1407–1414. http://dx.doi.org/10.1016/j.na.2008.02.021
19. X. He, W. Zou, Multiplicity of solutions for a class of Kirchhoff type problems, *Acta Math. Appl. Sin.-E.,* **26** (2010), 387–394. http://dx.doi.org/10.1007/s10255-010-0005-2
20. X. He, W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, *J. Differ. Equ.,* **252** (2012), 1813–1834. http://dx.doi.org/10.1016/j.jde.2011.08.035
21. M. Latifi, R. Bayat, Existence of nontrivial solution for a quasilinear elliptic equation with (p, q)-Laplacian in $\mathbb{R}^N$ involving critical Sobolev exponents, *J. Elliptic Parabol.,* **6** (2020), 733–749. http://dx.doi.org/10.1007/s41808-020-00081-x

22. G. Li, X. Liang, The existence of nontrivial solutions to nonlinear elliptic equation of p-q-Laplacian type on $\mathbb{R}^N$, *Nonlinear Anal. TMA,* **71** (2009), 2316–2334. http://dx.doi.org/10.1016/j.na.2009.01.066
23. G. Li, X. Zhu, Nontrivial solutions of nonlinear scalar field equations with strong nonlinearity, *Acta Math. Sci.,* **8** (1988), 431–448. http://dx.doi.org/10.1016/S0252-9602(18)30320-5
24. J. Lions, On some questions in boundary value problems of mathematical physics, *North-Holland Math. Stud.,* **30** (1978), 284–346. http://dx.doi.org/10.1016/S0304-0208(08)70870-3
25. P. Lions, The concentration-compacteness principle in the calculus of variations. The locally compact case, part 1, *Ann. I. H. Poincaré-Non.,* **1** (1984), 109–145. http://dx.doi.org/10.1016/S0294-1449(16)30428-0
26. S. Liu, K. Perera, Multiple solutions for (p, q)-Laplacian equations in $\mathbb{R}^N$ with critical or subcritical exponents, *Calc. Var. Part. Differ. Equ.,* **63** (2024), 199. http://dx.doi.org/10.1007/s00526-024-02811-8
27. A. Mao, Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, *Nonlinear Anal. TMA,* **70** (2009), 1275–1287. http://dx.doi.org/10.1016/j.na.2008.02.011
28. J. Ó. Solutions to perturbed eigenvalue problems of the p-Laplacian in $\mathbb{R}^N$, *Electron. J. Differ. Equ.,* **1997** (1997), 1–15. Available from: `http://eudml.org/doc/119277`.
29. K. Perera, Abstract multiplicity theorems and applications to critical growth problems, *J. Anal. Math.* **157** (2025), 211–223. http://dx.doi.org/10.1007/s11854-025-0389-9
30. P. Pucci, J. Serrin, Uniqueness of ground states for quasilinear elliptic operators, *Indiana Univ. Math. J.,* **47** (1998), 501–528. Available from: `http://www.jstor.org/stable/24899682`.
31. J. Serrin, M. Tang, Uniqueness of ground states for quasilinear elliptic equations, *Indiana Univ. Math. J.,* **49** (2000), 897–923. Available from: `http://www.jstor.org/stable/24901124`.
32. W. Strauss, Existence of solitary waves in higher dimensions, *Commun. Math. Phys.,* **55** (1977), 149–162. http://dx.doi.org/10.1007/BF01626517
33. A. Szulkin, T. Weth, *The method of Nehari manifold*, International Press, Boston, 2010, 597–632.
34. M. Wu, Z. Yang, A class of p-q-Laplacian type equation with potentials eigenvalue problem in $\mathbb{R}^N$, *Bound. Value Probl.,* **2009** (2009), 185319. http://dx.doi.org/10.1155/2009/185319
35. J. Yang, X. Zhu, On the existence of nontrivial solution of a quasilinear elliptic boundary value problem for unbounded Domains (I) Positive mass case, *Acta Math. Sci.,* **7** (1987), 341–359. http://dx.doi.org/10.1016/S0252-9602(18)30457-0
36. Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, *J. Math. Anal. Appl.,* **317** (2006), 456–463. http://dx.doi.org/10.1016/j.jmaa.2005.06.102
37. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.,* **29** (1987), 675–710. http://dx.doi.org/10.1070/IM1987v029n01ABEH000958

AIMS Press