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Research article

Numerical scheme for solving the time-fractional Huxley equation using shifted Dickson polynomials

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Abstract: This article introduces an efficient spectral collocation framework for numerically solving the time-fractional Huxley equation. New basis functions of shifted Dickson polynomials of the first kind are introduced and employed. To achieve this, new formulas for the shifted polynomials are derived, including a series representation, an inverse formula, and expressions for both integer and fractional derivatives, which together with the collocation method serve as the foundation of the proposed numerical algorithm for converting the equation with its governing conditions into a non-linear algebraic system. A convergence and error analysis of the proposed method is also provided. We present numerical results and compare them with existing methods to illustrate the high accuracy of the proposed algorithms and their applicability.

Keywords: time-fractional differential equations; Dickson polynomials; derivative expressions; collocation method; convergence analysis
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1. Introduction

Recently, fractional differential equations (FDEs) have become increasingly important in applications compared with traditional differential equations (DEs). This is due to their ability to model a broad spectrum of scientific phenomena. Such FDEs make it easier to analyze signals like fractal time series and self-similar signals that display non-integer power-law behaviors. They can simulate processes such as chemical reactions, heat transfer, and fluid dynamics. For some applications and theoretical studies of these equations, see [1–3]. Because the exact solutions of most FDEs are not available, numerical methods are necessary to solve these equations. There are many proposed numerical methods, including finite difference methods [4], high-order schemes for Caputo derivatives [5], spline and collocation techniques [6, 7], and operational matrix methods [8, 9]. Symmetry-based analytical numerical frameworks and power series methodologies have been utilized to examine fractional models [10, 11]. Iterative, matrix-based, and parallel-in-time methods have also been developed to enhance computational efficiency for FDEs [12–14]. In addition, fractional models have been widely applied in real-world problems such as epidemiological modeling [15].

A wide variety of domains use special functions and polynomials; see, for example, [16]. Approximation theory and numerical analysis extensively employ sequences of polynomials to obtain approximate solutions to various DEs. For instance, shifted Lucas polynomials were employed in [17] to handle the time-fractional FitzHugh–Nagumo equations. The classical Gegenbauer polynomials were utilized to handle the fractional Black–Scholes model in [18]. One of the most prominent families of polynomial sequences, which plays crucial roles across various disciplines, is the Dickson polynomials. These polynomials were employed in the numerical solutions of DEs. For example, they were utilized in [19] to treat a model of nonlinear corneal dynamics arising in eye surgery. They were also used to treat many models arising in the physical sciences in [20]. Furthermore, a matrix collocation algorithm based on Dickson polynomials was designed in [21] to treat particular fractional-order models. A combined Taylor–Dickson approximation scheme was presented in [22] for treating the nonlinear Kawahara DEs. The fractional-order logistic equations were handled through these polynomials in [23]. Another procedure based on a matrix system was followed in [24] to treat some nonlinear FDEs. For the numerical treatment of some types of integro-DEs, Dickson polynomials were used in [25]. Finally, these polynomials were employed in [26] to solve other integro-DEs.

An essential class of numerical methods for solving all types of DEs is spectral methods. These methods assume that the approximate solution can be expressed as a linear combination of selected special functions, possibly polynomials. These polynomials may be orthogonal or non-orthogonal. These methods have many advantages over other numerical methods. This is because the approximate solutions obtained by their application are global rather than local, unlike those of the finite difference and finite element methods. Furthermore, with these methods, highly accurate solutions are obtained using only a few terms of the basis functions. Some applications of spectral methods can be found in [27, 28]. Three main categories of spectral methods are used extensively. The most popular method is the collocation method. A hybrid collocation procedure was presented in [29] for the long-time heat conduction equation. In [30], a collocation approach was analyzed for treating the FitzHugh–Nagumo model, employing certain Chebyshev polynomials. In [31], the authors proposed a Bernoulli–barycentric rational matrix collocation method with preconditioning for efficiently solving a class of evolutionary partial differential equations. Compared with the collocation method, the tau and Galerkin approaches are more difficult to implement because they require evaluating specific integrals. Some contributions regarding these methods can be found in [32, 33].

The classical Huxley equation is used to model nerve impulse propagation, response and diffusion dynamics in two states, and population dynamics when thresholds are met. Its generalized forms further extend these applications by creating nonlinear reaction mechanisms that are more adaptable and better simulate complicated biological development and inhibition processes. Fractional versions of the Huxley and Burgers–Huxley equations are more useful because they incorporate memory and nonlocal effects. This makes them suitable for modeling anomalous diffusion, viscoelastic media, biological systems exhibiting hereditary behavior, , and plasma or wave propagation. Several numerical techniques have been developed to treat Huxley and Burgers–Huxley-type equations. For instance, in [34], a neural network framework was introduced to solve the Huxley equation. For Burgers–Huxley-type models, in [35], a neural network-based strategy was employed for the Burgers–Huxley equation. An iterative method was developed in [36] for the generalized Burgers–Huxley equation. A Taylor wavelet procedure was proposed in [37] for the generalized Burgers–Huxley equation, while a finite element method was introduced in [38] for a delayed generalized Burgers–Huxley equation. In addition, a Haar wavelet technique combined with a Runge–Kutta method was developed in [39] for generalized Burgers–Huxley-type equations. In [40], a septic Hermite collocation algorithm was presented for perturbed Huxley equations.

For fractional and time-fractional variants, which incorporate memory and nonlocal effects, several numerical techniques have also been developed. In [41], a decomposition method was introduced for the time-fractional Burgers–Huxley equation. A higher-order numerical approximation was proposed in [42]. The spline method was employed in [43] for inhomogeneous time-fractional Burgers–Huxley equations, while a cubic spline approach was used in [44]. A Galerkin scheme was proposed in [45]. More recently, a spectral algorithm based on generalized shifted Vieta–Fibonacci polynomials was developed in [46] for nonlinear variable-order time-fractional Burgers–Huxley equations.

This paper is devoted to the analysis of a collocation scheme to treat the following time-fractional Huxley equation (TFHE) [47]:

$$D_t^{\sigma} \mathrm{Z} = \nu \mathrm{Z}_{\zeta\zeta} - \beta_1 \mathrm{Z} \mathrm{Z}_{\zeta} + \beta_2 \mathrm{Z} (\mathrm{Z}^s - \beta_3)(1 - \mathrm{Z}^s) + g(\zeta, t), \quad (\zeta, t) \in]a, b] \times]0, T], \ \sigma \in (0, 1), \tag{1.1}$$

constrained with

$$\mathrm{Z}(\zeta, 0) = \mathrm{Z}_0(\zeta), \quad a < \zeta \le b, \tag{1.2}$$

$$\mathrm{Z}(a, t) = \mathrm{Z}_1(t), \quad \mathrm{Z}(b, t) = \mathrm{Z}_2(t), \quad 0 < t \le T, \tag{1.3}$$

where $\mathrm{Z} = \mathrm{Z}(\zeta, t)$, β_1, β_2, β_3, s are known parameters, $\nu > 0$ represents the kinematic viscosity, and $\beta_3 \in (0, 1)$. In addition, $g(\zeta, t)$ denotes the known source function of the problem, and $Z_0(\zeta)$, $Z_1(t)$, and $Z_2(t)$ are assumed to be sufficiently smooth functions represent the initial and boundary conditions.

In addition, we will give an account of the extension of the collocation scheme to solve the following two-dimensional TFHE:

$$\begin{aligned} D_t^{\sigma} \mathrm{Z} &= \nu \left(\mathrm{Z}_{\zeta\zeta} + \mathrm{Z}_{\eta\eta}\right) - \beta_1 \mathrm{Z} \left(\mathrm{Z}_{\zeta} + \mathrm{Z}_{\eta}\right) + \beta_2 \mathrm{Z} (\mathrm{Z}^s - \beta_3)(1 - \mathrm{Z}^s) + g(\zeta, \eta, t), \\ &(\zeta, \eta, t) \in]a, b] \times]c, d] \times]0, T], \sigma \in (0, 1), \end{aligned} \tag{1.4}$$

governed by the following conditions:

$$\mathrm{Z}(\zeta, \eta, 0) = \mathrm{Z}_0(\zeta, \eta), \quad a < \zeta \le b, \ c < \eta \le d, \tag{1.5}$$

$$\mathrm{Z}(a, \eta, t) = \mathrm{Z}_1(\eta, t), \quad \mathrm{Z}(b, \eta, t) = \mathrm{Z}_2(\eta, t), \quad c < \eta \le d, \ 0 < t \le T, \tag{1.6}$$

$$\mathrm{Z}(\zeta, c, t) = \mathrm{Z}_3(\zeta, t), \quad \mathrm{Z}(\zeta, d, t) = \mathrm{Z}_4(\zeta, t), \quad a < \zeta \le b, \ 0 < t \le T, \tag{1.7}$$

where $g(\zeta, \eta, t)$ is the source term and $Z_0(\zeta, \eta)$, $Z_1(\eta, t)$, $Z_2(\eta, t)$, $Z_3(\zeta, t)$, and $Z_4(\zeta, t)$ are assumed to be sufficiently smooth functions that represent the initial and boundary conditions.

In this paper, we apply a collocation method to obtain approximate solutions of the TFHE. It is important to note that the proposed method differs fundamentally from the commonly used time-stepping schemes for time-fractional partial DEs. In traditional time-stepping approaches, the temporal domain is discretized into successive time levels, and the solution is computed sequentially using some approximate schemes for the Caputo fractional derivative, combined with spatial discretization techniques such as finite difference or finite element methods. In contrast, the proposed method employs a space–time spectral collocation framework in which both spatial and temporal variables are approximated simultaneously using shifted Dickson polynomials. The main advantages of the proposed approach can be summarized as follows:

- The proposed method treats both spatial and temporal variables simultaneously within a unified space–time framework, providing a global approximation over the domain.
- The use of shifted Dickson polynomials combined with the collocation method introduces flexibility through a free parameter, enabling the construction of a family of approximations.
- The spectral nature of the method ensures rapid convergence for smooth solutions, leading to high accuracy with a relatively small number of degrees of freedom.
- To the best of our knowledge, this is the first time that shifted Dickson polynomials have been employed as basis functions for solving differential equations.
- The proposed shifted Dickson basis functions have potential applications in other applied science problems.

The core goals addressed in this paper are summarized below:

- Establishing new essential formulas for the shifted Dickson polynomials that are central to our analysis.
- Deriving the operational matrices associated with integer and fractional derivatives of the shifted Dickson polynomials.
- Transforming the TFHE into a system of equations by applying the collocation method.
- Conducting a convergence analysis of the double expansion involving the shifted Dickson polynomials.
- Validating the applicability and accuracy of the proposed algorithm via comparison with some existing methods.

The organization of the paper is outlined as follows: Section 2 introduces key definitions and formulas related to Dickson polynomials. New analytical formulas for the shifted Dickson polynomials, which are employed as basis functions, are established in Section 3. Section 4 introduces the analysis of the algorithm for solving the TFHE and an overview in its extension in two dimensions. The convergence analysis of the proposed Dickson expansion is investigated in Section 5. Some illustrative numerical examples, supported by comparisons, are presented in Section 6. Finally, some thoughts are presented in Section 7.

2. Some fundamental definitions and formulas

We introduce in this section some fundamental definitions and properties of the fractional calculus. In addition, some properties of generalized Lucas polynomials, Dickson polynomials of the first kind, and their shifted counterparts are presented.

2.1. The Caputo fractional derivative

Definition 2.1. *The Caputo fractional derivative is defined as [48]:*

$$D_z^{\sigma} Y(z) = \frac{1}{\Gamma(m-\sigma)} \int_0^z (z-t)^{m-\sigma-1} Y^{(m)}(t)\, dt, \quad \sigma > 0, \quad z > 0, \tag{2.1}$$

$$m-1 < \sigma < m, \quad m \in \mathbb{N},$$

and $\Gamma(\cdot)$ *denotes the classical gamma function.*

In addition, we have

$$D_z^{\sigma} C = 0, \ (C \text{ is a constant}), \tag{2.2}$$

$$D_z^{\sigma} z^{\epsilon} = \begin{cases} 0, & \text{if } \epsilon \in \mathbb{N}_0 \quad \textit{and} \quad \epsilon < \lceil \sigma \rceil, \\ \frac{\epsilon!}{\Gamma(\epsilon+1-\sigma)} z^{\epsilon-\sigma}, & \text{if } \epsilon \in \mathbb{N}_0 \quad \textit{and} \quad \epsilon \geq \lceil \sigma \rceil, \end{cases} \tag{2.3}$$

where $\mathbb{N} = \{1, 2, \ldots\}, \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\lceil \sigma \rceil$ represents the ceiling notation.

2.2. An overview of generalized Lucas polynomials

The generalized Lucas polynomials may be generated with the aid of the following recursive formula:

$$L_n^{(a,b)}(\zeta) = a\,\zeta\, L_{n-1}^{(a,b)}(\zeta) + b\, L_{n-2}^{(a,b)}(\zeta), \qquad n \geq 2, \tag{2.4}$$

with the starting values

$$L_0^{(a,b)}(\zeta) = 2, \qquad L_1^{(a,b)}(\zeta) = a\,\zeta. \tag{2.5}$$

The Binet form for these polynomials is

$$L_n^{(a,b)}(\zeta) = \frac{\left(a\zeta + \sqrt{a^2\zeta^2 + 4b}\right)^n + \left(a\zeta - \sqrt{a^2\zeta^2 + 4b}\right)^n}{2^n}, n \geq 0. \tag{2.6}$$

These polynomials can be expressed as [49]

$$L_n^{(a,b)}(\zeta) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{a^{n-2m}\, b^i \binom{n-i}{i}}{n-i}\, \zeta^{n-2m}, \tag{2.7}$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

The inverse formula to (2.7) can be written as [49]

$$\zeta^n = \frac{1}{a^n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} c_{n-2i}\, \frac{(-1)^i\,(n-i+1)_i\, b^i}{i!}\, L_{n-2i}^{(a,b)}(\zeta), \quad n \geq 0, \tag{2.8}$$

where

$$c_r = \begin{cases} \frac{1}{2}, & r = 0, \\ 1, & r \neq 0. \end{cases} \tag{2.9}$$

Remark 2.1. *It is worthy to mention here that many celebrated classes can be deduced as particular ones of the general class $L_n^{(a,b)}(\zeta)$. More precisely, Lucas, Pell-Lucas, Fermat-Lucas, the first kind of Chebyshev, and the first kind of Dickson polynomials can be deduced from $L_n^{(a,b)}(\zeta)$ in the sense that:*

$$L_n(\zeta) = L_n^{(1,1)}(\zeta), \qquad P_n(\zeta) = L_n^{(2,1)}(\zeta),$$
$$\mathcal{F}_n(\zeta) = L_n^{(3,-2)}(\zeta), \qquad T_n(\zeta) = L_n^{(2,-1)}(\zeta),$$
$$D_n^{(\alpha)}(\zeta) = L_n^{(1,-\alpha)}(\zeta).$$

2.3. An overview of Dickson polynomials of the first kind

In this section, we give an overview of Dickson polynomials of the first kind and some of their essential properties.

The Dickson polynomials $\left\{D_n^{(\alpha)}(\zeta)\right\}_{n\geq 0}$ are special cases of $L_n^{(a,b)}(\zeta)$. More precisely, we have

$$D_n^{(\alpha)}(\zeta) = L_n^{(1,-\alpha)}(\zeta). \tag{2.10}$$

The Dickson polynomials $\left\{D_n^{(\alpha)}(\zeta)\right\}_{n\geq 0}$ satisfy the following recursive formula:

$$D_n^{(\alpha)}(\zeta) = \zeta\, D_{n-1}^{(\alpha)}(\zeta) - \alpha\, D_{n-2}^{(\alpha)}(\zeta), \qquad n \geq 2, \tag{2.11}$$

with the starting values

$$D_0^{(\alpha)}(\zeta) = 2, \qquad D_1^{(\alpha)}(\zeta) = \zeta. \tag{2.12}$$

From (2.7), the polynomials $D_0^{(\alpha)}(\zeta)$ admit the following explicit series representation:

$$D_n^{(\alpha)}(\zeta) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-\alpha)^i\, \zeta^{n-2i}. \tag{2.13}$$

From (2.8), the inverse formula to (2.13) can be represented as

$$\zeta^m = \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} c_{m-2i}\, (-1)^i \binom{m}{i} (-\alpha)^i\, D_{m-2i}^{(\alpha)}(\zeta). \tag{2.14}$$

In addition, the Binet form for these polynomials is

$$D_m^{(\alpha)}(\zeta) = \left(\frac{\zeta + \sqrt{\zeta^2 - 4\alpha}}{2}\right)^m + \left(\frac{\zeta - \sqrt{\zeta^2 - 4\alpha}}{2}\right)^m, \qquad m \geq 0. \tag{2.15}$$

The first few Dickson polynomials are

$$D_0^{(\alpha)}(\zeta) = 2, \qquad D_1^{(\alpha)}(\zeta) = \zeta,$$
$$D_2^{(\alpha)}(\zeta) = \zeta^2 - 2\,\alpha, \qquad D_3^{(\alpha)}(\zeta) = \zeta^3 - 3\,\zeta\alpha,$$
$$D_4^{(\alpha)}(\zeta) = \zeta^4 - 4\,\zeta^2\alpha + 2\,\alpha^2, \qquad D_5^{(\alpha)}(\zeta) = \zeta^5 - 5\,\zeta^3\alpha + 5\,\zeta\,\alpha^2.$$

Remark 2.2. *To justify the use of Dickson polynomials, we first establish their completeness in* $L^2([-1,1])$*. In particular, for any* $f \in L^2([-1,1])$*, there exists a sequence of finite Dickson polynomial expansions that converges to* f *in the* L^2*-norm. We then introduce the shifted Dickson polynomials on* $[0,1]$ *via a linear transformation. Their completeness in* $L^2([0,1])$ *follows directly, ensuring that any function in* $L^2([0,1])$ *can be approximated arbitrarily well.*

Theorem 2.1. *The set of Dickson polynomials of the first kind* $\left\{D_n^{(\alpha)}(\zeta)\right\}_{n\geq 0}$ *forms a complete system in* $L^2([-1,1])$*.*

Proof. First, from the series form in (2.7), it follows that the term of highest degree in $D_n^{(\alpha)}(\zeta)$ is ζ^n, whose coefficient is 1. Hence $D_n^{(\alpha)}(\zeta)$ is a polynomial of exact degree n.

Now, consider a positive number N, and define the set

$$\mathcal{D}_N^{(\alpha)} = \{D_0^{(\alpha)}(\zeta), D_1^{(\alpha)}(\zeta), \ldots, D_N^{(\alpha)}(\zeta)\}.$$

Since the polynomials in $\mathcal{D}_N^{(\alpha)}$ have distinct degrees $0, 1, \ldots, N$, they are linearly independent.

Let $\mathbb{P}_N := \{p(\zeta) : \deg(p) \leq N\}$ denote the space of algebraic polynomials of degree at most N. It is well-known that $\dim(\mathbb{P}_N) = N + 1$. Moreover, the inverse formula in (2.14) shows that every monomial ζ^m, $0 \leq m \leq N$, can be written as a linear combination of $D_0^{(\alpha)}(\zeta), \ldots, D_m^{(\alpha)}(\zeta)$. Therefore, every polynomial in $\mathbb{P}_N$ can be written as a linear combination of the elements of $\mathcal{D}_N^{(\alpha)}$. Hence

$$\mathrm{span}\{D_0^{(\alpha)}(\zeta), \ldots, D_N^{(\alpha)}(\zeta)\} = \mathbb{P}_N. \tag{2.16}$$

Now, let $C[-1,1]$ denote the space of all continuous functions on $[-1,1]$. By the Weierstrass approximation theorem, algebraic polynomials are dense in $C([-1,1])$. In addition, it is a well-known result in real analysis that $C([-1,1])$ is dense in $L^2([-1,1])$, see, for example, [50]. Consequently, algebraic polynomials are dense in $L^2([-1,1])$.

Thus, for any $f \in L^2([-1,1])$ and any $\varepsilon > 0$, there exists a polynomial $p \in \mathbb{P}_N$ such that

$$\|f - p\|_{L^2([-1,1])} < \varepsilon.$$

Using (2.16), this polynomial p can be written as a finite linear combination of Dickson polynomials. Hence there exists a finite Dickson expansion $S_N(\zeta) = \sum_{n=0}^{N} c_n D_n^{(\alpha)}(\zeta)$ such that

$$\|f - S_N\|_{L^2([-1,1])} < \varepsilon.$$

Therefore, any function in $L^2([-1,1])$ can be approximated arbitrarily well by finite linear combinations of Dickson polynomials, which proves that $\{D_n^{(\alpha)}(\zeta)\}_{n\geq 0}$ is complete in $L^2([-1,1])$. □

2.4. Introducing shifted Dickson polynomials

The first kind of shifted Dickson polynomials, which are beneficial for our current goals, are defined as

$$K_j(\zeta) = D_j^{(\alpha)}(2\zeta - 1). \tag{2.17}$$

The following explicit recurrence is employed to generate the polynomials:

$$K_j(\zeta) = (2\,\zeta - 1)\,K_{j-1}(\zeta) - \alpha\,K_{j-2}(\zeta), \qquad j \geq 2. \tag{2.18}$$

Remark 2.3. *We will utilize the shifted Dickson polynomials as basis functions in the upcoming sections to design our proposed numerical algorithm. This requires deriving the basic properties of these polynomials, such as their series form, inversion formula, and integer and fractional derivatives. The following section derives these essential formulas.*

Remark 2.4. *As in Theorem 2.1, because $K_j(\zeta)$ is a polynomial of exact degree j for every $j \geq 0$, the set $\{K_0(\zeta), K_1(\zeta), \dots, K_N(\zeta)\}$ forms a basis of $\mathbb{P}_N$. Consequently, the linear span of the shifted Dickson polynomials coincides with the set of all algebraic polynomials on* $[0,1]$. *Since the space of algebraic polynomials is dense in $L^2([0,1])$, the shifted Dickson polynomials defined in* (2.17) *form a complete system in $L^2([0,1])$.*

3. Some new explicit representations of shifted Dickson polynomials of the first kind

In this section, we derive new fundamental formulas for the introduced polynomials, which are central to the design of our numerical approach. First, we present an analytic formula for these polynomials.

Theorem 3.1. *Consider a positive integer i. One can express $K_i(\zeta)$ as*

$$K_i(\zeta) = (-1)^i \sum_{m=0}^{i} (-2)^m \binom{i}{m} {}_2F_1\left(\left. \begin{matrix} \frac{m-i}{2}, \frac{m-i+1}{2} \\ 1-i \end{matrix} \right| 4\alpha \right) \zeta^m. \tag{3.1}$$

Proof. First, we make use of the analytic form of $D_i^{(\alpha)}(\zeta)$ in (2.13) to write the following representation of $K_i(\zeta)$:

$$K_i(\zeta) = i(-1)^i \sum_{k=0}^{\lfloor i/2 \rfloor} \frac{(-\alpha)^k}{i-k} \binom{i-k}{k} (1-2\zeta)^{i-2k}. \tag{3.2}$$

In virtue of the binomial theorem, the last formula turns into

$$K_i(\zeta) = i\,(-1)^i \sum_{k=0}^{\lfloor i/2 \rfloor} \frac{(-\alpha)^k}{i-k} \binom{i-k}{k} \sum_{r=0}^{i-2k} (-2)^r \binom{i-2k}{r} \zeta^r, \tag{3.3}$$

which can be written equivalently as follows:

$$K_i(\zeta) = (-1)^i\, i \sum_{m=0}^{i} (-2)^m \sum_{L=0}^{\lfloor i/2 \rfloor} \frac{(-\alpha)^L}{i-L} \binom{i-2L}{m} \binom{i-L}{L} \zeta^m. \tag{3.4}$$

Now, we write

$$K_i(\zeta) = (-1)^i\, i \sum_{m=0}^{i} (-2)^m\, G_{i,m}, \tag{3.5}$$

where

$$G_{i,m} = \sum_{L=0}^{\lfloor i/2 \rfloor} \frac{(-\alpha)^L}{i-L} \binom{i-2L}{m} \binom{i-L}{L} \zeta^m, \tag{3.6}$$

and define $H_{i,m} = G_{2i,m}$. Therefore, $H_{i,m}$ takes the following form:

$$H_{i,m} = \sum_{L=0}^{i} \frac{(-\alpha)^L}{2i-L} \binom{2i-2L}{m} \binom{2i-L}{L}. \tag{3.7}$$

Using the binomial definition and performing some simplifications, the last form can be converted into

$$H_{i,m} = \sum_{L=0}^{i} \frac{(-\alpha)^L \, (2i-L-1)!}{L! \, (2i-2L-m)! \, m!}, \tag{3.8}$$

which can also be converted into the following form based on the definition of the Pochhammer function:

$$H_{i,m} = \frac{(2i-1)!}{m! \, (2i-m)!} \sum_{L=0}^{i} \frac{\left(\frac{m-2i}{2}\right)_L \left(\frac{m-2i+1}{2}\right)_L (4\alpha)^L}{(1-2i)_L \, L!}. \tag{3.9}$$

Noting the definition of the hypergeometric function given by [51]

$$ {}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix} \,\middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(a)_k \, (b)_k}{(c)_k} \, \frac{z^k}{k!}, \tag{3.10}$$

it can be shown that $H_{i,m}$ can be written in the following hypergeometric form:

$$H_{i,m} = \frac{(2i-1)!}{m! \, (2i-m)!} \, {}_2F_1\left(\begin{matrix} \frac{m-2i}{2}, \ \frac{m-2i+1}{2} \\ 1-2i \end{matrix} \,\middle|\, 4\alpha \right), \tag{3.11}$$

and therefore, $G_{i,m}$ in (3.6) can be written as

$$G_{i,m} = \frac{(i-1)!}{(i-m)! \, m!} \, {}_2F_1\left(\begin{matrix} \frac{m-i}{2}, \ \frac{m-i+1}{2} \\ 1-i \end{matrix} \,\middle|\, 4\alpha \right). \tag{3.12}$$

Finally, the last expression, together with formula (3.5), leads to the following formula:

$$K_i(\zeta) = (-1)^i \sum_{m=0}^{i} (-2)^m \binom{i}{m} \, {}_2F_1\left(\begin{matrix} \frac{m-i}{2}, \ \frac{m-i+1}{2} \\ 1-i \end{matrix} \,\middle|\, 4\alpha \right) \zeta^m,$$

which completes the proof. □

Now, we derive the inverse formula to (3.1). The following theorem provides the explicit form of this formula.

Theorem 3.2. *For $r \geq 0$, ζ^r can be expanded as*

$$\zeta^r = \sum_{m=0}^{r} H_{m,r} \, K_m(\zeta), \tag{3.13}$$

where

$$H_{m,r}=\begin{cases}\dfrac{c_m\,2^{-r}r!\,(-1)^{\frac{r-m}{2}}(-\alpha)^{\frac{r-m}{2}}\;{}_2F_1\left(\begin{matrix}\frac{-m-r}{2},\ \frac{m-r}{2}\\ \frac{1}{2}\end{matrix}\middle|\frac{1}{4\alpha}\right)}{(\frac{r-m}{2})!(\frac{m+r}{2})!}, & (m+r)\ \textit{is even},\\ -\dfrac{c_m\,2^{-r}r!\ (-1)^{\frac{r-m+1}{2}}(-\alpha)^{\frac{r-m-1}{2}}\;{}_2F_1\left(\begin{matrix}\frac{-m-r+1}{2},\ \frac{m-r+1}{2}\\ \frac{3}{2}\end{matrix}\middle|\frac{1}{4\alpha}\right)}{(\frac{r-m-1}{2})!(\frac{m+r-1}{2})!}, & (m+r)\ \textit{is odd},\end{cases} \tag{3.14}$$

where c_m *is as defined in* (2.9)*.*

Proof. To prove (3.13), we prove its alternative formula written as

$$\begin{aligned}\zeta^j=&\sum_{s=0}^{\lfloor j/2\rfloor}\frac{(-1)^s\,2^{-j}\,(-\alpha)^s\,c_{j-2s}\,j!}{s!\,(j-s)!}\,{}_2F_1\left(\begin{matrix}-s,\ -j+s\\ \frac{1}{2}\end{matrix}\middle|\frac{1}{4\alpha}\right)K_{j-2s}(\zeta)\\&+\sum_{s=0}^{\lfloor (j-1)/2\rfloor}\frac{(-1)^s\,2^{-j}\,(-\alpha)^s\,c_{j-2s-1}\,j!}{s!\,(j-s-1)!}\,{}_2F_1\left(\begin{matrix}-s,\ 1-j+s\\ \frac{3}{2}\end{matrix}\middle|\frac{1}{4\alpha}\right)K_{j-2s-1}(\zeta).\end{aligned} \tag{3.15}$$

We will proceed by induction. The formula holds for $j=0$. Assume that (3.15) holds. We can write it in the following form:

$$\zeta^j=\sum_{s=0}^{\lfloor j/2\rfloor}U_{s,j}\,K_{j-2s}(\zeta)+\sum_{s=0}^{\lfloor (j-1)/2\rfloor}V_{s,j}\,K_{j-2s-1}(\zeta), \tag{3.16}$$

with

$$U_{s,j}=\frac{(-1)^s\,2^{-j}\,(-\alpha)^s\,c_{j-2s}\,j!\;{}_2F_1\left(\begin{matrix}-s,\ -j+s\\ \frac{1}{2}\end{matrix}\middle|\frac{1}{4\alpha}\right)}{s!\,(j-s)!}, \tag{3.17}$$

$$V_{s,j}=\frac{(-1)^s\,2^{-j}\,(-\alpha)^s\,c_{j-2s-1}\,j!\;{}_2F_1\left(\begin{matrix}-s,\ 1-j+s\\ \frac{3}{2}\end{matrix}\middle|\frac{1}{4\alpha}\right)}{s!\,(j-s-1)!}. \tag{3.18}$$

To complete the proof, we have to prove that

$$\zeta^{j+1}=\sum_{s=0}^{\lfloor (j+1)/2\rfloor}U_{s,j+1}\,K_{j-2s+1}(\zeta)+\sum_{s=0}^{\lfloor j/2\rfloor}V_{s,j+1}\,K_{j-2s}(\zeta), \tag{3.19}$$

which can be split into the following two identities:

$$\zeta^{2j+1}=\sum_{s=0}^{j}U_{s,2j+1}\,K_{2j-2s+1}(\zeta)+\sum_{s=0}^{j}V_{s,2j+1}\,K_{2j-2s}(\zeta), \tag{3.20}$$

$$\zeta^{2j} = \sum_{s=0}^{j} U_{s,2j}\, K_{2j-2s}(\zeta) + \sum_{s=0}^{j-1} V_{s,2j}\, K_{2j-2s-1}(\zeta). \tag{3.21}$$

Since the proofs of (3.20) and (3.21) are analogous, only the proof of (3.20) is provided. Multiplying formula (3.15) by ζ, we get

$$\zeta^{j+1} = \left(\sum_{s=0}^{\lfloor j/2 \rfloor} \zeta\, U_{s,j}\, K_{j-2s}(\zeta) + \sum_{s=0}^{\lfloor (j-1)/2 \rfloor} \zeta\, V_{s,j}\, K_{j-2s-1}(\zeta) \right). \tag{3.22}$$

Replacing j by $2j$ in (3.22), and utilizing the recursive formula (2.11) written as

$$\zeta\, K_s(\zeta) = \frac{1}{2}(K_s(\zeta) + K_{s+1}(\zeta) + \alpha\, K_{s-1}(\zeta)), \tag{3.23}$$

we obtain the following formula:

$$\zeta^{2j+1} = \sum_{s=0}^{j} G_{s,j}\, K_{2j-2s+1}(\zeta) + \sum_{s=0}^{j} H_{s,j}\, K_{2j-2s}(\zeta), \tag{3.24}$$

where

$$G_{s,j} = \frac{1}{2}(U_{s,2j} + \alpha\, U_{s-1,2j} + V_{s-1,2j}), \tag{3.25}$$

$$H_{s,j} = \frac{1}{2}(U_{s,2j} + V_{s,2j} + \alpha\, V_{s-1,2j}). \tag{3.26}$$

Some algebraic computations lead to the following two identities:

$$U_{s,2j+1} = \frac{1}{2}(U_{s,2j} + \alpha\, U_{s-1,2j} + V_{s-1,2j}), \tag{3.27}$$

$$V_{s,2j+1} = \frac{1}{2}(U_{s,2j} + V_{s,2j} + \alpha\, V_{s-1,2j}). \tag{3.28}$$

Accordingly, we can write

$$\zeta^{2j+1} = \left(\sum_{s=0}^{j} U_{s,2j+1}\, K_{2j-2s+1}(\zeta) + \sum_{s=0}^{j} V_{s,2j+1}\, K_{2j-2s}(\zeta) \right). \tag{3.29}$$

This ends the proof of (3.20). Formula (3.21) can be similarly proved. □

Theorem 3.3. *Consider two positive integers i, q, with $i \geq q$. We can write the following expression for the derivatives:*

$$\frac{d^q K_i(\zeta)}{d\zeta^q} = \sum_{s=0}^{i-q} U_{s,i,q}\, K_s(\zeta), \tag{3.30}$$

where

$$U_{s,i,q} = \frac{2^q\, i\, \alpha^{\frac{1}{2}(i-q-s)}\, c_s\, \left(\frac{i+q-s-2}{2}\right)!\, \left(\frac{i+q+s-2}{2}\right)!\, \eta_{s,i,q}}{(q-1)!\, \left(\frac{i-q-s}{2}\right)!\, \left(\frac{i-q+s}{2}\right)!}, \tag{3.31}$$

and $\eta_{s,i,q}$ is defined as

$$\eta_{s,i,q} = \begin{cases} 1, & i-q-s \equiv 0 \pmod 2, \\ 0, & \text{otherwise}. \end{cases} \tag{3.32}$$

Proof. We first derive the derivative expression for the polynomials $D_i^{(\alpha)}(\zeta)$. Based on the analytic form (2.13), we can write

$$\frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q} = i \sum_{k=0}^{\lfloor (i-q)/2 \rfloor} \frac{(-\alpha)^k \, (i-k-1)!}{k! \, (i-2k-q)!} \, \zeta^{\, i-2k-q}. \tag{3.33}$$

In view of the inversion formula (2.14), the above formula turns into

$$\begin{aligned} \frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q} = i \sum_{k=0}^{\lfloor (i-q)/2 \rfloor} & \frac{(-1)^k \, (i-k-1)!}{k! \, (i-2k-q)!} \\ \times & \sum_{r=0}^{\lfloor (i-q)/2 \rfloor - k} \frac{c_{i-2k-q-2r} \, \alpha^{r+k} \, (1+i-2k-q-r)_r}{r!} \, D_{i-2k-q-2r}^{(\alpha)}(\zeta), \end{aligned} \tag{3.34}$$

which, after rearrangement, leads to the following form:

$$\frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q} = i \sum_{m=0}^{\lfloor (i-q)/2 \rfloor} c_{i-2m-q} \, (-\alpha)^m \sum_{j=0}^{m} \frac{(-1)^{m-j} (i-j-1)!}{j!(m-j)!(i-j-m-q)!} \, D_{i-q-2m}^{(\alpha)}(\zeta). \tag{3.35}$$

In hypergeometric form, we can write

$$\frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q} = i! \sum_{m=0}^{\lfloor (i-q)/2 \rfloor} c_{i-2m-q} \frac{\alpha^m}{m!(i-m-q)!} \, {}_2F_1\!\left(\begin{matrix} -m, \; -i+m+q \\ 1-i \end{matrix} \,\middle|\, 1 \right) D_{i-q-2m}^{(\alpha)}(\zeta). \tag{3.36}$$

Chu Vandermond's identity [51] may be used to give a closed form of the ${}_2F_1(1)$ in (3.36), and thus the following formula can be obtained:

$$\frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q} = \frac{i}{(q-1)!} \sum_{m=0}^{\lfloor (i-q)/2 \rfloor} c_{i-2m-q} \frac{\alpha^m (i-m-1)!(m+q-1)!}{m!(i-m-q)!} \, D_{i-q-2m}^{(\alpha)}(\zeta). \tag{3.37}$$

Accordingly, we can express the $\frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q}$ as

$$\frac{d^q D_i^{(\alpha)}(\zeta)}{d\zeta^q} = \frac{i}{(q-1)!} \sum_{s=0}^{i-q} \frac{c_s \, \alpha^{\frac{i-q-s}{2}} \, \eta_{s,i,q} \left(\frac{i+q+s-2}{2}\right)! \left(\frac{i+q-s-2}{2}\right)!}{\left(\frac{i-q-s}{2}\right)! \left(\frac{i-q+s}{2}\right)!} \, D_s^{(\alpha)}(\zeta), \tag{3.38}$$

where $\eta_{s,i,q}$ is defined as

$$\eta_{s,i,q} = \begin{cases} 1, & i-q-s \equiv 0 \pmod 2, \\ 0, & \text{otherwise.} \end{cases}$$

The following desired formula is obtained by substituting $(2\zeta - 1)$ for ζ in (3.38):

$$\frac{d^q K_i(\zeta)}{d\zeta^q} = \sum_{s=0}^{i-q} U_{s,i,q} \, K_s(\zeta), \tag{3.39}$$

where $U_{s,i,q}$ can be expressed as

$$U_{s,i,q} = \frac{2^q\, i\, \alpha^{\frac{1}{2}(i-q-s)}\, c_s \left(\frac{i+q-s-2}{2}\right)! \left(\frac{i+q+s-2}{2}\right)! \, \eta_{s,i,q}}{(q-1)! \left(\frac{i-q-s}{2}\right)! \left(\frac{i-q+s}{2}\right)!}.$$

This ends the proof. □

Corollary 3.1. *The following derivative relations of $K_i(\zeta)$ are satisfied:*

$$\frac{d\,K_i(\zeta)}{d\,\zeta} = \sum_{s=0}^{i-1} U_{s,i,1}\, K_s(\zeta), \quad i \geq 1, \tag{3.40}$$

$$\frac{d^2\,K_i(\zeta)}{d\,\zeta^2} = \sum_{s=0}^{i-2} U_{s,i,2}\, K_s(\zeta), \quad i \geq 2. \tag{3.41}$$

Proof. The result is immediate upon substituting $q = 1, 2$ in Theorem 3.3. □

Theorem 3.4. *For $\sigma \in (0,1)$, the fractional derivative of $K_j(t)$ has the following form:*

$$D_t^{\sigma}\, K_j(t) = t^{-\sigma} \left(\sum_{p=0}^{j} \gamma_{p,j}^{\sigma}\, K_p(t) - o_j^{\sigma} \right), \tag{3.42}$$

where

$$\gamma_{p,j}^{\sigma} = \sum_{L=p}^{j} \frac{L!\, B_{L,j}\, H_{p,L}}{\Gamma(L-\sigma+1)}, \tag{3.43}$$

and

$$o_j^{\sigma} = \frac{(-1)^j}{\Gamma(1-\sigma)}\, {}_2F_1\left(\left.\begin{array}{c} \frac{1-j}{2}, -\frac{j}{2} \\ 1-j \end{array} \right| 4\,\alpha \right). \tag{3.44}$$

Proof. The application of Definition 2.1 using property (2.3) along with the series form in (3.1) enables one to write the following relation:

$$D_t^{\sigma}\, K_j(t) = \sum_{p=1}^{j} \frac{p!\, B_{p,j}}{\Gamma(p-\sigma+1)}\, t^{p-\sigma}, \tag{3.45}$$

where

$$B_{p,j} = (-2)^p (-1)^j \binom{j}{p}\, {}_2F_1\left(\left.\begin{array}{c} \frac{p-j}{2}, \frac{p-j+1}{2} \\ 1-j \end{array} \right| 4\,\alpha \right). \tag{3.46}$$

Equation (3.45) can be rewritten after using the inversion formula (3.13) as

$$D_t^{\sigma}\, K_j(t) = t^{-\sigma} \sum_{p=1}^{j} \sum_{L=0}^{p} \frac{p!\, B_{p,j}\, H_{L,p}}{\Gamma(p-\sigma+1)}\, K_L(t). \tag{3.47}$$

Upon expansion and rearrangement of the last equation, the following result is obtained:

$$D_t^{\sigma} K_j(t) = t^{-\sigma}\left(\sum_{p=0}^{j}\sum_{L=p}^{j}\frac{L!\, B_{L,j}\, H_{p,L}}{\Gamma(L-\sigma+1)}\; K_p(t) - \frac{H_{0,0}\, B_{0,j}}{\Gamma(1-\sigma)}\right). \tag{3.48}$$

Now, $H_{0,0}\, B_{0,j}$ can be simplified to give the following relation:

$$H_{0,0}\, B_{0,j} = (-1)^j\, {}_2F_1\left(\begin{array}{c} \frac{1-j}{2}, -\frac{j}{2} \\ 1-j \end{array}\middle| 4\,\alpha\right). \tag{3.49}$$

Therefore, we get the desired result

$$D_t^{\sigma} K_j(t) = t^{-\sigma}\left(\sum_{p=0}^{j}\gamma_{p,j}^{\sigma}\, K_p(t) - o_j^{\sigma}\right), \tag{3.50}$$

where

$$\gamma_{p,j}^{\sigma} = \sum_{L=p}^{j}\frac{L!\, B_{L,j}\, H_{p,L}}{\Gamma(L-\sigma+1)}, \tag{3.51}$$

and

$$o_j^{\sigma} = \frac{(-1)^j}{\Gamma(1-\sigma)}\, {}_2F_1\left(\begin{array}{c} \frac{1-j}{2}, -\frac{j}{2} \\ 1-j \end{array}\middle| 4\,\alpha\right). \tag{3.52}$$

This proves the theorem. □

Corollary 3.2. *Consider the following vector:*

$$\boldsymbol{K}(\zeta) = [K_0(\zeta), K_1(\zeta), \ldots, K_{\mathcal{J}}(\zeta)]^T. \tag{3.53}$$

The following expressions are valid:

$$\frac{d\,\boldsymbol{K}(\zeta)}{d\,\zeta} = \boldsymbol{D}^1\, \boldsymbol{K}(\zeta), \tag{3.54}$$

$$\frac{d^2\,\boldsymbol{K}(\zeta)}{d\,\zeta^2} = \boldsymbol{D}^2\, \boldsymbol{K}(\zeta), \tag{3.55}$$

$$D_t^{\sigma}\, \boldsymbol{K}(t) = t^{-\sigma}\left(\boldsymbol{D}^{\sigma}\, \boldsymbol{K}(t) - \boldsymbol{O}^{\sigma}\right), \tag{3.56}$$

where $\boldsymbol{D}^1 = (U_{L,i,1})$*,* $\boldsymbol{D}^2 = (U_{L,i,2})$*, and* $\boldsymbol{D}^{\sigma} = (\gamma_{p,j}^{\sigma})$ *are the operational matrices of derivatives of order* $(\mathcal{J}+1)^2$*. Also,* $\boldsymbol{O}^{\sigma} = [o_0^{\sigma}, o_1^{\sigma}, \ldots, o_{\mathcal{J}}^{\sigma}]^T$ *is a matrix of order* $(\mathcal{J}+1)\times 1$.

4. Collocation method for the TFHE

4.1. The numerical treatment of the one-dimensional TFHE

This part presents an analysis of a numerical scheme for the initial-boundary problem of the TFHE. In its general form, the problem is defined on the domain $(\zeta, t) \in\,]a, b] \times\,]0, T]$. Without loss of generality, this general domain can be mapped into the domain $]0,1]\times]0,1]$ via a suitable transformation. Accordingly, we consider the same TFHE in (1.1), governed by (1.2)-(1.3), but posed on $(\zeta,t)\in]0,1]\times]0,1]$.

The collocation method is employed to construct an efficient numerical scheme.

Define the following space function:

$$\mathcal{A}^{\mathcal{J}} = \operatorname{span}\{K_m(\zeta)\,K_n(t) : 0 \le m,n \le \mathcal{J}\}, \tag{4.1}$$

and then $Z_{\mathcal{J}} \in \mathcal{A}^{\mathcal{J}}$ can be represented as

$$Z_{\mathcal{J}} = \sum_{m=0}^{\mathcal{J}}\sum_{n=0}^{\mathcal{J}} \hat{z}_{mn}\,K_m(\zeta)\,K_n(t) = \boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t), \tag{4.2}$$

where $\boldsymbol{K}(\zeta) = [K_0(\zeta), K_1(\zeta),\ldots,K_{\mathcal{J}}(\zeta)]^T$, $\quad \boldsymbol{K}(t) = [K_0(t),K_1(t),\ldots,K_{\mathcal{J}}(t)]^T$, and $\hat{\mathbf{Z}} = (\hat{z}_{mn})_{0\le m,n\le\mathcal{J}}$ is the matrix, which will be determined, of order $(\mathcal{J}+1)^2$.

Now, the residual $R(\zeta,t)$ of Eq (1.1) can be written as

$$R(\zeta,t) = D_t^{\sigma} Z_{\mathcal{J}} - \nu\,(Z_{\mathcal{J}})_{\zeta\zeta} + \beta_1\,Z_{\mathcal{J}}\,(Z_{\mathcal{J}})_{\zeta} - \beta_2\,Z_{\mathcal{J}}\,(Z_{\mathcal{J}}^s - \beta_3)\,(1 - Z_{\mathcal{J}}^s) - g(\zeta,t). \tag{4.3}$$

Using Corollary 3.2 along with Eq (4.2), we can write

$$D_t^{\sigma} Z_{\mathcal{J}} = \boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,(t^{-\sigma}\,(\boldsymbol{D}^{\sigma}\,\boldsymbol{K}(t) - \boldsymbol{O}^{\sigma})), \tag{4.4}$$

$$(Z_{\mathcal{J}})_{\zeta\zeta} = (\boldsymbol{D}^2\,\boldsymbol{K}(\zeta))^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t), \tag{4.5}$$

$$Z_{\mathcal{J}}\,(Z_{\mathcal{J}})_{\zeta} = \left(\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right)\left([\boldsymbol{D}^1\,\boldsymbol{K}(\zeta)]^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right), \tag{4.6}$$

$$Z_{\mathcal{J}}\,(Z_{\mathcal{J}}^s - \beta_3)\,(1 - Z_{\mathcal{J}}^s) = \left(\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right)\left(\left[\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right]^s - \beta_3\right)\left(1 - \left[\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right]^s\right). \tag{4.7}$$

Inserting Eqs (4.4)–(4.7) into Eq (4.3), we can write

$$\begin{aligned} R(\zeta,t) = {} & \boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\left(t^{-\sigma}(\boldsymbol{D}^{\sigma}\,\boldsymbol{K}(t) - \boldsymbol{O}^{\sigma})\right) - \nu\,(\boldsymbol{D}^2\,\boldsymbol{K}(\zeta))^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t) \\ & + \beta_1\,(\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t))\,([\boldsymbol{D}^1\,\boldsymbol{K}(\zeta)]^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)) \\ & - \beta_2\left(\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right)\left(\left[\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right]^s - \beta_3\right)\left(1 - \left[\boldsymbol{K}(\zeta)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}(t)\right]^s\right) - g(\zeta,t). \end{aligned} \tag{4.8}$$

Here is how the collocation method is applied to get $\hat{z}_{mn}$ at certain points (ζ_m,t_n). The residual function $R(\zeta,t)$ is required to vanish identically, that is, $R(\zeta,t)\equiv 0$.

$$R\left(\frac{m}{\mathcal{J}+1},\frac{n}{\mathcal{J}+1}\right) = 0, \qquad 1 \le m \le \mathcal{J}-1, \quad 1 \le n \le \mathcal{J}. \tag{4.9}$$

Moreover, the constraints in (3.43) and (1.3) lead to

$$\begin{aligned} &\boldsymbol{K}\left(\frac{m}{\mathcal{J}+1}\right)^T\hat{\mathbf{Z}}\,\boldsymbol{K}(0) = Z_0\left(\frac{m}{\mathcal{J}+1}\right), \quad m = 1,2,\ldots,\mathcal{J}+1, \\ &\boldsymbol{K}(0)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}\left(\frac{n}{\mathcal{J}+1}\right) = Z_1\left(\frac{n}{\mathcal{J}+1}\right), \quad n = 1,2,\ldots,\mathcal{J}, \\ &\boldsymbol{K}(1)^T\,\hat{\mathbf{Z}}\,\boldsymbol{K}\left(\frac{n}{\mathcal{J}+1}\right) = Z_2\left(\frac{n}{\mathcal{J}+1}\right), \quad n = 1,2,\ldots,\mathcal{J}. \end{aligned} \tag{4.10}$$

Equations (4.9) and (4.10) collectively generate a nonlinear system of size $(\mathcal{J}+1)^2$, which is solved by means of Newton's iteration.

4.2. Extension to two-dimensional problems

In this part, we show that the proposed algorithm can be extended to treat two-dimensional models. We consider the two-dimensional TFHE introduced in (1.4), governed by the conditions (1.5)–(1.7), which is defined on the general domain $(\zeta,\eta,t)\in\,]a,b]\times]c,d]\times]0,T]$. Without loss of generality, this domain can be mapped into the reference domain $(\zeta,\eta,t)\in\,]0,1]^3$. Accordingly, we consider the same problem posed on $]0,1]^3$.

In this case, we may assume the approximate solution of the form

$$\mathrm{Z}_{\mathcal{J}}=\sum_{m=0}^{\mathcal{J}}\sum_{j=0}^{\mathcal{J}}\sum_{n=0}^{\mathcal{J}}\hat{z}_{mjn}\,K_m(\zeta)\,K_j(\eta)\,K_n(t).$$

We follow similar steps to those followed in Section 4.1 to get a nonlinear system of size $(\mathcal{J}+1)^3$, which is solved by means of Newton's iteration.

5. The convergence and error analysis

This section presents a detailed convergence analysis of the proposed shifted Dickson expansion.

Lemma 5.1. *The following inequality holds for $K_j(\zeta)$:*

$$|K_j(\zeta)|<(1+\alpha)^j,\quad \zeta\in[0,1]. \tag{5.1}$$

Proof. We argue by induction on j. Assuming that (5.1) holds for $(j-1)$ and $(j-2)$, we obtain

$$|K_{j-1}(\zeta)|<(1+\alpha)^{j-1}\quad\text{and}\quad |K_{j-2}(\zeta)|<(1+\alpha)^{j-2}. \tag{5.2}$$

Based on the recurrence relation (2.18), we can write

$$K_j(\zeta)=(2\,\zeta-1)\,K_{j-1}(\zeta)-\alpha\,K_{j-2}(\zeta),\qquad j\geq 2. \tag{5.3}$$

Invoking the previous relation, along with inequalities (5.2), we can write

$$\begin{aligned}|K_j(\zeta)|&=\left|(2\,\zeta-1)\,K_{j-1}(\zeta)\right|+\left|\alpha\,K_{j-2}(\zeta)\right|\\&<(1+\alpha)^{j-1}+\alpha\,(1+\alpha)^{j-2}=(1+\alpha)^{j-2}\,(1+2\,\alpha).\end{aligned} \tag{5.4}$$

Based on the following identity:

$$1+2\,\alpha\leq(1+\alpha)^2,\quad\forall\,\alpha\;\geq 0, \tag{5.5}$$

we obtain the estimation result (5.1). □

Lemma 5.2. *Consider an infinitely smooth function $\psi(\zeta)$ at the origin. It can be represented in the form*

$$\psi(\zeta)=\sum_{n=0}^{\infty}\sum_{s=n}^{\infty}\frac{\psi^{(s)}(0)\,H_{n,s}}{s!}\,K_n(\zeta), \tag{5.6}$$

where $H_{n,s}$ is defined in Eq (3.14).

Proof. The function $\psi(\zeta)$ may be expanded as

$$\psi(\zeta) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!}\, \zeta^n. \tag{5.7}$$

By invoking relation (3.13), one obtains the following identity:

$$\psi(\zeta) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{\psi^{(n)}(0)\, H_{r,n}}{n!}\, K_r(\zeta). \tag{5.8}$$

Through a sequence of algebraic manipulations, the preceding equation can be rewritten as follows:

$$\psi(\zeta) = \sum_{n=0}^{\infty} \sum_{s=n}^{\infty} \frac{\psi^{(s)}(0)\, H_{n,s}}{s!}\, K_n(\zeta). \tag{5.9}$$

The proof is complete. □

Theorem 5.1. *If* $\psi(\zeta)$ *is defined on* $[0,1]$ *and* $|\psi^{(i)}(0)| \le \mathcal{U}^i,\ i > 0$, *where* $\mathcal{U} > 0$ *and* $\psi(\zeta) = \sum_{i=0}^{\infty} c_i\, K_i(\zeta)$, *then we have*

$$|c_i| < \frac{e^{\frac{1}{2}(\alpha+2)\,\mathcal{U}} \left(\frac{1}{2}(\alpha+2)\,\mathcal{U}\right)^i}{i!}, \quad \forall\, i > 0. \tag{5.10}$$

Furthermore, the series converges absolutely.

Proof. Lemma 5.2 enables one to write

$$|c_i| = \left| \sum_{s=i}^{\infty} \frac{\psi^{(s)}(0)\, H_{i,s}}{s!} \right| \le \sum_{s=i}^{\infty} \frac{\left|\psi^{(s)}(0)\right| \left|H_{i,s}\right|}{s!}. \tag{5.11}$$

As a consequence of (3.14), the inequality below holds for $i \le s < \infty,\ i \in \mathbb{N}_0$:

$$\left|H_{i,s}\right| < \left(\frac{\alpha+2}{2}\right)^s, \tag{5.12}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Under the assumption $|\psi^{(i)}(0)| \le \mathcal{U}^i$ for $i > 0$, one may write

$$|c_i| \le \sum_{s=i}^{\infty} \frac{\left(\frac{\alpha+2}{2}\right)^s \mathcal{U}^s}{s!} = \frac{e^{\frac{1}{2}(\alpha+2)\,\mathcal{U}} \left((i-1)! - \Gamma\left(i, \frac{1}{2}(\alpha+2)\,\mathcal{U}\right)\right)}{(i-1)!}, \tag{5.13}$$

where $\Gamma\left(i, \frac{1}{2}(\alpha+2)\,\mathcal{U}\right)$ is the incomplete gamma function.

In virtue of the following inequality:

$$\frac{e^{\frac{1}{2}(\alpha+2)\,\mathcal{U}} \left((i-1)! - \Gamma\left(i, \frac{1}{2}(\alpha+2)\,\mathcal{U}\right)\right)}{(i-1)!} < \frac{e^{\frac{1}{2}(\alpha+2)\,\mathcal{U}} \left(\frac{1}{2}(\alpha+2)\,\mathcal{U}\right)^i}{i!}, \quad \forall\, i > 0, \tag{5.14}$$

the inequality in (5.13) leads to

$$|c_i| < \frac{e^{\frac{1}{2}(\alpha+2)\mathcal{U}}\left(\frac{1}{2}(\alpha+2)\,\mathcal{U}\right)^i}{i!}, \quad \forall\, i > 0. \tag{5.15}$$

Now, if we make use of the inequalities (5.1) and (5.10), we can write

$$\begin{aligned}\left|\sum_{i=0}^{\infty} c_i\, K_i(\zeta)\right| &= \sum_{i=0}^{\infty} |c_i|\; |K_i(\zeta)| \\ &< e^{\frac{1}{2}(\alpha+2)\mathcal{U}} \sum_{i=0}^{\infty} \frac{(\alpha+1)^i \left(\frac{1}{2}(\alpha+2)\,\mathcal{U}\right)^i}{i!} \\ &= e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}},\end{aligned} \tag{5.16}$$

so the series converges absolutely. □

Remark 5.1. *The exponential factor* $e^{\frac{1}{2}(\alpha+2)\mathcal{U}}$ *in inequality* (5.13) *arises from bounding the truncated exponential series using standard estimates. Moreover, inequality* (5.14) *is employed as an upper bound and has been verified symbolically and numerically using Mathematica for the range of parameters considered.*

Theorem 5.2. *Let* $\mathrm{Z} = \psi_1(\zeta)\,\psi_2(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{z}_{ij}\, K_i(\zeta)\, K_j(t)$, *with* $|\psi_1^{(i)}(0)| \le \mathcal{U}_1^i$ *and* $|\psi_2^{(i)}(0)| \le \mathcal{U}_2^i$, *where* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are positive constants. We get*

$$|\hat{z}_{ij}| < \frac{e^{\frac{1}{2}(\alpha+2)(\mathcal{U}_1+\mathcal{U}_1)}\left(\frac{1}{2}(\alpha+2)\,\mathcal{U}_1\right)^i\left(\frac{1}{2}(\alpha+2)\,\mathcal{U}_2\right)^j}{i!\; j!}, \quad \forall\, i,\ j > 0. \tag{5.17}$$

Furthermore, the series is absolutely convergent.

Proof. The application of Lemma 5.2, taking into consideration the assumption $\mathcal{Z} = \psi_1(\zeta)\,\psi_2(t)$, enables us to write

$$\hat{z}_{ij} = \sum_{s=i}^{\infty}\sum_{q=j}^{\infty} \frac{\psi_1^{(s)}(0)\,\psi_2^{(q)}(0)\,H_{j,q}\,H_{i,s}}{s!\,q!}. \tag{5.18}$$

Using the assumptions $|\psi_1^{(i)}(0)| \le \mathcal{U}_1^i$ and $|\psi_2^{(i)}(0)| \le \mathcal{U}_2^i$, then we can write

$$|\hat{z}_{ij}| \le \sum_{s=i}^{\infty} \frac{\mathcal{U}_1^s\, H_{i,s}}{s!} \times \sum_{q=j}^{\infty} \frac{\mathcal{U}_2^q\, H_{j,q}}{q!}. \tag{5.19}$$

Following arguments analogous to those employed in the proof of Theorem 5.1, the result below is derived.

$$|\hat{z}_{ij}| < \frac{e^{\frac{1}{2}(\alpha+2)(\mathcal{U}_1+\mathcal{U}_1)}\left(\frac{1}{2}(\alpha+2)\,\mathcal{U}_1\right)^i\left(\frac{1}{2}(\alpha+2)\,\mathcal{U}_2\right)^j}{i!\; j!}. \tag{5.20}$$

□

Theorem 5.3. *The following upper estimation holds if* Z *satisfies the assumptions of Theorem 5.2.*

$$|\mathrm{Z}-\mathrm{Z}_{\mathcal{J}}| < \frac{\varepsilon\, 2^{-\mathcal{J}-1}\left(\mathcal{U}_1((\alpha+1)(\alpha+2)\mathcal{U}_1)^{\mathcal{J}} + \mathcal{U}_2((\alpha+1)(\alpha+2)\mathcal{U}_2)^{\mathcal{J}}\right)}{\mathcal{J}!}, \tag{5.21}$$

where

$$\varepsilon = (\alpha+1)\,(\alpha+2)\, e^{\frac{1}{2}(\alpha+2)^2(\mathcal{U}_1+\mathcal{U}_2)}. \tag{5.22}$$

Proof. In view of the definitions of Z and $\mathrm{Z}_{\mathcal{J}}$, the following expression holds:

$$\begin{aligned}|\mathrm{Z}-\mathrm{Z}_{\mathcal{J}}| &< \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\hat{z}_{ij}\,K_i(\zeta)\,K_j(t) - \sum_{i=0}^{\mathcal{J}}\sum_{j=0}^{\mathcal{J}}\hat{z}_{ij}\,K_i(\zeta)\,K_j(t)\right| \\ &\le \left|\sum_{i=0}^{\mathcal{J}}\sum_{j=\mathcal{J}+1}^{\infty}\hat{z}_{ij}\,K_i(\zeta)\,K_j(t)\right| + \left|\sum_{i=\mathcal{J}+1}^{\infty}\sum_{j=0}^{\infty}\hat{z}_{ij}\,K_i(\zeta)\,K_j(t)\right|.\end{aligned} \tag{5.23}$$

The following conclusion follows from Theorem 5.2 and Lemma 5.1:

$$\begin{aligned}\sum_{i=0}^{\mathcal{J}}\frac{e^{\frac{1}{2}(\alpha+2)\mathcal{U}_1}(\alpha+1)^i\left(\frac{1}{2}(\alpha+2)\mathcal{U}_1\right)^i}{i!} &= \frac{e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_1}\Gamma\left(\mathcal{J}+1,\frac{1}{2}(\alpha+1)(\alpha+2)\mathcal{U}_1\right)}{\mathcal{J}!} \\ &< e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_1}, \\ \sum_{j=\mathcal{J}+1}^{\infty}\frac{e^{\frac{1}{2}(\alpha+2)\mathcal{U}_2}(\alpha+1)^j\left(\frac{1}{2}(\alpha+2)\mathcal{U}_2\right)^j}{j!} &= \frac{e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_2}\left(\mathcal{J}!-\Gamma\left(\mathcal{J}+1,\frac{1}{2}(\alpha+1)(\alpha+2)\mathcal{U}_2\right)\right)}{\mathcal{J}!} \\ &< \frac{e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_2}\left(\frac{1}{2}(\alpha+1)(\alpha+2)\mathcal{U}_2\right)^{\mathcal{J}+1}}{\mathcal{J}!}, \\ \sum_{i=\mathcal{J}+1}^{\infty}\frac{e^{\frac{1}{2}(\alpha+2)\mathcal{U}_1}(\alpha+1)^i\left(\frac{1}{2}(\alpha+2)\mathcal{U}_1\right)^i}{i!} &= \frac{e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_1}\left(\mathcal{J}!-\Gamma\left(\mathcal{J}+1,\frac{1}{2}(\alpha+1)(\alpha+2)\mathcal{U}_1\right)\right)}{\mathcal{J}!} \\ &< \frac{e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_1}\left(\frac{1}{2}(\alpha+1)(\alpha+2)\mathcal{U}_1\right)^{\mathcal{J}+1}}{\mathcal{J}!}, \\ \sum_{j=0}^{\infty}\frac{e^{\frac{1}{2}(\alpha+2)\mathcal{U}_2}(\alpha+1)^j\left(\frac{1}{2}(\alpha+2)\mathcal{U}_2\right)^j}{j!} &= e^{\frac{1}{2}(\alpha+2)^2\mathcal{U}_2}.\end{aligned} \tag{5.24}$$

By combining the derived bounds with the identity in (5.24), the following estimation is obtained:

$$|\mathrm{Z}-\mathrm{Z}_{\mathcal{J}}| < \frac{\varepsilon\, 2^{-\mathcal{J}-1}\left(\mathcal{U}_1((\alpha+1)(\alpha+2)\mathcal{U}_1)^{\mathcal{J}} + \mathcal{U}_2((\alpha+1)(\alpha+2)\mathcal{U}_2)^{\mathcal{J}}\right)}{\mathcal{J}!}, \tag{5.25}$$

where

$$\varepsilon = (\alpha+1)\,(\alpha+2)\, e^{\frac{1}{2}(\alpha+2)^2(\mathcal{U}_1+\mathcal{U}_2)}, \tag{5.26}$$

which is the desired result. □

6. Some numerical examples

To assess the effectiveness of the proposed algorithm, we present several test problems and compare the obtained results with those reported in the literature. After applying the proposed algorithm, the original problem is reduced to a nonlinear algebraic system of size $(\mathcal{J} + 1)^2$. This system is solved using Newton's method, where the initial guess is chosen as 10^{-i+j}, $i, j = 0, \ldots, \mathcal{J}$, and the iteration is terminated using a tolerance corresponding to an accuracy of approximately 10^{-6}. All computations are carried out using the `FindRoot` function in *Mathematica* 11 on an HP Z420 Workstation equipped with an Intel(R) Xeon(R) CPU E5-1620 v2 (3.70 GHz), 16 GB of DDR3 RAM, and 512 GB of storage.

Example 6.1 ([45]). *We consider the TFHE given in* (1.1)–(1.3) *with*

$$a = 0, \quad b = 1, \quad T = 1, \quad \nu = 1, \quad \beta_1 = 0.01, \quad \beta_2 = 1, \quad \beta_3 = 0.5, \quad s = 1.$$

The functions appearing on the right-hand side and the conditions are specified as

$$\mathrm{Z}_0(\zeta) = 0, \quad \mathrm{Z}_1(t) = 0, \quad \mathrm{Z}_2(t) = \frac{t^3}{2},$$

and the source term is given by

$$g(\zeta, t) = \frac{6}{\Gamma(4-\sigma)} t^{3-\sigma} \sin\left(\frac{5\pi\zeta}{6}\right) + t^9 \sin^3\left(\frac{5\pi\zeta}{6}\right) - 1.5\, t^6 \sin^2\left(\frac{5\pi\zeta}{6}\right) + 0.01309\, t^6 \sin\left(\frac{5\pi\zeta}{3}\right) + 7.35389\, t^3 \sin\left(\frac{5\pi\zeta}{6}\right).$$

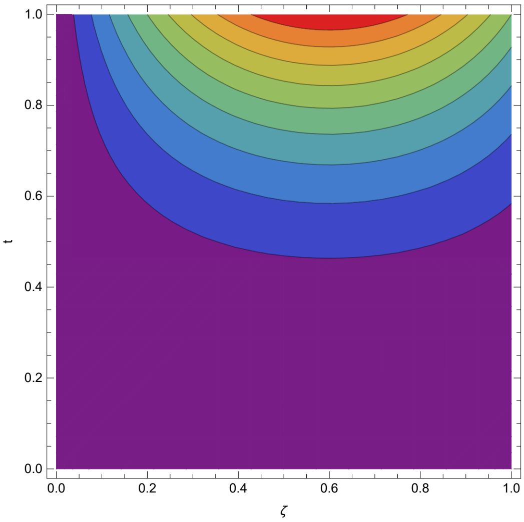
The exact solution is given by

$$\mathrm{Z} = t^3 \sin\left(\frac{5\pi\zeta}{6}\right).$$

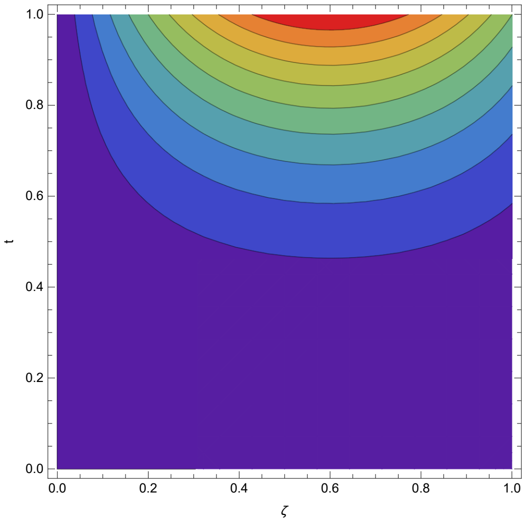
Table 1 presents a comparison of the L_2 and L_∞ errors between our method and the method in [47] at $\sigma = 0.5$. We see in this table that the results are accurate for small choices of $\mathcal{J}$. Also, this comparison reveals the superior performance of our technique over the method in [47]. Figure 1 shows the exact solution (contour), approximate solution (contour), and absolute contour at $\alpha = 0.8$ and $\sigma = 0.8$ when $\mathcal{J} = 14$. The accuracy of the suggested approach is demonstrated by Table 2, which shows the L_2 and L_∞ errors at different values of t when $\sigma = 0.3$ and $\mathcal{J} = 14$.

Table 1. Comparison of the L_2 and L_∞ errors of Example 6.1 at $\sigma = 0.5$.

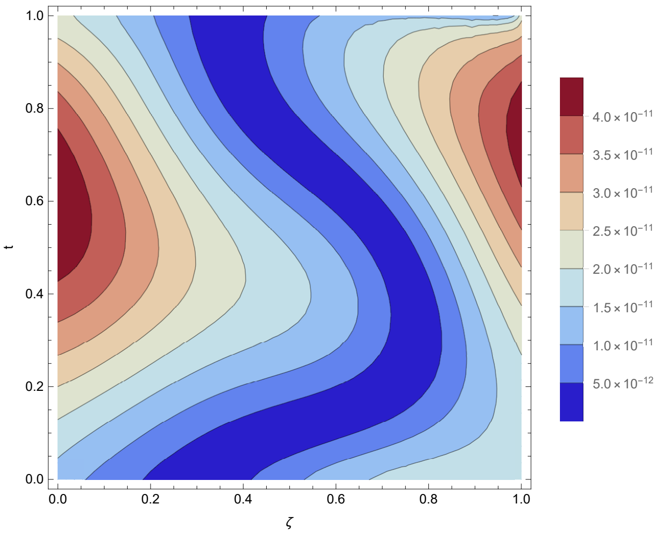
t	Method in [45] at $h = 0.1$		Proposed method at $\mathcal{J} = 14$ and $\alpha = 0.5$	
	L_2	L_∞	L_2	L_∞
0.25	5.68848×10^{-8}	7.83547×10^{-8}	1.30391×10^{-14}	2.80765×10^{-14}
0.5	1.57520×10^{-6}	2.23210×10^{-6}	1.06935×10^{-13}	2.13302×10^{-13}
0.75	8.59820×10^{-6}	1.19510×10^{-5}	3.90582×10^{-13}	7.74936×10^{-13}
1	1.15535×10^{-5}	1.81845×10^{-5}	6.42968×10^{-13}	1.91068×10^{-13}



(a) Exact solution (contour).



(b) Approximate solution (contour).



(c) Error contour.

Figure 1. The exact solution (contour), approximate solution (contour) and absolute contour of Example 6.1 at $\alpha = 0.8$ and $\sigma = 0.8$ when $\mathcal{J} = 14$.

Table 2. The L_2 and L_∞ errors of Example 6.1 at $\sigma = 0.3$ and $\mathcal{J} = 14$.

t	$\alpha = 0.3$		$\alpha = 0.7$		$\alpha = 0.9$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	1.36405×10^{-14}	2.74988×10^{-14}	5.06182×10^{-12}	6.0718×10^{-12}	3.92908×10^{-10}	6.19392×10^{-10}
0.5	1.08419×10^{-13}	2.09457×10^{-13}	4.62158×10^{-13}	7.44454×10^{-12}	8.89249×10^{-11}	1.93705×10^{-10}
0.75	3.96957×10^{-13}	7.58615×10^{-13}	4.65329×10^{-12}	6.24273×10^{-12}	1.53932×10^{-10}	2.24984×10^{-10}
1	1.03546×10^{-11}	2.05822×10^{-11}	4.42943×10^{-11}	9.09944×10^{-11}	1.59135×10^{-10}	3.23023×10^{-10}

Remark 6.1. *Table 3 demonstrates the agreement for both theoretical and numerical results at $\sigma = \alpha = 0.5$ for Example 6.1. As an example, if we set $\mathcal{U}_1 = 0.3$ and $\mathcal{U}_2 = 0.5$ in Theorem 5.3.*

Table 3. Theoretical and numerical results of Example 6.1.

$\mathcal{J}$	8	10	12	14
Theoretical error in Theorem 5.3	1.70731×10^{-4}	1.65665×10^{-6}	1.10051×10^{-8}	5.31008×10^{-11}
Numerical error	4.50572×10^{-6}	4.69553×10^{-8}	3.37259×10^{-10}	1.50147×10^{-12}

Example 6.2 ([52]). *We consider the TFHE given in* (1.1)–(1.3) *with*

$$a = 0, \quad b = 1, \quad T = 1, \quad \nu = 1, \quad \beta_1 = 0, \quad \beta_2 = 1, \quad \beta_3 = 1, \quad s = 1.$$

The functions appearing on the right-hand side and the conditions are specified as

$$Z_0(\zeta) = 0, \quad Z_1(t) = 0, \quad Z_2(t) = 0,$$

and the source term is given by

$$g(\zeta, t) = \frac{6}{\Gamma(4-\sigma)}(1-\zeta)\sin(\zeta)\, t^{3-\sigma} + t^3\Big((\zeta-1)\sin(\zeta)(t^3(-(\zeta-1))\sin(\zeta)(t^3(\zeta-1)\sin(\zeta)+2)-2)+2\cos(\zeta)\Big).$$

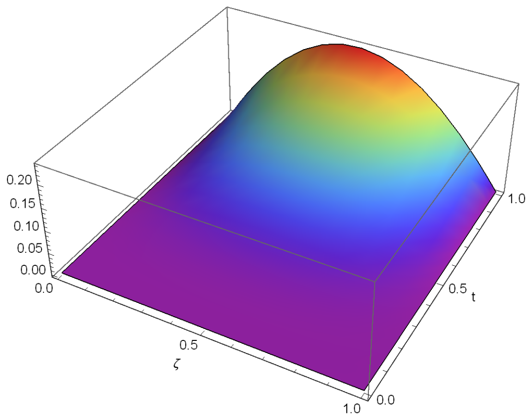
The exact solution is given by

$$Z = t^3\,(1-\zeta)\,\sin(\zeta).$$

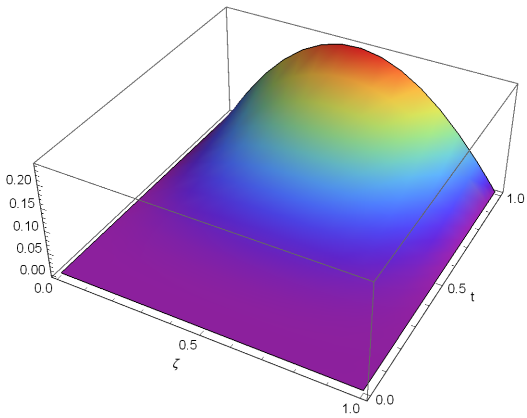
Table 4 introduces a comparison based on relative L_2 errors between our proposed scheme at $\mathcal{J} = 12$ and the approach in [52] at $\sigma = 0.9$. This table shows that, for small choices of $\mathcal{J}$, the results are accurate. Additionally, this comparison shows the superiority of our strategy over the strategy mentioned in [52]. Figure 2 shows the exact solution, approximate solution, and absolute errors at $\alpha = 0.5$ and $\sigma = 0.9$ when $\mathcal{J} = 12$. This figure verifies that the suggested approach reduces errors consistently throughout the domain and shows a good agreement of the approximate solution with the exact one. Figure 3 illustrates the absolute errors at different values of $\mathcal{J}$ when $\alpha = 1$ and $\sigma = 0.9$. Table 5 presents the absolute errors at different values of $\mathcal{J}$ and the CPU time (in seconds) when $\alpha = 0.25$ and $\sigma = 0.5$. Figure 4 illustrates the approximate solution, and absolute errors at $\alpha = 1$ and $\sigma = 0.5$ when $\mathcal{J} = 12$. Figure 5 shows the stability of the proposed technique $|Z_{\mathcal{J}+1} - Z_{\mathcal{J}}|$ at $\zeta = t$, $\alpha = 1$, and $\sigma = 0.9$.

Table 4. Comparison of the relative L_2 errors of Example 6.2.

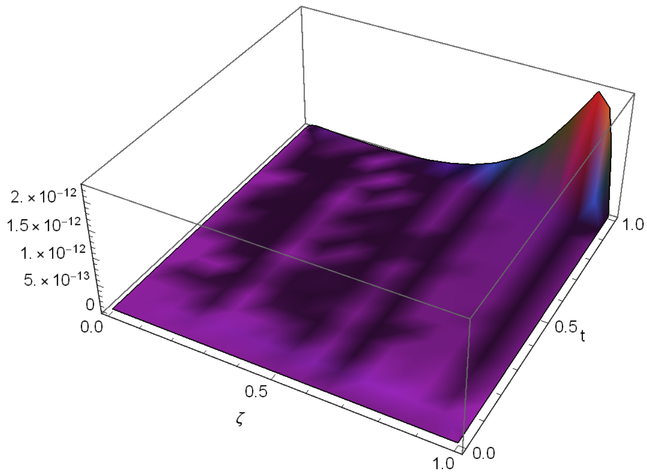
σ	Method in [52]		Proposed method at $\mathcal{J} = 12$	
	Mesh points	Error	$\alpha = 0.25$	$\alpha = 0.5$
0.9	10×10	9.30765×10^{-3}	2.42982×10^{-12}	3.05641×10^{-12}
	20×20	8.17603×10^{-3}		
	30×30	7.73402×10^{-3}		
	40×40	7.56828×10^{-3}		
	50×50	7.29012×10^{-3}		



(a) Exact solution.

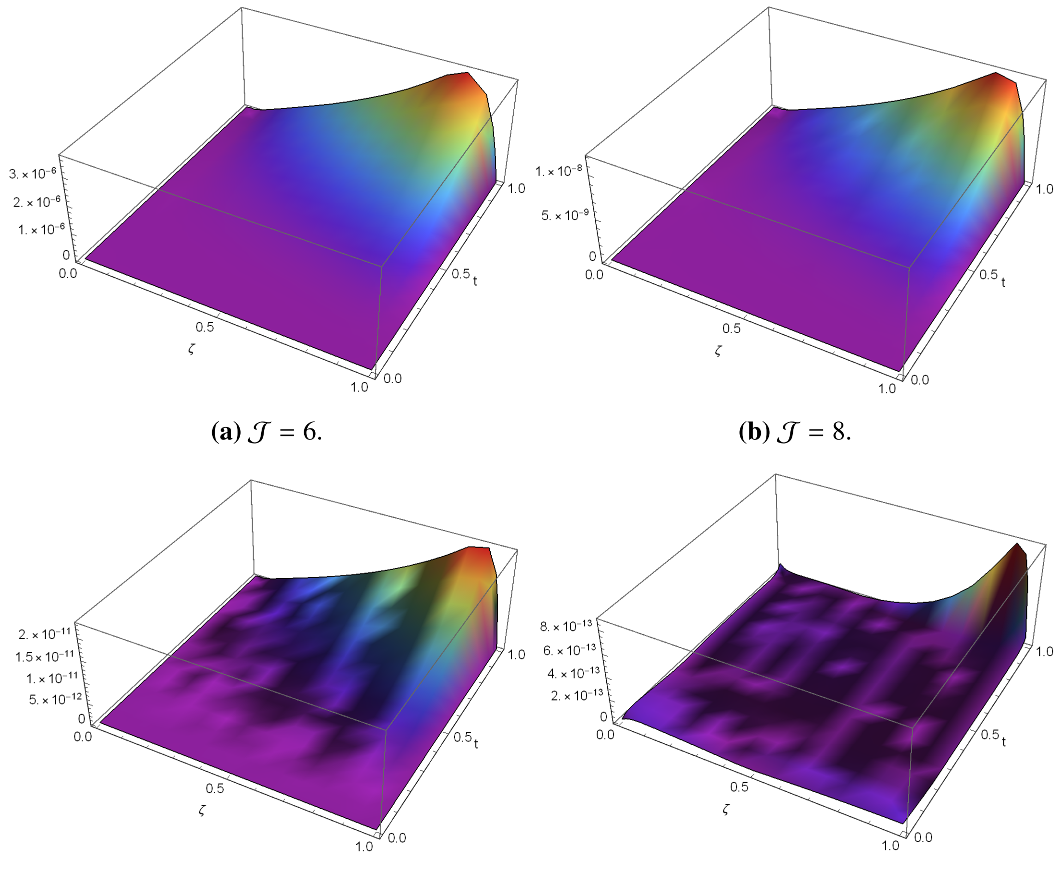


(b) Approximate solution.



(c) Absolute errors.

Figure 2. The exact solution, approximate solution, and absolute errors of Example 6.2 at $\alpha = 0.5$ and $\sigma = 0.9$ when $\mathcal{J} = 12$.



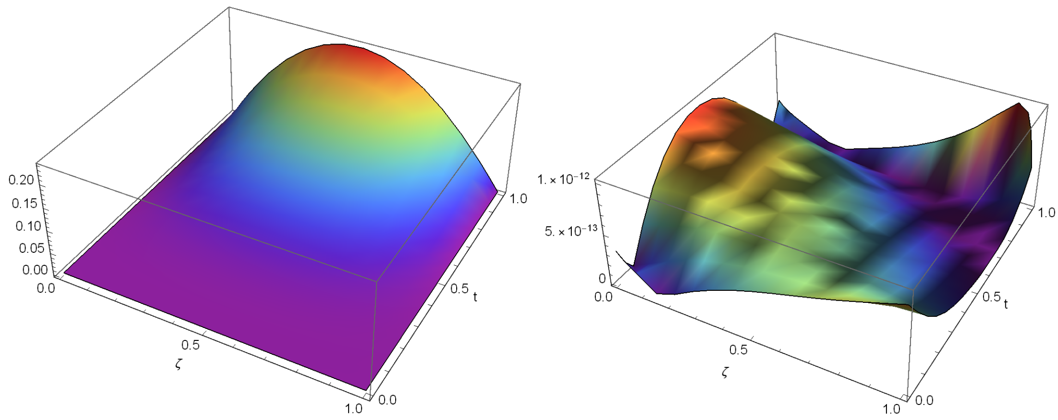
(a) $\mathcal{J} = 6$. **(b)** $\mathcal{J} = 8$.

(c) $\mathcal{J} = 10$. **(d)** $\mathcal{J} = 12$.

Figure 3. The absolute errors of Example 6.2 at $\alpha = 1$ and $\sigma = 0.9$.

Table 5. The absolute errors of Example 6.2 at $\alpha = 0.25$ and $\sigma = 0.5$.

(ζ, t)	$\mathcal{J} = 8$	CPU time	$\mathcal{J} = 10$	CPU time	$\mathcal{J} = 12$	CPU time
(0.1,0.1)	2.3567×10^{-13}		5.22518×10^{-16}		2.29038×10^{-17}	
(0.2,0.2)	1.02578×10^{-11}		1.88452×10^{-14}		1.22949×10^{-16}	
(0.3,0.3)	6.6037×10^{-11}		1.2096×10^{-13}		6.93889×10^{-17}	
(0.4,0.4)	2.41526×10^{-10}		4.40366×10^{-13}		5.6552×10^{-16}	
(0.5,0.5)	6.5823×10^{-10}	16.219	1.19966×10^{-12}	34.251	1.08247×10^{-15}	64.031
(0.6,0.6)	1.50373×10^{-19}		2.73605×10^{-12}		2.75474×10^{-15}	
(0.7,0.7)	3.03797×10^{-9}		5.53281×10^{-12}		6.73073×10^{-15}	
(0.8,0.8)	5.65039×10^{-9}		1.02468×10^{-11}		9.88098×10^{-15}	
(0.9,0.9)	9.03526×10^{-9}		1.72794×10^{-11}		9.70057×10^{-15}	



(a) Approximate solution. **(b)** Absolute errors.

Figure 4. The approximate solution, and absolute errors of Example 6.2 at $\alpha = 1$ and $\sigma = 0.5$ when $\mathcal{J} = 12$.

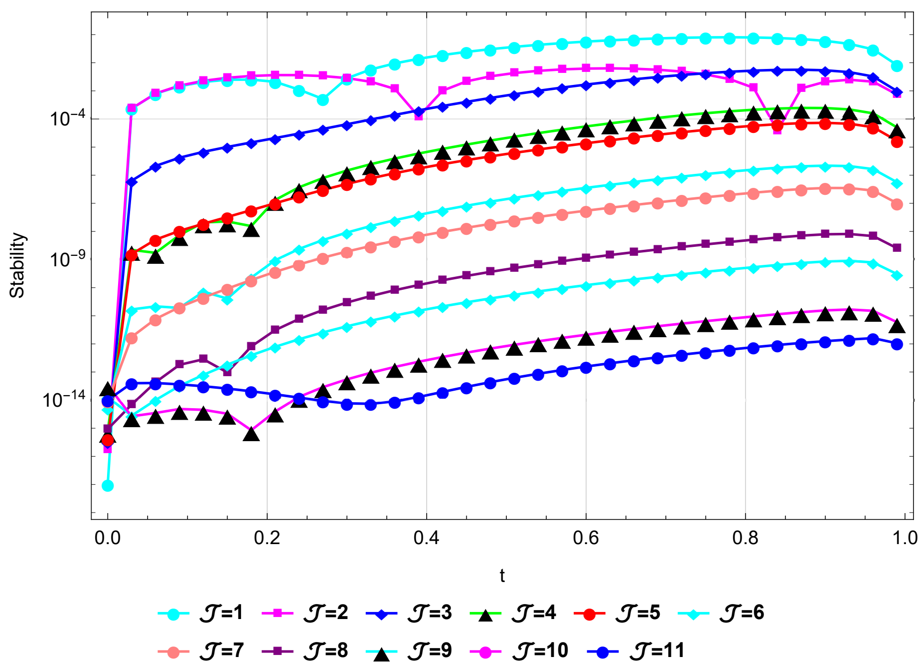


Figure 5. Stability $|Z_{\mathcal{J}+1} - Z_{\mathcal{J}}|$ at $\zeta = t$, $\alpha = 1$, and $\sigma = 0.9$ for Example 6.2.

Remark 6.2. *Table 6 demonstrates the agreement for both theoretical and numerical results in Figure 3 for Example 6.2. As an example, we set* $\mathcal{U}_1 = 0.1$, $\mathcal{U}_2 = 0.2$, *and* $\alpha = 1$ *in Theorem 5.3.*

Table 6. Theoretical error of Example 6.2.

$\mathcal{J}$	6	8	10	12
Error in Theorem 5.3	1.51148×10^{-4}	9.66019×10^{-7}	3.85843×10^{-9}	1.05191×10^{-11}

Example 6.3. *We consider the general TFHE given in* (1.1)–(1.3) *with*

$$a = 0, \quad b = 1, \quad T = 1, \quad \nu = 1, \quad \beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = 0.5, \quad s = 2.$$

The functions appearing on the right-hand side and the conditions are specified as

$$Z_0(\zeta) = \zeta, \quad Z_1(t) = t, \quad Z_2(t) = 0,$$

and the source term is given by

$$g(\zeta, t) = \zeta\, t.$$

Since the exact solution is not available, we will define the following absolute residual error norm:

$$RE = \max_{(\zeta,t)\in(0,1]\times(0,1]} \left| D_t^{\sigma} Z_{\mathcal{J}} - (Z_{\mathcal{J}})_{\zeta\zeta} + Z_{\mathcal{J}}\,(Z_{\mathcal{J}})_{\zeta} - Z_{\mathcal{J}}\,(Z_{\mathcal{J}}^2 - 0.5)\,(1 - Z_{\mathcal{J}}^2) - \zeta\, t \right|. \tag{6.1}$$

We apply the proposed method at $\sigma = 0.5$, $\alpha = 1$, *and* $\mathcal{J} = 12$ *to get Table 7, which illustrates the RE and CPU time (in seconds) used at different values of* $\mathcal{J}$. *We see in this table that the proposed method is accurate for small choices of* $\mathcal{J}$.

Table 7. The RE of Example 6.3 at $\sigma = 0.5$ and $\alpha = 1$.

(ζ, t)	$\mathcal{J} = 10$	CPU time	$\mathcal{J} = 12$	CPU time
(0.1,0.1)	1.08169×10^{-6}		1.07657×10^{-6}	
(0.2,0.2)	3.25836×10^{-7}		1.24941×10^{-7}	
(0.3,0.3)	1.2536×10^{-7}		8.74077×10^{-9}	
(0.4,0.4)	6.84586×10^{-8}		6.68989×10^{-9}	
(0.5,0.5)	5.23537×10^{-8}	93.485	7.87272×10^{-9}	194.718
(0.6,0.6)	5.52684×10^{-8}		5.31632×10^{-9}	
(0.7,0.7)	7.98698×10^{-8}		5.3254×10^{-9}	
(0.8,0.8)	1.58171×10^{-7}		5.26617×10^{-8}	
(0.9,0.9)	1.40407×10^{-6}		2.09091×10^{-7}	

Example 6.4. *We consider the general TFHE given in* (1.1)–(1.3) *with*

$$a = 0, \quad b = 1, \quad T = 1, \quad \nu = 1, \quad \beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = 0.5, \quad s = 2.$$

The functions appearing on the right-hand side and the conditions are specified as

$$Z_0(\zeta) = 0, \quad Z_1(t) = 0, \quad Z_2(t) = t^{1+\sigma},$$

and the source term is given by

$$g(\zeta, t) = \frac{\Gamma(\sigma + s + 1)}{s!}\, \zeta^3\, t^s + \zeta\, t^{\sigma+s} \left(\zeta^{14} t^{4(\sigma+s)} - 1.5\, \zeta^8 t^{2(\sigma+s)} + 3\, \zeta^4 t^{\sigma+s} + 0.5\, \zeta^2 - 6 \right).$$

The exact solution is given by

$$Z = t^{\sigma+s}\, \zeta^3.$$

Applying the proposed method at $\sigma = 0.75,\ s = 1, \alpha = 0.6$, *and* $\mathcal{J} = 12$, *we get Table 8, which shows the absolute errors at different values of t. Also, Table 9 presents the absolute errors at different values of t when* $\sigma = 0.75,\ s = 0, \alpha = 0.6$, *and* $\mathcal{J} = 16$. *The results in Tables 8 and 9 show a good agreement of the approximate solution with the exact one.*

Table 8. The absolute errors of Example 6.4 at $\sigma = 0.75,\ s = 1, \alpha = 0.6$, and $\mathcal{J} = 12$.

ζ	$t = 0.2$	$t = 0.5$	$t = 0.9$
0.1	3.66108×10^{-7}	5.60735×10^{-8}	8.32224×10^{-8}
0.2	7.07405×10^{-7}	1.08281×10^{-7}	1.57804×10^{-7}
0.3	9.99791×10^{-7}	1.5287×10^{-7}	2.11815×10^{-7}
0.4	1.22072×10^{-6}	1.86319×10^{-7}	2.26249×10^{-7}
0.5	1.35029×10^{-6}	2.05475×10^{-7}	1.70214×10^{-7}
0.6	1.37256×10^{-6}	2.07708×10^{-7}	5.36549×10^{-9}
0.7	1.27711×10^{-6}	1.91076×10^{-7}	3.75916×10^{-7}
0.8	1.06061×10^{-6}	1.54504×10^{-7}	1.05326×10^{-6}
0.9	7.28195×10^{-7}	9.79771×10^{-8}	2.19664×10^{-6}

Table 9. The absolute errors of Example 6.4 at $\sigma = 0.75,\ s = 0, \alpha = 0.6$, and $\mathcal{J} = 16$.

ζ	$t = 0.2$	$t = 0.5$	$t = 0.9$
0.1	1.50108×10^{-5}	2.35202×10^{-6}	9.27645×10^{-7}
0.2	2.89658×10^{-5}	4.53799×10^{-6}	1.84438×10^{-6}
0.3	4.08402×10^{-5}	6.40261×10^{-6}	2.72193×10^{-6}
0.4	4.96566×10^{-5}	7.79787×10^{-6}	3.48913×10^{-6}
0.5	5.4519×10^{-5}	8.58986×10^{-6}	3.98117×10^{-6}
0.6	5.46492×10^{-5}	8.66156×10^{-6}	3.84479×10^{-6}
0.7	4.94259×10^{-5}	7.91383×10^{-6}	2.36743×10^{-6}
0.8	3.84281×10^{-5}	6.26527×10^{-6}	1.82654×10^{-6}
0.9	2.14823×10^{-5}	3.6531×10^{-6}	1.13125×10^{-5}

Example 6.5. *We consider the two-dimensional TFHE given in* (1.4)–(1.7) *with*

$$a = 0, \quad b = 1, \quad c = 0, \quad d = 1, \quad T = 1, \quad \nu = 1, \quad \beta_1 = 0.01, \quad \beta_2 = 1, \quad \beta_3 = 0.5, \quad s = 1.$$

The functions appearing on the right-hand side and the conditions are specified as

$$\mathrm{Z}_0(\zeta,\eta) = 0, \quad \mathrm{Z}_1(\eta,t) = 0, \quad \mathrm{Z}_2(\eta,t) = \frac{1}{2}t^3 \sin\left(\frac{5\pi\eta}{6}\right),$$

$$\mathrm{Z}_3(\zeta,t) = 0, \quad \mathrm{Z}_4(\zeta,t) = \frac{1}{2}t^3 \sin\left(\frac{5\pi\zeta}{6}\right),$$

and the source term is given by

$$\begin{aligned} g(\zeta,\eta,t) = {} & \frac{6}{\Gamma(4-\sigma)}\, t^{3-\sigma} \sin\left(\frac{5\pi\zeta}{6}\right) \sin\left(\frac{5\pi\eta}{6}\right) + t^9 \sin^3\left(\frac{5\pi\zeta}{6}\right) \sin^3\left(\frac{5\pi\eta}{6}\right) \\ & + 0.01309\, t^6 \sin\left(\frac{5\pi\zeta}{3}\right) \sin^2\left(\frac{5\pi\eta}{6}\right) + t^6 \sin^2\left(\frac{5\pi\zeta}{6}\right)\left(0.01309 \sin\left(\frac{5\pi\eta}{3}\right) - 1.5 \sin^2\left(\frac{5\pi\eta}{6}\right)\right) \\ & + 14.2078\, t^3 \sin\left(\frac{5\pi\zeta}{6}\right) \sin\left(\frac{5\pi\eta}{6}\right). \end{aligned}$$

The exact solution is given by

$$Z = t^3 \sin\left(\frac{5\pi\zeta}{6}\right)\sin\left(\frac{5\pi\eta}{6}\right).$$

At $\sigma = 0.5$, $\alpha = 0.5$, *and* $\mathcal{J} = 9$, *Tables 10 and 11 show the absolute errors at different values of* η *and t, respectively. These tables demonstrate that the results of this method are extremely close to the exact solution.*

Table 10. The absolute errors of Example 6.5 at $\sigma = 0.5$, $\alpha = 0.5$, and $\mathcal{J} = 9$.

$\zeta = t$	$\eta = 0.1$	$\eta = 0.4$	$\eta = 0.7$	$\eta = 0.95$
0.1	5.08741×10^{-11}	1.48365×10^{-10}	2.83673×10^{-10}	3.8119×10^{-10}
0.2	7.23131×10^{-10}	2.08944×10^{-9}	4.19518×10^{-9}	5.81715×10^{-9}
0.3	3.60544×10^{-9}	1.04868×10^{-8}	2.05951×10^{-8}	2.79391×10^{-8}
0.4	1.14171×10^{-8}	3.35598×10^{-8}	6.29111×10^{-8}	8.19074×10^{-8}
0.5	2.82416×10^{-8}	8.41313×10^{-8}	1.4771×10^{-7}	1.80902×10^{-7}
0.6	6.00855×10^{-8}	1.81807×10^{-7}	2.94047×10^{-7}	3.30149×10^{-7}
0.7	1.15933×10^{-7}	3.5694×10^{-7}	5.25028×10^{-7}	5.2198×10^{-7}
0.8	2.10247×10^{-7}	6.60373×10^{-7}	8.77424×10^{-7}	7.35511×10^{-7}
0.9	3.47101×10^{-7}	1.11598×10^{-6}	1.35129×10^{-6}	9.13608×10^{-7}

Table 11. The absolute errors of Example 6.5 at $\sigma = 0.5$, $\alpha = 0.5$, and $\mathcal{J} = 9$.

$\zeta = \eta$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	4.43381×10^{-10}	3.8042×10^{-9}	1.33889×10^{-8}	3.28048×10^{-8}
0.3	2.23433×10^{-8}	1.98572×10^{-8}	7.13988×10^{-8}	1.77784×10^{-7}
0.5	5.88825×10^{-8}	5.13258×10^{-8}	1.82687×10^{-7}	4.50768×10^{-7}
0.7	1.11117×10^{-8}	9.35672×10^{-8}	3.26124×10^{-7}	7.92148×10^{-7}
0.9	1.26486×10^{-8}	1.02672×10^{-7}	3.49614×10^{-7}	8.34712×10^{-7}

7. Conclusions

In this paper, a spectral collocation algorithm based on a class of polynomials, namely shifted Dickson polynomials of the first kind, was developed to numerically solve the TFHE. Some properties of these polynomials were derived to serve in the numerical algorithm and to investigate the convergence analysis of the double expansion of the approximate solution. The expressions for the integer and fractional derivatives of the shifted polynomials were used to extract the operational matrices of the integer and fractional derivatives, which, together with the application of the typical collocation method, were employed to discretize the problem governed by its conditions. The resulting nonlinear systems were treated and effectively solved to produce the spectral solutions. The presented numerical results demonstrated the high accuracy of the spectral solutions. After establishing the fundamental formulas of the shifted Dickson polynomials of the first kind, these polynomials may be used as basis functions to treat other essential FDEs. Investigating this aspect is planned for future research.

Author contributions

Conceptualization, O.M.A., A.G.A. and W.M.A.-E.; Methodology, O.M.A., S.A.A., A.G.A. and W.M.A.-E.; Software, A.G.A. and W.M.A.-E.; Validation, O.M.A., S.A.A., A.G.A., A.K.A. and W.M.A.-E.; Formal analysis, O.M.A., A.G.A. and W.M.A.-E.; Investigation, O.M.A., S.A.A., A.G.A. and W.M.A.-E.; Data curation, A.G.A.; Writing—original draft, O.M.A., S.A.A., A.G.A. and W.M.A.-E.; Writing—review & editing, A.G.A., A.K.A. and W.M.A.-E.; Supervision, W.M.A.-E.; Funding acquisition, A.K.A.; All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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