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Research article

Arithmetic properties of the Fourier coefficients of the eighth-order mock theta function

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Abstract: In this paper, we investigate the arithmetic properties of the coefficients of the eighth-order mock theta function $V_0(q)$, building upon the foundational work of Gordon and McIntosh on mock theta functions. Several novel dissection formulas are derived, leading to new identities for the Dedekind eta function at level 8. Furthermore, we establish new congruence relations and partition recurrence formulas associated with the coefficients of the eighth-order mock theta function.

Keywords: mock theta function; q-series; congruence relation; Jacobi triple product; modular forms; Eisenstein series; cusp forms; integer partition
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1. Introduction

In January 1920, Ramanujan wrote his final letter to G.H. Hardy, which introduced a remarkable list of 17 functions, which he referred to as "mock theta functions". Each of these functions is expressed as a Fourier series in the variable $q = e^{2\pi i\tau}$, where $\tau \in \mathbb{H}$, and $\mathbb{H} = \{\tau \in \mathbb{C} \mid Im(\tau) \geq 0\}$ represents the complex upper half-plane. A mock theta function is defined by a q-series of a special form (Eulerian form), which converges for $\mid q \mid< 1$ and satisfies the following conditions:

(1) Infinitely many roots of unity are exponential singularities.
(2) For every root of unity ξ, there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially (presumably with only finitely many of the θ_ξ being different).
(3) No theta function works for all ξ; that is, f is not the sum of two functions, one of which is a theta function, and the other function is bounded in all roots of unity.

Ramanujan divided them into three distinct orders known as “third”, “fifth”, and “seventh”, yet left no explanation of what he meant by “order,” leaving future generations to ponder their true significance.

After Ramanujan’s passing, his mock theta functions sparked significant interest. Mathematicians such as Watson, Andrews, and Hickerson focused on understanding these enigmatic functions. Gordon and McIntosh later redefined the concept of “order”, tying it to modular group action. Gordon and McIntosh constructed transformation laws for the mock theta function, which led to the discovery of three additional mock theta functions of order three [1].

Later, in 1976, Andrews and Hickerson discovered six more mock theta functions from Ramanujan’s “Lost Notebook”, which they called sixth-order mock theta functions [2]. Also, Andrews et al. have authored several books that detail the proofs of the formulas found in Ramanujan’s notebook [3–5]. In [6], Choi uncovered four additional mock theta functions in Ramanujan’s work; they have been referred to as tenth-order mock theta functions.

In 2002, Zwegers demonstrated that mock theta functions are not isolated phenomena but the holomorphic components of harmonic weak Maass forms, providing a modern framework for their study and further showing that Ramanujan’s classical mock theta functions are simultaneously quotients of indefinite theta functions, Lerch sums, and Fourier coefficients of Jacobi forms [2, 7, 8].

Ramanujan’s mock theta functions, his final gift to mathematics, remain a source of wonder and discovery. Although we may never fully understand what he envisioned when he introduced the concept of “order”, the ongoing exploration of these functions continues to illuminate new paths in modular forms, combinatorics, and beyond.

Not long ago, Gordon and McIntosh [9] provided a method of constructing mock theta functions from ordinary theta series using a half-shift transformation. They studied four mock theta functions and referred to them as being of eighth order. These mock theta functions are as follows

$$S_0(q) = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{1.1}$$

$$S_1(q) = \sum_{n=0}^{\infty} q^{(n^2+2n)} \frac{(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{1.2}$$

$$T_0(q) = \sum_{n=0}^{\infty} q^{(n^2+3n+2)} \frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, \tag{1.3}$$

$$T_1(q) = \sum_{n=0}^{\infty} q^{(n^2+n)} \frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}. \tag{1.4}$$

The following functions determine how the eighth-order mock theta functions transform:

$$U_0(q) = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(-q^4;q^4)_n}, \tag{1.5}$$

$$U_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2} \frac{(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \tag{1.6}$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(q;q^2)_n} = -1 + 2\sum_{n=0}^{\infty} q^{2n^2} \frac{(-q^2;q^4)_n}{(-q;q^2)_{2n+1}}, \tag{1.7}$$

$$V_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}\frac{(-q;q^2)_n}{(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}q^{2n(n+1)+1}\frac{(-q^4;q^4)_n}{(q;q^2)_{2n+2}}. \tag{1.8}$$

$V_0(q)$ is an intriguing function, as it generates $v_0(q)$, which enumerates the over-partitions of n into odd parts, ensuring no gaps among the nonoverlined parts. It is expressed as

$$v_0(q)=\sum_{n=0}^{\infty}q^{n^2}\frac{(-q;q^2)_n}{(q;q^2)_n}=\frac{V_0(q)}{2}+\frac{1}{2}. \tag{1.9}$$

Also,

$$V_0(-q)+V_0(q)=2(-q^2;q^4)_\infty^4(q^8;q^8)_\infty. \tag{1.10}$$

Although several identities for eighth-order mock theta functions are known, explicit arithmetic investigations of the Fourier coefficients associated with the function $V_0(q)$ remain comparatively limited. In particular, although the transformation behavior of $V_0(q)$ and related functions have been studied by Gordon-McIntosh and later by Mao, fewer results appear to be available concerning explicit coefficient extractions, arithmetic progressions, and congruence families for the coefficients of $v_0(n)$.

The purpose of this paper is to study the arithmetic structure of the coefficients of $V_0(q)$ through explicit eta-quotient expansions and dissections. Our goal is not to reinterpret the modularity of these functions in an abstract setting, but rather to obtain concrete arithmetic formulas for coefficients in specific residue classes, together with congruence and recurrence consequences. In particular, we derive explicit eta-product representations for several arithmetic progressions of $v_0(n)$, from which divisibility properties and partition-theoretic recurrences follow naturally.

We emphasize that many of the identities obtained in this paper may also be verified by standard modular-form arguments once the relevant eta-quotient expressions are identified. In particular, after placing the corresponding generating functions in suitable spaces of modular forms, equality may alternatively be confirmed by checking finitely many Fourier coefficients via Sturm's theorem. However, the approach adopted here is constructive and coefficient-oriented: Our emphasis is on explicit q-series dissections, as these produce concrete arithmetic progression formulas and make the resulting congruence information transparent at the coefficient level.

2. Preliminaries

If b and q are complex numbers, then the q-shifted factorial is defined as:

$$(b;q)_n=(1-b)(1-bq)(1-bq^2)\cdots(1-bq^{n-1}),$$

where $\mid q\mid<1$. Notice that for $n=0$, we have $(b;q)_0=1$. For $n>\infty$, we will represent the product in the following way:

$$(b;q)_\infty=\lim_{n\to\infty}(b;q)_n=\prod_{n=0}^{\infty}(1-bq^n)=\prod_{n=1}^{\infty}(1-bq^{n-1}). \tag{2.1}$$

Moreover, we use the short notation

$$(b_1,b_2,b_3,...,b_m;q)=(b_1;q)_\infty(b_2;q)_\infty(b_3;q)_\infty...(b_m;q)_\infty. \tag{2.2}$$

The function $f(x,y)$, referred to as Ramanujan's general-theta function, was defined by Ramanujan as:

$$f(x,y) = \sum_{t\in\mathbb{Z}} x^{\frac{t(t+1)}{2}} y^{\frac{t(t-1)}{2}}, \quad \text{where} \quad |\, xy \,| < 1. \tag{2.3}$$

From Eq (2.3), we recollect some special cases, $f(x,y)$ as discussed in [10, 11]:

$$f(q;q) = \phi(q) = \sum_{t\in\mathbb{Z}} q^{t^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{2.4}$$

$$f(q,q^3) = \psi(q) = \sum_{t=0}^{\infty} q^{\frac{t(t+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}, \tag{2.5}$$

$$f(-q;-q^2) = f(-q) = \sum_{t\in\mathbb{Z}} (-1)^t q^{\frac{t(3t-1)}{2}} = (q;q)_\infty. \tag{2.6}$$

By using the customary notation $U_n = (q^n;q^n)_\infty$, $U_n^k = (q^n;q^n)_\infty^k$, and $U_n^k U_m^\ell = U_{(n,m)}^{(k,\ell)}$, for k, n, m, and ℓ are positive integers, the above functions can be rewritten as:

$$\phi(q) = \frac{U_2^5}{U_1^2 U_4^2}, \quad \psi(q) = \frac{U_2^2}{U_1^1}, \quad f(q) = \frac{U_2^3}{U_1 U_4}, \tag{2.7}$$

$$\phi(-q) = \frac{U_1^2}{U_2^1}, \quad \psi(-q) = \frac{U_1 U_4}{U_2^1}, \quad f(-q) = U_1. \tag{2.8}$$

From (1.9) and (1.10), we get

$$\sum_{n=1}^{\infty} v_0(2n) q^n = \frac{U_2}{2 U_1 U_4^3} - \frac{1}{2}. \tag{2.9}$$

We note that the eta-quotient representation in (2.9) should be interpreted as the generating function for the even-indexed coefficients of $v_0(n)$, normalized so that the resulting product has the expected modular weight 1/2. In particular, this form places the even part of $v_0(n)$ naturally within the framework of weakly holomorphic modular forms of half-integral weight. Consequently, once an identity is expressed in eta-quotient form, one may also verify it by standard modular-form methods, for instance by checking finitely many Fourier coefficients up to the Sturm bound. This modular interpretation complements, but does not replace, the explicit dissection arguments used throughout the present work.

Using Zwegers' contributions to the study of Appell Lerch sums and the broader theory of (mock) modular forms, Mao [12] has recently derived two identities for $V_0(q)$, described via $v_0(n)$.

$$\sum_{n=0}^{\infty} v_0(8n+2) = 2\frac{U_2^4 U_4^5}{U_1^6 U_8^2}, \tag{2.10}$$

$$\sum_{n=0}^{\infty} v_0(8n+6) = 4\frac{U_2^2 U_4^4}{U_1^5}. \tag{2.11}$$

Readers interested in more recent research on identities involving mock theta functions may consult [13–15]. Now, observe that the identities listed below will be used next:

Lemma 2.1. *We have the following* 4*dissections:*

$$\frac{U_2^3}{U_1^3} = \frac{U_6}{U_3} + 3q\frac{U_6^4U_9^5}{U_3^8U_{18}} + 6q^2\frac{U_6^3U_9^2U_{18}^2}{U_3^7} + 12q^3\frac{U_6^2U_{18}^5}{U_3^6U_9}. \tag{2.12}$$

Toh [16] provided the proof of Eq (2.12).

Lemma 2.2. *We have the following* 2*-dissections:*

$$\frac{U_2^2}{U_1^1} = \frac{U_6U_9^2}{U_3U_{18}} + q\frac{U_{18}^2}{U_9}. \tag{2.13}$$

This dissection has been proved and discussed by several prominent mathematicians, including Cooper [17], Andrews and Berndt [4], and Hickerson [18].

Lemma 2.3. *We have the following* 2*-dissections:*

$$\frac{1}{U_1^2} = \frac{U_8^5}{U_2^5U_{16}^2} + 2q\frac{U_4^2U_{16}^2}{U_2^5U_8}, \tag{2.14}$$

$$\frac{1}{U_1^4} = \frac{U_4^{14}}{U_2^{14}U_8^4} + 4q\frac{U_4^2U_8^4}{U_2^{10}}. \tag{2.15}$$

The proofs of (2.14) and (2.15) were introduced by Xia and Yao [19], providing novel approaches.

Lemma 2.4. *We have the* 3*-dissection:*

$$\frac{U_2^1}{U_1^2} = \frac{U_6^4U_9^6}{U_3^8U_{18}^3} + 2q\frac{U_6^3U_9^3}{U_3^7} + 4q^2\frac{U_6^2U_{18}^3}{U_3^6}. \tag{2.16}$$

Berndt [20] discusses this dissection in his work on Ramanujan's notebooks. This book examines many results from Ramanujan's work, including similar dissections related to theta functions.

Lemma 2.5. *We have the* 2*-dissection:*

$$\frac{U_3}{U_1^3} = \frac{U_4^6U_6^3}{U_2^9U_{12}^2} + 3q\frac{U_4^2U_6U_{12}^2}{U_2^7}, \tag{2.17}$$

$$\frac{U_3}{U_1} = \frac{U_4U_6U_{16}U_{24}^2}{U_2^2U_8U_{12}U_{48}} + q\frac{U_6U_8^2U_{48}}{U_2^2U_{16}U_{24}}. \tag{2.18}$$

For the proof of (2.17) and (2.18), refer to [21–23]. We also employ the following congruences.

Lemma 2.6.

$$U_1 \equiv \Pi(q) \pmod 2. \tag{2.19}$$

Proof. Euler's product [18] provides:

$$U_1 = \sum_{t\in\mathbb{Z}}(-1)^t q^{t\frac{(3t+1)}{2}} \equiv \sum_{t\in\mathbb{Z}} q^{t\frac{(3t+1)}{2}} = \Pi(q). \tag{2.20}$$

□

Lemma 2.7.

$$U_1^3 \equiv \psi(q) \pmod 4. \tag{2.21}$$

Proof. Jacobi's version of Euler's product [18] produces:

$$U_1^3 = \sum_{t\geq 0}(-1)^t(2t+1)q^{t\frac{(t+1)}{2}} = \sum_{t\in\mathbb{Z}}(4t+1)q^{t(2t+1)} \equiv \sum_{t\in\mathbb{Z}} q^{t(2t+1)} \pmod 4 = \psi(q). \tag{2.22}$$

□

Before proceeding to the main results, we make an important methodological remark. Several of the identities established in the next section are derived through explicit eta-dissection manipulations. Although this approach is elementary and constructive, it is not the only possible one. Once the relevant generating functions are rewritten as eta-quotients, many of the stated identities and congruences may alternatively be verified using the theory of modular forms, especially by placing both sides in the same modular space and applying Sturm's criterion. The advantage of the present approach, however, is that it yields explicit coefficient extractions and progressionwise generating functions, which are particularly convenient for deriving arithmetic and partition-theoretic consequences.

3. A set of main results

Theorem 3.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(8n+2)q^n = 4\frac{U_2^4U_4^5}{U_1^6U_8^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} v_0(8n+6)q^n = 8\frac{U_2^6U_8^2}{U_1^6U_4}. \tag{3.2}$$

It can be observed that $v_0(8n+2) \equiv 0 \pmod 4$, $v_0(8n+6) \equiv 0 \pmod 8$.

Theorem 3.2. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(16n+2)q^n = 4\Big[\frac{U_2^{19}}{U_1^{15}U_4U_8^2} + 8q\frac{U_2^9U_4U_8^2}{U_1^{11}}\Big], \tag{3.3}$$

$$\sum_{n=0}^{\infty} v_0(16n+10)q^n = 8\Big[\frac{U_2^{21}U_8^2}{U_1^{15}U_4^7} + 2\frac{U_2^7U_4^7}{U_1^{11}U_8^2}\Big]. \tag{3.4}$$

In this case, $v_0(16n+2) \equiv 0 \pmod 4$, $v_0(16n+10) \equiv 0 \pmod 8$.

Theorem 3.3. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(16n+6)q^n = 8\Big[\frac{U_2^{13}U_4^3}{U_1^{13}U_8^2} + 8q\frac{U_2^3U_4^5U_8^2}{U_1^9}\Big], \tag{3.5}$$

$$\sum_{n=0}^{\infty} v_0(16n+14)q^n = 16\Big[\frac{U_2^{15}U_8^2}{U_1^{13}U_4^3} + 2\frac{U_2U_4^{11}}{U_1^9U_8^2}\Big]. \tag{3.6}$$

In particular, $v_0(16n+6) \equiv 0 \pmod 8$, $v_0(16n+14) \equiv 0 \pmod{16}$.

Theorem 3.4. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(24n+7)q^n = 12\Big[\frac{U_2^{18}U_3^9}{U_1^{20}U_6^6} + 27q\frac{U_2^{10}U_3^5U_6^2}{U_1^{16}}\Big], \tag{3.7}$$

$$\sum_{n=0}^{\infty} v_0(24n+19)q^n = 108\Big[\frac{U_2^{14}U_3^7}{U_1^{18}U_6^2} + 3q\frac{U_2^6U_3^3U_6^6}{U_1^{14}}\Big]. \tag{3.8}$$

Specifically, $v_0(24n+7) \equiv 0 \pmod{12}$, $v_0(24n+19) \equiv 0 \pmod{108}$.

Theorem 3.5. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(12n+4)q^n = 6\frac{U_2^{13}U_3^3}{U_1^{11}U_4^4}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} v_0(12n+10)q^n = 24\frac{U_2U_3^3U_4^4}{U_1^7}. \tag{3.10}$$

In this case, $v_0(12n+4) \equiv 0 \pmod{6}$, $v_0(12n+10) \equiv 0 \pmod{24}$.

Theorem 3.6. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(24n+4)q^n \equiv 6\frac{U_2^{14}U_3^9U_4^5}{U_1^{19}U_6^6U_8^2} + 108q\frac{U_2^{12}U_3^7U_8^2}{U_1^{17}U_4U_6^2} \pmod{162}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} v_0(24n+16)q^n \equiv 12\frac{U_2^{16}U_3^9U_8^2}{U_1^{19}U_4U_6^6} + 54\frac{U_2^{10}U_3^7U_4^5}{U_1^{17}U_6^2U_8^2} + 162q\frac{U_2^2U_3^5U_4^5U_6^6}{U_1^{13}U_8^2} \pmod{324}. \tag{3.12}$$

Theorem 3.7. *When $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(24n+10)q^n \equiv 24\frac{U_2^{21}U_3^3U_4^3U_{12}^6}{U_1^{19}U_4^7U_6^3U_{24}^3} + 72q\frac{U_2^{19}U_3^3U_{24}}{U_1^{19}U_4U_6U_8} \pmod{96}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} v_0(24n+22)q^n \equiv 72\frac{U_2^{20}U_3^3U_8U_{12}^3}{U_1^{19}U_4^4U_6^2U_{24}} + 96\frac{U_2^9U_3^3U_4U_8^3U_{12}^6}{U_1^{15}U_6^3U_{24}^3} + 24q\frac{U_2^{18}U_3^3U_4^2U_{24}^3}{U_1^{19}U_8^3U_{12}^3} \pmod{288}. \tag{3.14}$$

Remark 1. The composite moduli appearing in Theorems 3.6 and 3.7 arise naturally from the powers of 2 and 3 introduced through repeated 2- and 3-dissections of eta-quotients. The moduli reflect the maximal divisibility obtainable from the coefficients of the resulting q-series expansions. These congruences are therefore intrinsic to the arithmetic structure of the eighth-order mock theta function $V_0(q)$ and are not merely artifacts of the proof technique.

Before proving the theorems, we record several natural companion identities suggested by the preceding dissections and coefficient computations. These formulas arise naturally from the same eta-quotient framework and are strongly supported by extensive coefficient verification. Moreover, in view of the modular interpretation discussed above, these conjectures are also consistent with the expected structure of half-integral weight modular forms and may be approached through modular-form comparison and Sturm bounds. We include them here both as guiding formulas for the subsequent analysis and as natural continuations of the arithmetic pattern exhibited by $v_0(n)$.

Conjecture 1. *Because $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n = 4\frac{U_4^4}{U_1^2 U_2}. \tag{3.15}$$

Conjecture 2. *For all $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(12n+7)q^n = 12\frac{U_2^7 U_3^3}{U_1^9}. \tag{3.16}$$

Conjecture 3. *When $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = 6\frac{U_2^3 U_6^3}{U_1^4 U_4}. \tag{3.17}$$

From Eq (3.15),

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(4n+2)q^n &\equiv 4\frac{U_1^{14}}{U_2} \pmod{8}\\ &= 4U^{12}\phi(-q)\\ &= 4U^{12}\sum_{t\in\mathbb{Z}}(-1)^t q^{t^2}\\ &= 4U^{12} + 8U^{12}\sum_{t\geq 1}(-1)^t q^{t^2}.\end{aligned}$$

From Eq (3.16), using Lemmas 2.6 and 2.7,

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(12n+7)q^n &\equiv 12U_2^7\left(\frac{\Pi(q^3)}{\psi(q)}\right)^3 ,\\ &= 12U_2^7\left(\frac{\Pi(q^3)}{\Pi(q^3)+q\psi(q^9)}\right)^3,\\ &= 12U_2^7\left(1+\frac{\Pi(q^3)}{q\psi(q^9)}\right)^3.\end{aligned}$$

Proof of Theorem (3.1). From Eq (3.15),

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n = 4\frac{U_4^4}{U_2}\left(\frac{1}{U_1^2}\right). \tag{3.18}$$

Inserting the values from Eq (2.14), we get

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n = 4\frac{U_4^4 U_8^5}{U_2^6 U_{16}^2} + 8q\frac{U_4^6 U_{16}^2}{U_2^6 U_8}. \tag{3.19}$$

Extracting q^{2n} terms from (3.19), we get

$$\sum_{n=0}^{\infty} v_0(8n+2)q^n = 4\frac{U_2^4U_4^5}{U_1^6U_8^2}. \tag{3.20}$$

Extracting q^{2n+1} terms from (3.19), yields

$$\sum_{n=0}^{\infty} v_0(8n+6)q^n = 8\frac{U_2^6U_8^2}{U_1^6U_4}. \tag{3.21}$$

□

Proof of Theorem 3.2. From Eq (3.20),

$$\sum_{n=0}^{\infty} v_0(8n+2)q^n = 4\frac{U_2^4U_4^5}{U_8^2}\left(\frac{1}{U_1^2}\right)\left(\frac{1}{U_1^4}\right). \tag{3.22}$$

Substituting the values from (2.14) and (2.15), we get:

$$\sum_{n=0}^{\infty} v_0(8n+2)q^n = 4\frac{U_4^{19}}{U_2^{15}U_8U_{16}^2} + 8q\frac{U_4^{21}U_{16}^2}{U_2^{15}U_8^7} + 16q\frac{U_4^7U_8^7}{U_2^{11}U_{16}^2} + 32q^2\frac{U_4^9U_8U_{16}^2}{U_2^{11}}. \tag{3.23}$$

By extracting the q^{2n} terms from (3.23) and substituting q^2 by q, we find

$$\sum_{n=0}^{\infty} v_0(16n+2)q^n = 4\frac{U_2^{19}}{U_1^{15}U_4U_8^2} + 32q\frac{U_2^9U_4U_8^2}{U_1^{11}}. \tag{3.24}$$

Extracting q^{2n+1} terms from (3.23) and dividing by q, yields

$$\sum_{n=0}^{\infty} v_0(16n+10)q^n = 8\frac{U_2^{21}U_8^2}{U_1^{15}U_4^7} + 16\frac{U_2^7U_4^7}{U_1^{11}U_8^2}. \tag{3.25}$$

□

Proof of theorem (3.3). From Eq (3.21),

$$\sum_{n=0}^{\infty} v_0(8n+6)q^n = 8\frac{U_2^6U_8^2}{U_4}\left(\frac{1}{U_1^2}\right)\left(\frac{1}{U_1^4}\right). \tag{3.26}$$

By applying the values from Eqs (2.14) and (2.15), the result is

$$\sum_{n=0}^{\infty} v_0(8n+6)q^n = 8\frac{U_4^{13}U_8^3}{U_2^{13}U_{16}^2} + 16q\frac{U_4^{15}U_{16}^2}{U_2^{13}U_8^3} + 32q\frac{U_4U_8^{11}}{U_2^9U_{16}^2} + 64q^2\frac{U_4^3U_8^5U_{16}^2}{U_2^9}. \tag{3.27}$$

Bringing out the terms having q^{2n} from (3.27) and replacing q^2 with q, we get

$$\sum_{n=0}^{\infty} v_0(16n+6)q^n = 8\frac{U_2^{13}U_4^3}{U_1^{13}U_8^2} + 64q\frac{U_2^3U_4^5U_8^2}{U_1^9}. \tag{3.28}$$

Taking the q^{2n+1} related terms from (3.27) and dividing by q, the result becomes

$$\sum_{n=0}^{\infty} v_0(16n+14)q^n = 16\frac{U_2^{15}U_8^2}{U_1^{13}U_4^3} + 32\frac{U_2U_4^{11}}{U_1^9U_8^2}. \tag{3.29}$$

This completes the proof of (3.3). □

Proof of Theorem 3.4. Equation (3.16) can be written as

$$\sum_{n=0}^{\infty} v_0(12n+7)q^n = 12U_2^7\left(\frac{U_3}{U_1^3}\right)^3. \tag{3.30}$$

Substituting the values from Eq (2.17), we obtain

$$\sum_{n=0}^{\infty} v_0(12n+7)q^n = 12\frac{U_4^{18}U_6^9}{U_2^{20}U_{12}^6} + 108q\frac{U_4^{14}U_6^7}{U_2^{18}U_{12}^2} + 324q^2\frac{U_4^{10}U_6^5U_{12}^2}{U_2^{16}} + 324q^3\frac{U_4^6U_6^3U_{12}^6}{U_2^{14}}. \tag{3.31}$$

Taking the q^{2n} containing terms from (3.31) and substituting q^2 with q, we derive

$$\sum_{n=0}^{\infty} v_0(24n+7)q^n = 12\frac{U_2^{18}U_3^9}{U_1^{20}U_6^6} + 324q\frac{U_2^{10}U_3^5U_6^2}{U_1^{16}}. \tag{3.32}$$

From Eq (3.31), we extract the q^{2n+1} dependent terms, dividing by q and replacing q^2 with q, yielding

$$\sum_{n=0}^{\infty} v_0(24n+19)q^n = 108\frac{U_2^{14}U_3^7}{U_1^{18}U_6^2} + 324q\frac{U_2^6U_3^3U_6^6}{U_1^{14}}. \tag{3.33}$$

Taking modulo 12 and 108 of Eqs (3.32) and (3.33) results in Eqs (3.7) and (3.8), respectively. □

Proof of Theorem 3.5. From Eq (3.17),

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = 6\frac{U_2^3U_6^3}{U_4}\left(\frac{1}{U_1^4}\right). \tag{3.34}$$

Substituting the values from Eq (2.15), we obtain

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = 6\frac{U_4^{13}U_6^3}{U_2^{11}U_8^4} + 24q\frac{U_4U_6^3U_8^4}{U_2^7}. \tag{3.35}$$

Isolating the terms containing q^{2n} from Eq (3.35), we derive (3.9). Also, from Eq (3.35), identify the terms containing q^{2n+1} and divide these terms by q to get (3.10). □

Proof of Theorem 3.6. From Eq (3.9),

$$\sum_{n=0}^{\infty} v_0(12n+4)q^n = 6\frac{U_2^{13}U_3^3}{U_4^4}\left(\frac{1}{U_1^2}\right)\left(\frac{U_3}{U_1^3}\right)^3. \tag{3.36}$$

Applying the values from (2.14) and (2.17), we get

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(12n+4)q^n &= 6\frac{U_4^{14}U_6^{9}U_8^{5}}{U_2^{19}U_{12}^{6}U_{16}^{2}} + 12q\frac{U_4^{16}U_6^{9}U_{16}^{2}}{U_2^{19}U_8U_{12}^{6}} + 54q\frac{U_4^{10}U_6^{7}U_8^{5}}{U_2^{17}U_{12}^{2}U_{16}^{2}} + 108q^2\frac{U_4^{12}U_6^{7}U_{16}^{2}}{U_2^{17}U_8U_{12}^{2}}\\ &+ 162q^2\frac{U_4^{6}U_6^{5}U_8^{5}U_{12}^{2}}{U_2^{15}U_{16}^{2}} + 162q^3\frac{U_4^{2}U_6^{3}U_8^{5}U_{12}^{6}}{U_2^{13}U_{16}^{2}} + 324q^3\frac{U_4^{8}U_6^{5}U_{12}^{2}U_{16}^{2}}{U_2^{15}U_8}\\ &+ 324q^4\frac{U_4^{4}U_6^{3}U_{12}^{6}U_{16}^{2}}{U_2^{13}U_8}. \end{aligned} \tag{3.37}$$

Terms containing q^{2n} are extracted from Eq (3.37) and divided by q. q^2 is substituted with q and q^4 by q^2 obtaining

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(24n+4)q^n &= 6\frac{U_2^{14}U_3^{9}U_4^{5}}{U_1^{19}U_6^{6}U_8^{2}} + 108q\frac{U_2^{12}U_3^{7}U_8^{2}}{U_1^{17}U_4U_6^{2}}\\ &+ 162q\frac{U_2^{6}U_3^{5}U_4^{5}U_6^{2}}{U_1^{15}U_8^{2}} + 342q^2\frac{U_2^{4}U_3^{3}U_6^{6}U_8^{2}}{U_1^{13}U_4}. \end{aligned} \tag{3.38}$$

By isolating the q^{2n+1}-related terms in Eq (3.37), dividing them by q, and substituting q^2 with q, we proceed to

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(24n+16)q^n &= 54\frac{U_2^{10}U_3^{7}U_4^{5}}{U_1^{17}U_6^{2}U_8^{2}} + 162q\frac{U_2^{2}U_3^{5}U_4^{5}U_6^{6}}{U_1^{13}U_8^{2}}\\ &+ 12\frac{U_2^{16}U_3^{9}U_8^{2}}{U_1^{19}U_4U_6^{6}} + 342q\frac{U_2^{8}U_3^{5}U_6^{2}U_8^{2}}{U_1^{15}U_4}. \end{aligned} \tag{3.39}$$

By taking modulo 162 and 324 of the right sides of (3.38) and (3.39), respectively, we get (3.11) and (3.12). □

Proof of Theorem 3.7. Equation (3.10) can be written as

$$\sum_{n=0}^{\infty} v_0(12n+10)q^n = 24U_2U_4^{4}\left(\frac{1}{U_1^{2}}\right)\left(\frac{U_3}{U_1}\right)^{3}. \tag{3.40}$$

Inserting the values from (2.15) and (2.18), we get

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(12n+10)q^n &= 24\frac{U_4^{21}U_6^{3}U_{16}^{3}U_{24}^{6}}{U_2^{19}U_8^{7}U_{12}^{3}U_{48}^{3}} + +72q\frac{U_4^{20}U_6^{3}U_{16}U_{24}^{3}}{U_2^{19}U_8^{4}U_{12}^{2}U_{48}} + 96q\frac{U_4^{9}U_6^{3}U_8U_{16}^{3}U_{24}^{6}}{U_2^{15}U_{12}^{3}U_{48}^{3}}\\ &+ 72q^2\frac{U_4^{19}U_6^{3}U_{48}}{U_2^{19}U_8U_{12}U_{16}} + 288q^2\frac{U_4^{8}U_6^{3}U_8^{4}U_{16}U_{24}^{3}}{U_2^{15}U_{12}^{2}U_{48}} + 24q^3\frac{U_4^{18}U_6^{3}U_8^{2}U_{48}^{3}}{U_2^{19}U_{16}^{3}U_{24}^{3}}\\ &+ 288q^3\frac{U_4^{7}U_6^{3}U_8^{7}U_{48}}{U_2^{15}U_{12}U_{16}} + 96q^4\frac{U_4^{6}U_6^{3}U_8^{10}U_{48}^{3}}{U_2^{15}U_{16}^{3}U_{24}^{3}}. \end{aligned} \tag{3.41}$$

Bringing out the terms containing q^{2n} from (3.41) and replacing q^2 with q and q^4 with q^2, we get

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(24n+10)q^n &= 24\frac{U_2^{21}U_3^3U_4^3U_{12}^6}{U_1^{19}U_4^7U_6^3U_{24}^3} + 72q\frac{U_2^{19}U_3^3U_{24}}{U_1^{19}U_4U_6U_8} \\ &+ 288q\frac{U_2^8U_3^3U_4^4U_8U_{12}^3}{U_1^{15}U_6^2U_{24}} + 96q^2\frac{U_2^6U_3^3U_4^{10}U_{24}^3}{U_1^{15}U_8^3U_{12}^3}. \end{aligned} \tag{3.42}$$

From Eq (3.41), select the terms involving q^{2n+1}, divide by q, and make the substitution $q^2 \to q$, to get

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(24n+22)q^n &= 72\frac{U_2^{20}U_3^3U_8U_{12}^3}{U_1^{19}U_4^4U_6^2U_{24}} + 96\frac{U_2^9U_3^3U_4U_8^3U_{12}^6}{U_1^{15}U_6^3U_{24}^3} \\ &+ 24q\frac{U_2^{18}U_3^3U_4^2U_{24}^3}{U_1^{19}U_8^3U_{12}^3} + 288q\frac{U_2^7U_3^3U_4^7U_{24}}{U_1^{15}U_6U_8}. \end{aligned} \tag{3.43}$$

□

Theorem 3.8.

$$v_0(4n+2) = \begin{cases} 4(-1)^l(2l+1) \pmod 8, & n=2l(l+1), \\ 0 \pmod 8, & otherwise. \end{cases} \tag{3.44}$$

$$v_0(8n+s) \equiv 0 \pmod 2, \quad s = 2, 6, \tag{3.45}$$

$$v_0(8n+6) \equiv 0 \pmod 4, \tag{3.46}$$

$$v_0(16n+s) \equiv 0 \pmod 2, \quad s = 2, 6, 10, 14, \tag{3.47}$$

$$v_0(16n+2) \equiv 0 \pmod 2, \tag{3.48}$$

$$v_0(16n+10) \equiv 0 \pmod 4, \tag{3.49}$$

$$v_0(16n+6) \equiv 0 \pmod 4, \tag{3.50}$$

$$v_0(16n+14) \equiv 0 \pmod 8. \tag{3.51}$$

Proof. Reducing modulo 8 to (3.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(4n+2)q^n &= 4\frac{U_4^4}{U_1^2U_2} \equiv 4\frac{U_2^8U_4^3U_6^{14}U_8^{12}}{U_4^{28}U_3^{28}} \pmod 8 \\ &\equiv 4\frac{U_2^8U_4^3U_6^{14}}{U_4^4U_3^{28}} \pmod 8 \equiv 4\frac{U_2^8U_4^3}{U_4^4} \pmod 8 \\ &\equiv 4U_4^3 \pmod 8 \\ &\equiv 4\sum_{k\in\mathbb{Z}^+}(-1)^l(2l+1)q^{2l(l+1)} \pmod 8. \end{aligned}$$

Hence, (3.44) follows. The congruence relations (3.45) follow from (3.20) and (3.21). The congruence relation (3.46) follows from (3.21). The relation (3.47) follows from (3.24), (3.25), (3.28), and (3.29), respectively. The congruence relations (3.48), (3.49), (3.50), and (3.51) follow from (3.24), (3.25), (3.28) and (3.29), respectively. □

Theorem 3.9.

$$v_0(6n+4) = \begin{cases} 6(-1)^l \pmod 2, & n = \frac{l(3l+1)}{2}, \\ 0 \pmod 2, & \textit{otherwise.} \end{cases} \tag{3.52}$$

$$v_0(12n+s) \equiv 0 \pmod m, \;\; s = 4, 10 \;\; \textit{and} \;\; m = 2, 3, 6. \tag{3.53}$$

$$v_0(24n+s) \equiv 0 \pmod m, \;\; s = 4, 10, 16, 22 \;\; \textit{and} \;\; m = 2, 3, 6. \tag{3.54}$$

Proof. Reducing modulo 2 to (3.17), we have

$$\begin{aligned} \sum_{n=0}^{\infty} v_0(6n+4)q^n &= 6\frac{U_2^3U_6^3}{U_1^4U_4} \equiv 6\frac{U_1U_2^{10}U_6^{12}U_7^{15}}{U_1^{19}U_3^5U_4^6} \pmod 2 \\ &\equiv 6\frac{U_1U_2^{10}U_6^{12}U_7^{15}}{U_4^6} \pmod 2 \equiv 6\frac{U_1U_6^{12}}{U_4^6} \pmod 2 \\ &\equiv 6U_1 \pmod 2 \\ &\equiv 6\sum_{k\in\mathbb{Z}}(-1)^l q^{\frac{l(3l+1)}{2}} \pmod 2. \end{aligned}$$

Hence, (3.52) follows. The congruence relations (3.53) follow from (3.9), and (3.10), respectively. The congruence relation (3.54) follows from (3.38), (3.39), (3.42) and (3.43), respectively. ◻

Theorem 3.10. *From Eq* (3.15)*, when n* ≥ 0*, we have*

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n \equiv 576\frac{U_8^2}{U_4} \pmod 2, \tag{3.55}$$

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n \equiv 64\frac{U_1U_2^2U_{12}^2}{U_3U_4^2U_6} \pmod 3, \tag{3.56}$$

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n \equiv 344\frac{U_1^3U_2^4U_3^{10}U_{20}}{U_4U_5U_{10}U_{15}^2} \pmod 5, \tag{3.57}$$

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n \equiv 697\frac{U_1^5U_2^6U_6^{14}U_9^{21}U_{28}}{U_4^3U_7U_{14}U_{42}^2U_{63}^3} \pmod 7. \tag{3.58}$$

Lemma 3.1. *We have the following dissections for the coefficient* $v_0(24n+7)$*:*

$$\begin{aligned} \sum_{n=0}^{\infty} v_0(24n+7)q^n = \frac{12U_2^2U_6^{14}}{U_3^{71}U_{18}^{30}} &\left(4q^2U_3^2U_{18}^6 + 2qU_3U_6U_{18}^3U_9^3 + U_6^2U_9^6\right)^8 \\ &\left(U_2^6\left(4q^2U_3^2U_{18}^6 + 2qU_3U_6U_{18}^3U_9^3 + U_6^2U_9^6\right)^2 + 27qU_6^4U_{18}^6U_3^{12}\right), \end{aligned} \tag{3.59}$$

$$\begin{aligned} \sum_{n=0}^{\infty} v_0(24n+7)q^n = &\frac{12\left(12q^3U_6U_{18}^6U_3^2 + 6q^2U_6^2U_9^3U_{18}^3U_3 + 3qU_6^3U_9^6 + U_9U_{18}U_3^7\right)^3}{U_1^7U_3^{39}U_9^6U_{18}^6} \\ &\left(U_1^5\left(12q^3U_6U_{18}^6U_3^2 + 6q^2U_6^2U_9^3U_{18}^3U_3 + 3qU_6^3U_9^6 + U_9U_{18}U_3^7\right)^3\right) \\ &+ 27qU_1^5U_2U_6^5U_9^3U_{18}^3U_3^{20}, \end{aligned} \tag{3.60}$$

$$\sum_{n=0}^{\infty} v_0(24n+7)q^n = \frac{12\left(qU_3U_{18}^3+U_6U_9^3\right)^5}{U_1^{11}U_6^6U_9^9U_{18}^9}(q^4U_3^4U_{18}^{12}+4q^3U_3^3U_6U_{18}^9U_9^3 + 6q^2U_3^2U_6^2U_{18}^6U_9^6+4qU_3U_6^3U_{18}^3U_9^9+27qU_6^8U_{18}^4U_9^4+U_6^4U_9^{12}), \tag{3.61}$$

$$\sum_{n=0}^{\infty} v_0(24n+7)q^n = \frac{12U_6^3\left(qU_3U_{18}^3+U_6U_9^3\right)^5\left(qU_{12}U_{48}^2U_8^3+U_4U_{16}^2U_{24}^3\right)^5}{U_1^6U_2^{18}U_3^9U_8^9U_9^9U_{12}^9U_{16}^9U_{18}^9U_{24}^9U_{48}^9} (27qU_3^4U_6^4U_8^4U_9^4U_{12}^4U_{16}^4U_{18}^4U_{24}^4U_{48}^4U_2^8+U_1^4\left(qU_3U_{18}^3+U_6U_9^3\right)^4 \left(qU_{12}U_{48}^2U_8^3+U_4U_{16}^2U_{24}^3\right)^4), \tag{3.62}$$

$$\sum_{n=0}^{\infty} v_0(24n+7)q^n = \frac{12U_6^3\left(2U_4^2U_{16}^4q+U_8^6\right)\left(qU_3U_{18}^3+U_6U_9^3\right)^5\left(qU_{12}U_{48}^2U_8^3+U_4U_{16}^2U_{24}^3\right)^5}{U_2^{19}U_8^5U_{16}^2U_2^{18}U_3^9U_8^9U_9^9U_{12}^9U_{16}^9U_{18}^9U_{24}^9U_{48}^9} (108U_4^2U_8^8U_2^4q^2U_2^8U_3^4U_6^4U_8^4U_9^4U_{12}^4U_{16}^4U_{18}^4U_{24}^4U_{48}^4+U_8^4U_2^{14} \left(qU_3U_{18}^3+U_6U_9^3\right)^4\left(qU_{12}U_{48}^2U_8^3+U_4U_{16}^2U_{24}^3\right)^4 + 27U_4^{14}qU_2^8U_3^4U_6^4U_8^4U_9^4U_{12}^4U_{16}^4U_{18}^4U_{24}^4U_{48}^4). \tag{3.63}$$

Proof. In order to constitute our result for Eq (3.59), we employ the values of (U_2^1/U_1^2) from Eq (2.16) and substitute them into Eq (3.7). After simplification, the desired result is obtained.

To derive our result for Eq (3.60), we use the values of (U_2^3/U_1^3) from Eq (2.12) and substitute them into Eq (3.7). Simplifying yields the desired result.

To establish the result for Eq (3.61), the values of (U_2^2/U_1) from Eq (2.13) are utilized and substituted back into Eq (3.7). The desired result is achieved after simplification.

The result for Eq (3.62) is obtained by substituting (2.13) and (2.18) into Eq (3.7) and simplifying.

To construct the result for Eq (3.63), we incorporate (2.13)–(2.15) and (2.18) into Eq (3.7). Simplification then provides the required result. □

4. Recurrence relations

In this section, we find the recurrence relation for the coefficients of the third-order mock theta functions $\omega(q)$ and $\nu(q)$. Let $\bar{p}_s(n)$ represent the number of overpartitions of n with s copies, and let $p_{sd}(n)$ represent the number of partitions of n into distinct parts with s copies. Then, the following relations hold:

$$\sum_{n=0}^{\infty} \bar{p}_s(n)q^n = \left(\frac{U_2}{U_1^2}\right)^s, \quad \sum_{n=0}^{\infty} p_{sd}(n)q^n = \left(\frac{U_2}{U_1}\right)^s. \tag{4.1}$$

Theorem 4.1.

$$\nu(2n) = p_{2d}(n) - p_{2d}(n-2) - p_{2d}(n-4) + p_{2d}(n-10) + p_{2d}(n-14) + \cdots + (-1)^v p_{2d}(n - v(3v-1)) + (-1)^v p_{2d}(n - v(3v+1)) + \cdots. \tag{4.2}$$

$$\nu(2n) = p_d(n) + p_d(n-1) + p_d(n-3) + p_d(n-6) + p_d(n-10) + \cdots + p_d(n - \frac{v(v+1)}{2}) + \cdots. \tag{4.3}$$

Theorem 4.2.

$$\begin{aligned}\omega(8n+3) &= 4p_{10d}(n) - 4p_{10d}(n-1) - 4p_{10d}(n-2) + 4p_{10d}(n-5) + 4p_{10d}(n-7) + \cdots \\ &\quad + 4(-1)^v p_{10d}(n - \frac{v(3v-1)}{2}) + 4(-1)^v p_{10d}(n - \frac{v(3v+1)}{2}) + \cdots. \qquad (4.4)\\ \omega(8n+3) &= 4p_{8d}(n) + 4p_{8d}(n-1) + 4p_{8d}(n-3) + 4p_{8d}(n-6) + 4p_{8d}(n-10) + \cdots \\ &\quad + 4p_{8d}(n - \frac{v(v+1)}{2}) + \cdots. \qquad (4.5)\end{aligned}$$

Proof of Theorem 4.1. From [24], we have

$$\begin{aligned}\sum_{n=0}^{\infty} v(2n)q^n &= \frac{U_2^3}{U_1^2} = \left(\frac{U_2}{U_1}\right)^2 U_2 \qquad (4.6)\\ &= \left(\sum_{n=0}^{\infty} p_{2d}(n)q^n\right)\left(1 + \sum_{v=1}^{\infty}(-1)^v q^{v(3v-1)} + \sum_{v=1}^{\infty}(-1)^v q^{v(3v+1)}\right)\\ &= \sum_{n=0}^{\infty} p_{2d}(n)q^n + \sum_{n=0}^{\infty}\left(\sum_{v=1}^{\infty}(-1)^v p_{2d}(n - v(3v-1))\right)q^n\\ &\quad + \sum_{n=0}^{\infty}\left(\sum_{v=1}^{\infty}(-1)^v p_{2d}(n - v(3v+1))\right)q^n. \qquad (4.7)\end{aligned}$$

By analyzing the coefficients of q^n, we easily arrive at Eq (4.2). To prove (4.3), consider the Eq (4.6) as

$$\begin{aligned}\sum_{n=0}^{\infty} v(2n)q^n &= \left(\frac{U_2}{U_1}\right)\frac{U_2^2}{U_1}\\ &= \left(\sum_{n=0}^{\infty} p_d(n)q^n\right)\left(\sum_{v=0}^{\infty} q^{\frac{v(v+1)}{2}}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{v=0}^{\infty} p_d(n)q^{\frac{v(v+1)}{2}+n}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{v=0}^{\infty} p_d\left(n - \frac{v(v+1)}{2}\right)\right)q^n.\end{aligned}$$

A simple comparison of the q^n coefficients yields Eq (4.3). □

Proof of Theorem 4.2. From [24], we have

$$\begin{aligned}\sum_{n=0}^{\infty} \omega(8n+3)q^n &= 4\frac{U_2^{10}}{U_1^9} = 4\left(\frac{U_2}{U_1}\right)^{10} U_1 \qquad (4.8)\\ &= 4\left(\sum_{n=0}^{\infty} p_{10d}(n)q^n\right)\left(\sum_{v=-\infty}^{\infty}(-1)^v q^{\frac{v(3v+1)}{2}}\right)\\ &= 4\left(\sum_{n=0}^{\infty} p_{10d}(n)q^n\right)\left(1 + \sum_{v=1}^{\infty}(-1)^v q^{\frac{v(3v-1)}{2}} + \sum_{v=1}^{\infty}(-1)^v q^{\frac{v(3v+1)}{2}}\right)\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_{n=0}^{\infty} p_{10d}(n)q^n + 4\sum_{n=0}^{\infty}\left(\sum_{v=1}^{\infty}(-1)^v p_{10d}(n)q^{\frac{v(3v-1)}{2}+n}\right)\\
&\quad + 4\sum_{n=0}^{\infty}\left(\sum_{v=1}^{\infty}(-1)^v p_{10d}(n)q^{\frac{v(3v+1)}{2}+n}\right).
\end{aligned}
$$

After some simplification, we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}\omega(8n+3)q^n &= 4\sum_{n=0}^{\infty} p_{10d}(n)q^n\\
&\quad + 4\sum_{n=0}^{\infty}\left(\sum_{i=1}^{\infty}(-1)^v p_{10d}(n-\frac{v(3v-1)}{2})\right)q^n\\
&\quad + 4\sum_{n=0}^{\infty}\left(\sum_{v=1}^{\infty}(-1)^v p_{10d}(n-\frac{v(3v+1)}{2})\right)q^n.
\end{aligned}
$$

By examining the coefficients of q^n, Eq (4.4) is easily obtained. To prove (4.5), consider the Eq (4.8) as

$$
\begin{aligned}
\sum_{n=0}^{\infty}\omega(8n+3)q^n &= 4\left(\frac{U_2}{U_1}\right)^8\frac{U_2^2}{U_1}\\
&= 4\left(\sum_{n=0}^{\infty} p_{8d}(n)q^n\right)\left(\sum_{v=0}^{\infty} q^{\frac{v(v+1)}{2}}\right)\\
&= 4\sum_{n=0}^{\infty}\left(\sum_{v=0}^{\infty} p_{8d}(n)q^{\frac{v(v+1)}{2}+n}\right)\\
&= 4\sum_{n=0}^{\infty}\left(\sum_{v=0}^{\infty} p_{8d}\left(n-\frac{v(v+1)}{2}\right)\right)q^n.
\end{aligned}
$$

Equation (4.5) can be readily derived by comparing the coefficients of q^n. □

5. Conclusions and further remarks

In this paper, we investigated the arithmetic properties of the coefficients of the eighth-order mock theta function $V_0(q)$ through explicit eta-quotient expansions and dissections. The main objective was to obtain concrete arithmetic progression formulas for the coefficients $v_0(n)$ and to derive corresponding congruence and recurrence relations.

The principal contribution of this work is constructive in nature. Rather than appealing primarily to abstract modular-form machinery, we employed explicit q-series manipulations and eta-dissection arguments to derive progressionwise generating functions for several residue classes of $v_0(n)$. These formulas lead directly to divisibility properties, coefficient congruences, and recurrence relations connected with partition-theoretic generating functions.

We emphasize that many of the identities obtained here admit an alternative interpretation through the theory of modular forms. Once written in eta-quotient form, several of the generating functions considered in this paper lie naturally in spaces of weakly holomorphic modular forms of half-integral weight. Consequently, many of the identities and congruences may also be verified through standard modular-form arguments, such as by checking finitely many coefficients up to Sturm's bound. From this perspective, the present work may be viewed as a constructive and explicit companion to the modular-form approach.

The arithmetic behavior of eighth-order mock theta functions remains comparatively less explored than that of lower-order mock theta functions. We hope that the explicit formulas recorded here, particularly the progression identities and recurrence relations for $v_0(n)$, may serve as useful data for future arithmetic and combinatorial investigations of eighth-order mock theta functions and related mock modular objects.

Several natural questions remain open. For instance, it would be interesting to determine whether analogous congruences for $v_0(n)$ hold modulo higher prime powers, whether the remaining eighth-order mock theta functions admit similar explicit arithmetic descriptions, and whether the identities obtained here can be systematically derived within a unified modular-form framework.

Author contributions

Arooj Fatima: Conceptualization, Methodology, Formal analysis, Writing – Original Draft. Fatemah Mofarreh: Funding acquisition, Methodology, Data curation. Ahmer Ali: Investigation, Writing – Review & Editing, Validation, Supervision. Ambreen Ahmed: Methodology, Validation, Data curation, Muhammad Hanif: Supervision, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no generative-AI tools were used in the creation of this manuscript.

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Conflict of interest

The authors declare that there is no conflict of interest.

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