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Research article

Symmetry breaking for a Schrödinger equation with van der Waals type potentials

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Abstract: We study the asymptotic behavior of ground state energy for a Schrödinger equation with van der Waals type potentials. We prove that when the mass c is sufficiently large, the ground state energy restricted to radially symmetric functions is greater than the ground state energy. This result implies that the symmetry of the ground state solution of the equation is broken.

Keywords: Schrödinger equations; van der Waals type potentials; symmetry breaking
Mathematics Subject Classification: 35B09, 35J20, 35Q40, 35Q55

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1. Introduction and main results

The aim of this paper is to show symmetry breaking phenomena for the ground state energy of the following nonlocal equation:

$$-\Delta u + \lambda u + \mu(I_\alpha * |u|^2)u = (I_\beta * |u|^2)u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^N} u^2 dx = c^2, \tag{1.2}$$

where $N \geq 3$, $\mu, c > 0$ and the frequency $\lambda \in \mathbb{R}$ is unknown and a Lagrange multiplier. Meanwhile, $I_s : \mathbb{R}^N \to \mathbb{R}$ ($s = \alpha, \beta$) is the Riesz potential of order $s \in (0, N)$, defined for $x \in \mathbb{R}^N \backslash \{0\}$ as

$$I_s(x) = \frac{M_s}{|x|^{N-s}}, \quad M_s = \frac{\Gamma(\frac{N-s}{2})}{\Gamma(\frac{s}{2})\pi^{\frac{N}{2}} 2^s}.$$

Problem (1.1) comes from the research of standing waves for the following Schrödinger equation:

$$i\varphi_t + \Delta\varphi + (W * |\varphi|^2)\varphi = 0, \text{ in } \mathbb{R}^N, \tag{1.3}$$

where $\varphi(t,x)$ is the complex wave function describing the quantum state of the system and the potential

$$W(x) = z_\alpha|x|^{-(N-\alpha)} + z_\beta|x|^{-(N-\beta)}.$$

Equation (1.3) is derived from a theoretical model designed to describe non-relativistic bosonic atoms and molecules. A key feature of this model is its inclusion of two interacting potentials-interactions that exhibit weaker strength and a longer range compared to the Dirac delta type potential. Further details regarding this model can be found in [4]. From a physical perspective, when the parameters satisfy $N-\alpha > 1$ and $N-\beta > 1$, the function $W(x)$ corresponds to the Van Der Waals type potential, commonly referred to as the Van Der Waals interaction. This interaction quantifies the potential energy between two neutral molecules, arising from the combined effects of attractive and repulsive forces acting between the molecules. In the fields of physics and chemistry, where intermolecular forces are a central area of study, the Van Der Waals interaction stands as a fundamental concept (see [15,17]). It is worth noting that the Van Der Waals coefficients z_6, z_8, and z_{10}, which are tightly linked to alkaline-earth interactions, have been calculated with exceptional accuracy, boasting a precision level of up to 1/100. Porsev and Derevianko [14] achieved this high-precision result by utilizing relativistic many-body perturbation theory in their calculations. Beyond its connection to intermolecular interactions, Eq (1.3) also holds significance in the context of bosonic systems. Specifically, it emerges as the mean-field limit of bosonic systems characterized by attractive two-body interactions. Importantly, this mean-field limit can be rigorously derived under a wide range of conditions, as documented in [9]. For the sake of simplifying subsequent analyses without compromising the generality of the conclusions, we may assume, without loss of generality, that $z_\alpha = \mu$, $z_\beta = -1$.

From a physical point of view, the most relevant objects in the study of Eq (1.1) are minimizers and critical points of the energy $\mathcal{I}$ corresponding to (1.1), defined by

$$\mathcal{I}(u) = \frac{1}{2}T(u) + \frac{\mu}{4}A(\alpha,u) - \frac{1}{4}A(\beta,u), \tag{1.4}$$

where

$$T(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx,\ A(\alpha,u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^2)u^2 dx,\ \ A(\beta,u) = \int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx,$$

subject to a prescribed mass constraint $|u|_2^2 = c^2$.

For Eq (1.3), in numerous existing articles, scholars primarily focus on studying normalized solutions, infinitely many solutions, asymptotic behavior of solutions, and orbital stability. For instance: When $\alpha = \beta$, Lieb [6] used symmetry techniques to prove the existence and uniqueness of normalized solutions, and Lions [10] studied the existence and stability of normalized solutions. While for the mass supercritical case $2 < \beta < \min\{N,4\}$, Luo [12] obtained a normalized solution through a constraint minimization method. When $\alpha \neq \beta$, the situation becomes much more complicated. In [2], the existence, multiplicity, and asymptotic behavior of solutions to the equation are proved under different assumptions on c, α, and β. Jia and Luo [5] extended main results in [2] to the energy critical case, i.e, $\beta = 4$ and $N \geq 5$. In addition, $\mu > 0$ and $2 < \beta < \alpha \leq \min\{N,4\}$, so the energy functional has a local minimum with a positive level and a global minimum with a negative level; see [16] for more details. The solutions of the Schrödinger equation have attracted significant interest from numerous scholars, and pertinent content can be referred to in [3,8,11].

However, due to the energy functional as well as the L^2 constraint being rotationally invariant does not imply that the ground state energy for radially symmetric functions coincides with ground state energy. Inspired by Reference [1], this paper mainly studies the occurrence of symmetry breaking when c is sufficiently large. Based on Theorem 4 in [13], we mainly discuss the following three cases:

(A) $\alpha \leq 1, 3 \leq N \leq 4 - \alpha, 0 < \beta < \frac{\alpha N}{4+\alpha}$.
(B) $\alpha \leq 1, 4 - \alpha < N < 4 + \alpha, \frac{\alpha+N-4}{2} < \beta < \frac{\alpha N}{4+\alpha}$.
(C) $\alpha > 1, 2 + \alpha < N < 4 + \alpha, \frac{\alpha+N-4}{2} < \beta < \frac{N^2+3\alpha N-4N}{5N-(4+\alpha)}$.

We define the ground state energy as the quantity I_{c^2}, which is given by

$$I_{c^2} = \inf_{u \in D_c} \mathcal{I}(u), \tag{1.5}$$

where $D_c := \left\{u \in H^1(\mathbb{R}^N) \mid |u|_2^2 = c^2\right\}$. We also define the radially symmetric ground state energy $I_{c^2}^{\mathrm{rad}}$ as

$$I_{c^2}^{\mathrm{rad}} = \inf_{u \in D_{c^2}^{\mathrm{rad}}} \mathcal{I}(u), \tag{1.6}$$

where $D_{c^2}^{\mathrm{rad}} := \left\{u \in H^1(\mathbb{R}^N) \mid u \text{ is radially symmetric and } |u|_2^2 = c^2\right\}$.

Our argument is based on two elements. The first element is that the following Coulomb-Sobolev interpolation inequality holds [13]:

$$||u||_q^q \leq C||\nabla u||_2^{\frac{q\phi}{2}} \, [A(\alpha, u)]^{\frac{q(1-\phi)}{2p}}, \tag{1.7}$$

where

$$\phi = \frac{q(N+\alpha) - 2pN}{q[(N+\alpha) - p(N-2)]}.$$

Let $N \in \mathbb{N}$, $\alpha \in (0, N)$ and $p, q \in [1, +\infty)$.

(i) If $\alpha \leq 1$, $p < \frac{N+\alpha}{N-2}$, and $\frac{2(\alpha+2p)}{\alpha+p+1} < q < \frac{2N}{N-2}$, then D_c^{rad} is embedded in $L^q(\mathbb{R}^N)$.
(ii) If $\alpha > 1$, $p < \frac{N+\alpha}{N-2}$, and $\frac{2[2p(N-1)+N-\alpha]}{3N+\alpha-4} < q \leq \frac{2N}{N-2}$, then D_c^{rad} is embedded in $L^q(\mathbb{R}^N)$.

So, in these cases, the interpolation estimate (1.7) is valid.

In this article, we take $p = 2$, leveraging inequality (1.7) in conjunction with the Hardy-Littlewood-Sobolev (HLS) inequality, and we derive an inequality that establishes a connection between $T(u)$, $A(\alpha, u)$, and $A(\beta, u)$—this derivation proceeds without relying on any information regarding the L^2-norm of the function. Notably, this resulting inequality entails the conclusion $\inf_{c>0} I_{c^2}^{rad} > -\infty$.

The second key element is the asymptotic behavior of the function $c \to \frac{I_{c^2}}{c^2}$ corresponding to the translation invariant energy functional. We will prove in Lemma 2.7 that this behavior of the function $c \to \frac{I_{c^2}}{c^2}$ implies two core conclusions: First, for all cases satisfying $0 < b < c$, it holds that $I_{c^2} < I_{b^2}$; second, $\inf_{c>0} I_{c^2} = -\infty$. With the help of these two conclusions, we can further derive our main results as follows.

Now, we state our main results. First, let's discuss the case where $\alpha \leq 1$.

Theorem 1.1. *Assume that* (A) *or* (B) *holds, then there exists* $c_0 > 0$ *such that* $I_{c^2} < I_{c^2}^{rad}$ *for all* $c > c_0$. *Namely, the symmetry of the ground state solution of systems* (1.1)-(1.2) *is broken.*

Next, we discuss the case where $\alpha > 1$.

Theorem 1.2. *Assume that* (C) *holds, then there exists* $c_0' > 0$ *such that* $I_{c^2} < I_{c^2}^{rad}$ *for all* $c > c_0'$*. That is, the symmetry of the ground state solution of systems* (1.1)-(1.2) *is broken.*

Remark 1.1. *Due to the fact that* D_c^{rad} *is embedded in* $L^q(\mathbb{R}^N)$ *with slightly different* q *values for different values of* α*, this article discusses the cases where* $\alpha \le 1$ *and* $\alpha > 1$ *separately. Meanwhile, to ensure that* $\beta > 0$*, we have carefully divided the relationship between* α *and* N*.*

2. Proof of Theorem 1.1

Proposition 2.1. (see [7]) *Let* $t, s > 1$ *and* $0 < \beta < N$ *with* $\frac{1}{s} + \frac{1}{t} + \frac{N-\beta}{N} = 2$*. There exists a sharp constant* $C(N, \beta, t, s) > 0$ *such that*

$$\int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f(x)||g(x)|}{|x-y|^{N-\beta}} dxdy \le C(N, \beta, t, s) \|f\|_{L^t(\mathbb{R}^N)} \|g\|_{L^s(\mathbb{R}^N)}, \quad \forall\, f \in L^t(\mathbb{R}^N), \text{ and } \forall\, g \in L^s(\mathbb{R}^N).$$

If $t = s = \frac{2N}{N+\beta}$*, then*

$$C(N, \beta, t, s) = C(N, \beta) = \pi^{\frac{N-\beta}{2}} \frac{\Gamma(\frac{\beta}{2})}{\Gamma(N + \frac{\beta}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{\frac{\beta}{N}}.$$

From Proposition 2.1, we have

$$\int_{\mathbb{R}^N} (I_\beta * |u|^2) u^2 dx \le C(N, \beta) \|u\|^4_{L^{\frac{4N}{N+\beta}}(\mathbb{R}^N)}. \tag{2.1}$$

Lemma 2.1. *Let* $u \in D_c^{rad}$ *and* (A) *or* (B) *hold. Then, there exists a constant* $K_1 > 0$*, which is independent of c, such that* $\mathcal{I}(u) > -K_1$*.*

Proof. First, when $\alpha \le 1$, $3 \le N < 4 + \alpha$, $q \in \left[\frac{2(4+\alpha)}{2+\alpha}, \frac{2N}{N-2}\right]$, we obtain through the Coulomb-Sobolev inequality that

$$\|u\|_q^q \le C \|\nabla u\|_2^{\frac{q(N+\alpha)-4N}{4+\alpha-N}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2) u^2 dx \right)^{\frac{2N-q(N-2)}{2(4+\alpha-N)}}, \tag{2.2}$$

with $N \ne 4 + \alpha$.

Therefore, when $q = \frac{4N}{N+\beta}$, i.e, $N - 4 \le \beta \le \frac{\alpha N}{4+\alpha}$, we have

$$\|u\|^{\frac{4N}{N+\beta}}_{L^{\frac{4N}{N+\beta}}} \le C \|\nabla u\|_2^{\frac{4N(\alpha-\beta)}{(N+\beta)(4+\alpha-N)}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2) u^2 dx \right)^{\frac{N(\beta-N+4)}{(N+\beta)(4+\alpha-N)}}.$$

Then,

$$\|u\|^4_{L^{\frac{4N}{N+\beta}}} \le C \|\nabla u\|_2^{\frac{4(\alpha-\beta)}{4+\alpha-N}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2) u^2 dx \right)^{\frac{\beta-N+4}{4+\alpha-N}}. \tag{2.3}$$

Combining (2.1) and (2.3), we obtain

$$A(\beta, u) \le C(N, \beta) [T(u)]^{\frac{2(\alpha-\beta)}{4+\alpha-N}} [A(\alpha, u)]^{\frac{\beta-N+4}{4+\alpha-N}}. \tag{2.4}$$

Next, let's set

$$\bar{x} = \|\nabla u\|^2_{L^2(\mathbb{R}^N)}, \quad \bar{y} = A(\alpha, u).$$

By (1.4) and (2.4), we have

$$\mathcal{I}(u) = \frac{1}{2}\overline{x} + \frac{\mu}{4}\overline{y} - \frac{C}{4}\overline{x}^{\frac{2(\alpha-\beta)}{4+\alpha-N}}\overline{y}^{\frac{\beta-N+4}{4+\alpha-N}}.$$

Since $\alpha \leq 1$, $3 \leq N \leq 4-\alpha$, $0 < \beta < \frac{\alpha N}{4+\alpha}$, it implies that $\mathcal{I}$ is bounded below by a constant independent of c. Similarly, when $\alpha \leq 1$, $4-\alpha < N < 4+\alpha$, $\frac{\alpha+N-4}{2} < \beta < \frac{\alpha N}{4+\alpha}$, we obtain that the lower bound of $\mathcal{I}(u)$ is bounded by a constant independent of c. □

Lemma 2.2. *If $u \in D_c$, then $\mathcal{I}$ is bounded from below.*

Proof. First, we claim that the following inequality

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-\beta}}dxdy \leq \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-\alpha}}dxdy + M\left(\int_{\mathbb{R}^N} u^2dx\right)^2$$

holds true for $\beta < \frac{\alpha N}{4+\alpha} < \alpha \leq 1$ and $M > 0$ is a constant. Indeed, when $|x-y| > 1$,

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-\beta}}dxdy \leq \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-\alpha}}dxdy.$$

If $|x-y| \leq 1$, we get

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-\beta}}dxdy \leq C(N,\beta)\int\int_{\mathbb{R}^N\times\mathbb{R}^N} u^2(x)u^2(y)dxdy.$$

Then, taking $M = C(N,\beta)$. By Fubini's theorem, we have

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^{N-\beta}}dxdy \leq M\left(\int_{\mathbb{R}^N} u^2dx\right)^2.$$

Therefore, applying our claim, one gets

$$\begin{aligned}
\mathcal{I}(u) &= \frac{1}{2}T(u) + \frac{\mu}{4}A(\alpha,u) - \frac{1}{4}A(\beta,u) \\
&\geq \frac{1}{2}T(u) + \frac{\mu}{4}A(\alpha,u) - \frac{1}{4}\left(A(\alpha,u) + M\left(\int_{\mathbb{R}^N} u^2dx\right)^2\right) \\
&\geq -\frac{Mc^4}{4}.
\end{aligned}$$

This implies that $I_{c^2} > -\infty$ for all $c > 0$.

However, since $\alpha > 1$ and $\beta < \frac{N^2+3\alpha N-4N}{5N-(4+\alpha)} < \alpha$, we can similarly prove that $\mathcal{I}$ is bounded from below when (C) holds. □

Lemma 2.3. *For any $0 < b < c$, we have $I_{c^2} \leq I_{b^2} + I_{c^2-b^2}$.*

Proof. We assume that $\eta > 1$ satisfies $c = \eta b$, then there exists $\{u_n\} \subset D_b$ such that $\mathcal{I}(u_n) \to I_{b^2}$, and for any $\varepsilon > 0$, we have

$$\mathcal{I}(u_n) \leq I_{b^2} + \varepsilon.$$

Let

$$v_n(x) = \eta^{\frac{\widehat{\alpha}+N}{\widehat{\alpha}}}u_n(\eta^{\frac{2}{\widehat{\alpha}}}x),$$

where $\widehat{\alpha} = \alpha - N$, $\widehat{\beta} = \beta - N$. Then $|v_n|_2^2 = \eta^2 |u_n|_2^2$, namely, $\{v_n\} \subset D_{\eta b}$, (A) or (B) holds, and $\beta < \frac{\alpha N}{4+\alpha} < \alpha \leq 1$. Thus,

$$\begin{aligned} I_{\eta^2 b^2} &\leq \mathcal{I}(v_n) = \eta^2 \mathcal{I}(u_n) + \frac{1}{2}(\eta^{2+\frac{4}{\widehat{\alpha}}} - \eta^2) T(u_n) + \frac{1}{4}(\eta^2 - \eta^{4-\frac{2\widehat{\beta}}{\widehat{\alpha}}}) A(\beta, u_n) \\ &< \eta^2 \mathcal{I}(u_n) \leq \eta^2 (I_{b^2} + \varepsilon). \end{aligned} \tag{2.5}$$

Since $\varepsilon > 0$ is arbitrary, the inequality $I_{\eta^2 b^2} \leq \eta^2 I_{b^2}$ holds, then

$$I_{c^2} = \frac{c^2 - b^2}{c^2} I_{c^2} + \frac{b^2}{c^2} I_{c^2} = \frac{c^2 - b^2}{c^2} I_{\left(\sqrt{\frac{c^2}{c^2-b^2}}\right)^2 \left(\sqrt{c^2-b^2}\right)^2} + \frac{b^2}{c^2} I_{\left(\sqrt{\frac{c^2}{b^2}}\right)^2 \left(\sqrt{b^2}\right)^2} \leq I_{c^2-b^2} + I_{b^2}.$$

This implies the desired conclusion. Under condition (C), when

$$\alpha > 1, 2 + \alpha < N < 4 + \alpha, \frac{\alpha + N - 4}{2} < \beta < \frac{N^2 + 3\alpha N - 4N}{5N - (4 + \alpha)} < \alpha,$$

we can similarly obtain (2.3). Then $I_{c^2} \leq I_{b^2} + I_{c^2-b^2}$ holds for any $0 < b < c$. □

Lemma 2.4. *$I_{c^2} < 0$ for all $\mu, c > 0$.*

Proof. For $u \in D_1$, let's consider the scaling function given by the following formula:

$$u_{\theta,t}(x) = \theta^{1-\frac{N}{2}t} u(\theta^{-t} x),$$

where $\theta, t \in \mathbb{R}$ are parameters. Then, we have $\overline{c} = |u_{\theta,t}|_2 = \theta |u|_2 = \theta$, and it is very easy for us to discover the following scaling rules:

$$\begin{aligned} T(u_{\theta,t}) &= \theta^{2-2t} T(u), \\ A(\alpha, u_{\theta,t}) &= \theta^{4+(\alpha-N)t} A(\alpha, u), \\ A(\beta, u_{\theta,t}) &= \theta^{4+(\beta-N)t} A(\beta, u). \end{aligned}$$

Notice that for $t = \frac{-2}{\alpha-N+2}$, we get

$$\begin{aligned} \mathcal{I}(u_{\theta,t}) &= \frac{1}{2}\theta^{\frac{2(\alpha-N+4)}{\alpha-N+2}} T(u) + \frac{\mu}{4}\theta^{\frac{2(\alpha-N+4)}{\alpha-N+2}} A(\alpha, u) - \frac{1}{4}\theta^{\frac{2(2\alpha-\beta-N+4)}{\alpha-N+2}} A(\beta, u) \\ &= \frac{1}{2}\theta^{\frac{2(\alpha-N+4)}{\alpha-N+2}} T(u) + \frac{\mu}{4}\theta^{\frac{2(\alpha-N+4)}{\alpha-N+2}} A(\alpha, u) - \frac{1}{4}\theta^{\frac{2(\alpha-N+4)}{\alpha-N+2} - \frac{2(\alpha-\beta)}{N-(\alpha+2)}} A(\beta, u). \end{aligned}$$

Due to $\alpha \leq 1$, $3 \leq N < 4 + \alpha$, $\beta < \frac{\alpha N}{4+\alpha} < \alpha$. Hence, for $\theta \to 0$, we have $\mathcal{I}(u_{\theta, \frac{-2}{\alpha-N+2}}) \to 0^-$. This shows that there is a small θ, and as a result, a small $\overline{c}$, for which

$$I_{s^2} < 0 \quad \text{for every} \quad s \in (0, \overline{c}].$$

By weak subadditivity (see Lemma 2.3), $I_{s^2} < 0$ for all s.

Similarly, when condition (C) is satisfied, since

$$\beta < \frac{N^2 + 3\alpha N - 4N}{5N - (4 + \alpha)} < \alpha,$$

it can be proven that the lemma is also valid. □

Lemma 2.5. *The function* $c \mapsto I_{c^2}$ *is continuous.*

Proof. Let $c > 0$ and $\{c_n\}$ be a sequence satisfying $c_n \geq 0$ and $\lim_{n\to\infty} c_n = c$. It follows from the definition of I_{c^2} that for any $\varepsilon > 0$, there exists $\{u_n\} \subset D_{c_n}$ such that $\mathcal{I}(u_n) \leq I_{c_n^2} + \varepsilon$. From Lemma 2.4, one has $I_{c_n^2} < 0$. If $c_n \geq c$, set $v_n = \frac{c}{c_n} u_n$, then we have $\{v_n\} \subset D_c$ and $\frac{c}{c_n} \to 1$ as $n \to \infty$. Hence,

$$\begin{aligned} I_{c^2} &\leq \mathcal{I}(v_n) = \mathcal{I}(u_n) + [\mathcal{I}(v_n) - \mathcal{I}(u_n)] \\ &= \mathcal{I}(u_n) + \frac{c^2 - c_n^2}{2c_n^2} T(u_n) + \frac{\mu(c^4 - c_n^4)}{4c_n^4} A(\alpha, u_n) - \frac{c^4 - c_n^4}{4c_n^4} A(\beta, u_n) \\ &= \mathcal{I}(u_n) + o_n(1) \leq I_{c_n^2} + \varepsilon + o_n(1). \end{aligned} \tag{2.6}$$

If $c_n < c$, then it follows from Lemma 2.3 that

$$I_{c^2} = \frac{c^2 - c_n^2}{c^2} I_{c^2} + \frac{c_n^2}{c^2} I_{c^2} \leq I_{c^2 - c_n^2} + I_{c_n^2} < I_{c_n^2}. \tag{2.7}$$

Combining (2.6) with (2.7), we have

$$I_{c^2} \leq I_{c_n^2} + \varepsilon + o_n(1).$$

Noting that for any $\varepsilon > 0$, there exists $\{u\} \in D_c$ such that $\mathcal{I}(u) \leq I_{c^2} + \varepsilon$. Set $w_n = \frac{c_n}{c} u$, then $\{w_n\} \subset D_{c_n}$ for n large enough. Since $\mathcal{I}(w_n) \to \mathcal{I}(u)$,

$$I_{c_n^2} \leq \mathcal{I}(w_n) = \mathcal{I}(u) + [\mathcal{I}(w_n) - \mathcal{I}(u)] = \mathcal{I}(u) + o_n(1) \leq I_{c^2} + \varepsilon + o_n(1).$$

Therefore, by the arbitrariness of $\varepsilon > 0$, we get $\lim_{n\to\infty} I_{c_n^2} = I_{c^2}$. The proof is complete. □

Lemma 2.6. *Asymptotic behavior at zero* $\lim_{c\to 0} \frac{I_{c^2}}{c^2} = 0$.

Proof. To demonstrate that $\lim_{c\to 0} \frac{I_{c^2}}{c^2} = 0$, we observe that

$$\frac{J_{c^2}}{c^2} \leq \frac{I_{c^2}}{c^2} < 0,$$

where J_{c^2} is defined by

$$J_{c^2} = \inf_{D_c} \mathcal{J}(u),$$

where

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^N} (I_\beta * |u|^2) u^2 dx.$$

To prove that $\lim_{c\to 0} \frac{I_{c^2}}{c^2} = 0$ holds, it suffices to prove that $\lim_{c\to 0} \frac{J_{c^2}}{c^2} = 0$ holds. Since the function $c \to \frac{J_{c^2}}{c^2}$ is monotonically decreasing, it is well established that minimizers for J_{c^2} exist for every c.

Now, we prove that the function $c \to \frac{J_{c^2}}{c^2}$ is monotone decreasing, i.e, for any $0 < c_1 < c_2$, there is

$$\frac{J_{c_1^2}}{c_1^2} > \frac{J_{c_2^2}}{c_2^2}.$$

For any $u \in D_{c_2}$, set $u_1(x) = \frac{c_1}{c_2}u(x)$, then $\|u_1\|_2^2 = \frac{c_1^2}{c_2^2}\|u\|_2^2$ and $\frac{c_1}{c_2} < 1$, thus we obtain $u_1 \in D_{c_1}$. Hence,

$$\mathcal{J}(u_1) = \frac{c_1^2}{2c_2^2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{c_1^4}{4c_2^4}\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx.$$

And because of $u_1 \in D_{c_1}$, we have $\mathcal{J}(u_1) \geq J_{c_1^2}$, thus

$$\frac{1}{2c_2^2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{c_1^2}{4c_2^4}\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx \geq \frac{J_{c_1^2}}{c_1^2}.$$

Furthermore, we obtain

$$\frac{1}{c_2^2}\left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{C}{4}\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx\right] \geq \frac{J_{c_1^2}}{c_1^2}, \quad \text{where } C = \frac{c_1^2}{c_2^4} > 0. \tag{2.8}$$

Due to

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{C}{4}\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx > \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx,$$

we have,

$$\frac{1}{c_2^2}\inf_{u\in D_{c_2}}\left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{C}{4}\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx\right] > \frac{J_{c_2^2}}{c_2^2}. \tag{2.9}$$

Combining (2.8) with (2.9), we get

$$\frac{J_{c_2^2}}{c_2^2} < \frac{J_{c_1^2}}{c_1^2}.$$

Let us denote $\overline{u} \in D_c$ as the minimizer for J_{c^2}, that is, the unique positive solution to

$$-\Delta u - \lambda_c u - (I_\beta * |u|^2)u = 0, \tag{2.10}$$

where $\lambda_c < 0$ is the corresponding Lagrange multiplier.

Since $\overline{u}$ satisfies the Pohozaev-type identity and $\overline{u}$ is solution of Eq (2.10), namely,

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla \overline{u}|^2 dx - \int_{\mathbb{R}^N} (I_\beta * |\overline{u}|^2)\overline{u}^2 dx = \lambda_c c^2, \\ \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla \overline{u}|^2 dx - \frac{\beta+N}{4}\int_{\mathbb{R}^N} (I_\beta * |\overline{u}|^2)\overline{u}^2 dx = \frac{N}{2}\lambda_c c^2, \end{cases}$$

there holds

$$\int_{\mathbb{R}^N} (I_\beta * |\overline{u}|^2)\overline{u}^2 dx = \frac{4c^2\lambda_c}{N-4-\beta}, \tag{2.11}$$

and

$$\int_{\mathbb{R}^N} |\nabla \overline{u}|^2 dx = \frac{c^2\lambda_c(N-\beta)}{N-4-\beta}. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\mathcal{I}(\overline{u}) = J_{c^2} = \frac{N-\beta-2}{2(N-4-\beta)}c^2\lambda_c = mc^2\lambda_c.$$

To conclude, we will prove that $\lim_{c\to 0}\lambda_c = 0$. Take $\widetilde{u}$, the unique positive solution to

$$-\Delta u + \lambda_c^2 u - (-\lambda_c)^{\frac{4-N+\beta}{2}}(I_\beta * |u|^2)u = 0,$$

by scaling $\overline{u}(x) = (-\lambda_c)^{\frac{2-N}{4}}\widetilde{u}((-\lambda_c)^{-\frac{1}{2}}x)$, and we have

$$c^2 = \|\overline{u}\|_2^2 = (-\lambda_c)\|\widetilde{u}\|_2^2.$$

Hence, $\lim_{c\to 0}(-\lambda_c) = \lim_{c\to 0}\frac{c^2}{\|\widetilde{u}\|_2^2} = 0$, namely, $\lim_{c\to 0}\lambda_c = 0$. So, we obtain

$$\lim_{c\to 0}\frac{J_{c^2}}{c^2} = 0 \quad\Rightarrow\quad \lim_{c\to 0}\frac{I_{c^2}}{c^2} = 0.$$

□

Lemma 2.7. *Let the function $c \to I_{c^2}$ satisfy the following conditions:*

(a) $-\infty < I_{c^2} < 0$ *for all* $c > 0$,
(b) $I_{c^2} \le I_{b^2} + I_{c^2-b^2}$ *for all* $0 < b < c$,
(c) the function $c \mapsto I_{c^2}$ *is continuous,*
(d) $\lim_{c\to 0}\frac{I_{c^2}}{c^2} = 0$.

Then,

$$I_{c^2} < I_{b^2} \text{ for all } 0 < b < c, \qquad \inf_{c>0} I_{c^2} = -\infty.$$

Proof. The function $c \to I_{c^2}$ exhibits strict monotonicity, which can be readily deduced from *(a)* and *(b)*. Let's now use proof by contradiction. Suppose there is a $K_2 > 0$ such that $I_{c^2} > -K_2$ for all $c > 0$. From *(a)*, we get $-K_2 < I_{c^2} < 0$, so $\frac{-K_2}{c^2} < \frac{I_{c^2}}{c^2} < 0$ is valid. In this case, we obtain

$$\lim_{c\to\infty}\frac{I_{c^2}}{c^2} = 0. \tag{2.13}$$

Based on conditions *(a)*, *(c)*, and *(d)*, the function $c \to \frac{I_{c^2}}{c^2}$ reaches a global minimum. In other words, there exists a c_0 such that

$$\frac{I_{c_0^2}}{c_0^2} \le \frac{I_{c^2}}{c^2} \quad \text{for every } c > 0.$$

On the other hand, based on the weak subadditivity condition *(b)*,

$$\frac{I_{2c_0^2}}{2c_0^2} \le \frac{I_{c_0^2} + I_{c_0^2}}{2c_0^2} = \frac{I_{c_0^2}}{c_0^2}.$$

This indicates that the function $c \to \frac{I_{c^2}}{c^2}$ also reaches a global minimum at $\sqrt{2}c_0$. Since c_0 is already the global minimum, it is impossible to have a smaller value, therefore

$$\frac{I_{(\sqrt{2}c_0)^2}}{(\sqrt{2}c_0)^2} = \frac{I_{c_0^2}}{c_0^2}.$$

The same logical reasoning illustrates that

$$\frac{I_{kc_0^2}}{kc_0^2} = \frac{I_{c_0^2}}{c_0^2} \quad \text{for all } k \in N,$$

which implies that $\liminf_{c\to\infty}\frac{I_{c^2}}{c^2} < 0$, contradicting (2.13). Therefore, the hypothesis is not valid, that is, $\inf_{c>0} I_{c^2} = -\infty$. □

Proof of Theorem 1.1. According to Lemma 2.7, the ground state energy I_{c^2} strictly decreases as a function of c, and $\inf_{c>0} I_{c^2} = -\infty$. Conversely, based on Lemma 2.1, for functions with radial symmetry, the $\inf_{c>0} I^{rad}_{c^2} > -\infty$. Hence, symmetry breaking takes place when c is sufficiently large. □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $u \in D^{rad}_c$ and (C) hold. Then, there exists a constant $L_1 > 0$, which is independent of c, such that $\mathcal{I}(u) > -L_1$.*

Proof. First, when $\alpha > 1$, $N < 4 + \alpha$, and $\frac{2(5N-4-\alpha)}{3N+\alpha-4} < q \leq \frac{2N}{N-2}$ hold, the following Coulomb-Sobolev inequality is valid. That is,

$$\|u\|_q^q \leq C\|\nabla u\|_2^{\frac{q(N+\alpha)-4N}{4+\alpha-N}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2)u^2 dx\right)^{\frac{2N-q(N-2)}{2(4+\alpha-N)}}. \tag{3.1}$$

Therefore, when $q = \frac{4N}{N+\beta}$, i.e, $N - 4 \leq \beta \leq \frac{\alpha N}{4+\alpha}$, we have

$$\|u\|^{\frac{4N}{N+\beta}}_{L^{\frac{4N}{N+\beta}}} \leq C\|\nabla u\|_2^{\frac{4N(\alpha-\beta)}{(N+\beta)(4+\alpha-N)}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2)u^2 dx\right)^{\frac{N(\beta-N+4)}{(N+\beta)(4+\alpha-N)}}.$$

Then,

$$\|u\|^4_{L^{\frac{4N}{N+\beta}}} \leq C\|\nabla u\|_2^{\frac{4(\alpha-\beta)}{4+\alpha-N}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2)u^2 dx\right)^{\frac{\beta-N+4}{4+\alpha-N}}. \tag{3.2}$$

Combining (2.1) and (3.2), we obtain

$$\int_{\mathbb{R}^N} (I_\beta * |u|^2)u^2 dx \leq C(N,\beta)\|\nabla u\|_2^{\frac{4(\alpha-\beta)}{4+\alpha-N}} \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^2)u^2 dx\right)^{\frac{\beta-N+4}{4+\alpha-N}}. \tag{3.3}$$

Next, let's set

$$\overline{x} = \|\nabla u\|^2_{L^2(\mathbb{R}^N)}, \quad \overline{y} = \int_{\mathbb{R}^N} (I_\alpha * |u|^2)u^2 dx.$$

By (1.4) and (3.3), we have

$$\mathcal{I}(u) = \frac{1}{2}\overline{x} + \frac{\mu}{4}\overline{y} - \frac{C}{4}\overline{x}^{\frac{2(\alpha-\beta)}{4+\alpha-N}}\overline{y}^{\frac{\beta-N+4}{4+\alpha-N}}.$$

Since

$$\alpha > 1, 2 + \alpha < N < 4 + \alpha, \frac{\alpha + N - 4}{2} < \beta < \frac{N^2 + 3\alpha N - 4N}{5N - (4 + \alpha)},$$

it implies that $\mathcal{I}$ is bounded below by a constant independent of c. □

Lemma 3.2. *Let the function $c \to I_{c^2}$ satisfy the following conditions:*

- *(a)* $-\infty < I_{c^2} < 0$ *for all* $c > 0$,
- *(b)* $I_{c^2} \leq I_{b^2} + I_{c^2-b^2}$ *for all* $0 < b < c$,
- *(c) the function* $c \mapsto I_{c^2}$ *is continuous,*

(d) $\lim_{c\to 0}\frac{I_{c^2}}{c^2}=0$.

Then, $I_{c^2}<I_{b^2}$, *for all* $0<b<c, \inf_{c>0} I_{c^2}=-\infty$.

Proof. According to the proofs of Lemmas 2.2–2.6, we obtain that *(a)*, *(b)*, *(c)*, and *(d)* are valid. Therefore, similar to the proof of lemma 2.7, we obtain $I_{c^2}<I_{b^2}$ for all $0<b<c$, $\inf_{c>0} I_{c^2}=-\infty$. □

Proof of Theorem 1.2. According to Lemma 3.2, the ground state energy I_{c^2} strictly decreases as a function of c, and $\inf_{c>0} I_{c^2}=-\infty$. Conversely, based on Lemma 3.1, for functions with radial symmetry, the $\inf_{c>0} I_{c^2}^{rad}>-\infty$. Hence, symmetry breaking takes place when c is sufficiently large. □

4. Conclusions

This paper is mainly based on two key ingredients. Firstly, by applying the Coulomb-Sobolev interpolation inequality and the Hardy-Littlewood-Sobolev inequality, we establish an inequality relating $\int_{\mathbb{R}^N}|\nabla u|^2dx$, $\int_{\mathbb{R}^N}\left(I_\alpha * |u|^2\right)u^2dx$, and $\int_{\mathbb{R}^N}\left(I_\beta * |u|^2\right)u^2dx$. Moreover, the derivation of this inequality does not require any assumptions on the L^2-norm of the function. Secondly, we analyze the asymptotic behavior of the mapping $c\mapsto I_c/c^2$ associated with the translation-invariant energy functional. Based on the above two crucial foundations, we further prove that when the mass parameter c is sufficiently large and the assumptions (A), (B) and (C) are satisfied, the ground state energy constrained to radially symmetric functions is greater than the ground state energy. This conclusion indicates that the ground state solutions of the equation undergo symmetry breaking. This paper enriches the research on the symmetry of ground state solutions in variational methods and PDEs, provides new perspectives for structural analysis of nonlinear equation solutions, and lays a solid theoretical foundation for subsequent similar studies.

Author contributions

Na Huang: Conceptualization, Formal analysis, Investigation, Writing–original draft, Validation; Yumi Shao: Formal analysis, Investigation, Writing–review and editing; Hongmin Suo: Conceptualization, Methodology, Project administration, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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