



Research article

Multiple solutions for discontinuous fractional p -Laplacian problems

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Abstract: In this paper, We studied a class of fractional p -Laplacian equations with nonsmooth potential. By nonsmooth analysis, nonsmooth critical point theory, and truncation techniques, we established two multiplicity theorems: One guaranteeing the existence of at least four solutions, and the other ensuring the existence of at least two solutions. In our approach, we addressed the challenges posed by the nonsmoothness of the potential, and the results extended the nonlocal problems with discontinuous nonlinearities.

Keywords: fractional p -Laplacian; nonsmooth critical point; locally Lipschitz; multiple solutions

Mathematics Subject Classification: 34A36, 35K57, 49J52

1. Introduction

This paper is focused on discussing the following fractional p -Laplacian equation

$$\begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where the operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy, \tag{1.2}$$

$0 < s < 1$, $p \in [2, +\infty)$, $sp < N$, $B_\epsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \epsilon\}$, and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory condition function.

Fractional elliptic problems are important in modern mathematical analysis. These problems have many real world applications. They appear in fields like quantum mechanics, social sciences, and material science. They also play a role in phase transitions, game theory, image processing, and Lévy

processes. References for these applications can be found in [3–5] and [17, 18]. The theoretical study of fractional Laplacian problems has also progressed via different methods. For example, variational methods have been employed to establish existence and multiplicity results for fractional p -Laplacian equations, as seen in [19, 25, 29]. Additionally, the connection between nonlocal operators and Lévy processes has been studied in works like [3, 7, 18]. Additional results on various aspects of these problems are provided in [12, 26, 30].

While the literature extensively covers fractional elliptic problems with continuous nonlinearities $f(x, u)$, significantly less attention has been given to discontinuous cases, despite their prevalence in physical phenomena. For instance, in electrorheological fluids, the potential $f(x, u)$ often exhibits discontinuous jumps due to phase transitions or material heterogeneities. Such discontinuities arise naturally in problems involving multiphase flows, composite materials, and biological systems with abrupt changes in constitutive properties.

Motivated by these applications, we consider the following discontinuous fractional p -Laplacian problem

$$\begin{cases} (-\Delta)_p^s u \in \partial F(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.3)$$

where $0 < s < 1$, $p \in [2, +\infty)$, $sp < N$, $B_\epsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \epsilon\}$, and $\partial F(x, u)$ is the partial generalized gradient of $F(x, \cdot)$ at point u .

It is well known that many free boundary problems and obstacle problems may be reduced to differential equations with nonsmooth potentials. As a result, the classical C^1 continuity theory is not suitable for studying these problems. Fortunately, the theory of differential inclusions has attracted the interest of researchers. This field offers new insights and methodologies to deal with the discontinuous and nonlocal elliptic problems. The area of nonsmooth analysis is closely related to the development of critical points theory for nondifferentiable functions, in particular, for locally Lipschitz continuous functionals based on Clarke's generalized gradient [8]. It provides an appropriate mathematical framework to extend the classic critical point theory for C^1 -functionals in a natural way. For a comprehensive treatment, we refer to the monographs by [21] and [11]. Further developments and applications can be found in, e.g., [22–24] for variational methods, [2, 15, 16] for related elliptic problems, and [1, 27, 28] for recent extensions.

In this paper, we investigate the multiplicity of solutions to problem (1.3). We establish two results. First, combining truncation techniques with the nonsmooth Mountain Pass theorem, we prove that problem (1.3) admits at least four constant sign solutions. Furthermore, by applying nonsmooth critical point theory and analysis techniques, we demonstrate the existence of at least two nontrivial solutions.

To give our main results, we give the hypotheses below. let $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $F(x, 0) = 0$ and $\partial F(x, 0) = \{0\}$ for almost every $x \in \Omega$.

- (H1) For all $u \in \mathbb{R}$, $\Omega \ni x \mapsto F(x, u)$ is measurable;
- (H2) For a.e. $x \in \Omega$, the function $u \mapsto F(x, u)$ is locally Lipschitz;
- (H3) For a.e. $x \in \Omega$, all $u \in \mathbb{R}$, $\zeta^* \in \partial F(x, u)$,

$$|\zeta^*| \leq \alpha(x) + C|u|^{p-1} \quad \text{with } \alpha \in L^\infty(\Omega)_+, C > 0;$$

- (H4) $\lim_{|u| \rightarrow +\infty} \frac{pF(x, u)}{|u|^p} = \lambda_1$ uniformly for a.e. $x \in \Omega$ and $\lim_{|u| \rightarrow +\infty} (\zeta^* u - pF(x, u)) = -\infty$ uniformly for a.e. $x \in \Omega$.

(H5) There exists $\sigma > 0$ such that $\lambda_1|u|^p \leq pF(x, u)$ for a.e. $x \in \Omega$ and all $|\zeta| \leq \sigma$;

(H6) There are $\alpha_- < 0 < \alpha_+$ such that $0 \in \partial F(x, \alpha_-)$ and $0 \in \partial F(x, \alpha_+)$ for a.e. $x \in \Omega$ and there exists $l > 0$ such that

$$0 \leq \zeta^* \leq l(\alpha_+ - u)^{p-1}$$

for a.e. $x \in \Omega$, all $u \in [0, \alpha_+)$ and all $\zeta^* \in \partial F(x, u)$, and

$$-l(u - \alpha_-) \leq \zeta^* \leq 0$$

for a.e. $x \in \Omega$, all $u \in (\alpha_-, 0]$ and all $\zeta^* \in \partial F(x, u)$;

(H7) There exists $\mu \in L^\infty(\Omega)$ such that $\mu(x) \leq 0$ for a.e. $x \in \Omega$ with strict inequality on a set of positive measure and

$$\lim_{|u| \rightarrow +\infty} \frac{pF(x, u)}{|u|^p} \leq \mu(x)$$

uniformly for a.e. $x \in \Omega$;

(H8) There exists $\omega > 0$ and for $u \in \mathbb{R}$ with $|u| \leq \omega$, such that

$$(\lambda_k - \lambda_1)|u|^p \leq pF(x, u) \leq (\lambda_{k+1} - \lambda_1)|u|^p \quad \forall k \geq 1$$

for a.e. $x \in \Omega$, where λ_i is the i -th eigenvalue of problem (2.1) ($i = 1, 2, \dots$).

Remark 1.1. It is straightforward to observe that numerous functions can fulfill the hypotheses. let's consider two examples. The first function, denoted as $F_1(u)$, is defined by

$$F_1(u) = \begin{cases} \frac{\lambda_1}{p} [1 - (1 - |u|^p)(1 + p|u|)], & \text{if } |u| \leq 1, \\ \frac{\lambda_1}{p} [|u|^p + \ln(1 + |u|) - 1], & \text{if } |u| > 1. \end{cases}$$

Then $F_1(u)$ satisfies hypotheses (H1)–(H6).

The second function $F_2(u)$ is defined as

$$F_2(u) = \begin{cases} \frac{\mu(x)}{p}|u|^p + \frac{\kappa}{p} - \frac{\mu(x)}{p}, & \text{if } |u| > 1, \\ \frac{\kappa}{p}|u|^p, & \text{if } |u| \leq 1, \end{cases}$$

where $\kappa \in [\lambda_k - \lambda_1, \lambda_{k+1} - \lambda_1]$. Function $F_2(u)$ complies with hypotheses (H1)–(H3), as well as (H7) and (H8).

Now we give our main results.

Theorem 1.1. If hypotheses (H1)–(H6) hold, then problem (1.3) has at least four nontrivial solutions $u_0, \hat{u}_0 \in \text{int}C_+$, $v_0, \hat{v}_0 \in -\text{int}C_+$ such that

$$u_0 \leq \hat{u}_0, \quad u_0 \neq \hat{u}_0 \quad \text{and} \quad \hat{v}_0 \leq v_0, \quad \hat{v}_0 \neq v_0.$$

If we replace the right side of (1.3) with $\lambda_1|u|^{p-2}u + \partial F(x, u)$, we can obtain the following multiplicity theorem:

Theorem 1.2. If hypotheses (H1)–(H3), (H7) and (H8) hold, then problem (1.3) has at least two nontrivial solutions.

The paper is organized as follows. In Section 2, we recall essential preliminary results and establish the functional framework necessary for our analysis. Section 3 is devoted to proving the existence of four distinct nontrivial constant sign solutions (two positive and two negative) to problem (1.3). Finally, in Section 4, we establish our second multiplicity result, demonstrating that problem (1.3) admits at least two additional nontrivial solutions.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space, and its topological dual as $(X^*, \|\cdot\|_*)$. The constants C and C_i (where $i = 1, 2, \dots$) represent estimated values that may vary from line to line. The symbol ' \rightarrow ' is employed to denote strong convergence in the space X , in contrast to ' \rightharpoonup ', which signifies weak convergence. The norm in the space $L^p(\Omega)$ is represented by $\|\cdot\|_p$. Last, $|\cdot|_m$ is used to denote the Lebesgue measure. Set

$$p_s^* = \begin{cases} \frac{Np}{N-sp}, & \text{if } N > sp, \\ \infty, & \text{if } N \leq sp. \end{cases}$$

The Gagliardo seminorm is defined for all measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}.$$

Subsequently, we introduce the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable and } [u]_{s,p} < \infty\},$$

equipped with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + [u]_{s,p}^p \right)^{1/p}.$$

For further properties of $W^{s,p}(\mathbb{R}^N)$, we recommend [10]. Denote $Q = \mathbb{R}^{2N} \setminus O$, where

$$O = C(\Omega) \times C(\Omega) \subset \mathbb{R}^{2N},$$

and $C(\Omega) = \mathbb{R}^N \setminus \Omega$. W is a linear space of Lebesgue measurable functions from \mathbb{R}^N to \mathbb{R} such that the restriction on Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\int_Q |u(x) - u(y)|^p \frac{1}{|x - y|^{N+ps}} dx dy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_p + \left(\int_Q |u(x) - u(y)|^p \frac{1}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}.$$

It is easy to see that $\|\cdot\|_W$ is a norm on W . In the following, we define the closed linear subspace

$$X(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

which can be endowed with the norm $[\cdot]_{s,p}$ (see [10, Theorem 7.1]). From [10, Theorems 6.5, 7.1], $(X(\Omega), \|\cdot\|)$ is a uniformly convex Banach space, and the embedding mapping $X(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, p_s^*]$ and compact for $q \in [1, p_s^*)$. Next, we introduce the fractional p -Laplacian operator, a nonlinear operator $A : X(\Omega) \rightarrow X(\Omega)^*$, defined for all $u, v \in X(\Omega)$ as follows:

$$\langle A(u), v \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} dx dy.$$

For any $v \in X(\Omega)$, we say $u \in X(\Omega)$ is weak solution of problem (1.3) if $u \in X(\Omega)$ satisfies the following equation

$$\langle A(u), v \rangle = \int_{\Omega} \zeta(x)v dx,$$

where $\zeta(x) \in \partial F(x, u)$. Now, let's consider the following nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (2.1)$$

depending on parameter $\lambda \in \mathbb{R}$. If problem (2.1) has a non-zero weak solution $u \in X(\Omega)$, then λ is an eigenvalue, and u is an eigenfunction corresponding to λ . We define the first eigenvalue, denoted as λ_1 , of the problem as the infimum of the ratio between the seminorm $[u]_{s,p}^p$ and the L^p -norm $\|u\|_p^p$ over all nonzero functions u in the space $X(\Omega)$. Specifically, we have

$$\lambda_1 = \inf_{u \in X(\Omega) \setminus \{0\}} \frac{[u]_{s,p}^p}{\|u\|_p^p}. \quad (2.2)$$

λ_1 is the first eigenvalue of problem (2.1), and the corresponding function u_1 is called the λ_1 -eigenfunction. According to Proposition 2.3 in [14], the λ_1 -eigenfunction u_1 maintains a consistent sign. Consequently, without loss of generality, we can assume that $u_1(x) \geq 0$ for every point x in the domain $\bar{\Omega}$. This assumption simplifies the analysis and interpretation of the eigenvalue problem.

Definition 2.1 ([11]) (i) I satisfies the nonsmooth $(PS)_c$ condition if every sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \rightarrow c \text{ and } m(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

has a strongly convergent subsequence, where $m(u_n) = \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*}$.

(ii) I satisfies the nonsmooth C-condition if every sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \rightarrow c \text{ and } (1 + \|u_n\|)m(u_n) \rightarrow 0,$$

has a strongly convergent subsequence, where $m(u_n) = \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*}$.

Definition 2.2 Suppose that $I : X \rightarrow \mathbb{R}$ is a locally Lipschitz function and $X = X_1 \oplus X_2$. We say that I has a local linking at 0, if for some $r > 0$, we have

$$\begin{cases} I(u) \geq 0, & u \in X_1, \|u\|_X \leq r, \\ I(u) \leq 0, & u \in X_2, \|u\|_X \leq r. \end{cases}$$

The next result is a nonsmooth version of a result due to Brézis-Nirenberg [6].

Theorem 2.1. *If X is a reflexive Banach space $X = Y \oplus V$ with $\dim Y < +\infty$, $I : X \rightarrow \mathbb{R}$ is a locally Lipschitz function which is bounded below, satisfies the nonsmooth C-condition, $I(0) = 0$, $\inf_X I < 0$ and there exists $r > 0$ such that*

$$\begin{cases} I(u) \geq 0, & u \in Y, \|u\|_X \leq r, \\ I(u) \leq 0, & u \in V, \|u\|_X \leq r. \end{cases} \quad (\text{local linking condition}),$$

then I has at least two nontrivial critical points.

3. The first multiplicity theorem

To give the proof of our results, we first present the following important lemma.

- Lemma 3.1.** (i) $A : X \rightarrow X^*$ is a continuous, bounded and strictly monotone operator;
(ii) A is a mapping of type (S_+) , i.e., if $u_n \rightarrow u$ in X and $\limsup_{n \rightarrow +\infty} \langle A(u_n), u_n - u \rangle \leq 0$ implies $u_n \rightarrow u$.

Proof. It is obvious to see that A is continuous and bounded. Note that

$$\begin{aligned}\langle A(u), u \rangle &= \int_Q |u(x) - u(y)|^p \frac{1}{|x - y|^{N+ps}} dx dy, \\ \langle A(u), v \rangle &= \int_Q |(u(x) - u(y))|^{p-2} (u(x) - u(y))(v(x) - v(y)) \frac{1}{|x - y|^{N+ps}} dx dy, \\ \langle A(v), v \rangle &= \int_Q |v(x) - v(y)|^p \frac{1}{|x - y|^{N+ps}} dx dy.\end{aligned}$$

Using Cauchy's inequality, we have

$$uv \leq |uv| \leq \frac{|u|^2 + |v|^2}{2}. \quad (3.1)$$

Then,

$$\begin{aligned}& \langle A(u) - A(v), u - v \rangle \\ &= \langle A(u), u \rangle - \langle A(u), v \rangle - \langle A(v), u \rangle + \langle A(v), v \rangle \\ &= \int_Q [|u(x) - u(y)|^p - |u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))] \frac{1}{|x - y|^{N+ps}} dx dy \\ &\quad - \int_Q [|v(x) - v(y)|^{p-2} (v(x) - v(y))(u(x) - u(y)) - |v(x) - v(y)|^p] \frac{1}{|x - y|^{N+ps}} dx dy \quad (3.2) \\ &\geq \int_Q [|u(x) - u(y)|^p - \frac{1}{2}|u(x) - u(y)|^{p-2} (|u(x) - u(y)|^2 + |v(x) - v(y)|^2)] \frac{1}{|x - y|^{N+ps}} dx dy \\ &\quad - \int_Q \left[\frac{|v(x) - v(y)|^{p-2}}{2} (|v(x) - v(y)|^2 + |u(x) - u(y)|^2) - |v(x) - v(y)|^p \right] \frac{1}{|x - y|^{N+ps}} dx dy \\ &= \int_Q \frac{1}{2} [|u(x) - u(y)|^{p-2} (|u(x) - u(y)|^2 - |v(x) - v(y)|^2)] \frac{1}{|x - y|^{N+ps}} dx dy \\ &\quad - \int_Q \frac{1}{2} |v(x) - v(y)|^{p-2} (|u(x) - u(y)|^2 - |v(x) - v(y)|^2) \frac{1}{|x - y|^{N+ps}} dx dy \\ &= \int_Q \frac{1}{2} (|u(x) - u(y)|^{p-2} - |v(x) - v(y)|^{p-2}) (|u(x) - u(y)|^2 - |v(x) - v(y)|^2) \frac{1}{|x - y|^{N+ps}} dx dy,\end{aligned}$$

i.e., A is monotone. In fact, A is strictly monotone. Indeed, if $\langle A(u) - A(v), u - v \rangle = 0$, then we have

$$\int_Q (|u(x) - u(y)|^{p-2} - |v(x) - v(y)|^{p-2}) (|u(x) - u(y)|^2 - |v(x) - v(y)|^2) \frac{1}{|x - y|^{N+ps}} dx dy = 0,$$

which deduces that $u = v$. Therefore, $\langle A(u) - A(v), u - v \rangle > 0$, which means that A is a strictly monotone operator in X .

(ii) From (i), if $u_n \rightarrow u$ in X and $\overline{\lim}_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $\lim_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle > 0$. By virtue of (3.2), one has $u_n \rightarrow u$ in Q . Thus, we obtain a subsequence, still denoted by itself, satisfying $u_n \rightarrow u$ a.e. $x \in Q$. From Fatou's lemma, we have

$$\underline{\lim}_{n \rightarrow +\infty} \int_Q |u_n(x) - u_n(y)|^p \frac{1}{|x - y|^{N+ps}} dx dy \geq \int_Q |u(x) - u(y)|^p \frac{1}{|x - y|^{N+ps}} dx dy. \quad (3.3)$$

By $u_n \rightarrow u$, we derive

$$\lim_{n \rightarrow +\infty} \langle A(u_n), u_n - u \rangle = \lim_{n \rightarrow +\infty} \langle A(u_n) - A(u), u_n - u \rangle = 0. \quad (3.4)$$

On the other hand, we have

$$\begin{aligned} & \langle A(u_n), u_n - u \rangle \\ &= \int_Q |u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) ((u_n(x) - u_n(y)) - (u(x) - u(y))) \frac{1}{|x - y|^{N+ps}} dx dy \\ &\geq \int_Q (|u_n(x) - u_n(y)|^p - |u_n(x) - u_n(y)|^{p-1} |u(x) - u(y)|) \frac{1}{|x - y|^{N+ps}} dx dy \\ &\geq \int_Q (|u_n(x) - u_n(y)|^p - \frac{p-1}{p} |u_n(x) - u_n(y)|^p - \frac{1}{p} |u(x) - u(y)|^p) \frac{1}{|x - y|^{N+ps}} dx dy \\ &= \frac{1}{p} \int_Q (|u_n(x) - u_n(y)|^p - |u(x) - u(y)|^p) \frac{1}{|x - y|^{N+ps}} dx dy. \end{aligned} \quad (3.5)$$

According to (3.3)–(3.5), we have

$$\lim_{n \rightarrow +\infty} \int_Q \frac{1}{|x - y|^{N+ps}} |u_n(x) - u_n(y)|^p dx dy = \int_Q \frac{1}{|x - y|^{N+ps}} |u(x) - u(y)|^p dx dy.$$

Therefore, $u_n \rightarrow u$ in X , i.e., A is of type (S_+) . \square

To prove the existence of solutions for problem (1.3), we give the following truncations of the nonsmooth potential $F(x, u)$. Set

$$\tilde{F}_+(x, u) = \begin{cases} 0, & \text{if } u \leq 0, \\ F(x, u), & \text{if } u \in [0, \alpha_+], \\ F(x, \alpha_+), & \text{if } \alpha_+ \leq u, \end{cases}$$

and

$$\tilde{F}_-(x, u) = \begin{cases} 0, & \text{if } u \geq 0, \\ F(x, u), & \text{if } u \in [\alpha_-, 0], \\ F(x, \alpha_-), & \text{if } u \leq \alpha_-. \end{cases}$$

It is evident that for every real number u , the mappings $u \mapsto \tilde{F}_\pm(x, u)$ are measurable. Moreover, for almost every point x in the domain Ω , these mappings exhibit local Lipschitz continuity in u .

By applying the chain rule for nonsmooth functions, as detailed in reference [8, p.42], we derive the following subdifferential conditions

$$\partial \tilde{F}_+(x, u) = \begin{cases} \{0\} & \text{if } u < 0, \\ \{\lambda \partial F(x, 0) : \lambda \in [0, 1]\} = \{0\} & \text{if } u = 0, \\ \partial F(x, u) & \text{if } u \in (0, \alpha_+), \\ \subset \partial F(x, \alpha_+) & \text{if } u = \alpha_+, \\ \{0\} & \text{if } u > \alpha_+. \end{cases} \quad (3.6)$$

Similarly,

$$\partial \tilde{F}_-(x, u) = \begin{cases} \{0\} & \text{if } u > 0, \\ \{\lambda \partial F(x, 0) : \lambda \in [0, 1]\} = \{0\} & \text{if } u = 0, \\ \partial F(x, u) & \text{if } u \in (\alpha_-, 0), \\ \subset \partial F(x, \alpha_-) & \text{if } u = \alpha_-, \\ \{0\} & \text{if } u < \alpha_-. \end{cases} \quad (3.7)$$

These equations delineate the subdifferential behavior of $\tilde{F}_\pm(x, u)$ under various conditions of u , ensuring that the functions are well-defined and their derivatives are appropriately constrained.

We define the functionals $I, \tilde{I}_\pm : X(\Omega) \rightarrow \mathbb{R}$ by

$$\tilde{I}_\pm(u) = \frac{1}{p} [u]_{s,p}^p - \int_{\Omega} \tilde{F}_\pm(x, u) dx, \quad \forall x \in X(\Omega), \quad (3.8)$$

and

$$I(u) = \frac{1}{p} [u]_{s,p}^p - \int_{\Omega} F(x, u) dx, \quad \forall x \in X(\Omega). \quad (3.9)$$

Since \tilde{I}_\pm and I are Lipschitz continuous on bounded set, they are locally Lipschitz (see [8, p.83]). Recall that

$$\partial \tilde{I}_\pm(u) \subset A(u) - \tilde{N}_\pm(u) \text{ and } \partial I(u) \subset A(u) - N(u), \quad \forall u \in X(\Omega),$$

where

$$\tilde{N}(u) = \{\zeta^* \in L^{p'}(\Omega) : \zeta^* \in \partial \tilde{F}_\pm(x, u(x)) \text{ a.e. on } \Omega\}$$

and

$$N(u) = \{\zeta^* \in L^{p'}(\Omega) : \zeta^* \in \partial F(x, u(x)) \text{ a.e. on } \Omega\}.$$

Denote by $C^0(\bar{\Omega})$ and $C^{0,\beta}(\bar{\Omega})$ ($0 < \beta < 1$) the usual Hölder spaces, endowed with the norms

$$\|u\|_{C^0(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |u(x)|, \quad \|u\|_{C^{0,\beta}(\bar{\Omega})} = \|u\|_{C^0(\bar{\Omega})} + \sup_{x,y \in \bar{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta},$$

respectively. Besides, set $d(x) = \text{dist}(x, \partial\Omega)$ for all $x \in \bar{\Omega}$. Fix $0 < \gamma < 1$ and introduce the weighted Hölder spaces

$$C_d^0(\bar{\Omega}) = \{u \in C^0(\bar{\Omega}) : ud^{-\gamma} \in C^0(\bar{\Omega})\}, \quad C_d^{0,\beta}(\bar{\Omega}) = \{u \in C^0(\bar{\Omega}) : ud^{-\gamma} \in C^{0,\beta}(\bar{\Omega})\},$$

respectively. As such, if $u \in C_d^0(\bar{\Omega})$, then $u = 0$ on $\partial\Omega$. In general, all functions that vanish at $\partial\Omega$ will be identified with their zero extensions to \mathbb{R}^N . From the Ascoli Theorem, the embedding $C_d^{0,\beta}(\bar{\Omega}) \hookrightarrow C_d^0(\bar{\Omega})$ is compact for all $0 < \beta < 1$. $C_d^0(\bar{\Omega})$ is an ordered Banach space with the order cone

$$C_+ = \{u \in C_d^0(\bar{\Omega}) : u(x) \geq 0 \text{ for all } x \in \bar{\Omega}\}.$$

By Lemma 5.1 in [14], we know that

$$\text{int}C_+ = \{u \in C_d^0(\bar{\Omega}) : \frac{u(x)}{d^\nu(x)} > 0 \text{ for all } x \in \bar{\Omega}\}.$$

Lemma 3.2. *If hypotheses (H1)–(H6) hold, then problem (1.3) has at least two solutions, $u_0 \in \text{int}C_+$ and $v_0 \in -\text{int}C_+$, u_0 is a minimizer of \tilde{I}_+ , v_0 is a minimizer of \tilde{I}_- , and both are local minimizers of I .*

Proof. We first prove that $u_0 \in \text{int}C_+$, which is a minimizer of \tilde{I}_+ and a local minimizer of I . Recall that

$$|\tilde{F}_\pm(x, u)| \leq \tilde{a}(x)$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$ and $\tilde{a} \in L^\infty(\Omega)_+$, from which it follows that \tilde{I}_+ is coercive. From $X(\Omega) \hookrightarrow L^p(\Omega)$, we have that \tilde{I}_+ is weakly lower semicontinuous. Hence, from the Weierstrass theorem, one can find $u_0 \in X(\Omega)$ such that

$$\tilde{I}_+(u_0) = \tilde{m}_+ = \inf\{\tilde{I}_+(u) : u \in X(\Omega)\}. \quad (3.10)$$

Next, we prove that $\tilde{m}_+ < 0$. Let $u_1 \in \text{int}C_+$ be the L^p -normalized principal eigenfunction. Take $\epsilon_0 > 0$ small enough such that

$$0 \leq \epsilon_0 u_1(x) \leq \delta < \min\{\sigma, \alpha_+\} \quad \forall x \in \bar{\Omega},$$

where σ and α_+ can be found in (H5) and (H6), respectively. Thus, from (H5), it follows that

$$\frac{\lambda_1}{p} \epsilon^p u_1(x)^p \leq F(x, \epsilon u_1(x)) = \tilde{F}_+(x, \epsilon u_1(x)) \quad (3.11)$$

for a.e. $x \in \Omega$, all $\epsilon \in (0, \epsilon_0]$. Therefore, by (2.2) and (3.11), one has

$$\begin{aligned} \tilde{I}_+(\epsilon u_1) &= \frac{\epsilon^p}{p} [u_1]_{s,p}^p - \int_{\Omega} \tilde{F}_+(x, \epsilon u_1(x)) dx \\ &\leq \frac{\epsilon^p}{p} [u_1]_{s,p}^p - \frac{\epsilon^p \lambda_1}{p} \|u_1\|_p^p \\ &= \left(\frac{\epsilon^p \lambda_1}{p} - \frac{\epsilon^p \lambda_1}{p} \right) \|u_1\|_p^p \\ &= 0, \end{aligned}$$

from which we infer that $\tilde{m}_+ \leq 0$. If $\tilde{m}_+ = 0$, then $\tilde{I}_+(\epsilon u_1) = 0$ for all $\epsilon \in (0, \epsilon_0]$. Hence,

$$0 \in \partial \tilde{I}_+(\epsilon u_1),$$

i.e.,

$$A(\epsilon u_1) = \zeta_\epsilon^*,$$

where $\zeta_\epsilon^* \in \tilde{N}_+(\epsilon u_1)$, which deduces that

$$(-\Delta)_p^s(\epsilon u_1)(x) = \zeta_\epsilon^*(x) \quad (3.12)$$

for a.e. $x \in \Omega$. Recalling that $0 < \epsilon u_1(x) \leq \delta < \alpha_+$ for a.e. $x \in \Omega$, by (3.6), we obtain

$$\partial \tilde{F}_+(x, \epsilon u_1(x)) = \partial F(x, \epsilon u_1(x))$$

for a.e. $x \in \Omega$. This, with (3.12), means that ϵu_1 is a solution of (1.3) and a minimizer of \tilde{I}_+ for any $\epsilon \in (0, \epsilon_0]$. Furthermore, since $\epsilon u_1 \in \text{int}C_+$ and $\epsilon u_1(x) \leq \delta < \alpha_+$ for all $x \in \bar{\Omega}$, there exists $\rho > 0$ small, such that

$$\tilde{I}_+ \big|_{\bar{B}_\rho^{C_d^0}}(\epsilon u_1(x)) = I_+ \big|_{\bar{B}_\rho^{C_d^0}}(\epsilon u_1(x)),$$

from which it follows that $\epsilon u_1(x)$ is a local $C_d^0(\bar{\Omega})$ -minimizer of I . By Proposition 5.4 in [14], we have that $\epsilon u_1(x)$ is a local $X(\bar{\Omega})$ -minimizer of I . Hence, if $\tilde{m}_+ = 0$, we derive a whole continuum of functions (the functions $\{\epsilon u_1\}_\epsilon \in (0, \epsilon_0]$) satisfying the properties claimed by the proposition. Thus, without loss of generality, we can assume that

$$\tilde{m}_+ < 0.$$

By (3.10), we have

$$\tilde{I}_+(u_0) < 0 = \tilde{I}_+(0),$$

from which we derive $u_0 \neq 0$. Thus,

$$0 \in \partial I_+(u_0),$$

i.e.,

$$A(u_0) = \zeta_0^*, \quad (3.13)$$

where $\zeta_0^* \in \tilde{N}_+(u_0)$. Acting on (3.13) with $-u_0^- \in X(\Omega)$ and by (3.6), one has $[u_0^-]_{s,p} = 0$, which infers that $u_0^- = 0$, and so $u_0 \geq 0$, $u_0 \neq 0$. From (3.13) it follows that

$$\begin{cases} (-\Delta)_p^s u_0(x) = \zeta_0^*(x) \in \partial \tilde{F}_+(x, u_0(x)) & \text{for a.e. } x \in \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.14)$$

From hypotheses (H3) and Theorem 1.1 in [13], we derive that $u_0 \in C_d^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, s]$. Recalling that

$$\zeta_0^*(x) = 0 \text{ for a.e. } x \in \{u > \alpha_+\},$$

if we act on (3.13) with $(u_0 - \alpha_+)^+ \in X(\Omega) \cap C(\bar{\Omega})$, we have

$$\int_{\{u_0 > \alpha_+\}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy = 0,$$

from which it follows that

$$|\{u_0 > \alpha_+\}|_m = 0.$$

Thus,

$$0 \leq u_0 \leq \alpha_+ \text{ for a.e. } x \in \bar{\Omega}.$$

From (3.14) and hypothesis (H6), we derive that

$$(-\Delta)_p^s u_0(x) \geq 0 \text{ for a.e. } x \in \Omega.$$

Invoking the strong maximum principle of [20, Lemma 2.3], we infer that $u_0 \in \text{int}C_+$. Furthermore, employing hypothesis (H6), we derive

$$(-\Delta)_p^s(\alpha_+ - u_0)(x) = (-\Delta)_p^s(-u_0)(x) = -\zeta_0^*(x) \geq -l(\alpha_+ - u_0)^{p-1}(x)$$

for a.e. $x \in \Omega$. Hence, using the minimum principle [9, Theorem 1.2], we have

$$u_0(x) < \alpha_+ \quad \forall x \in \Omega.$$

Using the definition of $\tilde{F}_+(x, u)$, by choosing $\rho > 0$ sufficiently small, we can establish that

$$\tilde{I}_+ \big|_{\tilde{B}_\rho^{C^0}}(u_0) = I_+ \big|_{\tilde{B}_\rho^{C^0}}(u_0(x)),$$

which implies that

$$u_0 \text{ is a local } C_d^0(\Omega) \text{ - minimizer of } I,$$

and hence

$$u_0 \text{ is a local } X(\Omega) \text{ - minimizer of } I,$$

(see [14, Proposition 5.4]). Similarly, by examining \tilde{I}_- , we deduce that $v_0 \in \text{int}C_+$ is a minimum point of \tilde{I}_- and a minimum point of I . Moreover, v_0 constitutes a solution to problem (1.3). \square

Now we begin to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.1, we already have two solutions, $u_0 \in \text{int}C_+$ and $v_0 \in -\text{int}C_+$. First, we prove that there exists a solution of \hat{u}_0 of problem (1.3) such that

$$u_0 \leq \hat{u}_0, \quad u_0 \neq \hat{u}_0.$$

With this aim in mind, we introduce the functional $\Upsilon_+ : X(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Upsilon_+(w) = \frac{1}{p}[u_0 + w]_{s,p}^p - \int_{\Omega} F(x, u_0 + w^+) dx + \int_{\Omega} \zeta_0^* w^- dx + \Lambda, \quad (3.15)$$

where $\zeta_0^* \in L^{p'}(\Omega)$, $\zeta_0^* \in \partial F(x, u_0(x))$ for a.e. $x \in \Omega$ satisfies

$$(-\Delta)_p^s u_0(x) = \zeta_0^*(x) \text{ for a.e. } x \in \Omega,$$

(see Lemma 3.1) and $\Lambda = \int_{\Omega} F(x, u_0) dx$. It is evident that Υ_+ is locally Lipschitz on $X(\Omega)$. Noting that

$$[u_0 + w]_{s,p}^p = [u_0 + w^+]_{s,p}^p + [u_0 - w^-]_{s,p}^p - [u_0]_{s,p}^p, \quad (3.16)$$

we can deduce that

$$\Upsilon_+(w) = I(u_0 + w^+) - I(u_0) + \frac{1}{p}[u_0 - w^-]_{s,p}^p + \int_{\Omega} \zeta_0^* w^- dx. \quad (3.17)$$

Since $\zeta_0^*(x) \in \partial F(x, u_0(x))$ for a.e. $x \in \Omega$, and $u_0(x) \in [0, \alpha_+]$ for all $x \in \Omega$, as indicated by (H6), we can conclude that

$$\zeta_0^*(x) \geq 0 \text{ for a.e. } x \in \Omega.$$

Furthermore, since $u_0 \in \text{int}C_+$ is a local minimizer of I , we can prove that the origin is a local minimizer of Υ_+ according to Eq (3.17) (cf [11, p.675]).

Next, we will utilize the Mountain Pass Theorem to prove the existence of a solution \hat{u}_0 to problem (1.3). This will be done in three steps.

Claim 1. There exists $r > 0$ small such that

$$\inf\{\Upsilon_+(w) : [w]_{s,p} = r\} =: \theta_+ \geq \frac{1}{p}[u_0]_{s,p}^p = \Upsilon_+(0).$$

Assume that $u_0 \in \text{int}C_+$ is an isolated local minimizer of I . Otherwise, we derive a whole sequence of distinct solutions of (1.3). So for $\rho > 0$ small enough, one has

$$I(u_0) < I(w) \text{ and } 0 \notin \partial I(w), \quad \forall w \in \bar{B}_\rho(u_0) \setminus \{u_0\}. \quad (3.18)$$

we show that for any $r \in (0, \rho)$,

$$m_r = \inf\{I(w) : w \in \bar{B}_\rho(u_0) \setminus B_r(u_0)\} > I(u_0). \quad (3.19)$$

Suppose, by contradiction, that there exists $r \in (0, \rho)$ and $\{w_n\}_{n \geq 1} \subset \bar{B}_\rho(u_0) \setminus B_r(u_0)$ such that

$$I(w_n) \searrow I(u_0). \quad (3.20)$$

Then we can conclude that the sequence $\{w_n\} \subset X(\Omega)$ is bounded. Consequently, we can assume that

$$w_n \rightharpoonup w \text{ in } X(\Omega) \text{ and } w_n \rightarrow w \text{ in } L^p(\Omega).$$

Because I is weakly lower semicontinuous, it follows from (3.20) that

$$I(w) \leq \liminf_{n \rightarrow \infty} I(w_n) = I(u_0).$$

Using (3.18), we can further deduce that $w = u_0$. By the nonsmooth mean value theorem (see [8, p.41]), one has

$$I(w_n) - I\left(\frac{w_n + u_0}{2}\right) = \left\langle w_n^*, \frac{w_n - u_0}{2} \right\rangle, \quad (3.21)$$

where $w_n^* \in \partial I(t_n w_n + (1 - t_n)\frac{w_n + u_0}{2})$, $t_n \in (0, 1)$, $n \geq 1$. Note that

$$w_n^* = A\left(t_n w_n + (1 - t_n)\frac{w_n + u_0}{2}\right) - \zeta_n^*, \quad (3.22)$$

with $\zeta_n^* \in \partial F(t_n w_n + (1 - t_n)\frac{w_n + u_0}{2})$ for a.e. $x \in \Omega$. Combining (3.21) and (3.22), we derive

$$I(w_n) - I\left(\frac{w_n + u_0}{2}\right) = \left\langle A\left(t_n w_n + (1 - t_n)\frac{w_n + u_0}{2}\right), \frac{w_n - u_0}{2} \right\rangle - \int_{\Omega} \zeta_n^* \frac{w_n - u_0}{2} dx. \quad (3.23)$$

Since

$$\int_{\Omega} \zeta_n^* \frac{w_n - u_0}{2} dx \rightarrow 0, \quad (3.24)$$

and

$$\frac{w_n + u_0}{2} \rightharpoonup u_0 \text{ in } X(\Omega),$$

we derive

$$I(u_0) \leq \liminf_{n \rightarrow +\infty} I\left(\frac{w_n + u_0}{2}\right). \quad (3.25)$$

Then, passing $n \rightarrow \infty$, from (3.20), (3.24) and (3.25), it follows that

$$\limsup_{n \rightarrow +\infty} \left\langle A\left(t_n w_n + (1 - t_n) \frac{w_n + u_0}{2}\right), \frac{w_n - u_0}{2} \right\rangle \leq 0. \quad (3.26)$$

Since

$$t_n w_n + (1 - t_n) \frac{w_n + u_0}{2} - u_0 = (1 + t_n) \frac{w_n - u_0}{2}$$

and

$$t_n w_n + (1 - t_n) \frac{w_n + u_0}{2} \rightharpoonup u_0 \text{ in } X(\Omega),$$

from (3.26) and Lemma 3.1, we infer that

$$t_n w_n + (1 - t_n) \frac{w_n + u_0}{2} \rightarrow u_0 \text{ in } X(\Omega). \quad (3.27)$$

However,

$$\left[t_n w_n + (1 - t_n) \frac{w_n + u_0}{2} - u_0 \right]_{s,p} = (1 + t_n) \left[\frac{w_n - u_0}{2} \right]_{s,p} \geq (1 + t_n) \frac{r}{2} > \frac{r}{2} > 0. \quad (3.28)$$

This contradicts (3.27), which proves (3.19).

Let us fix $r \in (0, \rho)$ and let $[w]_{s,p} = r$. Then, we have $[w^+]_{s,p} \geq \frac{r}{2}$ or $[w^-]_{s,p} \geq \frac{r}{2}$. First, let us assume that $\|w^+\| \geq \frac{r}{2}$. From (3.27) and (3.19), it follows that

$$\Upsilon_+(w) \geq m_{\frac{r}{2}} > 0.$$

Now, let us assume that $[w^-]_{s,p} \geq \frac{r}{2}$. From (3.17), $A(u_0) = \zeta_0^*$ and noting that $u \mapsto \frac{1}{p}[u_0 - w^-]_{s,p}^p$ is convex, we have

$$\begin{aligned} \Upsilon_+(w) &\geq m_{\frac{r}{2}} + \frac{1}{p}[u_0 - w^-]_{s,p}^p + \langle A(u_0), w^- \rangle \\ &\geq m_{\frac{r}{2}} + \frac{1}{p}[u_0]_{s,p}^p \\ &\geq \frac{1}{p}[u_0]_{s,p}^p, \end{aligned}$$

which proves Claim 1.

Claim 2. Υ_+ satisfies the nonsmooth C-condition.

Assume that $\{w_n\}_{n \geq 1} \subset X(\Omega)$ is a sequence, such that

$$\Upsilon_+(w_n) \rightarrow c \quad \text{and} \quad (1 + \|w_n\|)m(w_n) \rightarrow 0. \quad (3.29)$$

Suppose, by contradiction, that there exists sequence $\{w_n\}_{n \geq 1} \subset X(\Omega)$ is unbounded. Since $\partial \Upsilon_+(w_n) \subset X^*(\Omega)$ is weakly compact, and the norm functional in a Banach space is weakly lower semicontinuous,

according to the Weierstrass theorem, there exists $w_n^* \in \partial\Upsilon_+(w_n)$ such that $m(w_n) = \|w_n^*\|_{X^*}$ for $n \geq 1$. Let $\hat{F}_+ : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\hat{F}_+(x, \zeta) = F(x, u_0(x) + \zeta^+).$$

Thus, for all $\zeta \in \mathbb{R}$, the function $x \mapsto \hat{F}_+(x, \zeta)$ is measurable, and for almost every $x \in \Omega$, the function $\zeta \mapsto \hat{F}_+(x, \zeta)$ is locally Lipschitz. Additionally, let $j_-(\zeta) = \zeta^-$. Then we have

$$w_n^* = A(u_0 + w_n) - \hat{\zeta}_n^* + \zeta_0^* \eta_n, \quad (3.30)$$

where $\hat{\zeta}_n^* \in \partial\hat{F}_+(x, w_n(x))$ and $\eta_n \in \partial j_-(w_n(x))$ for almost every $x \in \Omega$ and all $n \geq 1$. Using the nonsmooth chain rule, we have

$$\partial\hat{F}_+(x, \zeta) = \begin{cases} \{0\} & \text{if } \zeta < 0, \\ \{\tau \partial F(x, u_0) : \tau \in [0, 1]\} = \{0\} & \text{if } \zeta = 0, \\ \partial F(x, u_0 + \zeta) & \text{if } \zeta > 0, \end{cases} \quad (3.31)$$

and

$$\partial j_-(\zeta) = \begin{cases} -1 & \text{if } \zeta < 0, \\ [-1, 0] & \text{if } \zeta = 0, \\ 0 & \text{if } \zeta > 0. \end{cases} \quad (3.32)$$

From the given choice of the sequence $\{w_n\} \subset X(\Omega)$, we have

$$|\langle w_n^*, v \rangle| \leq \epsilon_n, \quad \forall v \in X(\Omega),$$

where $\epsilon_n \searrow 0$. Let us take $v = -w_n^- \in X(\Omega)$, and note that

$$A(u_0 + w_n) = A(u_0 + w_n^+) + A(u_0 - w_n^-) - A(u_0). \quad (3.33)$$

Then

$$\begin{aligned} & \langle A(u_0 + w_n^+), -w_n^- \rangle + \langle A(u_0 - w_n^-), -w_n^- \rangle - \langle A(u_0), -w_n^- \rangle \\ & \quad + \int_{\Omega} \hat{\zeta}_n^* w_n^- dx - \int_{\Omega} \zeta_0^* \eta_n w_n^- dx \leq \epsilon_n. \end{aligned} \quad (3.34)$$

Since

$$\begin{aligned} & \langle A(u_0 + w_n^+), -w_n^- \rangle + \langle A(u_0 - w_n^-), -w_n^- \rangle - \langle A(u_0), -w_n^- \rangle \\ & = \langle A(u_0), -w_n^- \rangle + \langle A(u_0 - w_n^-), -w_n^- \rangle - \langle A(u_0), -w_n^- \rangle \\ & = \langle A(u_0 - w_n^-), -w_n^- \rangle \\ & \geq [u_0 - w_n^-]_{s,p}^p - [u_0 - w_n^-]_{s,p}^{p-1} [u_0]_{s,p}, \end{aligned} \quad (3.35)$$

also from (3.31), (3.32) and (H3), we derive

$$\left| \int_{\Omega} \hat{\zeta}_n^* w_n^- dx \right| \leq C [w_n^-]_{s,p}, \quad \forall n \geq 1, \quad (3.36)$$

and

$$\left| \int_{\Omega} \zeta_0^* \eta_n w_n^- dx \right| \leq C [w_n^-]_{s,p}, \quad \forall n \geq 1. \quad (3.37)$$

Substituting (3.35)–(3.37) into (3.34), we obtain

$$[u_0 - w_n^-]_{s,p}^p \leq C[u_0 - w_n^-]_{s,p}^{p-1} + C[w_n^-]_{s,p} + C,$$

from which it follows that the sequence $\{w_n^-\} \subset X(\Omega)$ is bounded, and therefore the sequence $\{w_n^+\} \subset X(\Omega)$ is unbounded. By passing to a subsequence if necessary, we assume that $[u_n^+]_{s,p} \rightarrow +\infty$, so,

$$[u_0 + w_n^+]_{s,p} \rightarrow +\infty.$$

Set

$$z_n = \frac{u_0 + w_n^+}{[u_0 + w_n^+]_{s,p}} \quad \forall n \geq 1. \quad (3.38)$$

Then $u_n \geq 0$ and $[z_n^+]_{s,p} = 1$ for all $n \geq 1$. Passing to a subsequence if necessary, one has

$$u_n \rightarrow u \text{ in } X(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^p(\Omega).$$

From hypothesis (H4), for a given $\epsilon > 0$, there exists $M_1 = M_1(\epsilon) > 0$ such that

$$\frac{1}{p}(\lambda_1 - \epsilon)|\zeta|^p \leq F(x, \zeta) \leq \frac{1}{p}(\lambda_1 + \epsilon)|\zeta|^p \quad (3.39)$$

for a.e. $x \in \Omega$, all $|\zeta| \geq M_1$. Noting that

$$\Upsilon_+(w_n) \rightarrow c,$$

we can find $\hat{M} > 0$ such that

$$|\Upsilon_+(w_n)| \leq \hat{M} \quad \forall n \geq 1.$$

From (3.16), one has

$$\begin{aligned} \frac{\Upsilon_+(w_n)}{[u_0 + w_n^+]_{s,p}^p} &= \left| \frac{1}{p}[z_n]_{s,p}^p + \frac{1}{p} \frac{[u_0 - w_n^-]_{s,p}^p}{[u_0 + w_n^+]_{s,p}^p} - \frac{1}{p} \frac{[u_0]_{s,p}^p}{[u_0 + w_n^+]_{s,p}^p} \right. \\ &\quad \left. - \int_{\Omega} \frac{F(x, u_0 + w_n^+)}{[u_0 + w_n^+]_{s,p}^p} dx + \int_{\Omega} \frac{\zeta_0^*}{[u_0 + w_n^+]_{s,p}^p} w_n^- dx + \frac{\Lambda}{[u_0 + w_n^+]_{s,p}^p} \right| \\ &\leq \frac{\hat{M}}{[u_0 + w_n^+]_{s,p}^p} \quad \forall n \geq 1. \end{aligned} \quad (3.40)$$

Since the sequence $\{w_n^-\}_{n \geq 1} \subset X(\Omega)$ is bounded, one has

$$\frac{[u_0 - w_n^-]_{s,p}^p}{[u_0 + w_n^+]_{s,p}^p} \rightarrow 0 \quad \text{and} \quad \int_{\Omega} \frac{\zeta_0^*}{[u_0 + w_n^+]_{s,p}^p} w_n^- dx \rightarrow 0.$$

Then, (3.40) can be transformed to

$$\left| \frac{1}{p}[z_n]_{s,p}^p - \int_{\Omega} \frac{F(x, u_0 + w_n^+)}{[u_0 + w_n^+]_{s,p}^p} dx \right| \leq \epsilon'_n \quad (3.41)$$

where $\epsilon'_n \downarrow 0$. From (3.39) and (H3), it can be inferred that

$$\begin{aligned} \int_{\Omega} \frac{F(x, u_0 + w_n^+)}{[u_0 + w_n^+]_{s,p}^p} dx &= \int_{\{u_0 + w_n^+ \geq \hat{M}\}} \frac{F(x, u_0 + w_n^+)}{[u_0 + w_n^+]_{s,p}^p} dx + \int_{\{u_0 + w_n^+ < \hat{M}\}} \frac{F(x, u_0 + w_n^+)}{[u_0 + w_n^+]_{s,p}^p} dx \\ &\leq \frac{\lambda_1 + \epsilon}{p} [z_n^+]_{s,p}^p + \frac{C}{[u_0 + w_n^+]_{s,p}^p} \quad \forall n \geq 1. \end{aligned} \quad (3.42)$$

Substituting (3.42) into (3.41) yields

$$\frac{1}{p}[z_n]_{s,p}^p - \frac{\lambda_1}{p}\|z_n\|_p^p - \frac{\epsilon}{p}\|z_n\|_p^p \leq \epsilon_n'', \quad (3.43)$$

where $\epsilon_n'' \downarrow 0$. As $n \rightarrow +\infty$ in (3.43) and letting $\epsilon \searrow 0$, we have

$$[z_n]_{s,p}^p \leq \lambda_1\|z_n\|_p^p,$$

which implies that

$$z = 0 \text{ or } z = u_1$$

(note that $u \geq 0$). If $z = 0$, then $z_n \rightarrow 0$ in $X(\Omega)$, which contradicts the fact that $[z_n]_{s,p} = 1$. Therefore, $z = u_1$. Consequently, using (3.38), we obtain

$$(u_0 + w_n^+)(x) \rightarrow +\infty \text{ for a.e. } x \in \Omega.$$

Furthermore, based on hypothesis (H4), we know that

$$\zeta_n^*(x)(u_0 + w_n^+)(x) - pF(x, (u_0 + w_n^+)(x)) \rightarrow -\infty \text{ for a.e. } x \in \Omega.$$

Then, from the Fatou Lemma, we derive

$$\int_{\Omega} [\zeta_n^*(x)(u_0 + w_n^+) - pF(x, (u_0 + w_n^+))] dx \rightarrow -\infty. \quad (3.44)$$

Furthermore, by (3.16), for all $n \geq 1$, we obtain

$$\begin{aligned} p\Upsilon_+(w_n) &= [u_0 + w_n^+]_{s,p}^p - \int_{\Omega} pF(x, u_0 + w_n^+) dx + \int_{\Omega} p\zeta_0^* w_n^- dx + p\Lambda \\ &= [u_0 + w_n^+]_{s,p}^p + [u_0 + w_n^-]_{s,p}^p - [u_0]_{s,p}^p - \int_{\Omega} pF(x, u_0 + w_n^+) dx + \int_{\Omega} p\zeta_0^* w_n^- dx + p\Lambda \\ &\leq [u_0 + w_n^+]_{s,p}^p - \int_{\Omega} pF(x, u_0 + w_n^+) dx + C. \end{aligned} \quad (3.45)$$

By (3.30) and (3.33), one has

$$\begin{aligned} -\langle w_n^*, u_0 + w_n^+ \rangle &= -\langle A(u_0 + w_n^+), u_0 + w_n^- \rangle - \langle A(u_0 - w_n^-), u_0 \rangle \\ &\quad + \int_{\Omega} \hat{\zeta}_n^*(u_0 + w_n^+) dx - \int_{\Omega} \zeta_0^* \eta_n(u_0 + w_n^+) dx \\ &\leq -[u_0 + w_n^-]_{s,p}^p - \langle A(u_0 - w_n^-), u_0 \rangle + \int_{\Omega} \hat{\zeta}_n^*(u_0 + w_n^+) dx + C. \end{aligned} \quad (3.46)$$

Adding (3.45) and (3.46), we obtain

$$p\Upsilon_+(w_n) - \langle w_n^*, u_0 + w_n^+ \rangle \leq \int_{\Omega} [\hat{\zeta}_n^*(u_0 + w_n^+) - pF(x, u_0 + w_n^+)] dx + C \quad \forall n \geq 1. \quad (3.47)$$

Using (3.29), we can deduce that

$$p\Upsilon_+(w_n) - \langle w_n^*, u_0 + w_n^+ \rangle \rightarrow pc. \quad (3.48)$$

Combining (3.44), (3.46), and (3.47), we arrive at a contradiction, which indicates that the sequence $\{w_n^+\}_{n \geq 1} \subset X(\Omega)$ is bounded, so the sequence $\{w_n\}_{n \geq 1} \subset X(\Omega)$ is also bounded. Consequently, by passing to a subsequence if necessary, we may assume that

$$\begin{aligned} w_n &\rightharpoonup w \text{ in } X(\Omega), \\ w_n &\rightarrow w \text{ in } L^p(\Omega), \\ w_n(x) &\rightarrow w(x) \text{ for a.e. } x \in \Omega, \end{aligned}$$

and

$$|w_n(x)| \leq l(x) \text{ for a.e. } x \in \Omega$$

with $l(x) \in L^p(\Omega)$. Hence, we have

$$|\langle w_n^*, w_n - w \rangle| \leq \epsilon_n,$$

which implies

$$\left| \langle A(u_0 + w_n), w_n - w \rangle - \int_{\Omega} \hat{\zeta}_n^*(w_n - w) dx + \int_{\Omega} \zeta_0^* \eta_n(w_n - w) dx \right| \leq \epsilon_n. \quad (3.49)$$

Considering that

$$\int_{\Omega} \hat{\zeta}_n^*(w_n - w) dx \rightarrow 0 \text{ and } \int_{\Omega} \zeta_0^* \eta_n(w_n - w) dx \rightarrow 0,$$

we can infer from (3.49) that

$$\langle A(u_0 + w_n), w_n - w \rangle \rightarrow 0. \quad (3.50)$$

As $w_n \rightharpoonup w$ in $X(\Omega)$, combining (3.50) and Lemma 3.1, we can conclude that

$$w_n \rightarrow w \text{ in } X(\Omega),$$

which demonstrates that Υ_+ satisfies the nonsmooth C-condition.

Claim 3. There exists $w \in X(\Omega)$ with $[w]_{s,p} > r$ such that

$$\Upsilon_+(w) < \Upsilon_+(0) \leq \theta_+$$

($r > 0$ and θ_+ as in Claim 1). From hypothesis (H4), for a given $\theta > 0$, we can find $\tilde{M} = \tilde{M}(\theta)$ such that

$$\zeta^* u - pF(x, u) \leq -\theta \quad (3.51)$$

for a.e. $x \in \Omega$, all $|u| \geq \tilde{M}$ and all $\zeta^* \in \partial F(x, u)$. From hypothesis (H2), for a.e. $x \in \Omega$, it follows that $u \mapsto \frac{F(x, u)}{u^p}$ is locally Lipschitz on $[\tilde{M}, +\infty)$ and

$$\partial \left(\frac{F(x, u)}{u^p} \right) \subset \frac{u^p \partial F(x, u) - pu^{p-1} F(x, u)}{u^{2p}} = \frac{u \partial F(x, u) - pu F(x, u)}{u^{p+1}} \quad (3.52)$$

for all $u \geq \tilde{M}$. By the Rademacher Theorem in [11, p716], we have that $\frac{d}{du} \left(\frac{F(x, u)}{u^p} \right)$ exists for a.e. $u \in \mathbb{R}$ and

$$\frac{d}{du} \left(\frac{F(x, u)}{u^p} \right) \in \partial \left(\frac{F(x, u)}{u^p} \right).$$

Consequently, from (3.51) and (3.52), one has

$$\frac{d}{du} \left(\frac{F(x, u)}{u^p} \right) \leq \frac{-\theta}{u^{p+1}} \quad \text{for a.e. } x \in \Omega.$$

For $a_1 < a_2$, integrating over $[a_1, a_2] \subset [\tilde{M}, +\infty)$, we derive

$$\frac{F(x, a_2)}{a_2^p} - \frac{F(x, a_1)}{a_1^p} \leq \frac{\theta}{p} \left(\frac{1}{a_2^p} - \frac{1}{a_1^p} \right) \quad \text{for a.e. } x \in \Omega.$$

Letting $a_2 \rightarrow +\infty$, by (H4), we have

$$\frac{\lambda_1}{p} - \frac{F(x, a_1)}{a_1^p} \leq -\frac{\theta}{pa_1^p}.$$

Then

$$pF(x, a_1) - \lambda_1 a_1^p \geq \theta \quad \text{for a.e. } x \in \Omega \text{ and all } a_1 \geq \tilde{M}. \quad (3.53)$$

Since $[u_1]_{s,p}^p = \lambda_1 \|u_1\|_p^p$, by (3.53), for $t > 0$, we can derive

$$\begin{aligned} pI(tu_1) &= t^p [u_1]_{s,p}^p - \int_{\Omega} pF(x, tu_1) dx \\ &= t^p \lambda_1 \|u_1\|_p^p - \int_{\Omega} pF(x, tu_1) dx \\ &= \int_{\Omega} [t^p \lambda_1 u_1^p - pF(x, tu_1)] dx \\ &= \int_{\{tu_1 \geq \tilde{M}\}} [t^p \lambda_1 u_1^p - pF(x, tu_1)] dx + \int_{\{tu_1 < \tilde{M}\}} [t^p \lambda_1 u_1^p - pF(x, tu_1)] dx \\ &\leq -\theta |\{tu_1 \geq \tilde{M}\}|_m + C |\{tu_1 < \tilde{M}\}|_m, \end{aligned} \quad (3.54)$$

where $|\cdot|_m$ denotes the Lebesgue measure on \mathbb{R}^N . Recall that

$$|\{tu_1 < \tilde{M}\}|_m \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Therefore, from (3.54), we can conclude

$$\limsup_{t \rightarrow +\infty} I(tu_1) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty. \quad (3.55)$$

Since $u_1 \in \text{int}C_+$, one can choose a sufficiently large $t > 0$ such that $tu_1 \geq u_0$. Let $w = tu_1 - u_0$, then $w \in C_+$. According to (3.17), we have

$$\Upsilon_+(tu_1 - u_0) = I(tu_1) - I(u_0) + \frac{1}{p} [u_0]_{s,p}^p.$$

From (3.55), it follows that

$$\Upsilon_+(tu_1 - u_0) < \Upsilon_+(0) < \theta_+,$$

which proves Claim 3. Therefore, by using the nonsmooth Mountain Pass Theorem (see [11, p.40]) with Claims 1–3, we can conclude that there exists $w_0 \in X(\Omega)$, $w_0 \neq 0$, such that

$$\theta_+ \leq \Upsilon_+(w_0) \quad \text{and} \quad 0 \in \partial \Upsilon_+(w_0). \quad (3.56)$$

Then, we have

$$A(u_0 + w_0) = \hat{\zeta}_0^* - \zeta_0^* \eta_0, \quad (3.57)$$

where $\hat{\zeta}_0^* \in \partial \hat{F}_+(x, w_0(x))$, $\zeta_0^* \in L^p(\Omega)$ for a.e. $x \in \Omega$, and $\zeta_0^* \eta_0 \in L^p(\Omega)$, $\eta_0^* \in \partial j_-(w_0(x))$ for almost every $x \in \Omega$. By multiplying both sides of (3.57) with $-w_0^- \in X(\Omega)$, we obtain

$$\langle A(u_0 + w_0), -w_0^- \rangle = \int_{\Omega} \hat{\zeta}_0^*(-w_0^-) dx - \int_{\Omega} \zeta_0^* \eta_0(-w_0^-) dx.$$

From (3.31), (3.32) and the fact that $A(u_0) = \zeta_0^*$, we can derive that

$$\langle A(u_0 - w_0^-), -w_0^- \rangle = \langle A(u_0), -w_0^- \rangle.$$

Consequently, we have

$$\langle A(u_0 - w_0^-) - A(u_0), -w_0^- \rangle = 0.$$

Based on Lemma 3.1 and the fact that A is strictly monotone, we combine it with (3.56) to deduce

$$-w_0 = 0, \quad \text{i.e., } w_0 \geq 0, \quad w_0 \neq 0.$$

Thus, from (3.57), we conclude that

$$\begin{cases} (-\Delta)_p^s(u_0 + w_0)(x) = \hat{\zeta}_0^*(x) - \zeta_0^*(x)\eta_0 & \text{for a.e. } x \in \Omega, \\ (u_0 + w_0) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (3.58)$$

Therefore, from Lemmas 2.1, 2.2 in [15], one has $u_0 + w_0 \in C_d^0(\bar{\Omega})$. Furthermore, by the fractional p -Laplacian Hopf's lemma [9, Theorem 1.5], we have, for all $x \in \partial\Omega$,

$$\lim_{\Omega \ni y \rightarrow x} \frac{(u_0 + w_0)(y)}{d_{\Omega}^s(y)} > 0,$$

which means that $u_0 + w_0 \in \text{int}C_+$ (see [15, Lemma 5.1]). Let $\hat{u}_0 = u_0 + w_0$, then we have

$$\hat{u}_0 \in \text{int}C_+, \quad u_0 \leq \hat{u}_0, \quad \text{and } u_0 \neq \hat{u}_0.$$

Furthermore, by applying the Stampacchia theorem, it is known that for a.e. $x \in \{w_0 = 0\}$,

$$(-\Delta)_p^s \hat{u}_0 = (-\Delta)_p^s u_0 = \zeta_0^*(x) \in \partial F(x, u_0(x)) = \partial F(x, \hat{u}(x)).$$

On the other hand, by (3.58), we can deduce that

$$(-\Delta)_p^s \hat{u}_0 = \hat{\zeta}_0^*(x) \in \partial F(x, \hat{u}(x)) \quad \text{for a.e. } x \in \{w_0 > 0\}.$$

Thus, it can be concluded that $\hat{u} \in \text{int}C_+$ is a solution to problem (1.3).

With the same method, we define the functional $\Upsilon_- : X(\Omega) \rightarrow \mathbb{R}$ as follows:

$$\Upsilon_-(w) = \frac{1}{p} [u_0 + w]_{s,p}^p - \int_{\Omega} F(x, u_0 - w^-) dx + \int_{\Omega} \zeta_0^* w^+ dx + \Lambda.$$

By repeating the same process, we obtain $\hat{v}_0 \in X(\Omega)$ such that

$$\hat{v}_0 \in -\text{int}C_+, \quad \hat{v}_0 \leq v_0, \quad \text{and } \hat{v}_0 \neq v_0.$$

Furthermore, \hat{v}_0 is a solution to problem (1.3). Thus, we have proved that problem (1.3) has at least four nontrivial smooth solutions of constant sign (two positive and two negative). \square

4. The second multiplicity theorem

In this section, we discuss the existence of problem (1.3), which can be restated as follows:

$$\begin{cases} (-\Delta)_p^s u \in \lambda_1 |u|^{p-2} u + \partial F(x, u) & \text{for a.e. } x \in \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (4.1)$$

where λ_1 represents the first eigenvalue associated with problem (2.1). We commence by establishing several foundational results, which are important in demonstrating the coercivity of the energy functional to problem (4.1).

Lemma 4.1. *Suppose that $\tilde{X} \subset X(\Omega)$ is a finite-dimensional subspace with a unique continuation property, and $\tilde{X} \subset L^\infty(\Omega)$. Then the following hold:*

(i) *For any $\epsilon_1 > 0$, there exists $\tau(\epsilon_1) > 0$ such that*

$$|\{x \in \Omega : |u(x)| < \tau(\epsilon_1)[u]_{s,p}\}| < \epsilon_1 \quad \forall u \in \tilde{X} \setminus \{0\}.$$

(ii) *For any $\epsilon_2 > 0$, there exists $\tau(\epsilon_2) > 0$ such that*

$$|\{x \in \Omega : |w(x)| > \tau(\epsilon_2)[w]_{s,p}\}| < \epsilon_2 \quad \forall w \in \tilde{X}^\perp.$$

Proof.(i) Noting that all norms on \tilde{X} are equivalent, we can choose the $\|\cdot\|_\infty$ -norm on \tilde{X} . Therefore, it suffices to show that for a given $\epsilon_1 > 0$, we can select $\tau(\epsilon_1) > 0$ such that

$$|\{x \in \Omega : |u(x)| < \tau(\epsilon_1)\}|_m < \epsilon_1 \quad \forall u \in \partial B_\rho.$$

To achieve this, we aim to prove that for every $u \in \partial B_\rho$, there exists $\tau(u, \epsilon_1) > 0$, satisfying

$$|\{x \in \Omega : |v(x)| < \tau(u, \epsilon_1)\}|_m < \epsilon_1 \quad \forall v \in \tilde{X}, [v - u]_{s,p} < \tau(u, \epsilon_1). \quad (4.2)$$

Suppose, by contradiction, that there exists $u_0 \in \partial B_\rho$ such that for all $n \geq 1$, we can find $v_n \in \tilde{X}$ such that

$$[v_n - u_0]_{s,p} < \frac{1}{2n} \quad \text{and} \quad |\{x \in \Omega : |v(x)| < \frac{1}{2n}\}|_m \geq \epsilon_1.$$

Let us define

$$\Omega_n = \{x \in \Omega : |v(x)| < \frac{1}{2n}\} \quad \forall n \geq 1.$$

Then

$$\Omega_n \subset \{x \in \Omega : |v(x)| < \frac{1}{n}\} \quad \forall n \geq 1.$$

Consequently,

$$|\{x \in \Omega : |u_0(x)| < \frac{1}{n}\}|_m \geq \epsilon_1 \quad \forall n \geq 1.$$

By utilizing the unique continuation property of \tilde{X} , we deduce that $u_0 \equiv 0$. However, this contradicts the fact that $u_0 \in \partial B_\rho$, and this contradiction verifies (4.2).

Next, let us examine the family $\{B_{\tau(u, \epsilon_1)}(u)\}$. It serves as an open cover for the compact set ∂B_1 (note that \tilde{X} is of finite dimension). Consequently, we can choose a finite subcover $\{B_{\tau(u_k, \epsilon_1)}(u_k)\}_{k=1}^m$. Let

$$\tau(\epsilon_1) = \min\{\tau(u_k, \epsilon_1)\}_{k=1}^m.$$

Then $\tau(\epsilon_1) > 0$ does the job.

(ii) Suppose that the result is not true. Then there exists $\epsilon_0 > 0$ such that for all $n \geq 1$, one can find $w_n \in \tilde{X}^\perp$ with $[w_n]_{s,p} = 1$, satisfying

$$|\{x \in \Omega : |w_n(x)| > n\}|_m \geq \epsilon_0.$$

By passing to a suitable subsequence if necessary, we assume that

$$w_n \rightharpoonup w \text{ in } X(\Omega).$$

Due to the compactness of the embedding $X(\Omega) \subset L^p(\Omega)$, we conclude that

$$w_n \rightarrow w \text{ in } L^p(\Omega).$$

Therefore,

$$\int_{\Omega} |w_n(x)|^p dx \geq \int_{\{x \in \Omega : |w_n(x)| \geq n\}} |w_n(x)|^p dx \geq \epsilon_0 n^p,$$

which implies that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |w_n(x)|^p dx = +\infty,$$

contradicting the fact that $w_n \rightharpoonup w$ in $L^p(\Omega)$. Thus, the proof is completed. \square

Let us consider the energy functional $\Xi : X(\Omega) \rightarrow \mathbb{R}$, defined by

$$\Xi(u) = \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \int_{\Omega} F(x, u) dx, \quad \forall u \in X(\Omega). \quad (4.3)$$

It is easy to observe from (H1)–(H3), that $\Xi(u)$ is locally Lipschitz.

Lemma 4.2. *If hypotheses (H1)–(H3), (H7) and (H8) hold, then Ξ is coercive.*

Proof. From (H7), for any given $\epsilon > 0$, it is possible to find $M_2 = M_2(\epsilon) > 0$ such that

$$pF(x, u) \leq \epsilon |u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } |u| \geq M_2. \quad (4.4)$$

Therefore, for all $u \in X(\Omega)$, we obtain

$$\begin{aligned} \Xi(u) &= \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \int_{\Omega} F(x, u) dx \\ &= \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \int_{\{|u| \leq M_2\}} F(x, u) dx - \int_{\{|u| > M_2\}} F(x, u) dx. \end{aligned} \quad (4.5)$$

By using (4.4), we deduce

$$\int_{\{|u| > M_2\}} F(x, u) dx \leq \frac{\epsilon}{p} \int_{\Omega} |u|^p dx \quad \forall u \in X(\Omega). \quad (4.6)$$

Moreover, from assumption (H3) and the mean value theorem for locally Lipschitz functions, it follows that

$$\int_{\{|u| \leq M_2\}} F(x, u) dx \leq C \quad \text{for all } x \in X(\Omega). \quad (4.7)$$

By substituting (4.6) and (4.7) into (4.5), we obtain

$$\Xi(u) \geq \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \frac{\epsilon}{p} \|u\|_p^p - C \quad \forall x \in X(\Omega). \quad (4.8)$$

For $u \in X(\Omega)$, let

$$u = u^1 + u^\perp \quad \text{with } u^1 \in E(\lambda_1), \quad u^\perp \in E(\lambda_1)^\perp,$$

i.e., $u^1 = tu_1$ for some $t \in \mathbb{R}$, where u_1 is the normalized principal eigenfunction of $((-\Delta)_p^s, X(\Omega))$. Using the orthogonality between $E(\lambda_1)$ and $E(\lambda_1)^\perp$ and the fact that $[u_1]_{s,p}^p = \lambda_1 \|u_1\|_p^p$, by (4.8), one has

$$\Xi(u) \geq \frac{1}{p} [u^\perp]_{s,p}^p - \frac{\lambda_1}{p} \|u^\perp\|_p^p - \frac{\epsilon}{p} \|u\|_p^p - C \quad \forall x \in X(\Omega). \quad (4.9)$$

Since

$$\begin{aligned} [y]_{s,p}^p &\geq \lambda_2 \|y\|_p^p \quad \forall y \in E(\lambda_1)^\perp, \\ [z]_{s,p}^p &\geq \lambda_1 \|z\|_p^p \quad \forall z \in X(\Omega), \end{aligned}$$

for all $u \in X(\Omega)$, we obtain

$$\Xi(u) \geq \frac{1}{p} \left(1 - \frac{\lambda_1}{\lambda_2} - \frac{\epsilon}{p\lambda_2}\right) [u^\perp]_{s,p}^p - \frac{\epsilon}{p\lambda_1} [u^1]_{s,p}^p - C \quad \forall u \in X(\Omega). \quad (4.10)$$

Then we will prove that Ξ is coercive. With this aim, we consider a sequence $\{u_n\} \subset X(\Omega)$ such that

$$[u_n]_{s,p} \rightarrow +\infty.$$

Since $u_n = u_n^1 + u_n^\perp$ with $u_n^1 = t_n u_1$, $t_n \in \mathbb{R}$ and $u_n^\perp \in E(\lambda_1)^\perp$ for all $n \geq 1$, if the sequence $\{u_n^1\}_{n \geq 1} \subset X(\Omega)$ is bounded, we must have

$$[u_n^\perp]_{s,p} \rightarrow +\infty,$$

this, combined with (4.10), follows that

$$\Xi(u_n) \rightarrow +\infty,$$

which means that Ξ is coercive. Thus, we suppose that the sequence $\{u_n^1\}_{n \geq 1} \subset X(\Omega)$ is unbounded. Passing to a subsequence if necessary, we assume that

$$[u_n^1]_{s,p} = |t_n| \rightarrow +\infty.$$

Note that the sequence $\{\frac{t_n^p}{[u_n^1]_{s,p}^p}\}_{n \geq 1}$ is within the interval $[0,1]$. By passing to a subsequence if necessary, we can assume that $\frac{t_n^p}{[u_n^1]_{s,p}^p} \rightarrow d$ as $n \rightarrow \infty$, where $d \in [0, 1]$. We will consider two cases.

Case 1. $0 \leq d < 1$. From (4.10), we can deduce that

$$\Xi(u) \geq \frac{1}{p} \left(1 - \frac{\lambda_1}{\lambda_2} - \frac{\epsilon}{\lambda_2}\right) [u_n^\perp]_{s,p}^p - \frac{\epsilon t_n^p}{p\lambda_1} - C \quad \forall n \geq 1.$$

Therefore,

$$\frac{\Xi(u_n)}{[u_n]_{s,p}^p} \geq \frac{1}{p} \left(1 - \frac{\lambda_1}{\lambda_2} - \frac{\varepsilon}{\lambda_2} \right) \frac{[u_n^+]_{s,p}^p}{[u_n]_{s,p}^p} - \frac{\varepsilon}{p\lambda_1} \frac{t_n^p}{[u_n]_{s,p}^p} - \frac{C}{[u_n]_{s,p}^p} \quad \forall x \in X(\Omega). \quad (4.11)$$

Since $X(\Omega) = E(\lambda_1) \oplus E(\lambda_1)^\perp$, we have

$$1 = \frac{[u_n]_{s,p}^p}{[u_n]_{s,p}^p} = \frac{[u_n^+]_{s,p}^p + t_n^p [u_1]_{s,p}^p}{[u_n]_{s,p}^p} = \frac{[u_n^+]_{s,p}^p + t_n^p}{[u_n]_{s,p}^p},$$

which implies

$$\frac{[u_n^+]_{s,p}^p}{[u_n]_{s,p}^p} \rightarrow 1 - d.$$

Taking the limit $n \rightarrow \infty$ in (4.11), we obtain

$$\liminf_{n \rightarrow +\infty} \frac{\Xi(u_n)}{[u_n]_{s,p}^p} \geq \frac{1}{p} \left(1 - \frac{\lambda_1}{\lambda_2} - \frac{\varepsilon}{\lambda_1} \right) (1 - d) - \frac{\varepsilon d}{p\lambda_1}.$$

By choosing $\varepsilon > 0$ to be sufficiently small such that $\frac{\lambda_1}{\lambda_2} + \frac{\varepsilon}{\lambda_2} < 1$, we can conclude that

$$\liminf_{n \rightarrow +\infty} \frac{\Xi(u_n)}{[u_n]_{s,p}^p} > 0,$$

which implies

$$\Xi(u_n) \rightarrow +\infty,$$

i.e., Ξ is coercive.

Case 2. $d = 1$. In this case, we have

$$\frac{t_n^p}{[u_n]_{s,p}^p} \rightarrow 1.$$

Then

$$\frac{[u_n^+]_{s,p}^p}{[u_n]_{s,p}^p} \rightarrow 0.$$

Now, we need to prove that

$$\limsup_{n \rightarrow +\infty} \frac{\int_{\Omega} F(x, u_n(x)) dx}{[u_n]_{s,p}^p} < 0. \quad (4.12)$$

From Lemma 4.1, for given $\varepsilon_1, \varepsilon_2 > 0$, one can find $\tau_1 = \tau_1(\varepsilon_1) \in (0, \varepsilon_1)$ and $\tau_2 = \tau_2(\varepsilon_2) \in (0, \varepsilon_2)$ such that

$$\{|x \in \Omega : |v(x)| < \tau_1 [v]_{s,p}\}_m < \varepsilon_1, \quad \forall v \in E(\lambda_1)$$

and

$$\{|x \in \Omega : |w(x)| > \tau_2 [w]_{s,p}\}_m < \varepsilon_2, \quad \forall w \in E(\lambda_1)^\perp.$$

We introduce the following two Lebesgue measurable subsets of Ω :

$$\begin{aligned} A_n &:= \{x \in \Omega : |u_n^1(x)| \geq \tau_1 [u_n^1]_{s,p}\}, \\ B_n &:= \{x \in \Omega : |u_n^\perp(x)| \leq \tau_2 [u_n^\perp]_{s,p}\} \end{aligned}$$

for all $n \geq 1$. As such, $|\Omega \setminus A_n|_m < \epsilon_1$ and $|\Omega \setminus B_n|_m < \epsilon_2$. Thus, if we choose $\epsilon_1, \epsilon_2 > 0$ small enough, we may derive $|A_n \cap B_n|_m \neq \emptyset$ for all $n \geq 1$. Since $E(\lambda_1) \subset C(\bar{\Omega})$ is finite dimensional, we have

$$|v(x)| \leq C[v]_{s,p} \quad \forall v \in E(\lambda_1), \quad x \in \Omega.$$

Consequently, for $x \in A_n \cap B_n$, and $n \geq 1$, one has

$$\frac{|u_n(x)|}{[u_n]_{s,p}} \leq \frac{|u_n^1(x)|}{[u_n]_{s,p}} + \frac{|u_n^\perp(x)|}{[u_n]_{s,p}} \leq C \frac{[u_n^1]_{s,p}}{[u_n]_{s,p}} + \tau_2 \frac{[u_n^\perp]_{s,p}}{[u_n]_{s,p}},$$

and so

$$\limsup_{n \rightarrow +\infty} \frac{|u_n(x)|}{[u_n]_{s,p}} \chi_{A_n \cap B_n}(x) \leq C \quad (4.13)$$

for a.e. $x \in \Omega$. Furthermore, for $x \in A_n \cap B_n$ and $n \geq 1$, we have

$$\frac{|u_n(x)|}{[u_n]_{s,p}} \geq \frac{|u_n^1(x)|}{[u_n]_{s,p}} - \frac{|u_n^\perp(x)|}{[u_n]_{s,p}} \geq \tau_1 \frac{[u_n^1]_{s,p}}{[u_n]_{s,p}} - \tau_2 \frac{[u_n^\perp]_{s,p}}{[u_n]_{s,p}},$$

from which it follows that

$$\liminf_{n \rightarrow +\infty} \frac{|u_n(x)|}{[u_n]_{s,p}} \chi_{A_n \cap B_n}(x) \geq \tau_1 \quad (4.14)$$

for a.e. $x \in \Omega$. Based on hypotheses (H3), (H7), and the mean value theorem for locally Lipschitz functions, we can deduce that for a given $\epsilon > 0$,

$$F(x, u) \leq \frac{1}{p}(\mu(x) + \epsilon)|u|^p + C \quad (4.15)$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. It follows from (4.13)–(4.15), for a given $\epsilon' > 0$, we can find $n_0 = n_0(\epsilon') \geq 1$, for all $n \geq n_0$, such that

$$\begin{aligned} & \int_{A_n \cap B_n} \frac{F(x, u_n(x))}{[u_n]_{s,p}^p} dx \\ & \leq \frac{1}{p} \int_{A_n \cap B_n} (\mu(x) + \epsilon) \frac{|u|^p}{[u_n]_{s,p}^p} dx + C \frac{|A_n \cap B_n|_m}{[u_n]_{s,p}^p} \\ & \leq \frac{1}{p} (C + \epsilon')^p \int_{\{\mu > -\epsilon\}} (\mu(x) + \epsilon) dx + \frac{1}{p} (\tau_1 - \epsilon')^p \int_{\{\mu < -\epsilon\}} (\mu(x) + \epsilon) dx + \epsilon' \\ & \leq \frac{1}{p} (C + \epsilon')^p \int_{\{\mu > -\epsilon\}} (\mu(x) + \epsilon)^+ dx - \frac{1}{p} (\tau_1 - \epsilon')^p \int_{\Omega} (\mu(x) + \epsilon)^- dx + \epsilon'. \end{aligned} \quad (4.16)$$

Furthermore, for $x \in B_n \setminus A_n$, the following inequality holds

$$\begin{aligned} \frac{|u_n(x)|}{[u_n]_{s,p}} & \leq \frac{|u_n^1(x)|}{[u_n]_{s,p}} + \frac{|u_n^\perp(x)|}{[u_n]_{s,p}} \leq \tau_1 \frac{[u_n^1]_{s,p}}{[u_n]_{s,p}} + \tau_2 \frac{[u_n^\perp]_{s,p}}{[u_n]_{s,p}} \\ & = \tau_1 \frac{t_n}{[u_n]_{s,p}} + \tau_2 \frac{[u_n^\perp]_{s,p}}{[u_n]_{s,p}}, \end{aligned}$$

thus we have

$$\limsup_{n \rightarrow +\infty} \frac{|u_n(x)|}{[u_n]_{s,p}} \chi_{A_n \cap B_n}(x) \leq \tau_1 \quad (4.17)$$

for a.e. $x \in \Omega$. Additionally, by

$$|F(x, u)| \leq \alpha_2(x) + C|u|^p,$$

where $\alpha_2 \in L^\infty(\Omega)_+$, for all $n \geq n_0$, we utilize Lemma 4.1 to obtain

$$\begin{aligned} \int_{B_n \setminus A_n} \frac{|F(x, u_n(x))|}{[u_n]_{s,p}^p} dx &\leq \int_{B_n \setminus A_n} \frac{\alpha_2(x) + C|u|^p}{[u_n]_{s,p}^p} dx \\ &\leq \epsilon' + C(\tau_1 + \epsilon')^p |\Omega \setminus A_n|_m \\ &\leq \epsilon' + C(\tau_1 + \epsilon')^p \epsilon_1. \end{aligned} \quad (4.18)$$

On the other hand, for all $n \geq n_0$, we have

$$\begin{aligned} \int_{\Omega \setminus B_n} \frac{|F(x, u_n(x))|}{[u_n]_{s,p}^p} dx &\leq \int_{\Omega \setminus B_n} \frac{\alpha_2(x) + C|u|^p}{[u_n]_{s,p}^p} dx \\ &\leq \epsilon' + C|\Omega \setminus B_n|_m \\ &\leq \epsilon' + C\epsilon_2. \end{aligned} \quad (4.19)$$

From (4.20), (4.18) and (4.19), we infer that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\Omega} \frac{F(x, u_n(x))}{[u_n]_{s,p}^p} dx \\ \leq \frac{1}{p}(C + \epsilon')^p \int_{\Omega} (\mu(x) + \epsilon)^+ dx - \frac{1}{p}(\tau_1 - \epsilon')^p \int_{\Omega} (\mu(x) + \epsilon)^- dx \\ + 3\epsilon' + C(\tau_1 + \epsilon')^p \epsilon_1 + C\epsilon_2. \end{aligned} \quad (4.20)$$

Since ϵ , ϵ' , ϵ_1 , and ϵ_2 are all arbitrary positive values and $\mu(x) \leq 0$ a.e. in Ω , letting ϵ , ϵ' , and ϵ_2 to zero and noting that $(\mu(x) + \epsilon)^+ \rightarrow 0$ and $(\mu(x) + \epsilon)^- \rightarrow \mu(x)$ for a.e. $x \in \Omega$, we take the limit to obtain

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\Omega} \frac{F(x, u_n(x))}{[u_n]_{s,p}^p} dx &\leq \frac{1}{p}\tau_1^p \int_{\Omega} \mu(x) dx + C\tau_1^p \epsilon_1 \\ &\leq \frac{1}{p}\epsilon_1^p \int_{\Omega} \mu(x) dx + C\epsilon_1^{p+1} \quad (\text{note that } \tau_1 \in (0, \epsilon_1)). \end{aligned}$$

Thus,

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} \frac{F(x, u_n(x))}{[u_n]_{s,p}^p} dx \leq \frac{1}{p}\epsilon_1^p \left(\int_{\Omega} \mu(x) dx + C\epsilon_1 \right).$$

Since $\int_{\Omega} \mu(x) dx < 0$ and $\epsilon_1 > 0$ is arbitrary, we infer that (4.12) is true. From $[u_n]_{s,p}^p \geq \lambda_1 \|u_n\|_p^p$ for all $n \geq 1$, it follows that

$$\begin{aligned} \Xi(u_n) &= \frac{1}{p}[u_n]_{s,p}^p - \frac{\lambda_1}{p}\|u_n\|_p^p - \int_{\Omega} F(x, u_n) dx \\ &\geq - \int_{\Omega} F(x, u_n) dx. \end{aligned} \quad (4.21)$$

Consequently,

$$\frac{\Xi(u_n)}{[u_n]_{s,p}^p} \geq - \int_{\Omega} \frac{F(x, u_n)}{[u_n]_{s,p}^p} dx,$$

from (4.12), we derive

$$\liminf_{n \rightarrow +\infty} \frac{\Xi(u_n)}{[u_n]_{s,p}^p} > 0,$$

from which it follows that there exists a subsequence such that $\Xi(u_n) \rightarrow +\infty$. Since every subsequence of $\{\Xi(u_n)\}_{n \geq 1}$ has a further subsequence such that $\Xi(u_n) \rightarrow +\infty$, we obtain that $\Xi(u_n) \rightarrow +\infty$, which means that Ξ is also coercive. \square

Remark 4.1. From Lemma 4.2, the coercivity of Ξ means that Ξ is bounded below and satisfies the nonsmooth PS-condition.

Let us define the sets $E_k = \bigoplus_{i=1}^k E(\lambda_i)$ and $\hat{E}_k = \bigoplus_{i=k+1}^{\infty} E(\lambda_i)$. It can be observed that $X(\Omega) = E_k \bigoplus \hat{E}_k$, where \hat{E}_k is the orthogonal complement of E_k , and the dimension of E_k is finite with $\dim E_k = k < \infty$.

In the following, we will demonstrate that Ξ has the local linking geometry for the direct sum decomposition $E_k \bigoplus \hat{E}_k$.

Lemma 4.3. If hypotheses (H1)–(H3), (H7), and (H8) hold, then there exists $\rho > 0$ such that

$$\begin{cases} \Xi(u) \leq 0, & \text{if } u \in E_k, \quad [u]_{s,p} \leq \rho, \\ \Xi(u) \leq 0, & \text{if } u \in \hat{E}_k, \quad [u]_{s,p} \leq \rho. \end{cases}$$

Proof. Since $E_k \subset C(\bar{\Omega})$ is finite dimensional, we can find $\rho_1 > 0$, such that

$$|u(x)| \leq \frac{\omega}{3} < \omega, \quad \forall x \in \Omega, \quad u \in E_k, \quad [u]_{s,p} \leq \rho_1,$$

where ω as in hypothesis (H8). Hence, for $u \in E_k$ with $[u]_{s,p} \leq \rho_1$,

$$\begin{aligned} \Xi(u) &= \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \int_{\Omega} F(x, u) dx \\ &\leq \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \frac{\lambda_k - \lambda_1}{p} \|u\|_p^p \\ &= \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_k}{p} \|u\|_p^p \leq 0. \end{aligned}$$

Now, set $u \in \hat{E}_k$ and decompose u as follows

$$u = \bar{u} + \tilde{u}, \quad \text{where } \bar{u} \in E(\lambda_{k+1}) \text{ and } \tilde{u} \in \bigoplus_{i=k+2}^{\infty} E(\lambda_i).$$

From hypothesis (H8) and the fact that $[\bar{u}]_{s,p}^p = \lambda_{k+1} \|\bar{u}\|_p^p$, one has

$$\begin{aligned}
 \Xi(u) &= \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \int_{\Omega} F(x, u) dx \\
 &= \frac{1}{p} [u]_{s,p}^p - \frac{\lambda_1}{p} \|u\|_p^p - \int_{\{|u| \leq \omega\}} F(x, u) dx - \int_{\{|u| > \omega\}} F(x, u) dx \\
 &\geq \frac{1}{p} [\tilde{u}]_{s,p}^p + \int_{\{|u| \leq \omega\}} \left[\frac{\lambda_{k+1} - \lambda_1}{p} |u(x)|^p - F(x, u) \right] dx \\
 &\quad + \int_{\{|u| > \omega\}} \left[\frac{\lambda_{k+1} - \lambda_1}{p} |u(x)|^p - F(x, u) \right] dx - \frac{\lambda_{k+1}}{p} \|\tilde{u}\|_p^p \\
 &\geq \frac{1}{p} [\tilde{u}]_{s,p}^p - \frac{\lambda_{k+1}}{p} \|\tilde{u}\|_p^p + \int_{\{|u| > \omega\}} \left[\frac{\lambda_{k+1} - \lambda_1}{p} |u(x)|^p - F(x, u) \right] dx.
 \end{aligned} \tag{4.22}$$

Furthermore, we claim that there exists $\rho_2 > 0$, such that

$$\frac{1}{p} |u(x)| \leq |\tilde{u}(x)| \tag{4.23}$$

with $[u]_{s,p} \leq \rho_2$ for a.e. $x \in \{|u| \geq \omega\}$ and all $u \in \hat{E}_k$. Indeed, leveraging the finite-dimensional space of $E(\lambda_{k+1}) \subset C(\bar{\Omega})$, we have

$$|u(x)| \leq C[u]_{s,p} \quad \forall x \in \Omega, \quad u \in E(\lambda_{k+1}).$$

Let $\rho_2 = \frac{\omega}{pC}$ and assume that $[u]_{s,p} \leq \rho_2$ for a.e. $x \in \{|u| \geq \omega\}$, one has

$$\begin{aligned}
 |u(x)| &\leq |\bar{u}(x)| + |\tilde{u}(x)| \leq C[\bar{u}(x)]_{s,p} + |\tilde{u}(x)| \\
 &\leq C[u(x)]_{s,p} + |\tilde{u}(x)| \leq \frac{\omega}{p} + |\tilde{u}(x)| \leq \frac{p-1}{p} |u(x)| + |\tilde{u}(x)|,
 \end{aligned}$$

from which we deduce (4.23). Since

$$|F(x, u(x))| \leq C|u(x)|^q \quad \text{for a.e. } x \in \{|u| > \omega\} \tag{4.24}$$

for $q \in (p, p^*)$. From (4.23), (4.24) and $X(\Omega) \subset L^q(\Omega)$, it follows that

$$\begin{aligned}
 &\left| \int_{\{|u| > \omega\}} \left[\frac{\lambda_{k+1} - \lambda_1}{p} |u(x)|^p - F(x, u) \right] dx \right| \\
 &\leq C \int_{\{|u| > \omega\}} |\tilde{u}(x)|^q \leq C[\tilde{u}]_{s,p}^q.
 \end{aligned} \tag{4.25}$$

Substituting (4.25) into (4.22), we obtain

$$\begin{aligned}
 \Xi(u) &\geq \frac{1}{p} [\tilde{u}]_{s,p}^p - \frac{\lambda_{k+1}}{p} \|\tilde{u}\|_p^p - C[\tilde{u}]_{s,p}^q \\
 &\geq \frac{1}{p} \left(1 - \frac{\lambda_{k+1}}{\lambda_{k+2}} \right) [\tilde{u}]_{s,p}^p - C[\tilde{u}]_{s,p}^q.
 \end{aligned}$$

Since $q > p$, one can find $\rho_3 > 0$ small enough, such that

$$\Xi(u) \geq 0 \quad \forall x \in \tilde{E}_k, \quad [u]_{s,p} \leq \rho_3.$$

Finally, by selecting $\rho = \min\{\rho_1, \rho_2, \rho_3\}$, we derive that Ξ has the local linking at 0. \square

Now, we are ready to prove the multiple results for problem (4.1).

Proof of Theorem 1.2. Noting that $F(x, 0) = 0$ for a.e. $x \in \Omega$, we have $\Xi(u) = 0$. If $v \in E(\lambda_{k-1}) \subset C(\bar{\Omega})$ and $|v(x)| \leq \omega$ for all $x \in \bar{\Omega}$, from hypothesis (H8), we derive

$$\begin{aligned}\Xi(u) &= \frac{1}{p}[u]_{s,p}^p - \frac{\lambda_1}{p}\|u\|_p^p - \int_{\Omega} F(x, u)dx \\ &\geq \frac{1}{p}[u]_{s,p}^p - \frac{\lambda_1}{p}\|u\|_p^p - \frac{\lambda_{k+1} - \lambda_1}{p}\|u(x)\|^p \\ &= \frac{1}{p}[u]_{s,p}^p - \frac{\lambda_k}{p}\|u\|_p^p \\ &\leq \frac{\lambda_{k-1}}{p}[u]_{s,p}^p - \frac{\lambda_k}{p}\|u\|_p^p < 0,\end{aligned}$$

so $\inf_{u \in X(\Omega)} \Xi(u) < 0$, and from the above inequality, Lemmas 4.2 and 4.3, and Theorem 2.1, we can derive two nontrivial critical points of Ξ , which are two nontrivial solutions of (4.1). \square

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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