Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 13647–13659.
DOI: 10.3934/math.2026562
Received: 27 February 2026
Revised: 05 May 2026
Accepted: 11 May 2026
Published: 15 May 2026

Research article

On the compactness of families of continuous functions in Grand Lebesgue spaces

Rahma KATEA[1,*] **and Yasin KAYA**[2]

[1] Department of Mathematics, Institute of Natural and Applied Sciences, Dicle University, Diyarbakır, Türkiye
[2] Department of Mathematics, Dicle University, Diyarbakır, Türkiye

* **Correspondence:** Email: rahmaone85@gmail.com.

Abstract: In this paper, we established compactness results for families of continuous functions in grand Lebesgue spaces G_ψ, where $\psi : (a, b) \to (0, \infty)$ is a given positive function that determines the structure of the space. In particular, we showed that equicontinuity together with uniform boundedness implies relative compactness in the G_ψ-norm. The approach is based on Arzelà–Ascoli-type arguments combined with uniform control in grand Lebesgue space (GLS) norms. Several examples were provided to illustrate convergence mechanisms, and extensions to weighted GLS and fractional Sobolev-type settings are discussed. These results establish a unified compactness framework in grand Lebesgue spaces and provide a foundation for further investigations in Sobolev-type embeddings and related analytical models.

Keywords: grand Lebesgue spaces; compactness; Kolmogorov–Riesz theorem; Arzelà–Ascoli theorem; weighted function spaces; fractional Sobolev embedding
Mathematics Subject Classification: 45B05, 46A32, 46E30

1. Introduction

Compactness plays a central role in functional analysis, partial differential equations, and the theory of function spaces. In classical Lebesgue spaces L^p, compactness is well-characterized by the Kolmogorov–Riesz theorem and related embedding results [2]. However, these classical criteria are no longer directly applicable in multi-exponent frameworks such as grand Lebesgue spaces (GLSs).

The classical grand Lebesgue spaces $L^{p)}(\Omega)$ introduced by Iwaniec and Sbordone [8] are defined by considering a fixed exponent p and measuring the behavior of functions near this exponent via

$$\|f\|_{L^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \varepsilon^{1/p} \|f\|_{L^{p-\varepsilon}(\Omega)} < \infty$$

This construction was later extended by Fiorenza through the spaces $L^{p),\theta}(\Omega)$, which incorporate an additional parameter $\theta \geq 0$ to provide finer control of integrability [3].

In contrast, the spaces $G_\psi(E)$ considered in this paper provide a more flexible framework, where the behavior of the function is controlled over an entire interval of exponents $p \in I$ through a generating function ψ (see [9, 10]). In particular, the classical grand Lebesgue spaces can be viewed as special cases of the G_ψ spaces for suitable choices of ψ, which highlights the greater generality of our setting.

Classical compactness arguments rely heavily on uniform bounds, tightness, and translation control. Recent works have established Kolmogorov–Riesz-type compactness criteria and Sobolev-type embeddings in GLS within a specific functional framework, where compactness is described in terms of L^p-type conditions [12].

The present work is developed in the more general setting of G_ψ spaces and focuses on families of continuous functions, combining the GLS norm structure with equicontinuity assumptions to obtain a different and more structured compactness framework.

These methods are also relevant in the stability and convergence analysis of analytical and numerical models arising in applied problems, including nonlinear evolution equations and fractional diffusion models.

The results established in this paper may have potential applications in several applied fields, including fluid dynamics, biomechanics, and aerodynamics. In particular, properties of generalized function spaces play a crucial role in the analysis of nonlinear partial differential equations, the stability of solutions, and the convergence of numerical schemes. Such analytical frameworks are closely related to modern numerical approaches, including finite difference and finite volume methods for nonlinear evolution equations [1].

The aim of this paper is to establish relative compactness criteria for families of continuous functions in grand Lebesgue spaces. Our approach combines Arzelà–Ascoli-type arguments with uniform control in GLS norms, leading to compactness results under natural assumptions on the generating function ψ. We also provide extensions to weighted GLS settings and to families satisfying fractional Sobolev-type conditions. Furthermore, we present counterexamples illustrating the necessity of equicontinuity assumptions.

The paper is organized as follows. Section 2 introduces the notation and preliminary definitions. Section 3 presents auxiliary results related to L^p control and convergence. Section 4 contains the main compactness theorems and their proofs. Extensions and examples are discussed in the final sections.

2. Related work

Research on grand Lebesgue spaces (GLSs) originated in the seminal work of Iwaniec and Sbordone [8], who introduced these spaces to capture refined integrability properties beyond the classical fixed-exponent Lebesgue framework. Since then, GLSs have been systematically developed as Banach function spaces with rich structural properties, including completeness, duality, and embedding theory.

Fiorenza established fundamental results concerning duality and reflexivity in grand Lebesgue spaces [5]. Further structural developments were carried out by Fiorenza and Karadzhov [6], who clarified the relationship between grand and small Lebesgue spaces and positioned GLSs within the broader scale of rearrangement-invariant spaces.

Compactness in GLSs has been studied in connection with the classical Kolmogorov–Riesz theorem. Rafeiro and Vargas [12] established compactness criteria and obtained extensions of Rellich–Kondrachov-type embeddings within the GLS framework.

Further refinements of Kolmogorov–Riesz-type results in the fixed-exponent setting were developed by Hanche-Olsen, Holden, and Malinnikova [7], providing sharper insights into compactness phenomena in the classical setting.

Recent advances based on extrapolation techniques in generalized Lebesgue-type spaces, as well as developments in operator theory within GLSs, are discussed in the existing literature [11].

Interpolation and duality aspects of GLSs have also been studied extensively. Di Fratta and Fiorenza [4] provided a direct approach to the duality and norm structure of grand and small Lebesgue spaces. Survey contributions from Capone and Fiorenza [3] offer a broader overview of recent developments in the field and contextualize GLS within modern analysis.

However, compactness criteria specialized to families of continuous equicontinuous functions in GLS remain less explored.

The present work addresses this gap by providing a direct and transparent compactness criterion adapted to continuous families in the G_ψ setting.

3. Preliminaries and notation

Throughout the paper, $(E, \mathcal{M}, \mu)$ denotes a measure space with finite measure $\mu(E) < \infty$, unless otherwise stated. The main results are formulated for subsets $E \subset \mathbb{R}^n$ endowed with the Lebesgue measure μ, but the analytic framework extends to general σ-finite measure spaces.

We adopt the following notation. A compact subset of E is denoted by K, and $C(K)$ denotes the space of continuous functions on K. When needed, each $f \in C(K)$ is identified with its zero extension to E, that is, $f(x) = 0$ for $x \in E \setminus K$.

The space $L^p(E)$ denotes the classical Lebesgue space. Unless otherwise indicated, constants may change from line to line.

Definition 3.1 (Grand Lebesgue space $G_\psi(E)$). Let $(E, \mathcal{M}, \mu)$ be a measure space with $\mu(E) < \infty$, and let $I = (a, b) \subset (1, \infty)$ be an open interval. Let $\psi : I \to (0, \infty)$ be a measurable function such that

$$\inf_{p \in I} \psi(p) > 0. \tag{3.1}$$

The grand Lebesgue space $G_\psi(E)$ is defined as the set of all measurable functions $f : E \to \mathbb{R}$ such that

$$\|f\|_{G_\psi(E)} := \sup_{p \in I} \frac{\|f\|_{L^p(E)}}{\psi(p)} < \infty, \tag{3.2}$$

where

$$\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad 1 \le p < \infty, \tag{3.3}$$

and

$$\|f\|_{L^\infty(E)} = \operatorname*{ess\,sup}_{x \in E} |f(x)|. \tag{3.4}$$

The function ψ is called the *generating function* of the space $G_\psi(E)$ (see [10]). It governs the growth of the L^p norms of f as p varies over the interval I, thereby encoding the integrability profile of functions in $G_\psi(E)$. Consequently, $G_\psi(E)$ provides a refinement of the classical Lebesgue scale through a ψ-modulated variable exponent structure. Moreover, $(G_\psi(E), \|\cdot\|_{G_\psi(E)})$ is a Banach space [9], whose completeness follows from the completeness of each $L^p(E)$ together with standard arguments for supremum norms.

The associate (Köthe dual) space of $G_\psi(E)$ consists of g such that $f \mapsto \int fg$ defines a bounded functional on $G_\psi(E)$.

Definition 3.2 (Köthe associate / small Lebesgue space). Let $X = G_\psi(E)$. The Köthe associate (also called the Köthe dual) of X is

$$(X)' := \{ g : E \to \mathbb{R} \text{ measurable} : \\ \|g\|_{(X)'} := \sup\left\{ \int_E |f(x)g(x)|\, d\mu(x) : f \in X,\ \|f\|_X \leq 1 \right\} < \infty \right\}.$$

We denote $S_\psi(E) := (G_\psi(E))'$ and call it the *small Lebesgue space*. Moreover, if $G_\psi(E)$ has an order-continuous norm, then $(S_\psi(E))' = G_\psi(E)$ (see [6]).

Embeddings and density. Several basic embeddings follow directly from the definition of $G_\psi(E)$:

- From the definition of the G_ψ-norm, we immediately have, for every $p \in I$,

$$\|f\|_{L^p(E)} \leq \psi(p)\, \|f\|_{G_\psi(E)}. \tag{3.5}$$

  This yields the continuous embedding $G_\psi(E) \hookrightarrow L^p(E)$ for each fixed $p \in I$, and in particular,

$$G_\psi(E) \subset \bigcap_{p \in I} L^p(E). \tag{3.6}$$

  The reverse embedding $L^p(E) \hookrightarrow G_\psi(E)$ does not hold in general without additional assumptions (e.g., simultaneous control in a range of exponents).

- A delicate feature of GLS is the non-density of smooth compactly supported functions in some cases. Capone and Fiorenza [3] provided explicit counterexamples, showing that standard mollification techniques do not always approximate arbitrary GLS elements. This motivates compactness proofs that avoid density arguments and instead rely on functional analytic methods.

For a family $F \subset C(K)$ of continuous functions on a compact set K, we recall some standard notions:

Equicontinuity means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, for all $f \in F$ and all $x, y \in K$.

Uniform boundedness refers to a common sup-norm bound

$$\sup_{f \in F} \|f\|_{L^\infty(K)} < \infty. \tag{3.7}$$

Translation operators are denoted by $(T_h f)(x) = f(x + h)$ when $x + h \in E$.

The modulus of continuity of f in L^p is

$$\omega_p(f, t) := \sup_{|h| \leq t} \|T_h f - f\|_{L^p(E)} \tag{3.8}$$

for equicontinuous families, where $\omega_p(f, t) \to 0$ as $t \to 0$, uniformly in f.

The following result is an immediate consequence of the definition of the norm in $G_\psi(E)$.

Lemma 3.3 (Uniform L^p control from G_ψ-boundedness). Let $I = (a, b) \subset (1, \infty)$ and let $\psi : I \to (0, \infty)$ be continuous. Let $F \subset G_\psi(E)$ be a family of functions such that

$$\sup_{f \in F} \|f\|_{G_\psi(E)} \le M < \infty. \tag{3.9}$$

Then: i. For any $p \in I$ and all $f \in F$, we have

$$\|f\|_{L^p(E)} \le M\,\psi(p). \tag{3.10}$$

ii. For any compact sub-interval $J \subset I$, there exists a constant C_J such that

$$\|f\|_{L^p(E)} \le C_J \quad \forall p \in J,\ \forall f \in F. \tag{3.11}$$

In particular, one may take

$$C_J := M \sup_{p \in J} \psi(p).$$

Proof. i. Let $p \in I$ be arbitrary and let $f \in F$. By the definition of the grand Lebesgue norm,

$$\|f\|_{G_\psi(E)} = \sup_{r \in I} \frac{\|f\|_{L^r(E)}}{\psi(r)}.$$

Since $p \in I$ is fixed, and the supremum is taken over all $r \in I$, we have:

$$\frac{\|f\|_{L^p(E)}}{\psi(p)} \le \sup_{r \in I} \frac{\|f\|_{L^r(E)}}{\psi(r)} = \|f\|_{G_\psi(E)}$$

$$\|f\|_{L^p(E)} \le \psi(p)\,\|f\|_{G_\psi(E)}$$

$$\|f\|_{L^p(E)} \le M\,\psi(p).$$

ii. Let $J \subset I$ be compact. Since ψ is continuous on J, it is bounded above there, and we set

$$B_J := \sup_{p \in J} \psi(p) < \infty.$$

Then by (i),

$$\|f\|_{L^p(E)} \le M\psi(p) \le MB_J = C_J, \qquad \forall p \in J,\ \forall f \in F$$

which proves (3.11). □

Lemma 3.4 (Uniform convergence implies L^p convergence). Let K be measurable with $\mu(K) < \infty$ and let $f_n, f \in C(K)$. If $f_n \to f$ uniformly on K, then $f_n \to f$ in $L^p(K)$ for every $p \in [1, \infty)$ (see, e.g., [2]).

Proof. Fix $1 \le p < \infty$. Uniform convergence gives $\|f_n - f\|_{L^\infty(K)} \to 0$. Hence

$$\|f_n - f\|^p_{L^p(K)} = \int_K |f_n - f|^p \, d\mu \le \mu(K)\,\|f_n - f\|^p_{L^\infty(K)} \to 0. \tag{3.12}$$

□

4. Main results

Our compactness arguments transfer convergence from the uniform norm on compact sets to the grand Lebesgue norm via a single quantitative estimate.

If K has finite measure and

$$C_{K,\psi} := \sup_{p\in I} \frac{\mu(K)^{1/p}}{\psi(p)} < \infty,$$

then every $g \in C(K)$ satisfies

$$\|g\|_{G_\psi(K)} \le C_{K,\psi}\, \|g\|_{L^\infty(K)}.$$

Therefore, whenever a subsequence converges uniformly on K (obtained from Arzelà–Ascoli under uniform boundedness and equicontinuity), it converges in $G_\psi(K)$.

When results are stated in $G_\psi(E)$ for functions originally defined on $K \subset E$, we identify $f \in C(K)$ with its zero extension to E. This reduces $L^p(E)$ estimates to integrals over K and yields the same transfer estimate.

Theorem 4.1 (Relative compactness in $G_\psi(K)$). Let $K \subset \mathbb{R}^n$ be compact with $\mu(K) < \infty$ and let $I = (a, b) \subset (1, \infty)$. Assume

$$C_{K,\psi} := \sup_{p\in I} \frac{\mu(K)^{1/p}}{\psi(p)} < \infty. \tag{4.1}$$

Let $F \subset C(K) \cap G_\psi(K)$ satisfy: (i). $\sup_{f\in F} \|f\|_{L^\infty(K)} < \infty$; (ii). F is equicontinuous on K.

Then F is relatively compact in $G_\psi(K)$.

Moreover, the estimate

$$\|g\|_{G_\psi(K)} \le C_{K,\psi}\, \|g\|_{L^\infty(K)} \tag{4.2}$$

holds for all $g \in C(K)$, and it holds with some finite constant for all $g \in C(K)$ if and only if $C_{K,\psi} < \infty$.

Proof. (Compactness.) Let $(f_n) \subset F$. By Arzelà–Ascoli, there exist (f_{n_k}) and $f \in C(K)$ with $\|f_{n_k} - f\|_{L^\infty(K)} \to 0$. For each $p \in I$,

$$\|f_{n_k} - f\|_{L^p(K)} \le \mu(K)^{1/p} \|f_{n_k} - f\|_{L^\infty(K)}.$$

Taking the supremum over $p \in I$ yields

$$\|f_{n_k} - f\|_{G_\psi(K)} \le C_{K,\psi} \|f_{n_k} - f\|_{L^\infty(K)} \to 0.$$

Hence F is relatively compact in $G_\psi(K)$.

(Necessity in the "iff".) If $\|g\|_{G_\psi(K)} \le C\|g\|_{L^\infty(K)}$ holds for all $g \in C(K)$ with some $C < \infty$, take $g \equiv 1$ to get $C_{K,\psi} = \|1\|_{G_\psi(K)} \le C$, hence $C_{K,\psi} < \infty$. □

The weighted version of the compactness result follows by adapting the argument of Theorem 4.1 to weighted norms.

Theorem 4.2 (Compactness in weighted grand Lebesgue spaces). Let $K \subset \mathbb{R}^n$ be compact with $\mu(K) < \infty$, $I = (a, b) \subset (1, \infty)$, and let $\omega : K \to (0, \infty)$ be a bounded weight satisfying

$$0 < \omega(x) \le M < \infty \quad \text{on } K.$$

Define

$$\|f\|_{G_{\psi,\omega}(K)} := \sup_{p\in I} \frac{\|f\|_{L^p_\omega(K)}}{\psi(p)}, \tag{4.3}$$

where

$$\|f\|_{L^p_\omega(K)} := \Big(\int_K |f|^p \omega \, d\mu \Big)^{1/p}. \tag{4.4}$$

Assume

$$C_{K,\psi} := \sup_{p \in I} \frac{\mu(K)^{1/p}}{\psi(p)} < \infty.$$

If $F \subset C(K) \cap G_{\psi,\omega}(K)$ is
(i). uniformly bounded in the sup norm on K, (ii). equicontinuous on K, then F is relatively compact in $G_{\psi,\omega}(K)$.

Proof. Let $(f_n) \subset F$. By (i) and (ii) and the Arzelà–Ascoli theorem, there exist a subsequence (f_{n_k}) and a function $f \in C(K)$ such that

$$\|f_{n_k} - f\|_{L^\infty(K)} \to 0.$$

For each $p \in I$, we have

$$\|f_{n_k} - f\|^p_{L^p_\omega(K)} = \int_K |f_{n_k} - f|^p \, \omega \, d\mu \le M \int_K |f_{n_k} - f|^p \, d\mu \le M \, \mu(K) \, \|f_{n_k} - f\|^p_{L^\infty(K)}.$$

Hence

$$\|f_{n_k} - f\|_{L^p_\omega(K)} \le M^{1/p} \mu(K)^{1/p} \|f_{n_k} - f\|_{L^\infty(K)}.$$

Dividing by $\psi(p)$ and taking the supremum over $p \in I$, we obtain

$$\|f_{n_k} - f\|_{G_{\psi,\omega}(K)} \le \Big(\sup_{p \in I} \frac{M^{1/p} \mu(K)^{1/p}}{\psi(p)} \Big) \|f_{n_k} - f\|_{L^\infty(K)}.$$

Since $M^{1/p} \le M^{1/a}$ for $p \in I$, it follows that

$$\|f_{n_k} - f\|_{G_{\psi,\omega}(K)} \le M^{1/a} C_{K,\psi} \, \|f_{n_k} - f\|_{L^\infty(K)} \to 0. \tag{4.5}$$

Therefore (f_{n_k}) converges to f in $G_{\psi,\omega}(K)$, and F is relatively compact in $G_{\psi,\omega}(K)$. □

Definition 4.3 (Fractional Sobolev seminorm). [1] Let $\Omega \subset \mathbb{R}^n$ be measurable and $|\Omega| < \infty$, $s \in (0, 1)$, and $p \in [1, \infty)$. The Slobodeckij seminorm is

$$[f]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}. \tag{4.6}$$

We set $\|f\|_{W^{s,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W^{s,p}(\Omega)}$.

Theorem 4.4 (Fractional smoothness and compactness). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $I = (a, b) \subset (1, \infty)$, and let $\psi : I \to (0, \infty)$ be an admissible function.

Fix $s \in (0, 1)$ and choose $p_0 \in I$ such that $sp_0 > n$. Assume that $F \subset G_\psi(\Omega)$ satisfies

(i). $\sup_{f \in F} [f]_{W^{s,p_0}(\Omega)} \le C < \infty$; (ii). $\sup_{f \in F} \|f\|_{G_\psi(\Omega)} \le M < \infty$.

Then F is relatively compact in $G_\psi(\Omega)$.

Proof. Since $sp_0 > n$, the fractional Morrey embedding theorem yields

$$W^{s,p_0}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \qquad \alpha = s - \frac{n}{p_0} > 0. \tag{4.7}$$

Hence, there exists $C_1 > 0$ such that

$$|f(x) - f(y)| \leq C_1 [f]_{W^{s,p_0}(\Omega)} |x - y|^{\alpha}, \quad x, y \in \Omega. \tag{4.8}$$

By assumption (i), the family F is uniformly Hölder equicontinuous. Moreover, (ii) yields uniform G_ψ-boundedness, hence uniform L^{p_0} bounds. Therefore F is uniformly bounded in $L^\infty(\Omega)$ by the Morrey embedding.

Applying the Arzelà–Ascoli theorem, every sequence in F admits a uniformly convergent subsequence on $\overline{\Omega}$. By Lemma 3.4, uniform convergence implies convergence in $L^p(\Omega)$ for every $p \in I$.

Finally, taking the supremum over $p \in I$ in the definition of the G_ψ norm yields

$$\|f_{n_k} - f\|_{G_\psi(\Omega)} \to 0.$$

Hence, F is relatively compact in $G_\psi(\Omega)$. □

Corollary 4.5 (Rellich–Kondrachov-type compactness in G_ψ). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary, and let $p > n$. Let $I = (a, b) \subset (1, \infty)$ and assume

$$C_{\Omega,\psi} := \sup_{q \in I} \frac{\mu(\Omega)^{1/q}}{\psi(q)} < \infty.$$

Then, every bounded set $B \subset W^{1,p}(\Omega)$ is relatively compact in $G_\psi(\Omega)$. Equivalently, the inclusion map

$$W^{1,p}(\Omega) \hookrightarrow G_\psi(\Omega)$$

is compact.

Proof. Since $p > n$ and Ω is bounded with a Lipschitz boundary, Morrey's inequality (see [1]) implies the continuous embedding

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \quad \alpha = 1 - \frac{n}{p} > 0. \tag{4.9}$$

Hence, every bounded subset of $W^{1,p}(\Omega)$ is uniformly bounded and equicontinuous on $\overline{\Omega}$. Because $\overline{\Omega}$ is compact, the Arzelà–Ascoli theorem yields a uniformly convergent subsequence in $C(\overline{\Omega})$. Applying Theorem 4.4, we conclude convergence in $G_\psi(\Omega)$. □

Counterexample 4.6 (Necessity of equicontinuity). This example shows that equicontinuity is a necessary condition. There exists a bounded family $F \subset C(K) \cap G_\psi(K)$ with $\sup_{f \in F} \|f\|_{G_\psi} < \infty$ which fails to be relatively compact in $G_\psi(K)$.

Proof. Let $K = [0, 2\pi]$ and let $\psi : (a, b) \to (0, \infty)$ be admissible with $p_0 = 2 \in (a, b)$ and $\psi(2) < \infty$. Define

$$f_n(x) = \sin(nx), \qquad x \in K, n \in \mathbb{N}. \tag{4.10}$$

Then, the family $F = \{f_n\}_{n \in \mathbb{N}} \subset C(K) \cap G_\psi(K)$ is bounded in $G_\psi(K)$ but is not relatively compact in $G_\psi(K)$.

$$\|f_n\|_{L^\infty(K)} \leq 1, \quad \forall n \in \mathbb{N}. \tag{4.11}$$

So F is uniformly bounded in $C(K)$.

For any $p \in (a, b)$,

$$\|f_n\|_{L^p(K)}^p = \int_0^{2\pi} |\sin(nx)|^p \, dx. \tag{4.12}$$

Using the change of variables $u = nx$ (so $du = n\,dx$), we obtain

$$\int_0^{2\pi} |\sin(nx)|^p \, dx = \frac{1}{n} \int_0^{2\pi n} |\sin u|^p \, du.$$

Since $|\sin u|^p$ is 2π-periodic, the interval $[0, 2\pi n]$ splits into n periods, hence

$$\int_0^{2\pi n} |\sin u|^p \, du = n \int_0^{2\pi} |\sin u|^p \, du.$$

Therefore,

$$\|f_n\|_{L^p(K)}^p = \int_0^{2\pi} |\sin u|^p \, du, \tag{4.13}$$

which is independent of n. Consequently,

$$\sup_{n \in \mathbb{N}} \|f_n\|_{G_\psi(K)} < \infty. \tag{4.14}$$

So F is bounded in $G_\psi(K)$.

But, the derivatives satisfy

$$f_n'(x) = n \cos(nx), \tag{4.15}$$

which shows that the oscillations increase with n, and therefore F is not equicontinuous.

To prove lack of compactness, observe that in $L^2(K)$, the functions $\{\sin(nx)\}_{n\in\mathbb{N}}$ are orthogonal. Hence, for $n \neq m$,

$$\|f_n - f_m\|_{L^2(K)}^2 = \|f_n\|_{L^2(K)}^2 + \|f_m\|_{L^2(K)}^2 = \int_0^{2\pi} \sin^2(nx) \, dx + \int_0^{2\pi} \sin^2(mx) \, dx = 2\pi.$$

Therefore,

$$\|f_n - f_m\|_{L^2(K)} = \sqrt{2\pi}.$$

Hence (f_n) has no Cauchy subsequence in $L^2(K)$.

Since

$$\|g\|_{G_\psi(K)} \geq \frac{\|g\|_{L^2(K)}}{\psi(2)},$$

we obtain

$$\|f_n - f_m\|_{G_\psi(K)} \geq \frac{\sqrt{2\pi}}{\psi(2)} > 0. \tag{4.16}$$

Therefore, no subsequence can converge in $G_\psi(K)$. Therefore, F is not relatively compact in $G_\psi(K)$.

Hence boundedness in $G_\psi(K)$ alone does not imply compactness. □

Corollary 4.7 (Zero-extension principle in $G_\psi(E)$). Under the assumptions of Theorem 4.1, let E be a set of finite measure, $K \subset E$ be compact, and let $F \subset C(K) \cap G_\psi(E)$.

Identifying each function $f \in C(K)$ with its zero extension to E, the family F is relatively compact in $G_\psi(E)$.

Proof. Since the zero extension preserves uniform boundedness on K and does not affect equicontinuity on K, the assumptions of Theorem 4.1 remain valid for the extended family.

Let $(f_n) \subset F$. By assumptions (i) and (ii) of Theorem 4.1 and the Arzelà–Ascoli theorem, there exist a subsequence (f_{n_k}) and a function $f \in C(K)$ such that

$$\|f_{n_k} - f\|_{L^\infty(K)} \to 0 \qquad \text{as } k \to \infty. \tag{4.17}$$

Let $\tilde{f}$ denote the zero extension of f to E. Since $f_{n_k} - \tilde{f} = 0$ on $E \setminus K$, for every $p \in I$, we have

$$\|f_{n_k} - \tilde{f}\|^p_{L^p(E)} = \int_E |f_{n_k} - \tilde{f}|^p \, d\mu = \int_K |f_{n_k} - f|^p \, d\mu \le \mu(K)\|f_{n_k} - f\|^p_{L^\infty(K)}.$$

Hence

$$\|f_{n_k} - \tilde{f}\|_{L^p(E)} \le \mu(K)^{1/p}\|f_{n_k} - f\|_{L^\infty(K)}. \tag{4.18}$$

Dividing by $\psi(p)$ and taking the supremum over $p \in I$, we obtain

$$\|f_{n_k} - \tilde{f}\|_{G_\psi(E)} = \sup_{p\in I} \frac{\|f_{n_k} - \tilde{f}\|_{L^p(E)}}{\psi(p)} \le \left(\sup_{p\in I} \frac{\mu(K)^{1/p}}{\psi(p)}\right)\|f_{n_k} - f\|_{L^\infty(K)}.$$

By the definition of the constant

$$C_{K,\psi} := \sup_{p\in I} \frac{\mu(K)^{1/p}}{\psi(p)} < \infty,$$

we conclude that

$$\|f_{n_k} - \tilde{f}\|_{G_\psi(E)} \le C_{K,\psi} \, \|f_{n_k} - f\|_{L^\infty(K)} \to 0. \tag{4.19}$$

Since $G_\psi(E)$ is complete, the limit belongs to $G_\psi(E)$. Therefore (f_{n_k}) converges in $G_\psi(E)$, and hence F is relatively compact in $G_\psi(E)$. □

5. Examples

Example 5.1 (Oscillatory family without equicontinuity). Let $\phi \in C_c^\infty([0, 1])$ with $\|\phi\|_{L^p} = 1$ for some $p \in I$, and set

$$f_n(x) = \phi(x)\sin(nx). \tag{5.1}$$

Then $f_n \in C([0, 1]) \cap G_\psi([0, 1])$ and $\sup_n \|f_n\|_{G_\psi} < \infty$. However, the sequence (f_n) is not equicontinuous for any fixed $\delta > 0$:

$$\sup_{|x-y|<\delta} |f_n(x) - f_n(y)| \longrightarrow 2\|\phi\|_{L^\infty} \qquad (n \to \infty). \tag{5.2}$$

Hence, no subsequence of (f_n) converges in $G_\psi([0, 1])$. It is enough to show non-compactness in L^2.

$$\|f_n\|^2_{L^2} = \int_0^1 \phi(x)^2 \sin^2(nx) \, dx = \frac{1}{2}\int_0^1 \phi^2 - \frac{1}{2}\int_0^1 \phi(x)^2 \cos(2nx) \, dx$$

$$\longrightarrow \frac{1}{2}\int_0^1 \phi^2 > 0 \tag{5.3}$$

by Riemann–Lebesgue (see, e.g., [2]). For $n \neq m$,

$$\|f_n - f_m\|_{L^2}^2 = \|f_n\|_{L^2}^2 + \|f_m\|_{L^2}^2 - 2\int_0^1 \phi^2 \sin(nx)\sin(mx)\,dx \tag{5.4}$$

and the cross-term equals

$$\frac{1}{2}\int_0^1 \phi^2\big(\cos((n-m)x) - \cos((n+m)x)\big)\,dx \quad \text{as } n, m \to \infty. \tag{5.5}$$

Hence

$$\|f_n - f_m\|_{L^2}^2 \longrightarrow \int_0^1 \phi^2 > 0. \tag{5.6}$$

Therefore, (f_n) is not Cauchy in L^2, thus there is no L^p (and no G_ψ) convergence.

Equicontinuous family. Fix $\phi \in C([0,1])$ with $\phi(0) = 0$ and define $f_n(x) = \phi(x/n)$. Then $\sup_n \|f_n\|_{L^\infty(K)} \leq \|\phi\|_{L^\infty(K)}$ and (f_n) is equicontinuous on $[0,1]$. Moreover,

$$\|f_n - \phi(0)\|_{L^\infty(K)} = \sup_{x\in[0,1]} |\phi(x/n) - \phi(0)| \;\longrightarrow\; 0, \tag{5.7}$$

so $f_n \to 0$ uniformly, hence in every L^p, and therefore in G_ψ:

$$\|f_n\|_{G_\psi} = \sup_{p\in I} \frac{\|f_n\|_{L^p}}{\psi(p)} \xrightarrow[n\to\infty]{} 0. \tag{5.8}$$

Example 5.2 (Poisson model with an explicit G_ψ rate). Consider

$$-u'' = f \quad \text{in } (0,1), \qquad u(0) = u(1) = 0 \tag{5.9}$$

with $f(x) = \sin(\pi x)$ and noisy data $f_n(x) = \sin(\pi x) + \varepsilon_n \sin(2\pi x)$, $\varepsilon_n \downarrow 0$. Since $-u'' = \sin(k\pi x)$ yields $u(x) = \frac{1}{(k\pi)^2}\sin(k\pi x)$, we have:

$$u(x) = \frac{1}{\pi^2}\sin(\pi x), \qquad u_n(x) = \frac{1}{\pi^2}\sin(\pi x) + \varepsilon_n \frac{1}{(2\pi)^2}\sin(2\pi x). \tag{5.10}$$

Thus

$$\|u_n - u\|_{L^p(0,1)} = \frac{\varepsilon_n}{4\pi^2}\|\sin(2\pi x)\|_{L^p(0,1)} \leq \frac{\varepsilon_n}{4\pi^2}\,\mu(0,1)^{1/p}. \tag{5.11}$$

If $C_{(0,1),\psi} := \sup_{p\in I} \mu(0,1)^{1/p}/\psi(p) < \infty$, then

$$\|u_n - u\|_{G_\psi(0,1)} \leq \frac{\varepsilon_n}{4\pi^2}\, C_{(0,1),\psi} \xrightarrow[n\to\infty]{} 0, \tag{5.12}$$

i.e., $u_n \to u$ in $G_\psi(0,1)$ and there exists a constant $C > 0$ such that

$$\|u_n - u\|_{G_\psi(0,1)} \leq C\,\varepsilon_n.$$

Example 5.3 (Galerkin scheme and G_ψ convergence). Let

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2}\sin(k\pi x), \qquad u(x) = \sum_{k=1}^{\infty} \frac{1}{k^2(1+(k\pi)^2)}\sin(k\pi x).$$

Let u_n denote the Galerkin approximation obtained by truncating the series at $k \leq n$. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2(1 + (k\pi)^2)} < \infty, \tag{5.13}$$

and $|\sin(k\pi x)| \leq 1$, the Weierstrass M-test yields uniform convergence of the series for u on $[0, 1]$. Hence $u_n \to u$ uniformly.

By Lemma 3.4, uniform convergence implies convergence in $L^p(0, 1)$ for every $p \in I$. If

$$C_{(0,1),\psi} := \sup_{p \in I} \frac{\mu(0, 1)^{1/p}}{\psi(p)} < \infty, \tag{5.14}$$

then Theorem 4.1 gives

$$\|u_n - u\|_{G_\psi(0,1)} \to 0. \tag{5.15}$$

Thus Galerkin approximations converge strongly in G_ψ, providing simultaneous control across all exponents $p \in I$.

6. Discussion

This work develops a compactness framework for families of continuous functions in grand Lebesgue spaces, providing results that extend classical compactness principles such as the Kolmogorov–Riesz and Rellich–Kondrachov theorems to the GLS setting [1, 7, 12].

The approach relies on uniform boundedness and equicontinuity, allowing compactness to be characterized without invoking reflexivity or density arguments typically used in classical settings [5]. Explicit counterexamples further clarify the necessity of the equicontinuity assumption.

The results also extend naturally to weighted and fractional settings [4], offering a unified compactness perspective across several functional frameworks. Applications to elliptic and fractional PDEs suggest that the developed theory is not only structurally robust but also relevant for stability and convergence analysis in modern analytical models [1].

Future research may focus on extending these compactness criteria to nonlinear problems, time-dependent equations in GLS, and nonlocal operators on unbounded domains.

7. Conclusions

We have established a compactness criterion for families of continuous functions in grand Lebesgue spaces under a quantitative structural condition on the generating function ψ. The results provide a direct extension of classical compactness principles to the multi-exponent GLS framework and clarify the precise role of equicontinuity in this setting.

The developed theory extends naturally to weighted and fractional contexts and yields strong convergence results relevant to approximation schemes and PDE models. Further investigations may explore nonlinear, time-dependent, and nonlocal extensions within the GLS setting.

Author contributions

Rahma Katea contributed to the main ideas, proofs, manuscript preparation, and revision. Yasin Kaya supervised the research and contributed to the mathematical analysis and final revision of the manuscript.

Use of Generative-AI tools declaration

The authors did not use Artificial Intelligence (AI) tools for scientific content generation, mathematical results, proofs, or research analysis in this article. Limited AI-assisted tools were used only for language improvement, translation, and LaTeX formatting.

Acknowledgments

This work is based on the first author's Ph.D. thesis.

Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, 2Eds., Academic Press, 2003.
2. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York: Springer, 2011. https://doi.org/10.1007/978-0-387-70914-7
3. C. Capone, A. Fiorenza, On small Lebesgue spaces, *J. Funct. Spaces Appl.*, **3** (2005), 73–89. https://doi.org/10.1155/2005/192538
4. G. Di Fratta, A. Fiorenza, A direct approach to the duality of grand and small Lebesgue spaces, *Nonlinear Anal.-Theor.*, **70** (2009), 2582–2592. https://doi.org/10.1016/j.na.2008.03.044
5. A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces, *Collect. Math.*, **51** (2000), 131–148.
6. A. Fiorenza, G. Karadzhov, Grand and small Lebesgue spaces and their analogs, *Z. Anal. Anwend.*, **23** (2004), 657–681. https://doi.org/10.4171/ZAA/1215
7. H. Hanche-Olsen, H. Holden, The Kolmogorov–Riesz compactness theorem, *Expo. Math.*, **28** (2010), 385–394. https://doi.org/10.1016/j.exmath.2010.03.001
8. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, *Arch. Ration. Mech. Anal.*, **119** (1992), 129–143. https://doi.org/10.1007/BF00375119
9. E. Ostrovsky, L. Sirota, Moment Banach spaces: Theory and applications, *HAIT J. Sci. Eng. C*, **4** (2007), 233–262.
10. E. Ostrovsky, L. Sirota, Composed grand Lebesgue spaces, 2011, arXiv: 1110.4880. https://doi.org/10.48550/arXiv.1110.4880
11. E. Ostrovsky, L. Sirota, Composition operators between two different bilateral grand Lebesgue spaces, 2013, arXiv: 1308.1706v1. https://doi.org/10.48550/arXiv.1308.1706
12. H. Rafeiro, A. Vargas, On the compactness in grand spaces, *Georgian Math. J.*, **22** (2015), 141–152. https://doi.org/10.1515/gmj-2014-0060