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Research article

Several boundary value problems arising from the extensions of analytic functions

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Abstract: In this paper, with the properties of β-Cauchy type integrals and the Plemelj formula for the corresponding singular integrals, the Hilbert boundary value problems for β-analytic functions were first discussed on the unit disc in $\mathbb{C}$, and the concrete representations of the solutions were obtained. Thereafter, on the basis of the results for the Riemann boundary value problems for analytic functions, several Riemann problems for higher-order complex partial differential systems and $(\lambda, 1)$ bi-analytic functions were investigated for the bicylinder in $\mathbb{C}^2$. The solutions to the problems and the corresponding solvable conditions were obtained.

Keywords: complex partial differential equations; Hilbert problems; Riemann problems; analytic functions; Cauchy type integral
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1. Introduction

The boundary value problem of analytic functions serve as a core research direction at the intersection of complex analysis and mathematical physics. It has a wide range of applications in elasticity, fluid mechanics, and complex system modeling. For example, in the field of elasticity, the boundary value problem for analytic functions is the core method for solving the plane elasticity or anti plane elasticity problems. In fluid mechanics, the flow characteristics of incompressible fluids can be described through the boundary value analysis of analytic functions. In electromagnetics, it can be used to solve the boundary value distribution of the electromagnetic fields. The theoretical system for boundary value problems of analytic functions is relatively complete [1–5]. However, in some practical engineering applications, classical boundary value problems for analytic functions are often difficult to handle complex boundaries and high-dimensional nonlinear scenarios. This motivates the development of multifaceted generalizations for analytic functions. There are many types of the extensions of analytic functions, such as generalized analytic functions, bianalytic functions, and weighted analytic functions [6–10].

One type of the important extension of analytic functions is obtained by weakening the constraints of their Cauchy-Riemann systems. For example, the Beltrami equation,

$$(\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)f(z) = 0, \quad z = x + iy, \quad 0 \le \beta < 1, \tag{1.1}$$

which was proposed by Tungatarov [11], is one of the most essential and intrinsic generalizations of the classical Cauchy-Riemann system. Here,

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y), \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y). \tag{1.2}$$

Let

$$(\partial_{\bar{z}}^{\beta})f(z) = (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)f(z).$$

The solutions to Eq (1.1) are called β-analytic functions. This kind of function retains some of the core properties of analytic functions while giving them greater flexibility, and can describe the distribution of physical fields in non-uniform media. It has demonstrated the significant application value in fields such as elasticity, fluid mechanics, and composite material mechanics [12, 13]. In [11], a β-generalized Cauchy-Pompeiu representation formula and the Cauchy integral formula for β-analytic functions were obtained. Iwaniec and Martin [14] studied another more extensive Beltrami equation:

$$\partial_{\bar{z}}f(z) = \mu(z)\partial_z f(z), \quad |\mu(z)| \le q < 1,$$

where $\mu(z)$ is a measurable function. With the existence theorem for quasiconformal mappings, they gave the existence and uniqueness theory of the Beltrami equation, and various properties of the solutions to this equation. The Beltrami equation is a complex form of many first-order linear elliptic equation systems and has attracted the attention of many scholars. By the β-Cauchy integral formula, Blaya, Reyes, and Dixan [15] developed the Plemelj-Sokhotski formulas and the Liouville theorem for β-analytic functions, and obtained the unique solution to the Riemann jump boundary value problem. Thereafter, they studied the properties of the β-Cauchy-type integral with continuous density, and based on that, they discussed the Riemann boundary value problem for β-analytic functions on regular curves [16,17]. Later, by the properties of β-singular integral operators and β-Cauchy type integral on a domain with d-summable boundary, they derived the solutions and the solvability conditions for a jump problem of β-analytic functions in domains with fractal boundaries, as well as for the Riemann boundary value problem on d-summable closed curves [18,19]. Reyes, Blaya, and Rosa [20] obtained the Hilbert formulas for β-analytic functions on the unit circle and on the upper half-plane by the singular β-Cauchy integral operator. Furthermore, the Poincaré-Bertrand formula and the corresponding Schwarz and Poisson formulas were derived. Using the orthogonal decomposition of the Sobolev space with respect to operator $\partial_{\bar{z}}^{\beta}$, Reyes, Barseghyan, and Schneider [21] discussed a type of Dirichlet problem for β-analytic functions and obtained the specific representation of the solution. With the measurable Riemann mapping theorem and the Hilbert problem for analytic functions on the disc, Gutlyanskii, Ryazanov, and Yakubov et al. [22] studied the existence of solutions to a type of Hilbert boundary value problem for the Beltrami equation (1.2), though they did not provide the specific form of the solution. Inspired by this, in this article, we will first seek the specific form of the solution to the Hilbert problem for β-analytic functions on the unit disc in $\mathbb{C}$. We will obtain the solution to the problem and the solvability conditions. The conclusions obtained will further deepen the study of the boundary value problems for β-analytic functions and provide new methods for dealing with problems of mixed boundary and crack surface in orthotropic elastic mechanics.

Bi-analytic functions are another extension of analytic functions, which were proposed by Sander [23] when studying the partial differential system

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \omega, \\ (k+1)\frac{\partial \theta}{\partial x} + \frac{\partial \omega}{\partial y} = 0, \quad (k+1)\frac{\partial \theta}{\partial y} - \frac{\partial \omega}{\partial x} = 0, \end{cases}$$

where $k \in \mathbb{R}$, $k \neq -1$. Sander obtained some analogous properties of bi-analytic functions similar to analytic functions. The widespread application of partial differential equations in dynamics and image processing [24–27] has prompted researchers into the use of bi-analytic functions in the plane strain, the generalized plane stress, the flow of viscous fluids, and image processing.

Lin and Wu [28] introduced bi-analytic functions of type (λ, k), which are defined by the system:

$$\begin{cases} \frac{1}{k}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, \quad \frac{\partial u}{\partial y} + \frac{1}{k}\frac{\partial v}{\partial x} = \omega, \\ k\frac{\partial \theta}{\partial x} + \lambda\frac{\partial \omega}{\partial y} = 0, \quad k\frac{\partial \theta}{\partial y} - \lambda\frac{\partial \omega}{\partial x} = 0, \end{cases}$$

where $f(z) = u(x,y) + iv(x,y)$, and λ and k are real constants with $\lambda \neq 0, 1, k^2$ and $0 < k \leq 1$, respectively. The corresponding complex form is

$$\frac{k+1}{2}\frac{\partial f}{\partial \bar{z}} - \frac{k-1}{2}\frac{\partial f}{\partial z} = \frac{\lambda - k}{4\lambda}\varphi(z) + \frac{\lambda + k}{4\lambda}\overline{\varphi(z)}, \quad \frac{\partial \varphi(z)}{\partial \bar{z}} = 0.$$

They obtained the Cauchy integral expression and some properties of bi-analytic functions of type (λ, k). Thus, bi-analytic functions of type (λ, k) provide a powerful tool to deal with problems of plane elasticity [29,30], and become the research focus. Applying the higher-order Cauchy-Pompeiu representations, Karaca [31] derived the explicit solutions to some mixed boundary value problems for bi-polyanalytic functions and higher-order boundary value problems for model equations in the upper half plane. By the properties of complex and quaternionic polyanalytic Fock spaces, Gal and Sabadini [32] obtained quantitative approximation results by polyanalytic polynomials. Applying the method of Riemann boundary value problem for analytic functions, Lin [33] obtained the conditions of solvability and the expression of the solutions for a class of inverse boundary value problems for $(\lambda, 1)$ bi-analytic functions. These conclusions about bi-analytic functions are in the complex plane, yet there are few in $\mathbb{C}^n$. In view of this, in this article, we will explore the solution to the boundary value problems for bi-analytic functions of type $(\lambda, 1)$ and higher-order complex partial differential systems on the bicylinder in $\mathbb{C}^2$. The conclusions drawn will lay a necessary foundation for further research on boundary value problems of bi-analytic functions of type (λ, k) in high-dimensional complex spaces or the corresponding high-order partial differential equations, and will provide new tools for handling complex image and signal problems.

The outline of the article is as follows: In Section 2, we list the definitions and lemmas to be used in the article. In Section 3, we discuss the Hilbert boundary value problems for β-analytic functions. In the last section, we study several Riemann problems for higher-order complex partial differential equations and $(\lambda, 1)$ bi-analytic functions on the bicylinder in $\mathbb{C}^2$.

2. Some definitions and lemmas

To get the results, we need the following definition and lemmas.

Definition 2.1. *[15] Let Ω be an open domain in the complex plane, and let $f(z)$ be a continuously differentiable complex-valued function. $f(z)$ is called a β-analytic function on Ω if it is the solution of the Beltrami equation*

$$(\partial^{\beta}_{\bar{z}})f(z) := (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)f(z) = 0, \quad z \in \Omega, \quad 0 \leq \beta < 1.$$

$z|z|^{\theta}$ is an example of β-analytic functions where $\theta = \frac{2\beta}{1-\beta}$. For $\beta = 0$, β-analytic functions are well-known analytic functions.

Lemma 2.2. *[15] Let $f \in C^1(\Omega; \mathbb{C}) \bigcap C(\overline{\Omega}; \mathbb{C})$, where Ω is an open domain in the complex plane, and the boundary Γ of Ω consists of a simple smooth Jordan curve. Then*

$$(C^{\beta}_{\Gamma}f)(z) := \frac{1}{2(1-\beta)\pi i}\int_{\Gamma}\frac{f(\zeta)d\zeta}{\zeta - z|z/\zeta|^{\theta}} + \frac{\beta}{2(1-\beta)\pi i}\int_{\Gamma}\frac{\zeta f(\zeta)d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^{\theta})}, \quad z \notin \Gamma$$

is a β-analytic function in $\mathbb{C} \setminus \Gamma$, where $\theta = \frac{2\beta}{1-\beta}$ $(0 \leq \beta < 1)$. The above integral plays the role of the Cauchy type integral in the theory of β-analytic functions and is called the β-Cauchy type integral.

Lemma 2.3. *[15] Let $f \in C^1(\Omega; \mathbb{C}) \bigcap C(\overline{\Omega}; \mathbb{C})$, where Ω is an open domain in the complex plane, and the boundary Γ of Ω consists of a simple smooth Jordan curve. If the β-Cauchy type integral $(C^{\beta}_{\Gamma}f)(z)$ has limit values $(C^{\beta}_{\Gamma}f)^{\pm}(t)$ when z approaches t from $\Omega^{\pm}$, almost everywhere on Γ, then $(S^{\beta}_{\Gamma}f)(t)$ exists for almost every $t \in \Gamma$, and the formula*

$$(C^{\beta}_{\Gamma}f)^{\pm}(t) = \frac{1}{2}((S^{\beta}_{\Gamma}f)(t) \pm f(t))$$

holds, where $(S^{\beta}_{\Gamma}f)(t)$ is the singular β-Cauchy integral operator:

$$(S^{\beta}_{\Gamma}f)(t) := \frac{1}{(1-\beta)\pi i}\int_{\Gamma}\frac{(f(\zeta) - f(t))d\zeta}{\zeta - t|t/\zeta|^{\theta}} + \frac{\beta}{(1-\beta)\pi i}\int_{\Gamma}\frac{\zeta(f(\zeta) - f(t))d\bar{\zeta}}{\bar{\zeta}(\zeta - t|t/\zeta|^{\theta})} + f(t), \quad z \in \Gamma,$$

which is taken in the sense of Cauchy's principal value, and $\theta = \frac{2\beta}{1-\beta}$ $(0 \leq \beta < 1)$.

Conversely, if $(S^{\beta}_{\Gamma}f)(t)$ exists for almost every $t \in \Gamma$, then the β-Cauchy type integral $(C^{\beta}_{\Gamma}f)(z)$ has limit values $(C^{\beta}_{\Gamma}f)^{\pm}(t)$ almost everywhere on Γ and satisfies the above formulas.

Lemma 2.4. *[17] (β-Analytic Generalized Liouville Theorem) Let $F(z)$ be a β-analytic function in the whole complex plane $\mathbb{C}$. For $r > 0$, suppose that*

$$\lim_{R\to\infty}\frac{M(R)}{R^r} = 0, \quad M(R) := \max_{|\zeta|=R}|F(\zeta)|.$$

Let l be the least positive integer such that $l(1+\theta) \geq r$, then F is a β-polynomial $P_{l-1}(z|z|^{\theta})$ whose degree is not greater than $l-1$, that is

$$F(z) = a_0 + a_1 z|z|^{\theta} + a_2(z|z|^{\theta})^2 + \cdots + a_{l-1}(z|z|^{\theta})^{l-1},$$

where $\theta = \frac{2\beta}{1-\beta}$ $(0 \leq \beta < 1)$.

Definition 2.5. *Let G be the generalized bicylinder in $\mathbb{C}^2$, and let $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$. Then $f(z)$ is called a bi-analytic function of type $(\lambda, 1)$ on G if*

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda - 1}{4\lambda}\varphi(z) + \frac{\lambda + 1}{4\lambda}\overline{\varphi(z)}, \quad \partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0, \quad z = (z_1, z_2),$$

and $\varphi(z)$ is called the associate function of $f(z)$.

3. Hilbert BVP

In this section, we discuss the Hilbert boundary value problems for β-analytic functions. In the following, let D be the unit disc in $\mathbb{C}$ with the boundary L, and D^+ be the interior region bounded by L, and $D^- = \mathbb{C} \setminus \overline{D^+}$. Let $f^+(t)$ and $f^-(t)$ denote the limit values of $f(z)$ as $z \to t (t \in L)$ along the side of D^+ and D^-, respectively. Let $C_\alpha(\overline{G})$ represent the set of Hölder continuous functions on G.

Problem H1 Let $a = a(t), b = b(t), c = c(t) \in C_\mu(L)$ be the known Hölder continuous real functions defined on L, and let $\overline{\lambda(t)} = a(t) + ib(t)$ with $|\overline{\lambda(t)}|^2 \neq 0$. Find the β-analytic function $\Phi^+(z)$ in D^+ being continuous on $\overline{D^+}$, such that

$$\begin{cases} \partial_{\bar{z}}^{\beta}\Phi^+(z) = (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)\Phi^+(z) = 0, \quad z \in D^+, \\ \Re[\overline{\lambda(t)}\Phi^+(t)] = c(t), \quad t \in L, \end{cases} \tag{3.1}$$

where $0 \leq \beta < 1$.

Theorem 3.1. *Let $a = a(t), b = b(t), c = c(t) \in C_\mu(L)$ be the known Hölder continuous real functions defined on L. Let $\overline{\lambda(t)} = a(t) + ib(t)$ with $|\overline{\lambda(t)}|^2 \neq 0$, and let*

$$\kappa = \frac{1}{2\pi}\triangle_L \arg\frac{[a(t) - ib(t)]^2}{a^2(t) + b^2(t)}.$$

Then the solution of problem H1 can be expressed as the following:

(1) For $\kappa \geq 0$,

$$\Phi(z) = \Phi_0(z) + X(z)[c_0(z|z|^\theta)^\kappa + c_1(z|z|^\theta)^{\kappa-1} + \cdots + c_{\frac{\kappa}{2}}(z|z|^\theta)^{\frac{\kappa}{2}} + \overline{c_{\frac{\kappa}{2}-1}}(z|z|^\theta)^{\frac{\kappa}{2}-1} + \cdots + \overline{c_0}],$$

where $\theta = \frac{2\beta}{1-\beta}$ $(0 \leq \beta < 1)$, c_j $(j = 0, 1, \cdots, \frac{\kappa}{2} - 1)$ are arbitrary complex constants and $c_{\frac{\kappa}{2}}$ is arbitrary real constant, and

$$\begin{aligned} \Phi_0(z) = \frac{X(z)}{2\pi i}\Big\{ &\int_L \frac{c(t)(1 + (z|z|^\theta)^\kappa t^{-\kappa})}{[a(t) + ib(t)]X^+(t)}\frac{dt}{t - z|z|^\theta} \\ &- (z|z|^\theta)^\kappa \int_L t^{-\kappa-1}\frac{c(t)}{[a(t) + ib(t)]X^+(t)}dt\Big\}, \end{aligned}$$

$$X(z) = \begin{cases} C\exp\Big\{\frac{1}{2\pi i}\int_L \frac{\ln[t^{-\kappa}\frac{ib(t)-a(t)}{a(t)+ib(t)}]dt}{t-z|z|^\theta}\Big\}, \quad z \in D^+, \\ C(z|z|^\theta)^{-\kappa}\exp\Big\{\frac{1}{2\pi i}\int_L \frac{\ln[t^{-\kappa}\frac{ib(t)-a(t)}{a(t)+ib(t)}]dt}{t-z|z|^\theta}\Big\}, \quad z \in D^-, \end{cases}$$

$$C = \exp\Big\{\frac{1}{4\pi i}\int_0^{2\pi} \arg\Big(e^{-\kappa i\theta}\frac{ib(e^{i\theta}) - a(e^{i\theta})}{a(e^{i\theta}) + ib(e^{i\theta})}\Big)d\theta\Big\}.$$

(2) For $\kappa < 0$, $\Phi(z) = \Phi_0(z)$ which is represented in (1), if and only if

$$\int_L t^m\frac{c(t)}{[a(t) + ib(t)]X^+(t)}dt = 0 \quad (m = 0, 1, \cdots, -\kappa - 2).$$

Proof. (i) First, we transform Problem H1.

Let

$$\Phi_*(z) = \overline{\Phi(\frac{1}{\bar{z}})},\ z \in D^-,$$

in which $D^- = \mathbb{C} \setminus \overline{D^+}$ and Φ_* is the symmetric expansion of Φ in D^-. Let

$$\Psi(z) = \begin{cases} \Phi^+(z) = \Phi(z), & z \in D^+, \\ \Phi_*(z), & z \in D^-. \end{cases} \tag{3.2}$$

Suppose that Φ^+ is β-analytic in D^+ and $\frac{1}{\bar{z}} \doteq w$ for $z \in D^-$, then $w \in D^+$ and $\partial^\beta_{\bar{z}}\Phi^+(w) = 0$. Since $\Phi^+ = \Phi$ in D^+, then $\partial^\beta_{\bar{z}}\Phi^+(w) = \partial^\beta_{\bar{z}}\Phi(w) = \partial_{\bar{w}}\Phi(w) - \beta\frac{w}{\bar{w}}\partial_w\Phi(w) = 0$. Therefore, for $z \in D^-$,

$$\begin{aligned}
\partial^\beta_{\bar{z}}\Phi_*(z) &= \partial^\beta_{\bar{z}}\overline{\Phi(w)} = (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)\overline{\Phi(w)} \\
&= \partial_{\bar{z}}\overline{\Phi(w)} - \beta\frac{z}{\bar{z}}\partial_z\overline{\Phi(w)} = \frac{\partial}{\partial w}\overline{\Phi(w)} \cdot \frac{\partial w}{\partial \bar{z}} - \beta\frac{z}{\bar{z}}\frac{\partial}{\partial \bar{w}}\overline{\Phi(w)} \cdot \frac{\partial \bar{w}}{\partial z} \\
&= \partial_w\overline{\Phi(w)} \cdot \partial_{\bar{z}}w - \beta\frac{z}{\bar{z}}\partial_{\bar{w}}\overline{\Phi(w)}\partial_z\bar{w} \\
&= \overline{[\partial_{\bar{w}}\Phi(w) - \beta\frac{w}{\bar{w}}\partial_w\Phi(w)]}\partial_{\bar{z}}w + \beta\frac{\bar{w}}{w}\overline{\partial_w\Phi(w)}\partial_{\bar{z}}w - \beta\frac{z}{\bar{z}}\overline{\partial_w\Phi(w)}\partial_z\bar{w} \\
&= \beta\overline{\partial_w\Phi(w)}\Big[\frac{\bar{w}}{w}\partial_{\bar{z}}w - \frac{z}{\bar{z}}\partial_z\bar{w}\Big] \\
&= 0,
\end{aligned}$$

which means that $\Phi_*(z)$ is β-analytic in D^-. Similarly, if $\Phi_*(z)$ is β-analytic in D^-, then $\Phi(z)$ is β-analytic in D^+. From (3.2),

$$\begin{cases} \Psi_*(z) = \Psi(z), & z \in D^\pm, \\ \overline{\Psi^+(t)} = \overline{\Phi^+(t)} = \overline{\Phi^+(\frac{1}{\bar{t}})} = \Phi^-_*(t) = \Psi^-(t), & t \in L. \end{cases} \tag{3.3}$$

By (3.2) and (3.3), the boundary condition in (3.1) is transformed into

$$(a + ib)\Psi^+(t) + (a - ib)\Psi^-(t) = 2c, \quad t \in L. \tag{3.4}$$

Since $\Phi(z)$ is bounded in $z = 0$, then $\Psi(\infty)$ is bounded. Thus, Problem H1 is transformed into the following:

Problem R: Find the piecewise β-analytic function $\Psi(z)$ in $D^\pm$ such that $\Psi(z)$ is bounded in ∞ with $\Psi_*(z) = \Psi(z)$, and satisfies the boundary condition:

$$\Psi^+(t) = G(t)\Psi^-(t) + g(t), \tag{3.5}$$

in which

$$G(t) = -\frac{a(t) - ib(t)}{a(t) + ib(t)}, g(t) = \frac{2c(t)}{a(t) + ib(t)}, \tag{3.6}$$

and the index

$$\kappa = \frac{1}{2\pi}\triangle_L \arg G(t) = \frac{1}{2\pi}\triangle_L \arg\frac{a(t) - ib(t)}{a(t) + ib(t)} = \frac{1}{2\pi}\triangle_L \arg\frac{[a(t) - ib(t)]^2}{a^2(t) + b^2(t)}.$$

κ is an even number. Let

$$X(z) = \begin{cases} Ce^{\Gamma(z)}, & z \in D^+, \\ C(z|z|^\theta)^{-\kappa} e^{\Gamma(z)}, & z \in D^-, \end{cases} \tag{3.7}$$

where C is arbitrary non-zero complex constant and

$$\begin{aligned} \Gamma(z) &= \frac{1}{2(1-\beta)\pi i} \int_L \frac{\ln[(t|t|^\theta)^{-\kappa} G(t)]dt}{t - z|z/t|^\theta} + \frac{\beta}{2(1-\beta)\pi i} \int_L \frac{t \ln[(t|t|^\theta)^{-\kappa} G(t)]d\bar{t}}{\bar{t}(t - z|z/t|^\theta)} \\ &= \frac{1}{2(1-\beta)\pi i} \int_L \frac{\ln[t^{-\kappa} G(t)]dt}{t - z|z|^\theta} + \frac{\beta}{2(1-\beta)\pi i} \int_L \frac{t \ln[t^{-\kappa} G(t)]d\bar{t}}{\bar{t}(t - z|z|^\theta)} \\ &= \frac{1}{2(1-\beta)\pi i} \int_L \frac{\ln[t^{-\kappa} G(t)]dt}{t - z|z|^\theta} + \frac{\beta}{2(1-\beta)\pi i} \int_L \frac{t \ln[t^{-\kappa} G(t)]}{\frac{1}{t}(t - z|z|^\theta)} d\frac{1}{t} \\ &= \frac{1}{2(1-\beta)\pi i} \int_L \frac{\ln[t^{-\kappa} G(t)]dt}{t - z|z|^\theta} + \frac{\beta}{2(1-\beta)\pi i} \int_L \frac{-\ln[t^{-\kappa} G(t)]dt}{t - z|z|^\theta} \\ &= \frac{1}{2\pi i} \int_L \frac{\ln[t^{-\kappa} G(t)]dt}{t - z|z|^\theta}. \end{aligned} \tag{3.8}$$

By Lemma 2.2 and the β-analyticity of $z|z|^\theta$ ($\theta = \frac{2\beta}{1-\beta}$), $X(z)$ is β-analytic on $D^\pm$. Applying Lemma 2.3, we get that

$$\Gamma^+(t) - \Gamma^-(t) = \ln[(t|t|^\theta)^{-\kappa} G(t)],$$

which leads to

$$Ce^{\Gamma^+(t)} = Ce^{\Gamma^-(t) + \ln[(t|t|^\theta)^{-\kappa} G(t)]} = C(t|t|^\theta)^{-\kappa} G(t) e^{\Gamma^-(t)}.$$

That is

$$X^+(t) = G(t) X^-(t),$$

which means that $X(z)$ is a particular solution to $\Psi^+(t) = G(t)\Psi^-(t)$.

Since

$$\ln[t^{-\kappa} G(t)] = \ln \left| t^{-\kappa} \frac{ib(t) - a(t)}{a(t) + ib(t)} \right| + i \arg \left(t^{-\kappa} \frac{ib(t) - a(t)}{a(t) + ib(t)} \right) = i \arg \left(t^{-\kappa} \frac{ib(t) - a(t)}{a(t) + ib(t)} \right),$$

let

$$\arg \left(t^{-\kappa} \frac{ib(t) - a(t)}{a(t) + ib(t)} \right) = \widetilde{G}(t),$$

then

$$\Gamma(z) = \frac{1}{2\pi} \int_L \frac{\widetilde{G}(t)dt}{t - z|z|^\theta}. \tag{3.9}$$

Therefore,

$$\begin{aligned} \Gamma_*(z) = \overline{\Gamma(\frac{1}{\bar{z}})} &= \frac{1}{2\pi} \int_L \frac{\widetilde{G}(t)d\bar{t}}{\bar{t} - \frac{1}{z}|\frac{1}{z}|^\theta} = \frac{1}{2\pi} \int_L \frac{z|z|^\theta \widetilde{G}(t)dt}{t(t - z|z|^\theta)} \\ &= \frac{1}{2\pi} \int_L \frac{\widetilde{G}(t)dt}{t - z|z|^\theta} - \frac{1}{2\pi} \int_L \frac{\widetilde{G}(t)dt}{t} \\ &= \Gamma(z) - i\alpha, \end{aligned}$$

where

$$\alpha = \frac{1}{2\pi i}\int_L \frac{\widetilde{G}(t)dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{G}(e^{i\theta})d\theta$$

is a real number for $\widetilde{G} \in \mathbb{R}$. As a result,

$$X_*(z) = \begin{cases} \overline{C}e^{\Gamma_*(z)} = \overline{C}e^{-i\alpha}e^{\Gamma(z)}, & z \in D^-, \\ \overline{C}(z|z|^\theta)^\kappa e^{\Gamma_*(z)} = \overline{C}e^{-i\alpha}(z|z|^\theta)^\kappa e^{\Gamma(z)}, & z \in D^+. \end{cases} \tag{3.10}$$

By (3.9) and (3.10), we get that

$$X_*(z) = \frac{\overline{C}}{C}e^{-i\alpha}(z|z|^\theta)^\kappa X(z), \quad z \in D^\pm.$$

Choose

$$C = e^{-i\frac{\alpha}{2}} = \exp\Big\{\frac{1}{4\pi i}\int_0^{2\pi} \widetilde{G}(e^{i\theta})d\theta\Big\},$$

then we have

$$X_*(z) = (z|z|^\theta)^\kappa X(z), \quad z \in D^\pm. \tag{3.11}$$

(ii) In the case of $c(t) = 0$ (i.e., $g(t) = 0$), to obtain the solution to Problem H1, we first solve the corresponding homogeneous Problem R1, where the boundary condition is

$$\Psi^+(t) = G(t)\Psi^-(t).$$

For $\kappa \geq 0$, as $X^+(t) = G(t)X^-(t)$, then we have

$$\frac{\Psi^+(t)}{X^+(t)} = \frac{\Psi^-(t)}{X^-(t)}, \quad t \in L.$$

Thus, $\frac{\Psi}{X}$ is β-analytic in the whole complex plane. Since $\Psi(\infty)$ is bounded and X has a zero point of κ order at $z|z|^\theta = \infty$, then $\frac{\Psi}{X}$ has a pole of κ order at $z|z|^\theta = \infty$. Applying Lemma 2.4,

$$\frac{\Psi}{X} = P_\kappa(z|z|^\theta),$$

where $P_\kappa(z|z|^\theta)$ is an arbitrary β-polynomial whose degree is not greater than κ. Therefore,

$$\Psi(z) = X(z)P_\kappa(z|z|^\theta), \tag{3.12}$$

where

$$P_\kappa(z|z|^\theta) = c_0(z|z|^\theta)^\kappa + c_1(z|z|^\theta)^{\kappa-1} + \cdots + c_\kappa,$$

and c_j $(j = 1, \cdots, \kappa)$ are arbitrary complex constants.

By (3.11) and (3.12), we get that

$$\begin{aligned} \Psi_*(z) &= X_*(z)[P_\kappa(z|z|^\theta)]_* \\ &= (z|z|^\theta)^\kappa X(z)[\overline{c_0}(z|z|^\theta)^{-\kappa} + \overline{c_1}(z|z|^\theta)^{-(\kappa-1)} + \cdots + \overline{c_\kappa}] \\ &= X(z)[\overline{c_0} + \overline{c_1}(z|z|^\theta) + \cdots + \overline{c_\kappa}(z|z|^\theta)^\kappa]. \end{aligned}$$

To make $\Psi_*(z) = \Psi(z)$,

$$\overline{c_0} + \overline{c_1}(z|z|^\theta) + \cdots + \overline{c_\kappa}(z|z|^\theta)^\kappa = c_0(z|z|^\theta)^\kappa + c_1(z|z|^\theta)^{\kappa-1} + \cdots + c_\kappa$$

is required, that is

$$c_l = \overline{c_{k-l}} \quad (l = 0, 1, \cdots, \kappa).$$

Since κ is an even number, only

$$c_l = \overline{c_{k-l}} \quad (l = 0, 1, \cdots, \frac{\kappa}{2})$$

is needed. Thus,

$$P_\kappa(z|z|^\theta) = c_0(z|z|^\theta)^\kappa + c_1(z|z|^\theta)^{\kappa-1} + \cdots + c_{\frac{\kappa}{2}}(z|z|^\theta)^{\frac{\kappa}{2}} + \overline{c_{\frac{\kappa}{2}-1}}(z|z|^\theta)^{\frac{\kappa}{2}-1} + \cdots + \overline{c_0}, \tag{3.13}$$

and c_j $(j = 0, 1, \cdots, \frac{\kappa}{2} - 1)$ are arbitrary complex constants, and $c_{\frac{\kappa}{2}}$ is arbitrary real constant. Therefore, the solution to the homogeneous Hilbert Problem H1 is $\Phi(z) = \Psi(z)$ represented by (3.12), in which $P_\kappa(z|z|^\theta)$ is represented by (3.13).

For $\kappa < 0$, $P_\kappa(z|z|^\theta) \equiv 0$. Therefore, $\Psi(z) \equiv 0$, so that $\Phi(z)$ has only zero solution.

(iii) In the case of $c(t) \not\equiv 0$ (i.e., $g(t) \not\equiv 0$), in the following, we seek the solution to the non-homogeneous Problem H1. We first seek a specific solution to the non-homogeneous Problem R1 under the boundary condition (3.5).

For $\kappa \geq 0$ and by (ii), we have

$$X^+(t) = G(t)X^-(t).$$

Plugging the above equation into (3.5), we get that

$$\Psi^+(t) = \frac{X^+(t)}{X^-(t)}\Psi^-(t) + g(t),$$

so

$$\frac{\Psi^+(t)}{X^+(t)} = \frac{\Psi^-(t)}{X^-(t)} + \frac{g(t)}{X^+(t)}.$$

Let

$$F(z) = \frac{\Psi(z)}{X(z)}.$$

As Ψ and X are β-analytic on $D^\pm$, then F is β-analytic on $D^\pm$ with the the jumping condition

$$F^+(t) = F^-(t) + \frac{g(t)}{X^+(t)}.$$

As X has an order of $-\kappa$ at $z|z|^\theta = \infty$ and $\Psi(\infty)$ is bounded, then the maximum order of F at $z|z|^\theta = \infty$ is κ.

Applying the Plemelj formula for β-analytic functions, we get that

$$\begin{aligned}(C_L^\beta \frac{g}{X^+})(z) &= \frac{1}{2(1-\beta)\pi i}\int_L \frac{g(t)}{X^+(t)}\frac{dt}{t - z|z|^\theta} + \frac{\beta}{2(1-\beta)\pi i}\int_L \frac{g(t)}{X^+(t)}\frac{t d\bar{t}}{\bar{t}(t - z|z|^\theta)} \\ &= \frac{1}{2\pi i}\int_L \frac{g(t)}{X^+(t)}\frac{dt}{t - z|z|^\theta},\end{aligned}$$

and

$$\Big(C_L^\beta \frac{g}{X^+}\Big)^+(t) = \Big(C_L^\beta \frac{g}{X^+}\Big)^-(t) + \frac{g(t)}{X^+(t)}.$$

Thus, $C_L^\beta \frac{g}{X^+}$ is a special solution to

$$F^+(t) = F^-(t) + \frac{g(t)}{X^+(t)}.$$

Therefore,

$$\Psi(z) = X(z)(C_L^\beta \frac{g}{X^+})(z) = \frac{X(z)}{2\pi i}\int_L \frac{g(t)}{X^+(t)}\frac{dt}{t - z|z|^\theta} \tag{3.14}$$

is a special solution to (3.5). Thus,

$$\Phi_0(z) = \frac{1}{2}[\Psi(z) + \Psi_*(z)]$$

is a special solution to Problem H1.

Since $X_*(z) = (z|z|^\theta)^\kappa X(z)$, then

$$\overline{X^+(t)} = X_*^-(t) = (t|t|^\theta)^\kappa X^-(t).$$

Considering

$$X^+(t) = G(t)X^-(t) = \frac{ib(t) - a(t)}{a(t) + ib(t)}X^-(t),$$

then $(a(t) + ib(t))X^+(t) = (ib(t) - a(t))X^-(t)$, and thus

$$\begin{aligned}(a(t) - ib(t))\overline{X^+(t)} &= -(a(t) + ib(t))\overline{X^-(t)} \\ &= -(a(t) + ib(t))\overline{(t|t|^\theta)^{-\kappa}}X^+(t) \\ &= -(a(t) + ib(t))(\frac{|t|^\theta}{t})^{-\kappa}X^+(t) \\ &= -(a(t) + ib(t))t^\kappa X^+(t).\end{aligned}$$

Therefore,

$$\overline{\Big(\frac{g(t)}{X^+(t)}\Big)} = \frac{2c(t)}{(a(t) - ib(t))\overline{X^+(t)}} = \frac{2c(t)}{-(a(t) + ib(t))t^\kappa X^+(t)} = \frac{-g(t)}{t^\kappa X^+(t)}. \tag{3.15}$$

By (3.14) and (3.15), we get that

$$\begin{aligned}\Psi_*(z) = X_*(z)(C_L^\beta \frac{g}{X^+})_*(z) &= X_*(z)\frac{-1}{2\pi i}\int_L \frac{\overline{g(t)}}{\overline{X^+(t)}}\frac{d\bar{t}}{\bar{t} - \frac{1}{\bar{z}|z|^\theta}} \\ &= (z|z|^\theta)^\kappa X(z)\frac{1}{2\pi i}\int_L \frac{\overline{g(t)}}{\overline{X^+(t)}}[\frac{1}{t} - \frac{1}{t - z|z|^\theta}]dt \\ &= \frac{(z|z|^\theta)^\kappa X(z)}{2\pi i}\int_L \frac{g(t)}{t^\kappa X^+(t)}\Big[\frac{1}{t - z|z|^\theta} - \frac{1}{t}\Big]dt,\end{aligned}$$

which leads to

$$\begin{aligned}\Phi_0(z) &= \frac{1}{2}[\Psi(z) + \Psi_*(z)]\\ &= \frac{1}{2}\Big[\frac{X(z)}{2\pi i}\int_L \frac{g(t)}{X^+(t)}\frac{dt}{t - z|z|^\theta} + \frac{(z|z|^\theta)^\kappa X(z)}{2\pi i}\int_L \frac{g(t)}{t^\kappa X^+(t)}\Big[\frac{1}{t - z|z|^\theta} - \frac{1}{t}\Big]dt\Big] \\ &= \frac{X(z)}{4\pi i}\Big[\int_L \frac{g(t)}{X^+(t)}(1 + (z|z|^\theta)^\kappa t^{-\kappa})\frac{dt}{t - z|z|^\theta} - (z|z|^\theta)^\kappa \int_L t^{-\kappa-1}\frac{g(t)}{X^+(t)}dt\Big].\end{aligned} \tag{3.16}$$

The general solution $\Phi(z)$ of the nonhomogeneous Problem H1 can be obtained from the general solution of the corresponding homogeneous problem and a particular solution of the nonhomogeneous problem, that is

$$\Phi(z) = \Phi_0(z) + X(z)[c_0(z|z|^\theta)^\kappa + c_1(z|z|^\theta)^{\kappa-1} + \cdots + c_{\frac{\kappa}{2}}(z|z|^\theta)^{\frac{\kappa}{2}} + \overline{c_{\frac{\kappa}{2}-1}}(z|z|^\theta)^{\frac{\kappa}{2}-1} + \cdots + \overline{c_0}],$$

in which c_j $(j = 0, 1, \cdots, \frac{\kappa}{2} - 1)$ are arbitrary complex constants and $c_{\frac{\kappa}{2}}$ is arbitrary real constant.

For $\kappa < 0$, the β-analytic function F satisfying

$$F^+(t) = F^-(t) + \frac{g(t)}{X^+(t)}$$

has an $-\kappa$-order zero point at $z|z|^\theta = \infty$. Expanding the special solution $(C_L^\beta \frac{g}{X^+})(z)$ of

$$F^+(t) = F^-(t) + \frac{g(t)}{X^+(t)}$$

into a Laurent series near $z|z|^\theta = \infty$, we get that

$$\begin{aligned}(C_L^\beta \frac{g}{X^+})(z) &= \frac{1}{2\pi i}\int_L \frac{g(t)}{X^+(t)}\frac{dt}{t - z|z|^\theta}\\ &= -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\Big[\int_L t^m \frac{g(t)}{X^+(t)}dt\Big](z|z|^\theta)^{-m-1}.\end{aligned}$$

To ensure that $(C_L^\beta \frac{g}{X^+})(z)$ has $-\kappa$-order zero at $z|z|^\theta = \infty$, if and only if

$$\int_L t^m \frac{g(t)}{X^+(t)}dt = 0 \quad (m = 0, 1, \cdots, -\kappa - 2). \tag{3.17}$$

At this point, the problem for β-analytic functions satisfying

$$F^+(t) = F^-(t) + \frac{g(t)}{X^+(t)}$$

has a unique solution $(C_L^\beta \frac{g}{X^+})(z)$. Thus, the non-homogeneous Problem H1 has a unique solution $\Phi_0(z)$ if and only if (3.17) holds. □

Remark 3.2. *Setting $\beta = 0$, we get the result of the corresponding Hilbert problem for analytic functions.*

4. The Riemann problems

In this section, we discuss several Riemann problems for higher-order complex partial differential equations and (λ, k) bi-analytic functions on the bicylinder in $\mathbb{C}^2$.

In the following, let $G = D_1 \times D_2$ be the bicylinder in $\mathbb{C}^2$, where $D_1 = D_2 = D$. Let $L = L_1 \times L_2$ in which L_m is the boundary of D_m $(m = 1, 2)$, and let D_m^+ and D_m^- denote bounded domains and unbounded domains, respectively, enclosed by L_m. Let $f^{++}(t)$, $f^{+-}(t)$, $f^{-+}(t)$, $f^{--}(t)$ denote the limit values of $f(z)$ as $z \to t(t \in L)$ along the side of $D_1^+ \times D_2^+$, $D_1^+ \times D_2^-$, $D_1^- \times D_2^+$, $D_1^- \times D_2^-$, respectively. Let $H_\alpha(\overline{G})$ represent the set of Hölder continuous functions on G with $\alpha \in (0, 1]$, and let $G^{++} = D_1^+ \times D_2^+$, $G^{+-} = D_1^+ \times D_2^-$, $G^{-+} = D_1^- \times D_2^+$, $G^{--} = D_1^- \times D_2^-$.

Problem R1 Let G be the bicylinder in $\mathbb{C}^2$. Find the function $F(z)$ in $G^{++} \cup G^{+-} \cup G^{-+} \cup G^{--}$ such that $\partial_{z_1}\partial_{z_2}F(z)$ has finite limit in infinity, and such that

$$\begin{cases} \alpha\overline{\partial_{z_1}^2\partial_{z_2}^2 F(z)} + \beta\partial_{\bar{z}_1}\partial_{z_1}\partial_{\bar{z}_2}\partial_{z_2}F(z) = 0, \\ F^{++}(t) - t_1^{-p_{21}}t_2^{-p_{22}}F^{+-}(t) - t_1^{-p_{31}}t_2^{-p_{32}}F^{-+}(t) - t_1^{-p_{41}}t_2^{-p_{42}}F^{--}(t) = f_0(t), \quad t \in L, \\ \partial_{t_1}\partial_{t_2}F^{++} - t_1^{-k_{21}}t_2^{-k_{22}}\partial_{t_1}\partial_{t_2}F^{+-} - t_1^{-k_{31}}t_2^{-k_{32}}\partial_{t_1}\partial_{t_2}F^{-+} - t_1^{-k_{41}}t_2^{-k_{42}}\partial_{t_1}\partial_{t_2}F^{--} = f_1(t), \\ \partial_{t_1}\partial_{t_2}F^{++} - t_1^{k_{21}}t_2^{k_{22}}\partial_{t_1}\partial_{t_2}F^{+-} - t_1^{k_{31}}t_2^{k_{32}}\partial_{t_1}\partial_{t_2}F^{-+} - t_1^{k_{41}}t_2^{k_{42}}\partial_{t_1}\partial_{t_2}F^{--} = f_2(t), \end{cases} \tag{4.1}$$

where α and β are real constants, p_{ij} and k_{ij} $(j = 1, 2,\ i = 2, 3, 4)$ are integers, and the functions f_0, f_1, f_2 are known Hölder continuous functions.

Theorem 4.1. *Let G be the bicylinder in $\mathbb{C}^2$, and let $f_0, f_1, f_2 \in H_\alpha(L)$ ($\alpha \in (0, 1]$). Then the solution to Problem R1 is given by*

$$F(z) = \frac{\alpha^2}{\beta^2 - \alpha^2}\Big[\frac{\beta}{\alpha}\varphi(z) - z_1 z_2\overline{\partial_{z_1}\partial_{z_2}\varphi(z)} + \overline{\psi(z)}\Big], \tag{4.2}$$

where

$$\varphi(z) = \frac{1}{\alpha}\int_0^{z_1}\int_0^{z_2} E(\zeta)d\zeta, \tag{4.3}$$

and $E(z)$ is expressed as the following in different situations, and $\psi(z)$ is equivalent to replacing k_{ij} and g in $E(z)$ with p_{ij} and g_0, in which

$$\begin{aligned} g_0(t) = {} & \frac{\beta^2 - \alpha^2}{\alpha^2} f_0(t) \\ & - \frac{\beta}{\alpha}\overline{[\varphi^{++}(t) - t_1^{-p_{21}}t_2^{-p_{22}}\varphi^{+-}(t) - t_1^{-p_{31}}t_2^{-p_{32}}\varphi^{-+}(t) - t_1^{-p_{41}}t_2^{-p_{42}}\varphi^{--}(t)]} \\ & + \overline{t_1 t_2}[\partial_{t_1}\partial_{t_2}\varphi^{++}(t) - t_1^{-p_{21}}t_2^{-p_{22}}\partial_{t_1}\partial_{t_2}\varphi^{+-}(t) \\ & - t_1^{-p_{31}}t_2^{-p_{32}}\partial_{t_1}\partial_{t_2}\varphi^{-+}(t) - t_1^{-p_{41}}t_2^{-p_{42}}\partial_{t_1}\partial_{t_2}\varphi^{--}(t)]. \end{aligned}$$

(1) For $k_{21}, k_{32} \leq 0$*:* $E(z)$ *can be expressed as*

$$E(z) = \begin{cases} \frac{1}{(2\pi i)^2} \int_L \frac{g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n, \quad z \in G^{++}, \\ z_1^{-k_{21}} z_2^{-k_{22}} [\frac{1}{(2\pi i)^2} \int_L \frac{g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{+-}, \\ z_1^{-k_{31}} z_2^{-k_{32}} [\frac{1}{(2\pi i)^2} \int_L \frac{g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{-+}, \\ -z_1^{-k_{41}} z_2^{-k_{42}} [\frac{1}{(2\pi i)^2} \int_L \frac{g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{--}, \end{cases}$$

in which c_{mn} *are arbitrary complex constants, and* $l_1 = \min\{k_{31}, k_{41}\}, l_2 = \min\{k_{22}, k_{42}\}$. *In the case of* $\min\{k_{22}, k_{31}, k_{41}, k_{42}\} < 0$, $\sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n = 0$ *with the solvable condition*

$$\begin{cases} \int_L \frac{\zeta_1^m g(\zeta)d\zeta}{\zeta_2 - z_2} = 0 \ \ (m = 0, 1, \cdots, M_1 - 2), \\ \int_L \frac{\zeta_2^n g(\zeta)d\zeta}{\zeta_1 - z_1} = 0 \ \ (n = 0, 1, \cdots, N_1 - 2), \end{cases} \tag{4.4}$$

where

$$M_1 = \frac{1}{2} \max\{|k_{31}| - k_{31}, |k_{41}| - k_{41}\}, \quad N_1 = \frac{1}{2} \max\{|k_{22}| - k_{22}, |k_{42}| - k_{42}\},$$

and no solvable conditions are required when $M_1, N_1 = 0, 1$.

(2) For $k_{21} > 0, k_{32} \leq 0$*:*

$$E(z) = \begin{cases} z_1^{k_{21}} [\frac{1}{(2\pi i)^2} \int_L \frac{\zeta_1^{-k_{21}} g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{++}, \\ z_2^{-k_{22}} [\frac{1}{(2\pi i)^2} \int_L \frac{\zeta_1^{-k_{21}} g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{+-}, \\ z_1^{k_{21} - k_{31}} z_2^{-k_{32}} [\frac{1}{(2\pi i)^2} \int_L \frac{\zeta_1^{-k_{21}} g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{-+}, \\ -z_1^{k_{21} - k_{41}} z_2^{-k_{42}} [\frac{1}{(2\pi i)^2} \int_L \frac{\zeta_1^{-k_{21}} g(\zeta)d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], \quad z \in G^{--}, \end{cases}$$

in which c_{mn} *are arbitrary complex constants and* $l_1 = \min\{k_{31} - k_{21}, k_{41} - k_{21}\}, l_2 = \min\{k_{22}, k_{42}\}$. *In the case of* $\min\{k_{22}, k_{42}, k_{31} - k_{21}, k_{41} - k_{21}\} < 0$, $\sum\limits_{m=0}^{l_1} \sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n = 0$ *with the solvable condition*

$$\begin{cases} \int_L \frac{\zeta_1^{m - k_{21}} g(\zeta)d\zeta}{\zeta_2 - z_2} = 0 \ \ (m = 0, 1, \cdots, M_2 - 2), \\ \int_L \frac{\zeta_1^{-k_{21}} \zeta_2^n g(\zeta)d\zeta}{\zeta_1 - z_1} = 0 \ \ (n = 0, 1, \cdots, N_2 - 2), \end{cases} \tag{4.5}$$

where

$$M_2 = \frac{1}{2} \max\{|k_{31} - k_{21}| - k_{31} + k_{21}, |k_{41} - k_{21}| - k_{41} + k_{21}\}, \quad N_2 = \frac{1}{2} \max\{|k_{22}| - k_{22}, |k_{42}| - k_{42}\},$$

and no solvable conditions are required when $M_2, N_2 = 0, 1$.

(3) For $k_{21} \leq 0, k_{32} > 0$*:*

$$E(z) = \begin{cases} z_2^{k_{32}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{++}, \\ z_1^{-k_{21}} z_2^{k_{32}-k_{22}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{+-}, \\ z_1^{-k_{31}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{-+}, \\ -z_1^{-k_{41}} z_2^{k_{32}-k_{42}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{--}, \end{cases}$$

in which c_{mn} *are arbitrary complex constants and* $l_1 = \min\{k_{31}, k_{41}\}, l_2 = \min\{k_{22} - k_{32}, k_{42} - k_{32}\}$. *In the case of* $\min\{k_{22} - k_{32}, k_{31}, k_{41}, k_{42} - k_{32}\} < 0$, $\sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n = 0$ *with the solvable condition*

$$\begin{cases} \int_L \frac{\zeta_1^m \zeta_2^{-k_{32}} g(\zeta)d\zeta}{\zeta_2-z_2} = 0 \ \ (m = 0, 1, \cdots, M_3 - 2), \\ \int_L \frac{\zeta_2^{n-k_{32}} g(\zeta)d\zeta}{\zeta_1-z_1} = 0 \ \ (n = 0, 1, \cdots, N_3 - 2), \end{cases} \tag{4.6}$$

where

$$M_3 = \frac{1}{2}\max\{|k_{31}| - k_{31}, |k_{41}| - k_{41}\}, \quad N_3 = \frac{1}{2}\max\{|k_{22} - k_{32}| - k_{22} + k_{32}, |k_{42} - k_{32}| - k_{42} + k_{32}\},$$

and no solvable conditions are required when $M_3, N_3 = 0, 1$.

(4) For $k_{21} > 0, k_{32} > 0$*:*

$$E(z) = \begin{cases} z_1^{k_{21}} z_2^{k_{32}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_1^{-k_{21}}\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{++}, \\ z_2^{k_{32}-k_{22}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_1^{-k_{21}}\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{+-}, \\ z_1^{k_{21}-k_{31}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_1^{-k_{21}}\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{-+}, \\ -z_1^{k_{21}-k_{41}} z_2^{k_{32}-k_{42}}[\frac{1}{(2\pi i)^2}\int_L \frac{\zeta_1^{-k_{21}}\zeta_2^{-k_{32}} g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)} + \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n], & z \in G^{--}, \end{cases}$$

in which c_{mn} *are arbitrary complex constants and* $l_1 = \min\{k_{31} - k_{21}, k_{41} - k_{21}\}, l_2 = \min\{k_{22} - k_{32}, k_{42} - k_{32}\}$. *In the case of* $\min\{k_{22} - k_{32}, k_{31} - k_{21}, k_{41} - k_{21}, k_{42} - k_{32}\} < 0$, $\sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn} z_1^m z_2^n = 0$ *with the solvable condition*

$$\begin{cases} \int_L \frac{\zeta_1^{m-k_{21}} \zeta_2^{-k_{32}} g(\zeta)d\zeta}{\zeta_2-z_2} = 0 \ \ (m = 0, 1, \cdots, M_4 - 2), \\ \int_L \frac{\zeta_1^{-k_{21}}\zeta_2^{n-k_{32}} g(\zeta)d\zeta}{\zeta_1-z_1} = 0 \ \ (n = 0, 1, \cdots, N_4 - 2), \end{cases} \tag{4.7}$$

where

$$\begin{cases} M_4 = \frac{1}{2}\max\{|k_{31}-k_{21}|-k_{31}+k_{21}, |k_{41}-k_{21}|-k_{41}+k_{21}\}, \\ N_4 = \frac{1}{2}\max\{|k_{22}-k_{32}|-k_{22}+k_{32}, |k_{42}-k_{32}|-k_{42}+k_{32}\}, \end{cases}$$

and no solvable conditions are required when $M_4, N_4 = 0, 1$.

Proof. To solve problem R1, we denote

$$\alpha\overline{\partial_{z_1}\partial_{z_2}F(z)}+\beta\partial_{z_1}\partial_{z_2}F(z) = E(z), \tag{4.8}$$

so

$$\alpha\overline{\partial^2_{z_1}\partial^2_{z_2}F(z)}+\beta\partial_{\bar z_1}\partial_{z_1}\partial_{\bar z_2}\partial_{z_2}F(z) = \partial_{\bar z_1}\partial_{\bar z_2}E(z).$$

Then, by (4.1), we get $\partial_{\bar z_1}\partial_{\bar z_2}E(z) = 0$. In addition, (4.8) follows that

$$\begin{aligned}
&E^{++}-t_1^{k_{21}}t_2^{k_{22}}E^{+-}-t_1^{k_{31}}t_2^{k_{32}}E^{-+}-t_1^{k_{41}}t_2^{k_{42}}E^{--}\\
=&[\alpha\overline{\partial_{z_1}\partial_{z_2}F^{++}}+\beta\partial_{z_1}\partial_{z_2}F^{++}]-t_1^{k_{21}}t_2^{k_{22}}[\alpha\overline{\partial_{z_1}\partial_{z_2}F^{+-}}+\beta\partial_{z_1}\partial_{z_2}F^{+-}]\\
&-t_1^{k_{31}}t_2^{k_{32}}[\alpha\overline{\partial_{z_1}\partial_{z_2}F^{-+}}+\beta\partial_{z_1}\partial_{z_2}F^{-+}]-t_1^{k_{41}}t_2^{k_{42}}[\alpha\overline{\partial_{z_1}\partial_{z_2}F^{--}}+\beta\partial_{z_1}\partial_{z_2}F^{--}]\\
=&\alpha[\overline{\partial_{t_1}\partial_{t_2}F^{++}}-t_1^{k_{21}}t_2^{k_{22}}\overline{\partial_{t_1}\partial_{t_2}F^{+-}}-t_1^{k_{31}}t_2^{k_{32}}\overline{\partial_{t_1}\partial_{t_2}F^{-+}}-t_1^{k_{41}}t_2^{k_{42}}\overline{\partial_{t_1}\partial_{t_2}F^{--}}]\\
&+\beta[\partial_{t_1}\partial_{t_2}F^{++}-t_1^{k_{21}}t_2^{k_{22}}\partial_{t_1}\partial_{t_2}F^{+-}-t_1^{k_{31}}t_2^{k_{32}}\partial_{t_1}\partial_{t_2}F^{-+}-t_1^{k_{41}}t_2^{k_{42}}\partial_{t_1}\partial_{t_2}F^{--}]\\
=&\alpha\overline{f_1(t)}+\beta f_2(t)\\
\doteq&\, g(t),
\end{aligned}$$

Therefore, $\partial_{z_1}\partial_{z_2}F(z)$ has a finite limit in infinity and satisfies (4.1), which follows that $E(z)$ has a finite limit in infinity and satisfies

$$\begin{cases} \partial_{\bar z_1}\partial_{\bar z_2}E(z) = 0, \\ E^{++}-t_1^{k_{21}}t_2^{k_{22}}E^{+-}-t_1^{k_{31}}t_2^{k_{32}}E^{-+}-t_1^{k_{41}}t_2^{k_{42}}E^{--} = g(t), \end{cases} \tag{4.9}$$

Applying the results in [34], we get the solution $E(z)$ to problem (4.9).

(1) For $k_{21}, k_{32} \leq 0$:

(i) In the case of $\min\{k_{22}, k_{31}, k_{41}, k_{42}\} \geq 0$, $E(z)$ can be expressed as

$$E(z) = \begin{cases} E_1(z)+Q_1(z), \quad z \in G^{++}, \\ z_1^{-k_{21}}z_2^{-k_{22}}[E_1(z)+Q_1(z)], \quad z \in G^{+-}, \\ z_1^{-k_{31}}z_2^{-k_{32}}[E_1(z)+Q_1(z)], \quad z \in G^{-+}, \\ -z_1^{-k_{41}}z_2^{-k_{42}}[E_1(z)+Q_1(z)], \quad z \in G^{--}, \end{cases} \tag{4.10}$$

where

$$\begin{cases} E_1(z) = \frac{1}{(2\pi i)^2}\int_L \frac{g(\zeta)d\zeta}{(\zeta_1-z_1)(\zeta_2-z_2)}, \\ Q_1(z) = \sum\limits_{m=0}^{l_1}\sum\limits_{n=0}^{l_2} c_{mn}z_1^m z_2^n \ \ (l_1 = \min\{k_{31}, k_{41}\}, l_2 = \min\{k_{22}, k_{42}\}), \end{cases} \tag{4.11}$$

in which c_{mn} are arbitrary complex constants;

(ii) In the case of $\min\{k_{22}, k_{31}, k_{41}, k_{42}\} < 0$, $E(z)$ is given by (4.10), except that $Q_1(z) = 0$ and the solvable condition is (4.4).

(2) For $k_{21} > 0, k_{32} \leq 0$:

(i) In the case of $\min\{k_{22}, k_{42}, k_{31} - k_{21}, k_{41} - k_{21}\} \geq 0$, $E(z)$ can be expressed as

$$E(z) = \begin{cases} z_1^{k_{21}}[E_2(z) + Q_2(z)], & z \in G^{++}, \\ z_2^{-k_{22}}[E_2(z) + Q_2(z)], & z \in G^{+-}, \\ z_1^{k_{21}-k_{31}} z_2^{-k_{32}}[E_2(z) + Q_2(z)], & z \in G^{-+}, \\ -z_1^{k_{21}-k_{41}} z_2^{-k_{42}}[E_2(z) + Q_2(z)], & z \in G^{--}, \end{cases} \tag{4.12}$$

where

$$\begin{cases} E_2(z) = \frac{1}{(2\pi i)^2} \int_L \frac{\zeta_1^{-k_{21}} g(\zeta) d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)}, \\ Q_2(z) = \sum_{m=0}^{l_1} \sum_{n=0}^{l_2} c_{mn} z_1^m z_2^n \;\; (l_1 = \min\{k_{31} - k_{21}, k_{41} - k_{21}\}, l_2 = \min\{k_{22}, k_{42}\}), \end{cases} \tag{4.13}$$

in which c_{mn} are arbitrary complex constants;

(ii) In the case of $\min\{k_{22}, k_{42}, k_{31} - k_{21}, k_{41} - k_{21}\} < 0$, $E(z)$ is given by (4.12), except that $Q_2(z) = 0$ and the solvable condition is (4.5).

(3) For $k_{21} \leq 0, k_{32} > 0$:

(i) In the case of $\min\{k_{22} - k_{32}, k_{31}, k_{41}, k_{42} - k_{32}\} \geq 0$, $E(z)$ can be expressed as

$$E(z) = \begin{cases} z_2^{k_{32}}[E_3(z) + Q_3(z)], & z \in G^{++}, \\ z_1^{-k_{21}} z_2^{k_{32}-k_{22}}[E_3(z) + Q_3(z)], & z \in G^{+-}, \\ z_1^{-k_{31}}[E_3(z) + Q_3(z)], & z \in G^{-+}, \\ -z_1^{-k_{41}} z_2^{k_{32}-k_{42}}[E_3(z) + Q_3(z)], & z \in G^{--}, \end{cases} \tag{4.14}$$

where

$$\begin{cases} E_3(z) = \frac{1}{(2\pi i)^2} \int_L \frac{\zeta_2^{-k_{32}} g(\zeta) d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)}, \\ Q_3(z) = \sum_{m=0}^{l_1} \sum_{n=0}^{l_2} c_{mn} z_1^m z_2^n \;\; (l_1 = \min\{k_{31}, k_{41}\}, l_2 = \min\{k_{22} - k_{32}, k_{42} - k_{32}\}), \end{cases} \tag{4.15}$$

in which c_{mn} are arbitrary complex constants;

(ii) In the case of $\min\{k_{22} - k_{32}, k_{31}, k_{41}, k_{42} - k_{32}\} < 0$, $E(z)$ is given by (4.14), except that $Q_3(z) = 0$ and the solvable condition is (4.6).

(4) For $k_{21} > 0, k_{32} > 0$:

(i) In the case of $\min\{k_{22} - k_{32}, k_{31} - k_{21}, k_{41} - k_{21}, k_{42} - k_{32}\} \geq 0$, $E(z)$ can be expressed as

$$E(z) = \begin{cases} z_1^{k_{21}} z_2^{k_{32}}[E_4(z) + Q_4(z)], & z \in G^{++}, \\ z_2^{k_{32}-k_{22}}[E_4(z) + Q_4(z)], & z \in G^{+-}, \\ z_1^{k_{21}-k_{31}}[E_4(z) + Q_4(z)], & z \in G^{-+}, \\ -z_1^{k_{21}-k_{41}} z_2^{k_{32}-k_{42}}[E_4(z) + Q_4(z)], & z \in G^{--}, \end{cases} \tag{4.16}$$

where

$$\begin{cases} E_4(z) = \frac{1}{(2\pi i)^2} \int_L \frac{\zeta_1^{-k_{21}} \zeta_2^{-k_{32}} g(\zeta) d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)}, \\ Q_4(z) = \sum_{m=0}^{l_1} \sum_{n=0}^{l_2} c_{mn} z_1^m z_2^n \;\; (l_1 = \min\{k_{31} - k_{21}, k_{41} - k_{21}\}, l_2 = \min\{k_{22} - k_{32}, k_{42} - k_{32}\}), \end{cases} \tag{4.17}$$

in which c_{mn} are arbitrary complex constants;

(ii) In the case of $\min\{k_{22}-k_{32}, k_{31}-k_{21}, k_{41}-k_{21}, k_{42}-k_{32}\} < 0$, $E(z)$ is given by (4.16), except that $Q_4(z) = 0$ and the solvable condition is (4.7).

In the following, we seek the solution to (4.1). Since

$$\alpha\overline{\partial_{\bar{z}_1}^2\partial_{\bar{z}_2}^2 F(z)} + \beta\partial_{\bar{z}_1}\partial_{z_1}\partial_{\bar{z}_2}\partial_{z_2}F(z) = 0$$

implies that

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}[\overline{\partial_{z_1}\partial_{z_2}F} + \frac{\beta}{\alpha}\partial_{z_1}\partial_{z_2}F] = 0,$$

then

$$\overline{\partial_{z_1}\partial_{z_2}F} + \frac{\beta}{\alpha}\partial_{z_1}\partial_{z_2}F = \phi(z), \tag{4.18}$$

where $\phi(z)$ is analytic on G. From (4.18), we get that

$$(\frac{\beta^2}{\alpha^2} - 1)\partial_{z_1}\partial_{z_2}F = \frac{\beta}{\alpha}\phi(z) - \overline{\phi(z)},$$

which yields that

$$(\frac{\beta^2}{\alpha^2} - 1)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\overline{F} = \frac{\beta}{\alpha}\overline{\phi(z)} - \phi(z).$$

Let $\phi(z) = \partial_{z_1}\partial_{z_2}\varphi(z)$, so we have that

$$\begin{aligned}(\frac{\beta^2}{\alpha^2} - 1)\overline{F} &= \frac{\beta}{\alpha}\overline{\int_0^{z_1}\int_0^{z_2}\phi(\zeta)d\zeta} - \bar{z}_1\bar{z}_2\phi(z) + \psi(z)\\ &= \frac{\beta}{\alpha}\overline{\varphi(z)} - \bar{z}_1\bar{z}_2\partial_{z_1}\partial_{z_2}\varphi(z) + \psi(z),\end{aligned}$$

which follows that

$$F(z) = \frac{\alpha^2}{\beta^2-\alpha^2}\Big[\frac{\beta}{\alpha}\varphi(z) - z_1z_2\overline{\partial_{z_1}\partial_{z_2}\varphi(z)} + \overline{\psi(z)}\Big],$$

in which φ and ψ are undetermined analytic functions. By (4.8) and (4.2), we get that

$$\begin{aligned}E(z) &= \alpha\overline{\partial_{z_1}\partial_{z_2}F(z)} + \beta\partial_{z_1}\partial_{z_2}F(z)\\ &= \alpha\cdot\frac{\alpha^2}{\beta^2-\alpha^2}\Big[\frac{\beta}{\alpha}\overline{\partial_{z_1}\partial_{z_2}\varphi(z)} - \partial_{z_1}\partial_{z_2}\varphi(z)\Big] + \beta\cdot\frac{\alpha^2}{\beta^2-\alpha^2}\Big[\frac{\beta}{\alpha}\partial_{z_1}\partial_{z_2}\varphi(z) - \overline{\partial_{z_1}\partial_{z_2}\varphi(z)}\Big]\\ &= \alpha\partial_{z_1}\partial_{z_2}\varphi(z),\end{aligned}$$

so

$$\varphi(z) = \frac{1}{\alpha}\int_0^{z_1}\int_0^{z_2}E(\zeta)d\zeta.$$

In addition, plugging (4.2) into (4.1), we obtain that

$$\begin{aligned}\frac{\beta^2-\alpha^2}{\alpha^2}f_0(t) &= \frac{\beta^2-\alpha^2}{\alpha^2}[F^{++}(t) - t_1^{-p_{21}}t_2^{-p_{22}}F^{+-}(t) - t_1^{-p_{31}}t_2^{-p_{32}}F^{-+}(t) - t_1^{-p_{41}}t_2^{-p_{42}}F^{--}(t)]\\ &= \Big[\frac{\beta}{\alpha}\varphi^{++}(t) - t_1t_2\overline{\partial_{t_1}\partial_{t_2}\varphi^{++}(t)} + \overline{\psi^{++}(t)}\Big]\\ &\quad - t_1^{-p_{21}}t_2^{-p_{22}}\Big[\frac{\beta}{\alpha}\varphi^{+-}(t) - t_1t_2\overline{\partial_{t_1}\partial_{t_2}\varphi^{+-}(t)} + \overline{\psi^{+-}(t)}\Big]\\ &\quad - t_1^{-p_{31}}t_2^{-p_{32}}\Big[\frac{\beta}{\alpha}\varphi^{-+}(t) - t_1t_2\overline{\partial_{t_1}\partial_{t_2}\varphi^{-+}(t)} + \overline{\psi^{-+}(t)}\Big]\\ &\quad - t_1^{-p_{41}}t_2^{-p_{42}}\Big[\frac{\beta}{\alpha}\varphi^{--}(t) - t_1t_2\overline{\partial_{t_1}\partial_{t_2}\varphi^{--}(t)} + \overline{\psi^{--}(t)}\Big],\end{aligned}$$

consequently,

$$\psi^{++}(t) - t_1^{p_{21}}t_2^{p_{22}}\psi^{+-}(t) - t_1^{p_{31}}t_2^{p_{32}}\psi^{-+}(t) - t_1^{p_{41}}t_2^{p_{42}}\psi^{--}(t) = g_0(t), \tag{4.19}$$

in which

$$\begin{aligned}g_0(t) = {}&\frac{\beta^2-\alpha^2}{\alpha^2}f_0(t)\\ &- \frac{\beta}{\alpha}\overline{[\varphi^{++}(t) - t_1^{-p_{21}}t_2^{-p_{22}}\varphi^{+-}(t) - t_1^{-p_{31}}t_2^{-p_{32}}\varphi^{-+}(t) - t_1^{-p_{41}}t_2^{-p_{42}}\varphi^{--}(t)]}\\ &+ \overline{t_1t_2}[\partial_{t_1}\partial_{t_2}\varphi^{++}(t) - t_1^{-p_{21}}t_2^{-p_{22}}\partial_{t_1}\partial_{t_2}\varphi^{+-}(t)\\ &- t_1^{-p_{31}}t_2^{-p_{32}}\partial_{t_1}\partial_{t_2}\varphi^{-+}(t) - t_1^{-p_{41}}t_2^{-p_{42}}\partial_{t_1}\partial_{t_2}\varphi^{--}(t)].\end{aligned}$$

As ψ is analytic on G, applying Theorem 2.5 in [34], we obtain that $\psi(z)$ satisfying (4.19) can be expressed as $E(z)$ in (4.10), (4.12), (4.14), and (4.16) (in which k_{ij} and g are replaced by p_{ij} and g_0, respectively) in different situations. With the above $\varphi(z)$ and $\psi(z)$, we obtain the solution $F(z)$ to Problem R1. □

Problem R2 Let G be the bicylinder in $\mathbb{C}^2$. Find $(\lambda, 1)$ bi-analytic function $W(z)$ in $G^{++} \cup G^{+-} \cup G^{-+} \cup G^{--}$ such that $W(z)$ and $\phi(z)$ have finite limits in infinity, and such that

$$\begin{cases}\partial_{\bar z_1}\partial_{\bar z_2}W(z) = \frac{\lambda^{++}-1}{4\lambda^{++}}\phi^{++}(z) + \frac{\lambda^{++}+1}{4\lambda^{++}}\overline{\phi^{++}(z)}, & z \in G^{++},\\ \partial_{\bar z_1}\partial_{\bar z_2}W(z) = \frac{\lambda^{+-}-1}{4\lambda^{+-}}\phi^{+-}(z) + \frac{\lambda^{+-}+1}{4\lambda^{+-}}\overline{\phi^{+-}(z)}, & z \in G^{+-},\\ \partial_{\bar z_1}\partial_{\bar z_2}W(z) = \frac{\lambda^{-+}-1}{4\lambda^{-+}}\phi^{-+}(z) + \frac{\lambda^{-+}+1}{4\lambda^{-+}}\overline{\phi^{-+}(z)}, & z \in G^{-+},\\ \partial_{\bar z_1}\partial_{\bar z_2}W(z) = \frac{\lambda^{--}-1}{4\lambda^{--}}\phi^{--}(z) + \frac{\lambda^{--}+1}{4\lambda^{--}}\overline{\phi^{--}(z)}, & z \in G^{--},\\ \partial_{\bar z_1}\partial_{\bar z_2}\phi^{++}(z) = \beta_1\partial_{\bar z_1}\partial_{\bar z_2}\theta_1(z), & z \in G^{++},\\ \partial_{\bar z_1}\partial_{\bar z_2}\phi^{+-}(z) = \beta_2\partial_{\bar z_1}\partial_{\bar z_2}\theta_2(z), & z \in G^{+-},\\ \partial_{\bar z_1}\partial_{\bar z_2}\phi^{-+}(z) = \beta_3\partial_{\bar z_1}\partial_{\bar z_2}\theta_3(z), & z \in G^{-+},\\ \partial_{\bar z_1}\partial_{\bar z_2}\phi^{--}(z) = 0, & z \in G^{--},\\ W^{++}(t) = t_1^{c_{21}}t_2^{c_{22}}W^{+-}(t) + t_1^{c_{31}}t_2^{c_{32}}W^{-+}(t) + t_1^{c_{41}}t_2^{c_{42}}W^{--}(t) + f(t), & t \in L,\\ \phi^{++}(t) = t_1^{k_{21}}t_2^{k_{22}}\phi^{+-}(t) + t_1^{k_{31}}t_2^{k_{32}}\phi^{-+}(t) + t_1^{k_{41}}t_2^{k_{42}}\phi^{--}(t) + g(t), & t \in L,\end{cases} \tag{4.20}$$

where $\lambda^{\pm\pm} \in \mathbb{R}\backslash\{0, 1, -1\}$, θ_j $(j = 1, 2, 3)$ are known functions and c_{ij}, k_{ij} $(j = 1, 2,\ i = 2, 3, 4)$ are integers, and the functions $\beta_1, \beta_2, \beta_3, f$, and g are known Hölder continuous functions.

Theorem 4.2. *Let G be the bicylinder in $\mathbb{C}^2$, and let $\lambda^{\pm\pm} \in \mathbb{R}\backslash\{0, 1, -1\}$. Let θ_j ($j = 1, 2, 3$) be known functions and c_{ij}, k_{ij} ($j = 1, 2,\ i = 2, 3, 4$) be integers, and let the functions β_1,β_2,β_3, f, and g be known Hölder continuous functions. Then the solution to Problem R2 is given by*

$$W(z) = \frac{\lambda - 1}{4\lambda}\phi(z)\bar{z}_1\bar{z}_2 + \frac{\lambda + 1}{4\lambda}\overline{\Phi(z)} + \Psi(z), \tag{4.21}$$

where $\Phi(z)$ is determined by $\partial_{z_1}\partial_{z_2}\Phi(z) = \phi(z)$ and

$$\phi(z) = \begin{cases} \phi^{++}(z) = \phi_1^{++}(z) + \beta_1\theta_1(z), & z \in G^{++}, \\ \phi^{+-}(z) = \phi_1^{+-}(z) + \beta_2\theta_2(z), & z \in G^{+-}, \\ \phi^{-+}(z) = \phi_1^{-+}(z) + \beta_3\theta_3(z), & z \in G^{-+}, \\ \phi^{--}(z) = \phi_1^{--}(z), \quad z \in G^{--}, \end{cases}$$

in which $\phi_1 = E(z)$ is expressed as in Theorem 4.1, and f is replaced by

$$g^*(t) = g(t) + t_1^{k_{11}}t_2^{k_{21}}\beta_2\theta_2(t) + t_1^{k_{12}}t_2^{k_{22}}\beta_3\theta_3(t) - \beta_1\theta_1(t), \tag{4.22}$$

and $\Psi(z) = E(z)$ is expressed as in Theorem 4.1 in which k_{ij} and g are replaced by c_{ij} and f^*, respectively, in which

$$\begin{aligned} f^*(t) = f(t) - \Big[&\frac{\lambda^{++} - 1}{4\lambda^{++}}\phi^{++}(t)\bar{t}_1\bar{t}_2 + \frac{\lambda^{++} + 1}{4\lambda^{++}}\overline{\Phi^{++}(t)}\Big] \\ &+ t_1^{c_{11}}t_2^{c_{21}}\Big[\frac{\lambda^{+-} - 1}{4\lambda^{+-}}\phi^{+-}(t)\bar{t}_1\bar{t}_2 + \frac{\lambda^{+-} + 1}{4\lambda^{+-}}\overline{\Phi^{+-}(t)}\Big] \\ &+ t_1^{c_{12}}t_2^{c_{22}}\Big[\frac{\lambda^{-+} - 1}{4\lambda^{-+}}\phi^{-+}(t)\bar{t}_1\bar{t}_2 + \frac{\lambda^{-+} + 1}{4\lambda^{-+}}\overline{\Phi^{-+}(t)}\Big] \\ &+ t_1^{c_{13}}t_2^{c_{23}}\Big[\frac{\lambda^{--} - 1}{4\lambda^{--}}\phi^{--}(t)\bar{t}_1\bar{t}_2 + \frac{\lambda^{--} + 1}{4\lambda^{--}}\overline{\Phi^{--}(t)}\Big]. \end{aligned} \tag{4.23}$$

Proof. To solve Problem R2, we first discuss the problem:

$$\begin{cases} \partial_{\bar{z}_1}\partial_{\bar{z}_2}W(z) = \frac{\lambda^{++}-1}{4\lambda^{++}}\phi^{++}(z) + \frac{\lambda^{++}+1}{4\lambda^{++}}\overline{\phi^{++}(z)}, \quad z \in G^{++}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}W(z) = \frac{\lambda^{+-}-1}{4\lambda^{+-}}\phi^{+-}(z) + \frac{\lambda^{+-}+1}{4\lambda^{+-}}\overline{\phi^{+-}(z)}, \quad z \in G^{+-}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}W(z) = \frac{\lambda^{-+}-1}{4\lambda^{-+}}\phi^{-+}(z) + \frac{\lambda^{-+}+1}{4\lambda^{-+}}\overline{\phi^{-+}(z)}, \quad z \in G^{-+}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}W(z) = \frac{\lambda^{--}-1}{4\lambda^{--}}\phi^{--}(z) + \frac{\lambda^{--}+1}{4\lambda^{--}}\overline{\phi^{--}(z)}, \quad z \in G^{--}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{--}(z) = 0, \quad z \in G^{++} \bigcup G^{+-} \bigcup G^{-+} \bigcup G^{--}, \\ W^{++}(t) = t_1^{c_{21}}t_2^{c_{22}}W^{+-}(t) + t_1^{c_{31}}t_2^{c_{32}}W^{-+}(t) + t_1^{c_{41}}t_2^{c_{42}}W^{--}(t) + f(t), \quad t \in L, \\ \phi^{++}(t) = t_1^{k_{21}}t_2^{k_{22}}\phi^{+-}(t) + t_1^{k_{31}}t_2^{k_{32}}\phi^{-+}(t) + t_1^{k_{41}}t_2^{k_{42}}\phi^{--}(t) + g(t), \quad t \in L. \end{cases} \tag{4.24}$$

Applying the results in [34], we get the solution to the problem

$$\begin{cases} \partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi(z) = 0, \\ \phi^{++}(t) = t_1^{k_{21}}t_2^{k_{22}}\phi^{+-}(t) + t_1^{k_{31}}t_2^{k_{32}}\phi^{-+}(t) + t_1^{k_{41}}t_2^{k_{42}}\phi^{--}(t) + g(t), \quad t \in L, \end{cases}$$

which can be expressed as $\phi(z) = E(z)$ in Theorem 4.1 in different situations.

Moreover, $\partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi(z) = 0$ implies that $\phi(z)$ is analytic on G. Let

$$\partial_{z_1}\partial_{z_2}\Phi(z) = \phi(z). \tag{4.25}$$

Thus, for

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}W(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\overline{\phi(z)},$$

we have that

$$W(z)=\frac{\lambda-1}{4\lambda}\phi(z)\bar{z}_1\bar{z}_2+\frac{\lambda+1}{4\lambda}\overline{\Phi(z)}+\Psi(z),$$

in which $\Psi(z)$ is analytic on G. Let $z\to t$, and we get that

$$\begin{cases} W^{++}(t)=\frac{\lambda^{++}-1}{4\lambda^{++}}\phi^{++}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{++}+1}{4\lambda^{++}}\overline{\Phi^{++}(t)}+\Psi^{++}(t),\\ W^{+-}(t)=\frac{\lambda^{+-}-1}{4\lambda^{+-}}\phi^{+-}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{+-}+1}{4\lambda^{+-}}\overline{\Phi^{+-}(t)}+\Psi^{+-}(t),\\ W^{-+}(t)=\frac{\lambda^{-+}-1}{4\lambda^{-+}}\phi^{-+}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{-+}+1}{4\lambda^{-+}}\overline{\Phi^{-+}(t)}+\Psi^{-+}(t),\\ W^{--}(t)=\frac{\lambda^{--}-1}{4\lambda^{--}}\phi^{--}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{--}+1}{4\lambda^{--}}\overline{\Phi^{--}(t)}+\Psi^{--}(t),\end{cases}$$

which leads to

$$\begin{aligned} &W^{++}(t)-t_1^{c_{11}}t_2^{c_{21}}W^{+-}(t)-t_1^{c_{12}}t_2^{c_{22}}W^{-+}(t)-t_1^{c_{13}}t_2^{c_{23}}W^{--}(t)\\ =&\Big[\frac{\lambda^{++}-1}{4\lambda^{++}}\phi^{++}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{++}+1}{4\lambda^{++}}\overline{\Phi^{++}(t)}\Big]\\ &-t_1^{c_{11}}t_2^{c_{21}}\Big[\frac{\lambda^{+-}-1}{4\lambda^{+-}}\phi^{+-}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{+-}+1}{4\lambda^{+-}}\overline{\Phi^{+-}(t)}\Big]\\ &-t_1^{c_{12}}t_2^{c_{22}}\Big[\frac{\lambda^{-+}-1}{4\lambda^{-+}}\phi^{-+}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{-+}+1}{4\lambda^{-+}}\overline{\Phi^{-+}(t)}\Big]\\ &-t_1^{c_{13}}t_2^{c_{23}}\Big[\frac{\lambda^{--}-1}{4\lambda^{--}}\phi^{--}(t)\bar{t}_1\bar{t}_2+\frac{\lambda^{--}+1}{4\lambda^{--}}\overline{\Phi^{--}(t)}\Big]\\ &+\Psi^{++}(t)-t_1^{c_{11}}t_2^{c_{21}}\Psi^{+-}(t)-t_1^{c_{12}}t_2^{c_{22}}\Psi^{-+}(t)-t_1^{c_{13}}t_2^{c_{23}}\Psi^{--}(t)\\ =&f(t),\end{aligned}$$

so we have that

$$\Psi^{++}(t)=t_1^{c_{11}}t_2^{c_{21}}\Psi^{+-}(t)+t_1^{c_{12}}t_2^{c_{22}}\Psi^{-+}(t)+t_1^{c_{13}}t_2^{c_{23}}\Psi^{--}(t)+f^*(t), \tag{4.26}$$

where $f^*(t)$ is represented by (4.23). Reapplying the conclusions in [34], we can find the solution $\Psi(z)$ to the problem (4.26), that is, $\Psi(z)=E(z)$ determined in Theorem 4.1 (in which k_{ij} and f are replaced by c_{ij} and f^*, respectively) in different situations. By (4.21), with $\phi(z)$, $\Psi(z)$ and (4.23), we can get the solution $W(z)$ to (4.24).

Let

$$\phi_1(z)=\begin{cases}\phi_1^{++}(z)=\phi^{++}(z)-\beta_1\theta_1(z), & z\in G^{++},\\ \phi_1^{+-}(z)=\phi^{+-}(z)-\beta_2\theta_2(z), & z\in G^{+-},\\ \phi_1^{-+}(z)=\phi^{-+}(z)-\beta_3\theta_3(z), & z\in G^{-+},\\ \phi_1^{--}(z)=\phi^{--}(z), & z\in G^{--}.\end{cases} \tag{4.27}$$

By (4.20) we get $\partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi_1(z)=0$ and

$$\begin{aligned}\phi_1^{++}(t)+\beta_1\theta_1(t)=&\,t_1^{k_{11}}t_2^{k_{21}}[\phi_1^{+-}(t)+\beta_2\theta_2(t)]+t_1^{k_{12}}t_2^{k_{22}}[\phi_1^{-+}(t)+\beta_3\theta_3(t)]\\ &+t_1^{k_{13}}t_2^{k_{23}}\phi_1^{--}(t)+g(t),\end{aligned}$$

that is

$$\phi_1^{++}(t)=t_1^{k_{11}}t_2^{k_{21}}\phi_1^{+-}(t)+t_1^{k_{12}}t_2^{k_{22}}\phi_1^{-+}(t)+t_1^{k_{13}}t_2^{k_{23}}\phi_1^{--}(t)+g^*(t), \tag{4.28}$$

in which $g^*(t)$ is represented by (4.22). Therefore, problem (4.20) is transformed into question (4.24).

Replacing g by g^* in the solution $W(z)$ to (4.24), we can get the solution to (4.20). □

Remark 4.3. *On the basis of Theorem 4.2, we can investigate higher order complex partial differential systems of* $(\lambda, 1)$ *bi-analytic functions.*

5. Conclusions

We first discuss a type of Hilbert boundary value problems for β-analytic functions on the unit disc in $\mathbb{C}$. Applying the generalized Liouville theorem for β-analytic functions, the properties of β-Cauchy type integral and the Plemelj formula for the corresponding singular integral, we get the concrete representations of the solutions. In addition, on the basis of the results for the Riemann boundary value problems for analytic functions, we investigate two types of boundary value problems with Riemann conditions for higher-order complex partial differential systems and $(\lambda, 1)$ bi-analytic functions on the bicylinder in $\mathbb{C}^2$. With the help of the boundary properties of Cauchy type integrals, we obtain the solutions and the corresponding solvable conditions. With the method used in this paper, we can investigate the higher order complex partial differential systems of $(\lambda, 1)$ bi-analytic functions. The conclusions obtained here lay the foundations for studying other boundary value problems for β-analytic functions or the corresponding Beltrami equations.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Chaojun Wang: Conceptualization, project administration, writing original draft, writing-review and editing. Yanyan Cui: Investigation, writing original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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