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Research article

Soliton structures of the modified unstable nonlinear Schrödinger equation and their applications in nonlinear optical communication systems

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Abstract: This work focuses on the modified unstable nonlinear Schrödinger equation. It is an important model for forecasting the evolution of unstable nonlinear wave packets in dispersive media. We derive several new solitary wave solutions, including periodic, dark, explosive, and singular waveforms, utilizing a new closed-form analytical technique. The proposed solutions demonstrate the complex interactions of wave amplitude, velocity, and localization with the system's nonlinear and dispersive properties, providing clear insight into instability causes. MATLAB is employed to generate two-dimensional, three-dimensional, and trajectory plots of the selected solutions, thereby facilitating the visualization of solitary wave propagation in the modified unstable nonlinear Schrödinger equation. This deeper analytical insight allows for better management of dispersion, nonlinearity, and instability evolution, resulting in increased transmission distance, spectrum efficiency, and signal robustness. Finally, the study demonstrates that accurately constructed soliton structures, formulated within the modified unstable framework, offer a robust and scalable approach for next-generation optical communication systems operating in highly nonlinear and ultrafast regimes.

Keywords: closed form framework; solitary waves; mUNLSE model; modern fiber-optic systems
Mathematics Subject Classification: 35B40, 35C05, 35Q35, 35Q55

1. Introduction

The nonlinear Schrödinger equation (NLSE) provides a fundamental mathematical model for describing wave propagation phenomena, particularly in nonlinear media. It is essential to many branches of applied physics and engineering, especially the analysis of nonlinear material reactions and optical communication systems. The NLSE offers crucial insight into the behavior of wave envelopes in intricate propagation environments by capturing the interplay between dispersion and nonlinearity [1, 2]. The modified unstable nonlinear Schrödinger equation (mUNLSE) is a significant extension of the standard nonlinear Schrödinger equation (NLSE). It incorporates additional higher-order nonlinear, dispersive, or instability-inducing terms to more accurately describe complex wave propagation in realistic physical systems. In optical fiber communications, the standard NLSE successfully models the balance of chromatic dispersion and Kerr nonlinearity; however, in high-power transmission, ultrashort pulse propagation, or strongly nonlinear media, additional effects such as higher-order dispersion, nonlinear gain or loss, and perturbation-induced instabilities become significant [3]. These events cannot be adequately described by the standard NLSE, necessitating other formulations such as the mUNLSE. The modified unstable form more accurately represents signal evolution in optical fibers, especially in regimes where modulation instability, pulse amplification, or nonlinear perturbations affect signal stability and integrity. Consequently, the mUNLSE plays a critical role in the analysis and design of advanced optical communication systems, including ultrafast fiber lasers, optical amplifiers, and high-bit-rate transmission networks, where instability mechanisms must be understood and controlled to ensure reliable signal propagation. Beyond optical communications, the mUNLSE also arises in plasma physics, Bose–Einstein condensates, and nonlinear waveguides, where instability and higher-order nonlinear interactions govern wave dynamics. Therefore, investigating exact and analytical solutions of the modified unstable nonlinear Schrödinger equation is essential for understanding instability mechanisms, predicting nonlinear wave behavior, and improving the performance and stability of modern photonic and engineering communication systems [4–7].

Because they accurately depict the evolution of signals under actual physical influences, nonlinear partial differential equations (NPDEs) serve as the essential mathematical foundation for both optical fiber communications and contemporary wireless systems [8–10]. By capturing the combined effects of dispersion, Kerr nonlinearity, attenuation, noise, and higher-order effects, NPDEs—most notably the nonlinear Schrödinger equation and its higher-dimensional or stochastic extensions—control the propagation of ultrashort pulses in optical fiber communications. Critical phenomena that directly affect data rate, transmission distance, and signal integrity in high-capacity fiber lines, such as soliton creation, pulse compression, modulation instability, and nonlinear crosstalk, are explained by these equations. The modeling of electromagnetic wave propagation across nonlinear, dispersive, and randomly fluctuating media in wireless communications also gives rise to NPDEs, which take into consideration multipath fading, interference, nonlinear amplifier effects, and stochastic environmental disturbances. NPDEs are essential for studying signal distortion, stability, and energy localization at various scales because they may contain both nonlinearity and randomness. Thus, in both fiber-optic and wireless communication technologies, developments in analytical and numerical methods for NPDEs immediately result in better system design, increased spectrum efficiency, and reliable signal processing procedures [11–13].

Solitary waves are one of the most potent concepts connecting nonlinear research to practical engineering applications because they explain stable, isolated wave packets that maintain their shape and speed over extended distances thanks to a precise balance of nonlinearity and dispersion [14]. Because of their extraordinary durability, solitary waves are useful in engineering systems requiring signal integrity and energy localization. In optical and microwave engineering, soliton-based transmission uses solitary waves to allow long-distance, high-speed communication with little distortion, even in the face of noise and system flaws. Solitary waves simulate internal waves, hydraulic bores, and tsunami propagation in fluid and coastal engineering, offering crucial information for structure design and hazard prediction. Solitary waves occur in nonlinear lattices, granular chains, and smart materials in mechanical and civil engineering, where they are used for energy harvesting, impact mitigation, and nondestructive testing [15, 16]. Furthermore, coherent structures that control energy transfer and stability in nonlinear media are explained by solitary waves in plasma, electrical, and power engineering. The engineering significance of solitary waves is highlighted by their universality across a variety of physical platforms, since comprehending and managing these nonlinear structures immediately results in more effective, durable, and high-performing technological systems [17, 18].

Alsisi developed a closed-form analytical framework [19], based on the extended tanh approach [20, 21], to obtain solitary wave solutions for nonlinear partial differential equations (NPDEs). In the current study, this methodology is used for the first time to solve the modified unsteady nonlinear Schrödinger equation (mUNLSE), extending its applicability to a broader class of nonlinear wave models. This closed-form formulation's main feature is its ability to accommodate free physical parameters, which yield numerous families of accurate solitary wave solutions with different amplitudes, widths, and propagation velocities. These closed-form solitary waves are fundamentally important because they represent stable, localized energy packets that retain their shape during propagation and interaction, making them crucial for modeling real physical phenomena. In particular, such solutions provide vital insight into pulse dynamics in nonlinear optical fibers, energy localization in plasma physics, signal transmission in optical communication systems, and wave propagation in hydrodynamics and condensed matter physics. Unlike purely numerical approaches, the proposed closed-form structure offers explicit analytical expressions that reveal the intrinsic relationship between physical parameters and wave characteristics, enabling precise prediction, stability analysis, and physical interpretation. Moreover, the method significantly reduces computational complexity, avoids iterative procedures, and ensures high analytical accuracy and efficiency. Therefore, this approach constitutes a powerful, reliable, and systematic toolbox for mathematicians, physicists, and engineers, facilitating deeper understanding of nonlinear wave mechanisms and offering practical analytical solutions applicable to a wide spectrum of nonlinear models in modern applied science and engineering.

The rest of this article is structured as follows. In Section 2, we provide the derivation of the modified unstable nonlinear Schrödinger equation. Section 3 is devoted to the construction of exact closed-form solutions of the mUNLSE model. The dynamical characteristics of selected solutions are illustrated graphically in Section 4. To further evaluate the efficiency and reliability of the proposed approach, a detailed analysis and interpretation of the obtained results are carried out in Section 5. Finally, a summary of the key findings and closing thoughts are provided in Section 6.

2. Modified unstable nonlinear Schrödinger equation

The nonlinear Schrödinger equation (NLSE) arises as a fundamental model describing the evolution of slowly varying wave envelopes in weakly nonlinear dispersive media. In optical communication systems, it can be rigorously derived from Maxwell's equations governing electromagnetic wave propagation in nonlinear dielectric media. Starting from the vector wave equation obtained from Maxwell's equations,

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}, \tag{2.1}$$

where $\mathbf{E}$ is the electric field, ϵ is the permittivity of the medium, μ_0 is the magnetic permeability, and $\mathbf{P}_{NL}$ represents the nonlinear polarization, which for Kerr-type media is given by

$$\mathbf{P}_{NL} = \epsilon_0 \chi^{(3)} |\mathbf{E}|^2 \mathbf{E}, \tag{2.2}$$

with $\chi^{(3)}$ denoting the third-order nonlinear susceptibility, and ϵ_0 the vacuum permittivity (electric constant). Assuming propagation along one spatial direction and applying the slowly varying envelope approximation, the electric field can be expressed as

$$E(x,t) = \frac{1}{2}\left[Q(x,t)e^{i(kx-\omega t)} + Q^*(x,t)e^{-i(kx-\omega t)}\right], \tag{2.3}$$

where $Q(x,t)$ is the complex envelope function, and $Q^*(x,t)$ is the complex conjugate. Substituting this form into the wave equation and applying standard perturbation methods leads to the classical nonlinear Schrödinger equation [4,5]:

$$iQ_t + Q_{xx} + 2\alpha|Q|^2 Q = 0. \tag{2.4}$$

However, this classical form assumes ideal conditions, including weak nonlinearity, slowly varying envelopes, and negligible higher-order effects. In practical optical communication systems involving ultrashort pulses, strong nonlinearities, or rapidly varying wave packets, higher-order corrections must be included. The modified unstable nonlinear Schrödinger equation (mUNLSE) considered in this study is given by [2]:

$$iQ_t + Q_{xx} + 2\alpha \mid Q \mid^2 \; Q - \beta Q_{xt} = 0, \tag{2.5}$$

where $Q = Q(x,t)$ denotes the complex wave envelope, α represents the nonlinear coefficient associated with the cubic Kerr-type nonlinearity, and β is a real parameter that characterizes the strength of the mixed space–time derivative term. The classical NLSE can be seen as a modified and generalized version of this equation, obtained by introducing the additional term Q_{xt}, which accounts for higher-order dispersive and instability effects not captured by the standard model. In optical fibers, such effects become significant during ultrashort pulse propagation, where the classical NLSE fails to fully describe instability and pulse evolution [4,6]. Mathematically, this term modifies the dispersion relation and introduces instability mechanisms that influence wave propagation. A temporal development of disturbances in unstable medium is shown by Eq (2.5). The mUNLSE identifies clear instabilities in modulated wave trains. A number of single points between stability and instability are significant to Eq (2.5). There are many different kinds of solutions in these areas, including soliton, periodic, localized-shock, blowup, rogue, dark, and bright waves. The behavior of solutions near these places varies greatly.

To derive the modified unstable nonlinear Schrödinger equation, we apply the multiple-scale expansion method. Consider a nonlinear dispersive wave field expressed as

$$u(x,t) = \epsilon\left(Q(X,T)e^{i(kx-\omega t)} + Q^*(X,T)e^{-i(kx-\omega t)}\right) + O(\epsilon^2), \tag{2.6}$$

where ϵ is a small parameter, $Q^*(X,T)$ is the complex conjugate $Q(X,T)$ ensuring that the physical field $u(x,t)$ remains real-valued, and the slow variables are defined as

$$X = \epsilon(x - v_g t), \quad T = \epsilon^2 t. \tag{2.7}$$

Applying the chain rule, derivatives transform as

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} = -\epsilon v_g \frac{\partial}{\partial X} + \epsilon^2 \frac{\partial}{\partial T}. \tag{2.8}$$

Substituting into the governing wave equation and collecting terms of equal order in ϵ, the classical NLSE emerges at order $O(\epsilon^3)$. However, when higher-order corrections are retained, mixed derivative terms naturally appear:

$$\frac{\partial^2 Q}{\partial X \partial T}, \tag{2.9}$$

and transforming back to the original variables yields the mUNLSE (2.5).

To examine stability, consider the continuous wave solution

$$Q_0(x,t) = A e^{i(2\alpha A^2 t)}. \tag{2.10}$$

Introducing a small perturbation,

$$Q(x,t) = [A + \epsilon(x,t)]\, e^{i(2\alpha A^2 t)}, \tag{2.11}$$

and substituting into the governing equation yields the modified dispersion relation:

$$(\Omega + \beta k \Omega)^2 = k^2 \left(k^2 - 4\alpha A^2\right). \tag{2.12}$$

Here, Ω denotes the temporal frequency associated with the perturbation mode introduced during the linear stability analysis. This demonstrates that the mixed derivative term modifies instability growth rates and instability bandwidth. To understand the influence of the mixed derivative term, we compare the stability properties of the classical nonlinear Schrödinger equation and the modified unstable nonlinear Schrödinger equation.

In the classical case ($\beta = 0$), the governing equation reduces to (2.4). Performing a standard linear stability analysis around the continuous wave solution gives

$$Q(x,t) = A e^{i(2\alpha A^2 t)}, \tag{2.13}$$

and introducing a small perturbation leads to the dispersion relation

$$\Omega^2 = k^2 \left(k^2 - 4\alpha A^2\right). \tag{2.14}$$

This relation shows that instability occurs when

$$k^2 < 4\alpha A^2. \tag{2.15}$$

Thus, in the focusing regime ($\alpha > 0$), perturbations grow exponentially, leading to modulation instability and the formation of localized nonlinear wave structures such as solitons. However, in the modified unstable nonlinear Schrödinger equation (2.5), the presence of the mixed derivative term modifies the dispersion relation. The resulting relation becomes

$$(\Omega + \beta k\Omega)^2 = k^2\left(k^2 - 4\alpha A^2\right). \tag{2.16}$$

This expression shows that the instability characteristics depend explicitly on the parameter β. In particular, the mixed derivative term produces the following effects:

- It modifies the instability growth rate,
- it alters the instability bandwidth,
- it introduces asymmetry in the instability spectrum,
- it enhances or suppresses instability depending on the sign and magnitude of β.

Physically, this term represents higher-order coupling between spatial dispersion and temporal evolution. Such effects become significant in ultrashort pulse propagation, nonlinear optical fibers, and nonlinear transmission systems, where higher-order dispersive corrections cannot be neglected. Mathematically, the mixed derivative term enriches the dynamical behavior of the system and allows for a broader class of exact solutions, including unstable solitons and nonlinear wave patterns. These solutions can be constructed analytically using methods such as the extended tanh method, which is employed in this work.

3. Solutions of mUNLSE

Using the wave transformation

$$Q(x,t) = e^{i(px+vt)}q(\xi), \qquad \xi = c\,x + \omega t, \tag{3.1}$$

where p, r, v and ω are constants, in Eq (2.5) gives

$$\lambda_1\, q'' + \lambda_2\, q^3 + \lambda_3\, q = 0, \tag{3.2}$$

from the real part, where $\lambda_1 = c(c - \beta\omega), \lambda_2 = 2\alpha, \lambda_3 = \left(p\beta v - v - p^2\right)$. Indeed, the imaginary part gives $\omega = \frac{c(\beta v - 2p)}{1-\beta p}$. According to the closed-form analytical framework [19], the solutions of (3.2) are given in the cases below.

Case 1.

- For $\frac{\lambda_3}{\lambda_2} < 0$, $\frac{\lambda_3}{\lambda_1} > 0$,

$$q_{1,2}(\xi) = \pm\sqrt{\frac{-\lambda_3}{\lambda_2}}\,tanh(\sqrt{\frac{\lambda_3}{2\lambda_1}}\,\xi), \tag{3.3}$$

$$q_{3,4}(\xi) = \pm\sqrt{\frac{-\lambda_3}{\lambda_2}}\,coth(\sqrt{\frac{\lambda_3}{2\lambda_1}}\,\xi). \tag{3.4}$$

Thus, the solutions of Eq (2.5) are

$$Q_{1,2}(x,t) = \pm\sqrt{\frac{-\lambda_3}{\lambda_2}}\,tanh(\sqrt{\frac{\lambda_3}{2\lambda_1}}\,(c\,x + \omega t))e^{i(px+vt)}, \tag{3.5}$$

$$Q_{3,4}(x,t) = \pm\sqrt{\frac{-\lambda_3}{\lambda_2}}\, coth(\sqrt{\frac{\lambda_3}{2\lambda_1}}\,(c\,x+\omega t))e^{i(px+vt)}. \tag{3.6}$$

- For $\frac{\lambda_3}{\lambda_2} > 0$, $\frac{\lambda_3}{\lambda_1} < 0$,

$$\begin{aligned} q_{5,6}(\xi) &= \pm\sqrt{\frac{\lambda_3}{\lambda_2}}\, tan(\sqrt{\frac{-\lambda_3}{2\lambda_1}}\,\xi), \\ q_{7,8}(\xi) &= \mp\sqrt{\frac{\lambda_3}{\lambda_2}}\, cot(\sqrt{\frac{-\lambda_3}{2\lambda_1}}\,\xi). \end{aligned} \tag{3.7}$$

Thus, the solutions of Eq (2.5) are

$$\begin{aligned} Q_{5,6}(x,t) &= \pm\sqrt{\frac{\lambda_3}{\lambda_2}}\, tan(\sqrt{\frac{-\lambda_3}{2\lambda_1}}\,(c\,x+\omega t))e^{i(px+vt)}, \\ Q_{7,8}(x,t) &= \mp\sqrt{\frac{\lambda_3}{\lambda_2}}\, cot(\sqrt{\frac{-\lambda_3}{2\lambda_1}}\,(c\,x+\omega t))e^{i(px+vt)}. \end{aligned} \tag{3.8}$$

- For $\lambda_3 = 0$, $\frac{\lambda_1}{\lambda_2} < 0$,

$$q_{9,10}(\xi) = \pm\sqrt{\frac{-2\lambda_1}{\lambda_2}}\frac{1}{\xi}. \tag{3.9}$$

Thus, the solutions of Eq (2.5) are

$$Q_{9,10}(x,t) = \pm\sqrt{\frac{-2\lambda_1}{\lambda_2}}\,\frac{1}{c\,x+\omega t}\,e^{i(px+vt)}. \tag{3.10}$$

Case 2.

- For $\frac{\lambda_3}{\lambda_2} > 0$, $\frac{\lambda_3}{\lambda_1} < 0$,

$$q_{11,12}(\xi) = \pm\sqrt{\frac{\lambda_3}{2\lambda_2}}\left(tanh\left(\sqrt{\frac{-\lambda_3}{4\lambda_1}}\,\xi\right) - coth\left(\sqrt{\frac{-\lambda_3}{4\lambda_1}}\,\xi\right)\right), \tag{3.11}$$

Thus, the solutions of Eq (2.5) are

$$Q_{11,12}(x,t) = \pm\sqrt{\frac{\lambda_3}{2\lambda_2}}\left(tanh\left(\sqrt{\frac{-\lambda_3}{4\lambda_1}}\,(c\,x+\omega t)\right) - coth\left(\sqrt{\frac{-\lambda_3}{4\lambda_1}}\,(c\,x+\omega t)\right)\right)e^{i(px+vt)}. \tag{3.12}$$

- For $\frac{\lambda_3}{\lambda_2} < 0$, $\frac{\lambda_3}{\lambda_1} > 0$,

$$q_{13,14}(\xi) = \pm\sqrt{\frac{-\lambda_3}{2\lambda_2}}\left(tan\left(\sqrt{\frac{\lambda_3}{4\lambda_1}}\,\xi\right) + cot\left(\sqrt{\frac{\lambda_3}{4\lambda_1}}\,\xi\right)\right). \tag{3.13}$$

Thus, the solutions of Eq (2.5) are

$$Q_{13,14}(x,t) = \pm\sqrt{\frac{-\lambda_3}{2\lambda_2}}\left(tan\left(\sqrt{\frac{\lambda_3}{4\lambda_1}}\,(c\,x+\omega t)\right) + cot\left(\sqrt{\frac{\lambda_3}{4\lambda_1}}\,(c\,x+\omega t)\right)\right)e^{i(px+vt)}. \tag{3.14}$$

Case 3.

- For $\frac{\lambda_3}{\lambda_2} > 0$, $\frac{\lambda_3}{\lambda_1} < 0$, the solutions of (3.2) are

$$q_{15,16}(\xi) = \pm\sqrt{\frac{\lambda_3}{4\lambda_2}}\left(tan\left(\sqrt{\frac{-\lambda_3}{8\lambda_1}}\,\xi\right) - cot\left(\sqrt{\frac{-\lambda_3}{8\lambda_1}}\,\xi\right)\right). \tag{3.15}$$

Thus, the solutions of Eq (2.5) are

$$Q_{15,16}(x,t) = \pm\sqrt{\frac{\lambda_3}{4\lambda_2}}\left(tan\left(\sqrt{\frac{-\lambda_3}{8\lambda_1}}\,(c\,x+\omega t)\right) - \sqrt{\frac{\lambda_3}{4\lambda_2}}\,cot\left(\sqrt{\frac{-\lambda_3}{8\lambda_1}}\,(c\,x+\omega t)\right)\right)e^{i(px+vt)}. \tag{3.16}$$

- For $\frac{\lambda_3}{\lambda_2} < 0$, $\frac{\lambda_3}{\lambda_1} > 0$,

$$q_{17,18}(\xi) = \pm\sqrt{\frac{-\lambda_3}{4\lambda_2}}\left(tanh\left(\sqrt{\frac{\lambda_3}{8\lambda_1}}\,\xi\right) + coth\left(\sqrt{\frac{\lambda_3}{8\lambda_1}}\,\xi\right)\right). \tag{3.17}$$

Thus, the solutions of Eq (2.5) are

$$Q_{17,18}(x,t) = \pm\sqrt{\frac{-\lambda_3}{4\lambda_2}}\left(tanh\left(\sqrt{\frac{\lambda_3}{8\lambda_1}}\,(c\,x+\omega t)\right) + coth\left(\sqrt{\frac{\lambda_3}{8\lambda_1}}\,(c\,x+\omega t)\right)\right)e^{i(px+vt)}. \tag{3.18}$$

4. Graphical representation

The results of applying the closed-form technique on the mUNLSE are shown visually in this section using MATLAB. To help explain the answers we gave, here are some two- and three-dimensional graphs. The physical characteristics of the gain solution provided in (3.5) are shown in Figures 1–3. The physical properties of the gain solution given in Eqs (3.8) and (3.10) are shown in Figures 4 and 5, respectively.

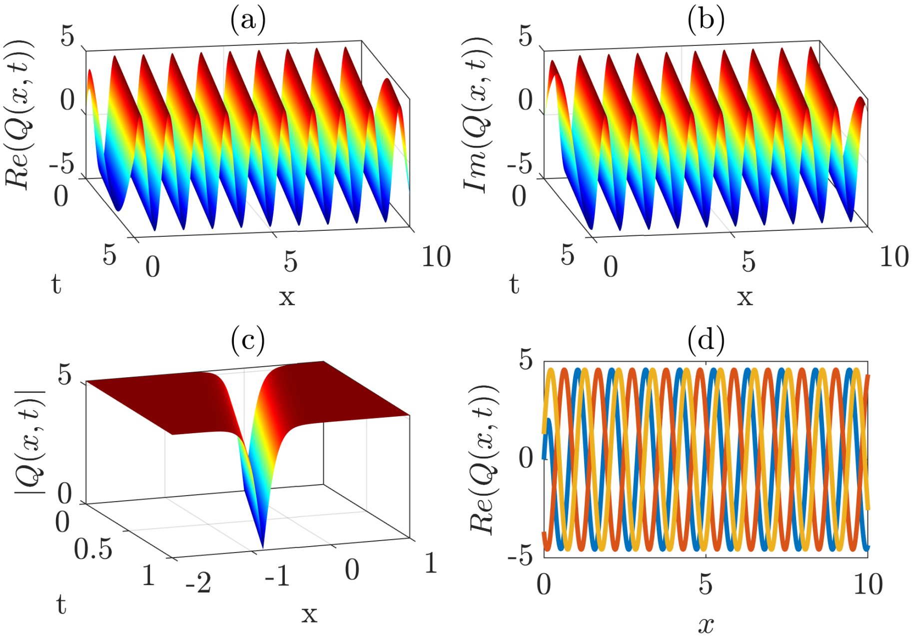

Figure 1. The figure illustrates the absolute value, imaginary part, real part, and temporal evolution of the solution. Panels (a)–(c) present three-dimensional surface plots of the solution defined by Eq (3.5). Specifically, panel (a) depicts the real part, and panel (b) shows the imaginary part of the solution for $x \in [0, 10]$ and $t \in [0, 5]$, with the parameters $c = 1$, $\beta = 2.2857$, $\nu = 1$, $\alpha = 0.5536$, and $p = 6$. Panel (c) illustrates the modulus of the same solution over the domain $x \in [-2, 1]$ and $t \in [0, 1]$, with $c = 1$, $\beta = 2.2857$, $\nu = 0.5$, $\alpha = 0.5536$, and $p = 6$. Panel (d) shows the temporal evolution of the periodic wave over a broader time range, for $t \in \{0, 2.5, 5\}$, under the same parameter set $c = 1, \beta = 2.2857$, $\nu = 1$, $\alpha = 0.5536$, and $p = 6$, and spatial domain $x \in [0, 10]$.

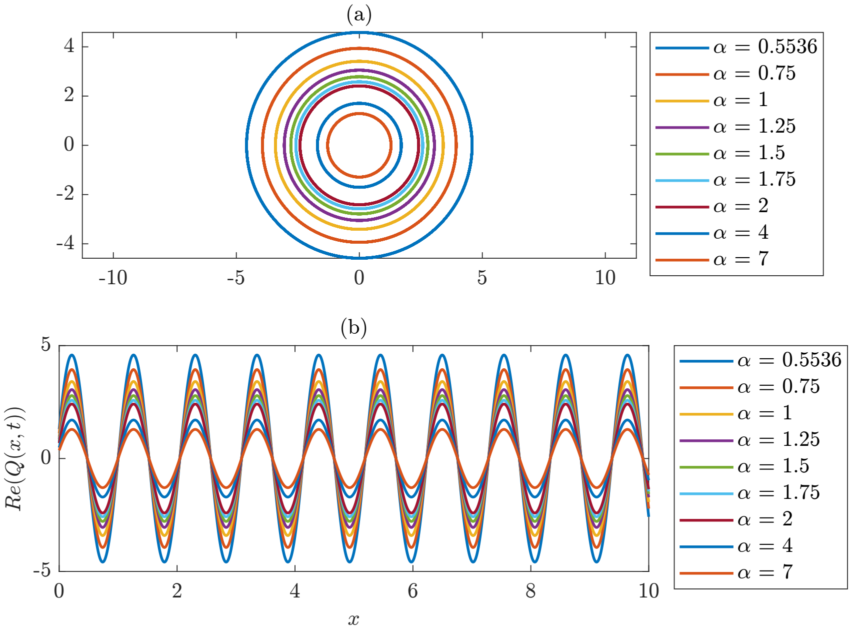

Figure 2. Panel (a) trajectory profiles of the solution given in Eq (3.5). Panel (b) shows the real part of the solution given in Eq (3.5). Fixed parameters $c = 1, \beta = 2.2857, \nu = 1$, and $p = 6$ were used for both panels. Both panels illustrate the system's sensitivity to variations in α, where $\alpha \in \{0.5536, 0.75, 1, 1.25, 1.5, 1.75, 2, 4, 7\}$.

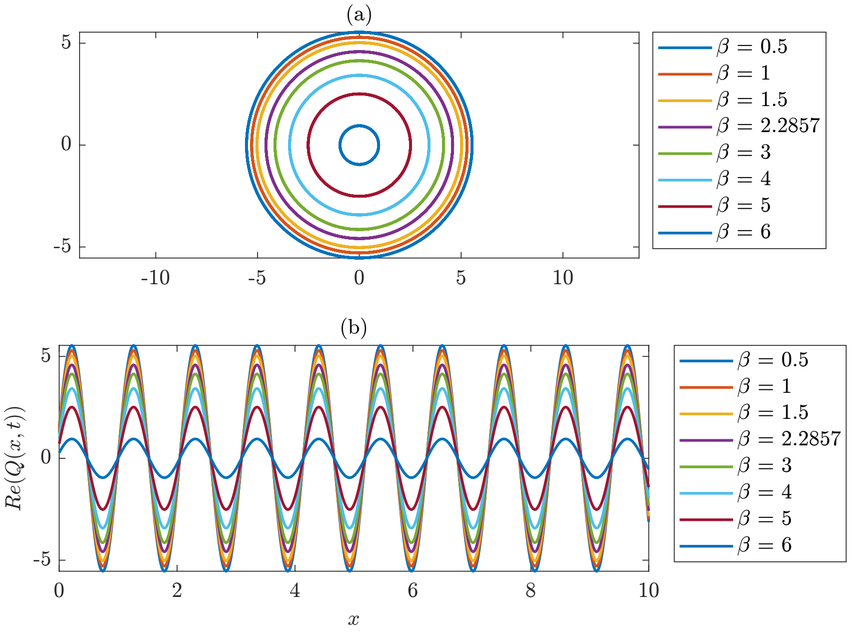

Figure 3. The dynamical characteristics of the solution defined in Eq (3.5) are presented. Specifically, Panel (a) depicts the trajectory profiles, while Panel (b) captures the real component of the solution. This analysis investigates the impact of varying the parameter β within the set $\{0.5, 1, 1.5, 2.2857, 3, 4, 5, 6\}$. Throughout this comparison, the coefficients are held constant at $c = 1$, $\alpha = 0.5536$, $\nu = 1$, and $p = 6$ to isolate the effects of β on the system behavior.

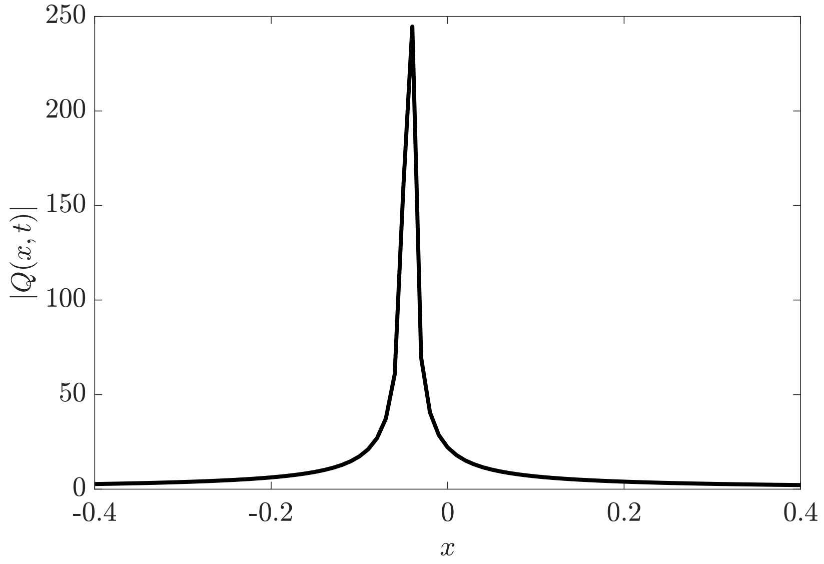

Figure 4. The graph represents the solution of Eq (3.8). The figure represents the solution's absolute value for $x \in [-0.4, 0.4]$ and $t = 0.4$, with the parameters $c = 0.2$, $\beta = 0.5$, $\nu = 0.001$, $\alpha = -6$, and $p = 1.4$.

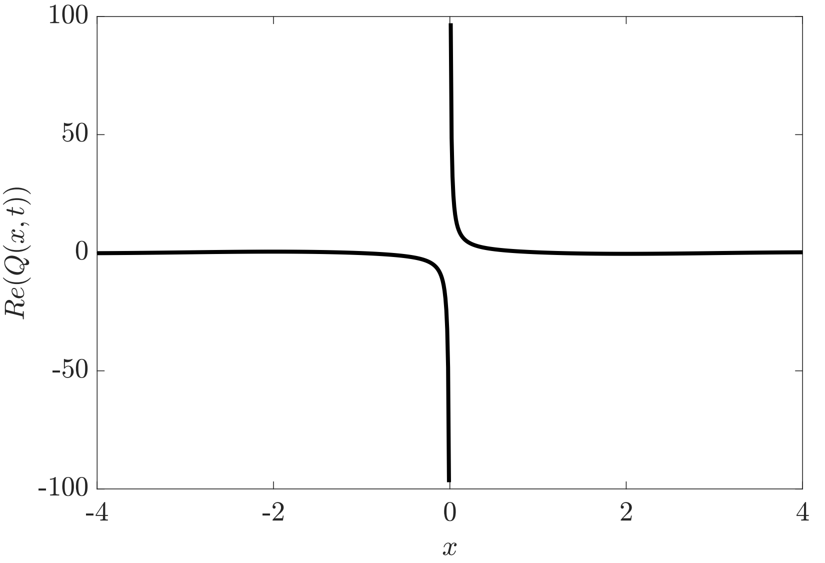

Figure 5. The graph represents the solution of Eq (3.10) for $x \in [-4, 4]$ and $t = 0$, with the parameters $c = 0.2, \beta = 0.5$, $\nu = 0.001$, $\alpha = -6$, and $p = 1.4$.

5. Results and discussion

A sophisticated physical framework for characterizing wave systems where intrinsic nonlinearity, dispersion, and destabilizing factors coexist and compete is offered by the modified unstable nonlinear Schrödinger equation (mUNLSE). The modified unstable form includes additional correction terms that explicitly account for energy exchange with the surrounding medium, such as higher-order dispersion, nonlinear gain/loss, saturation, or external forcing, in contrast to the classical NLS equation, which balances dispersion and nonlinearity to support stable solitons. This alteration physically represents the beginning of modulational instability, which is characterized by the exponential growth of minor perturbations of a continuous wave, which can result in the production of chaotic dynamics, wave collapse, or localized high-intensity structures.

Physically speaking, the mUNLSE simulates true nonequilibrium settings where ideal stability presumptions fail. It explains the formation of rogue optical pulses and filamentation in nonlinear optics by describing pulse propagation in optical fibers and waveguides exposed to gain, Raman scattering, self-steepening, or noise-induced amplification. It controls the evolution of Langmuir and ion-acoustic wave packets in plasma physics under unstable background conditions, where turbulent transfer and energy localization are caused by wave–particle interactions. The equation describes the dynamics of deep-water wave trains in hydrodynamics and offers a mechanism for the spontaneous emergence of severe ocean waves as a result of instability-driven focusing.

We proposed a new closed-form analytical framework for the mUNLSE. By implementing an appropriate traveling-wave transformation, the mUNLSE is reduced to an ordinary differential equation, which is subsequently treated using the proposed closed-form methodology to derive substantial, physically meaningful, and more general solitary-wave solutions. The developed approach significantly reduces computational cost and complexity by avoiding iterative or purely numerical procedures, while preserving high analytical accuracy. Moreover, it demonstrates strong robustness and versatility in handling a broad class of nonlinear physical systems. Owing to its precision, efficiency, and straightforward implementation, the proposed closed-form scheme can serve as a unified “box-solver” for investigating increasingly sophisticated nonlinear models arising in diverse areas of applied science.

We present two-dimensional (2D) and three-dimensional (3D) visualizations of the selected solution to illustrate and analyze its dynamical behavior graphically. Figure 1(a),(b) depicts the 3D periodic wave solution (3.5). Figure 1(c) shows the 2D localized soliton wave solution. Figure 1(d) shows the 2D periodic wave solution for varying t. Figure 2(a) shows the trajectory profile of solution (3.5). Figure 2(b) shows the 2D periodic wave solution with different values of α. It shows that the single wave amplitude decreases as α increases. Figure 3(a) also shows the trajectory profile of solution (3.5). Figure 3(b) shows the 2D periodic wave solution with different values of β. It shows that the single wave amplitude decreases and collapses as β increases. Figure 4 shows a blowup wave profile. Lastly, Figure 5 shows a singular rational wave profile.

Overall, the modified unstable nonlinear Schrödinger equation provides a theoretical basis for diagnosing, forecasting, and managing nonlinear instability occurrences in optical fibres. It directly contributes to improving optical communication performance by providing analytical and numerical insight into solitary waves, instability thresholds, and nonlinear energy dynamics, resulting in increased stability, longer transmission distances, higher bandwidth efficiency, and lower error rates.

6. Conclusions

In summary, an analytical study of soliton dynamics within the context of the mUNLSE is presented in this paper. The work offers a set of closed-form solitary wave solutions that represent pulse propagation under more general settings by expanding the classical model to incorporate instability-inducing and higher-order nonlinear phenomena. The derived solutions show the circumstances in which various kinds of wave structures may emerge and provide insight into the interaction between dispersion, nonlinearity, and instability. These findings advance our theoretical knowledge of nonlinear wave behavior in intricate optical media. However, it should be noted that the present study is primarily analytical in nature. It does not include quantitative comparisons with existing numerical or experimental approaches in terms of accuracy, computational efficiency, or practical system performance. Therefore, while the findings highlight the potential relevance of the proposed framework, further investigation is required to assess its applicability in realistic optical communication settings.

Even though the current study makes substantial analytical and practical contributions, there are still a number of interesting research avenues that could be explored further. In order to simulate more diverse and realistic physical settings, future research might concentrate on expanding the modified unstable nonlinear Schrödinger equation to incorporate stochastic perturbations, variable coefficients, and higher-dimensional formulations. In addition, from a methodological standpoint, the suggested closed-form framework can be generalized and combined with numerical and machine learning techniques to improve solution flexibility and predictive performance.

Author contributions

Abdulhamed Alsisi: Methodology, software, conceptualization, formal analysis, writing-original draft; Rayan Hamza Alsisi: Conceptualization, visualization, data curation, validation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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