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Research article

On progressive first-failure reliability analysis: Classical and Bayesian approaches

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Abstract: The progressive first-failure censoring (PF-FC) plan is widely used in reliability settings where test items are arranged into groups of size k, and only the earliest failure in each group is observed. In this study, statistical inference for the Gompertz–Lindley distribution (GLD) under PF-FC was considered, with emphasis on estimating the model parameters together with the reliability and hazard rate functions (HRFs). Classical inference was performed via the maximum likelihood method (MLE), and confidence intervals (CIs) were formed using the large-sample behavior of the estimators. A Bayesian framework was also constructed using independent gamma priors and non-informative priors (NIPs) under loss structures. Markov Chain Monte Carlo (MCMC) algorithms were used to generate Bayesian estimates (BEs) and credible intervals (CRIs). Reliability measures are examined from both classical and BE. To evaluate the proposed procedures, a MCMC simulation study was carried out to examine their precision and robustness. The practical relevance of the developed methodology was illustrated using a real lifetime dataset.

Keywords: MLEs; progressive first-failure censoring; reliability inference; Gompertz-Lindley distribution; Bayesian estimation
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1. Introduction

The preponderance of modern products are designed for prolonged durability. Consequently, additional time and financial resources are necessary for conducting reliability testing under normal operational settings. Under regular conditions, longevity studies with a fixed testing time have very low failure rates. The experimenter must conclude the experiment prior to detecting any failures due to the expenses and time necessary for life testing investigations. Consequently, censoring schemes (CSs) are frequently employed to reduce the time and financial resources allocated to testing. The conventional Type-I (TI) and Type-II (TII) censoring techniques are the most often utilized for laboratory analyses. The progressive Type-II censoring (PT-II) system has gained significant popularity in recent applications.

The fundamental difference between TI and TII CSs lies in the fact that TI CSs are dictated by the timing of experiment termination, whereas TII CSs are dictated by the number of observed failures. The experimental units from both CSs cannot be excluded from consideration until the completion of the test. Moreover, the TII censoring method can be extended into a PT-II method, which prompts the individual experimenting to eliminate the units under examination at various intervals throughout the test. For more reading on PT-II CS, one may consult [1, 2].

Several researchers have created a life test that enables the experimenter to partition the test units into many groups, each serving as a collection of test units. Subsequently, all test units are subject to simultaneous evaluation until one group fails. This CS is referred to as "first-failure censoring". It was initially created by the researchers in [3]. Various research has been undertaken on the notion of first-failure censoring, as Wu et al. [4], which estimated the parameters of the Gompertz distribution under the first failure censored sampling, Wu and Yu [5] made statistical inference about the shape parameter of the Burr type-XII distribution, and Alotaibi et al. [6] worked on the length-bias exponential model and its optimal plans.

Wu and Kus [7] created a hybrid of the progressive first-failure censoring (PF-FC), which integrates the first-failure censoring system with the PT-II censoring method. Most researchers have concentrated on the examination and analysis of the PF-FC scheme. For example, Alsadat et al. [8] and Abu-Moussa et al. [9] investigated recurrence relations, characterizations, and reliability function estimation for the extended Rayleigh distribution under progressive first-failure censoring. Similarly, Soliman et al. [10] and Soliman et al. [11] studied parameter estimation for the Gompertz and Burr type XII distributions, respectively, using the same censoring scheme. In the context of multicomponent stress-strength models, Fayomi et al. [12] and Alsadat et al. [13] provided reliability inference based on the inverted exponentiated Pareto and exponentiated Pareto distributions. Dube et al. [14] and Kayal et al. [15] focused on the Lindley and Chen distributions under progressive first-failure censoring. Furthermore, Ramadan et al. [16] and Mahmoud et al. [17]). conducted statistical inference and Bayesian estimation for the inverse Lomax and inverted generalized linear exponential distributions. Finally, Yusuf and Barakat [18], estimated parameters of the generalized inverted Kumaraswamy distribution using a first failure-censored sampling plan.

The combination of the GLD with PF-FC leads to a highly non-linear likelihood function due to the joint effect of grouped failures and progressive unit removal, with no closed-form solutions for parameter estimation. Despite this complexity, the flexible hazard structure of the model enhances parameter identifiability and enables accurate modeling under heavily censored data. This makes the proposed framework mathematically non-trivial and practically effective for reliability analysis.

From a reliability perspective, the adoption of the Gompertz–Lindley distribution (GLD) is supported by its interpretable hazard rate structure rather than solely its empirical fit. In particular, the GLD is capable of modeling monotonically increasing hazard rates, which are commonly associated with aging, cumulative damage, and wear-out failure mechanisms in engineering systems. This behavior is consistent with progressive first-failure settings, where the failure risk escalates over time due to degradation effects. Moreover, the Lindley component introduces an implicit mixing structure that accounts for unobserved heterogeneity among units, reflecting variability in material properties or operating environments. Therefore, the GLD provides a flexible yet physically plausible framework for representing failure-time data under progressive first-failure censoring schemes.

For the GLD, the functional representations of its probability density function (PDF), along with the survival function (SF), and HRF are delineated following:

$$f(t) = \frac{\alpha^2 \lambda e^{\lambda t}[e^{\lambda t} + \alpha + 1]}{(\alpha + 1)[e^{\lambda t} + \alpha - 1]^3}, \quad t > 0, \ \alpha, \lambda > 0, \tag{1.1}$$

$$S(t) = \overline{F}(t) = 1 - F(t) = \frac{\alpha^2[e^{\lambda t} + \alpha]}{(\alpha + 1)[e^{\lambda t} + \alpha - 1]^2}, \quad t > 0, \tag{1.2}$$

and

$$H(t) = \frac{\lambda e^{\lambda t}[e^{\lambda t} + \alpha + 1]}{(e^{\lambda t} + \alpha)[e^{\lambda t} + \alpha - 1]}, \quad t > 0. \tag{1.3}$$

Figure 1 illustrates the PDF, survival, and HRF of the GLD for different values of α and λ. The curves (a)–(d) show how parameter changes influence the shape, tail behavior, and reliability properties of the distribution. According to the study conducted by Ghitany et al. [19], the GLD is superior to several standard models, including the Gamma, Weibull, Lognormal, Inverse-Gaussian, and the baseline Gompertz distributions, providing a more flexible and robust fit for lifetime datasets. Later, Abu-Moussa et al. [20] investigated the GLD competing risks model under progressively censored data.

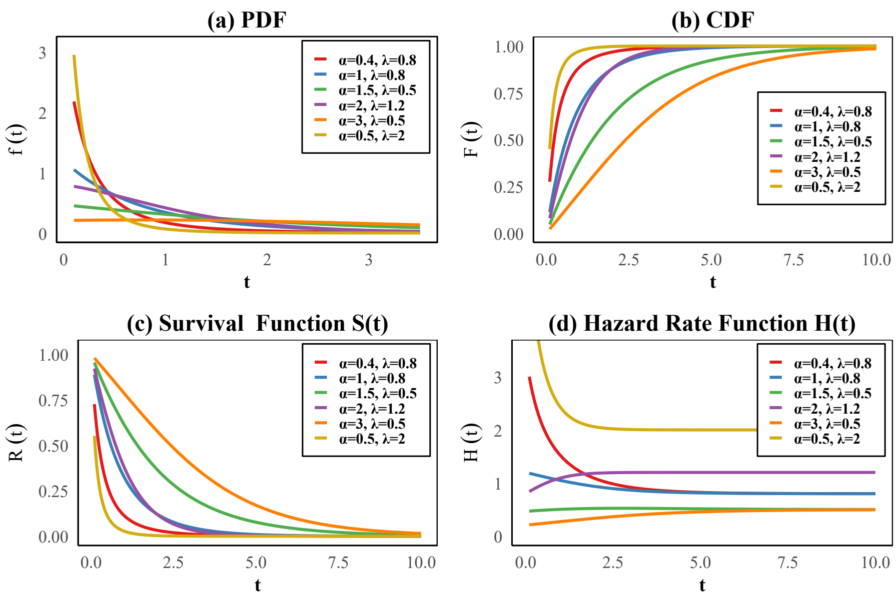


Figure 1. PDF, CDF, survival, and HRF of the GLD for different values of α and λ.

This paper was written to draw a statistical inference on the GLD distribution as a viable match for lifetime models with ascending, declining, and bathtub failure rates. The development of this study was driven by focused efforts to achieve the stated research objectives. The objective of this inference was to ascertain whether or not the distribution is a suitable match for the findings of the investigation. Additionally, methods for parameter estimation that use PF-FC data were developed. Moreover, the MLE method and the Bayesian estimate (BE) method are two examples of these types of analysis techniques. Furthermore, in real-world applications, the GLD distribution has demonstrated that it is superior to competing distributions in its ability to match real-world data more closely.

Our major objective of this work is to focus on the challenge of developing a PF-FC life test with a GLD. The subsequent sections of this article are organized as follows: In Section 2, we delineate the concept and formulation of PF-FC. The MLEs and asymptotic CIs for the parameters, as well as for the SF and HRF of the GLD, contingent upon the PF-FC, are delineated in Section 3. In Section 4, we examine the Bayes estimators and their corresponding MCMC-based approximations, along with the associated CRIs derived from several loss functions, including squared error loss (SEL), linear-exponential loss (LINEX), generalized entropy loss (GEL), and the Al-Bayatti loss (ALB). In Section 5, we present a concrete numerical illustration utilizing an authentic data set. In Section 6, we present a simulation study to assess the efficiency of the estimating methodologies. In Section 7, we conclude the article.

2. Model formulation

The PF-FC scheme involves organizing n independent groups, each consisting of k units, for a lifetime testing process. Upon observing the first failure $X_{1:m:n:k}$, a specified number R_1 of the remaining groups are randomly withdrawn from the test, along with the $k-1$ units in the group containing the failed item. This procedure continues such that, at the second failure time $X_{2:m:n:k}$, an additional R_2 group is randomly withdrawn along with the remaining $k-1$ units from the corresponding group. This process is repeated until the mth failure $X_{m:m:n:k}$ is observed for $(m < n)$. At that juncture, R_m denotes that the residual groups with their corresponding $k-1$ units are eliminated from the test; see Wu and Kus [7].

Assuming $f(x,\Theta)$ and $F(x,\Theta)$ denote the PDF and SF, respectively, the joint density of the observed failure times $(X_{1:m:n:k},\ldots,X_{m:m:n:k}) := (x_1,\ldots,x_m)$ is given by:

$$L(\Theta|\mathbf{X}) = C\,k^m \prod_{i=1}^{m} f(x_i)\left[\overline{F}(x_i)\right]^{kR_i+k-1}, \tag{2.1}$$

where, $C = n(n-R_1-1)(n-R_1-R_2-2)(n-R_1-R_2-2)\cdots(n-R_1-R_2-\cdots-R_{m-1}-m+1)$.

3. Maximum likelihood estimation

The MLEs of the parameters of GLD are derived in this section.

$$L(\alpha,\lambda;\underline{\mathbf{x}}) = \frac{Ck^m\alpha^{2m}\lambda^m}{(\alpha+1)^m}\left(\frac{\alpha^2}{\alpha+1}\right)^{\sum_{i=1}^{m}(kR_i+k-1)}\prod_{i=1}^{m}\frac{e^{\lambda x_i}+\alpha+1}{(e^{\lambda x_i}+\alpha-1)^3}\left(\frac{e^{\lambda x_i}+\alpha}{(e^{\lambda x_i}+\alpha-1)^2}\right)^{(kR_i+k-1)}. \tag{3.1}$$

Based on expression (3.1), the corresponding log-likelihood function is derived as

$$\begin{aligned}\ell &\propto 2m\log(\alpha)-m\log(\alpha+1)+m\log(\lambda)+\sum_{i=1}^{m}(k(R_i+1)-1)\log\left(\frac{\alpha^2\left(\alpha+e^{\lambda x_i}\right)}{(\alpha+1)(\alpha+e^{\lambda x_i}-1)^2}\right)\\&- 3\sum_{i=1}^{m}\log\left(\alpha+e^{\lambda x_i}-1\right)+\sum_{i=1}^{m}\log\left(\alpha+e^{\lambda x_i}+1\right)+\lambda\sum_{i=1}^{m}x_i. \qquad (3.2)\end{aligned}$$

By computing the partial derivatives of (3.2) with respect to α and λ, and equating them to zero, the corresponding likelihood equations are obtained as follows:

$$\begin{aligned}\frac{(\alpha+2)m}{\alpha(\alpha+1)} &+ \sum_{i=1}^{m}(k(R_i+1)-1)\left(-\frac{1}{\alpha+1}+\frac{2}{\alpha}-\frac{2}{\alpha+e^{\lambda x_i}-1}+\frac{1}{\alpha+e^{\lambda x_i}}\right)\\&= 3\sum_{i=1}^{m}\frac{1}{\alpha+e^{\lambda x_i}-1}-\sum_{i=1}^{m}\frac{1}{\alpha+e^{\lambda x_i}+1}, \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}\frac{m}{\lambda} &+ \sum_{i=1}^{m}(k(R_i+1)-1)\left(\frac{x_ie^{\lambda x_i}}{\alpha+e^{\lambda x_i}}-\frac{2x_ie^{\lambda x_i}}{\alpha+e^{\lambda x_i}-1}\right)+\sum_{i=1}^{m}x_i\\&= 3\sum_{i=1}^{m}\frac{x_ie^{\lambda x_i}}{\alpha+e^{\lambda x_i}-1}-\sum_{i=1}^{m}\frac{x_ie^{\lambda x_i}}{\alpha+e^{\lambda x_i}+1}. \qquad (3.4)\end{aligned}$$

To obtain the estimates $\hat{\alpha}$ and $\hat{\lambda}$, Eqs (3.3) and (3.4) are resolved utilizing the Newton-Raphson technique. For more reading about estimation techniques, see [21, 22].

3.1. Asymptotic confidence intervals

In this subsection, the asymptotic CIs the parameters α and λ are constructed using the Fisher information matrix, defined as follows:

$$I(\widehat{\alpha},\widehat{\lambda})=\left.\begin{pmatrix}-\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\lambda}\\ -\frac{\partial^2\ell}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell}{\partial\lambda^2}\end{pmatrix}\right|_{\alpha=\hat{\alpha},\lambda=\hat{\lambda}}, \qquad (3.5)$$

where

$$\begin{aligned}\frac{\partial^2\ell}{\partial\alpha^2} &= \sum_{i=1}^{m}(kR_i+k-1)\left(-\frac{2}{\alpha^2}+\frac{1}{(\alpha+1)^2}+\frac{2}{(\alpha+e^{\lambda x_i}-1)^2}-\frac{1}{(\alpha+e^{\lambda x_i})^2}\right)\\&- 3\sum_{i=1}^{m}-\frac{1}{(\alpha+e^{\lambda x_i}-1)^2}+\sum_{i=1}^{m}-\frac{1}{(\alpha+e^{\lambda x_i}+1)^2}-\frac{2m}{\alpha^2}+\frac{m}{(\alpha+1)^2},\end{aligned}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\lambda} = \frac{\partial^2\ell}{\partial\lambda\partial\alpha}=\sum_{i=1}^{m}(kR_i+k-1)\left(\frac{2x_ie^{\lambda x_i}}{(\alpha+e^{\lambda x_i}-1)^2}-\frac{x_ie^{\lambda x_i}}{(\alpha+e^{\lambda x_i})^2}\right)$$

$$- \quad 3\sum_{i=1}^{m} -\frac{x_i e^{\lambda x_i}}{(\alpha + e^{\lambda x_i} - 1)^2} + \sum_{i=1}^{m} -\frac{x_i e^{\lambda x_i}}{(\alpha + e^{\lambda x_i} + 1)^2},$$

and

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \lambda^2} &= \sum_{i=1}^{m}(kR_i + k - 1)\left(-\frac{2x_i^2 e^{\lambda x_i}}{\alpha + e^{\lambda x_i} - 1} + \frac{x_i^2 e^{\lambda x_i}}{\alpha + e^{\lambda x_i}} + \frac{2x_i^2 e^{2\lambda x_i}}{(\alpha + e^{\lambda x_i} - 1)^2} - \frac{x_i^2 e^{2\lambda x_i}}{(\alpha + e^{\lambda x_i})^2}\right) \\ &- 3\sum_{i=1}^{m}\left(\frac{x_i^2 e^{\lambda x_i}}{\alpha + e^{\lambda x_i} - 1} - \frac{x_i^2 e^{2\lambda x_i}}{(\alpha + e^{\lambda x_i} - 1)^2}\right) + \sum_{i=1}^{m}\left(\frac{x_i^2 e^{\lambda x_i}}{\alpha + e^{\lambda x_i} + 1} - \frac{x_i^2 e^{2\lambda x_i}}{(\alpha + e^{\lambda x_i} + 1)^2}\right) - \frac{m}{\lambda^2}.\end{aligned}$$

The two-sided asymptotic CIs at the $100(1-\gamma)\%$ level for parameters α and λ are derived as follows:

$$\left(\widehat{\alpha} - z_{\gamma/2}\sqrt{\widehat{V}(\widehat{\alpha})}, \widehat{\alpha} + z_{\gamma/2}\sqrt{\widehat{V}(\widehat{\alpha})}\right), \tag{3.6}$$

and

$$\left(\widehat{\lambda} - z_{\gamma/2}\sqrt{\widehat{V}\left(\widehat{\lambda}\right)}, \widehat{\lambda} + z_{\gamma/2}\sqrt{\widehat{V}\left(\widehat{\lambda}\right)}\right). \tag{3.7}$$

Here, $\widehat{V}(\widehat{\alpha})$ and $\widehat{V}\left(\widehat{\lambda}\right)$ denote the estimated variances of $\widehat{\alpha}$ and $\widehat{\lambda}$, respectively, obtained from the diagonal elements of the inverse Fisher information matrix. The term $z_{\gamma/2}$ represents the top $\gamma/2$ quantile of the standard normal distribution.

3.2. Approximate confidence intervals for $S(t)$ and $H(t)$

The delta approach is utilized to generate approximate confidence intervals for both the survival function and hazard rate function, as referenced in [23].

$$G_1 = \begin{bmatrix} \frac{\partial S(t)}{\partial \alpha} & \frac{\partial S(t)}{\partial \lambda} \end{bmatrix} \text{ and } G_2 = \begin{bmatrix} \frac{\partial H(t)}{\partial \alpha} & \frac{\partial H(t)}{\partial \lambda} \end{bmatrix}, \tag{3.8}$$

where

$$\frac{\partial S(t)}{\partial \alpha} = \frac{\alpha e^{\lambda t}[\alpha^2 + 3\alpha + e^{\lambda t}(2\alpha + 3) + 1]}{(\alpha + 1)^2 (e^{\lambda t} + \alpha - 1)^3}, \tag{3.9}$$

$$\frac{\partial S(t)}{\partial \lambda} = -\frac{\alpha^2 t e^{\lambda t}(e^{\lambda t} + \alpha + 1)}{(\alpha + 1)(e^{\lambda t} + \alpha - 1)^3}, \tag{3.10}$$

$$\frac{\partial H(t)}{\partial \alpha} = -\frac{\lambda e^{\lambda t}[e^{2\lambda t} + 2\alpha e^{\lambda t} + 2e^{\lambda t} + \alpha^2 + 2\alpha - 1]}{(e^{\lambda t} + \alpha - 1)^2 (e^{\lambda t} + \alpha)^2}, \tag{3.11}$$

$$\frac{\partial H(t)}{\partial \lambda} = -\frac{\lambda t e^{2\lambda t}\left(e^{\lambda t} + \alpha + 1\right)\left(3e^{\lambda t} + 3\alpha - 1 - e^{2\lambda t} - 2\alpha e^{\lambda t} - \alpha^2\right)}{(e^{\lambda t} + \alpha - 1)^2 (e^{\lambda t} + \alpha)^2}. \tag{3.12}$$

The approximate variance estimates for $V\left(\widehat{S}(t)\right)$ and $V\left(\widehat{H}(t)\right)$ are obtained as follows:

$$V\left(\widehat{S}(t)\right) \simeq \left[G_1' I^{-1}(\alpha, \lambda) G_1\right]_{(\widehat{\alpha}_{ML}, \widehat{\lambda}_{ML})},$$

$$V\left(\widehat{H}(t)\right) \simeq \left[G_2' I^{-1}(\alpha,\lambda) G_2\right]_{(\widehat{\alpha}_{ML},\widehat{\lambda}_{ML,})},$$

where G_i^T is the transpose of G_i, $i = 1, 2$. Based on these results, the approximate CIs for $S(t)$ and $H(t)$ are the following:

$$\left(\widehat{S}(t) - z_{\gamma/2}\sqrt{V\left(\widehat{S}(t)\right)}, \widehat{S}(t) + z_{\gamma/2}\sqrt{V\left(\widehat{S}(t)\right)}\right) \tag{3.13}$$

and

$$\left(\widehat{H}(t) - z_{\gamma/2}\sqrt{V\left(\widehat{H}(t)\right)}, \widehat{H}(t) + z_{\gamma/2}\sqrt{V\left(\widehat{H}(t)\right)}\right). \tag{3.14}$$

4. Bayesian estimation

The BS of parameters α and λ is presented in this section under symmetric and asymmetric loss functions, including SEL, LINEX, GEL, and ALB. Parameters α and λ are presumed to be separately distributed, following gamma prior distributions characterized by shape parameters a_i and scale parameters b_i, for $i = 1, 2$. A detailed discussion of the rationale behind this choice can be found in [24, 25]. The joint prior density function of α and λ is expressed as follows:

$$\pi(\alpha,\lambda) \propto \alpha^{a_1-1}\lambda^{a_2-1}\exp(-b_1\alpha - b_2\lambda), a_i, b_i > 0, \text{ for } i = 1, 2. \tag{4.1}$$

More researchers have used Gamma as a prior distribution, such as [26, 27].

The posterior of α and λ is given from (3.1) and (4.1), as follows.

$$\begin{aligned}\pi^*(\alpha,\lambda|\underline{\mathbf{x}}) &= I^{-1}(\alpha+1)^{-m}\alpha^{a_1+2m-1}\lambda^{a_2+m-1}\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}+1\right)e^{\lambda\sum_{i=1}^{m}x_i-\alpha b_1-\lambda b_2}\\ &\times \frac{\left(\prod_{i=1}^{m}\left(\frac{\alpha^2\left(\alpha+e^{\lambda x_i}\right)}{(\alpha+1)\left(\alpha+e^{\lambda x_i}-1\right)^2}\right)^{k(R_i+1)-1}\right)}{\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}-1\right)^3},\end{aligned} \tag{4.2}$$

where I denotes the normalizing constant, defined as follows:

$$\begin{aligned}I &= \int_0^\infty\int_0^\infty(\alpha+1)^{-m}\alpha^{a_1+2m-1}\lambda^{a_2+m-1}\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}+1\right)e^{\lambda\sum_{i=1}^{m}x_i-\alpha b_1-\lambda b_2}\\ &\times \frac{\left(\prod_{i=1}^{m}\left(\frac{\alpha^2\left(\alpha+e^{\lambda x_i}\right)}{(\alpha+1)\left(\alpha+e^{\lambda x_i}-1\right)^2}\right)^{k(R_i+1)-1}\right)}{\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}-1\right)^3}d\alpha d\lambda.\end{aligned} \tag{4.3}$$

The Bayes estimator of any given function $g(\alpha,\lambda)$ using the SEL function corresponds to its posterior expectation and is given by:

$$\begin{aligned}\hat{g}(\alpha,\lambda)_{BS} &= E(g(\alpha,\lambda)|\underline{\mathbf{x}})\\ &\simeq \int_0^\infty\int_0^\infty g(\alpha,\lambda)\pi^*(\alpha,\lambda|\underline{\mathbf{x}})d\alpha d\lambda.\end{aligned} \tag{4.4}$$

The Bayes estimator under the LINEX is expressed as follows:

$$\begin{aligned}\hat{g}(\alpha,\lambda)_{BL} &= -\frac{1}{h}\log\left[E(e^{-hg(\alpha,\lambda)}|\underline{\mathbf{x}})\right]\\ &\simeq -\frac{1}{h}\log\left[\int_0^\infty\int_0^\infty e^{-hg(\alpha,\lambda)}\pi^*(\alpha,\lambda|\underline{\mathbf{x}})d\alpha d\lambda\right].\end{aligned} \tag{4.5}$$

The Bayes estimator under the GEL is expressed as follows:

$$\hat{g}(\alpha,\lambda)_{BG} = \left(E(g^{-q}(\alpha,\lambda)|\underline{\mathbf{x}})\right)^{-1/q} \simeq \left(\int_0^\infty\int_0^\infty g^{-q}(\alpha,\lambda)\pi^*(\alpha,\lambda|\underline{\mathbf{x}})d\alpha d\lambda\right)^{-1/q}. \tag{4.6}$$

The Bayes estimator under ALB, as referenced in [28], is articulated as follows:

$$\begin{aligned}\hat{g}(\alpha,\lambda)_{BB} &= \frac{E(g^{c+1}(\alpha,\lambda)|\underline{\mathbf{x}})}{E(g^{c}(\alpha,\lambda)|\underline{\mathbf{x}})}\\ &\simeq \frac{\int_0^\infty\int_0^\infty g^{c+1}(\alpha,\lambda)\pi^*(\alpha,\lambda|\underline{\mathbf{x}})d\alpha d\lambda}{\int_0^\infty\int_0^\infty g^{c}(\alpha,\lambda)\pi^*(\alpha,\lambda|\underline{\mathbf{x}})d\alpha d\lambda}.\end{aligned} \tag{4.7}$$

Unfortunately, based on (4.4)–(4.7), the acquisition of Bayes estimators in an explicit form is challenging; therefore, approximation techniques such as the MCMC approach are advised. The subsequent part delineates the MCMC approach utilized to approximate the Bayes estimators.

4.1. MCMC technique

The conditional posteriors $\pi_1^*(\alpha \mid \lambda; \underline{\mathbf{x}})$ and $\pi_2^*(\lambda \mid \alpha; \underline{\mathbf{x}})$ can be derived from the posterior distribution and are expressed as follows:

$$\begin{aligned}\pi_1^*(\alpha|\lambda;\underline{\mathbf{x}}) &\propto e^{-\alpha b_1}(\alpha+1)^{-m}\alpha^{a_1+2m-1}\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}+1\right)\\ &\times \frac{\prod_{i=1}^{m}\left(\frac{\alpha^2\left(\alpha+e^{\lambda x_i}\right)}{(\alpha+1)\left(\alpha+e^{\lambda x_i}-1\right)^2}\right)^{k(R_i+1)-1}}{\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}-1\right)^3}\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\pi_2^*(\lambda|\alpha;\underline{\mathbf{x}}) &\propto \lambda^{a_2+m-1}e^{\lambda\sum_{i=1}^{m}x_i-\lambda b_2}\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}+1\right)\\ &\times \frac{\prod_{i=1}^{m}\left(\frac{\alpha+e^{\lambda x_i}}{\left(\alpha+e^{\lambda x_i}-1\right)^2}\right)^{k(R_i+1)-1}}{\prod_{i=1}^{m}\left(\alpha+e^{\lambda x_i}-1\right)^3}.\end{aligned} \tag{4.9}$$

Since the conditional distributions in (4.8) and (4.9) are not of standard form, the Metropolis-Hastings algorithm is employed to generate samples of (α,λ) within MCMC Algorithm 1 (see [29,30]).

Algorithm 1 MCMC technique.

Step 1: Initialize the parameter vector as $\Theta^{(0)} = (\hat{\alpha}_{\mathrm{ML}}, \hat{\lambda}_{\mathrm{ML}})$, where $\Theta = (\alpha, \lambda)$.
Step 2: Set $i = 1$.
Step 3: For each parameter θ_k $(k = 1, 2)$, where $\theta_1 \equiv \alpha$ and $\theta_2 \equiv \lambda$, generate a candidate value θ_k^* from the normal distribution:

$$\theta_k^* \sim \mathcal{N}(\theta_k^{(i-1)}, \widehat{V}(\hat{\theta}_k)).$$

Step 4: Compute the acceptance probability:

$$\rho_{\theta_k} = \min\left[1, \frac{\pi_k^*(\theta_k^* \mid \underline{\mathbf{x}})}{\pi_k^*(\theta_k^{(i-1)} \mid \underline{\mathbf{x}})}\right], \quad k = 1, 2.$$

Step 5: Get a random value u distributed uniformly in the interval $(0, 1)$. If $u \leq \rho_{\theta_k}$, accept the proposal by setting $\theta_k^{(i)} = \theta_k^*$. Otherwise, retain the previous value: $\theta_k^{(i)} = \theta_k^{(i-1)}$.
Step 6: Increment the iteration counter: $i = i + 1$.
Step 7: Repeat Steps 3 to 6 for N iterations to obtain the samples $\left(\alpha^{(i)}, \lambda^{(i)}\right)$, for $i = 1, 2, \ldots, N$.

Let $g(\alpha, \lambda)$ be a function for the model parameters. The Bayes estimators of g can be approximated utilizing MCMC-provided samples $\left(\alpha^{(i)}, \lambda^{(i)}\right)$ for $i = B + 1, B + 2, B + 3, \ldots, N$, with B representing the burn-in period. The approximations are calculated using the SEL, LINEX, GEL, and ALB functions in the following order:

$$\hat{g}(\alpha, \lambda)_{BS} = \frac{1}{N - B} \sum_{i=B+1}^{N} g(\alpha^{(i)}, \lambda^{(i)}), \tag{4.10}$$

$$\hat{g}(\alpha, \lambda)_{BL} = \frac{-1}{h} \log\left[\frac{1}{N - B} \sum_{i=B+1}^{N} \exp\{h\, g(\alpha^{(i)}, \lambda^{(i)})\}\right], \tag{4.11}$$

$$\hat{g}(\alpha, \lambda)_{BG} = \left[\frac{1}{N - B} \sum_{i=B+1}^{N} [g(\alpha^{(i)}, \lambda^{(i)})]^{-q}\right]^{-1/q}, \tag{4.12}$$

$$\hat{g}(\alpha, \lambda)_{BB} = \frac{\sum_{i=B+1}^{N} [g(\alpha^{(i)}, \lambda^{(i)})]^{c+1}}{\sum_{i=B+1}^{N} [g(\alpha^{(i)}, \lambda^{(i)})]^{c}}. \tag{4.13}$$

The performance of the Metropolis–Hastings algorithm is assessed by monitoring the acceptance rate. Across simulation settings, the average acceptance rate lies between 0.25 and 0.38, indicating efficient mixing of the Markov chain. The proposal variance, based on the inverse Fisher information matrix, is slightly adjusted when necessary to maintain acceptance rates within a desirable range.

4.2. Bayesian credible intervals

The $100(1-\gamma)\%$ Bayesian CRI for a parameter η, where η is either α or λ, is derived from (4.14),

$$\int_{L}^{U} \pi^{*}(\eta|\underline{\mathbf{x}})d\eta = 1-\gamma. \tag{4.14}$$

Here, L and U denote the lower and upper bounds of the CRI, respectively.

Due to the analytical complexity of the integral in (4.14), the MCMC estimate for the CRIs is employed, utilizing the $(N-B)$ produced parameter values. By arranging the produced values in ascending sequence as shown below, $\left(\alpha^{(B+1)},...,\alpha^{(N)}\right)$ and $\left(\lambda^{(B+1)},...,\lambda^{(N)}\right)$, the $100(1-\gamma)\%$ CIs are as follows:

$$(\alpha^{(w_1)},\alpha^{(w_2)}) \quad and \quad (\lambda^{(w_1)},\lambda^{(w_2)}),$$

where $w_1 = ((N-B)\gamma/2)$ and $w_2 = ((N-B)(1-\gamma/2))$.

5. Simulation study

Inside this section, a numerical study depending on simulated data is executed to examine the efficiency of the proposed methods of estimation and to compare them. The process of generating data from the GLD has been operated under PF-FC with different values of (n,m,k), and the used values are $(20,15,1), (20,15,2), (30,20,1), (30,20,2), (50,30,1)$, and $(50,30,2)$. Three schemes of PF-FC are used as follows:

1. Scheme 1: $R_i = 0,\ i = 2,...,m$ and $R_1 = n-m$.
2. Scheme 2: $R_i = 0,\ i = 1,2,...,m-1$ and $R_m = n-m$.
3. Scheme 3: $R_i = 1,\ i = 1,2,...,n-m$ and $R_j = 0,\ j = n-m+1,...,m$.

Through the simulation study, 1000 samples are generated from the $GLD(\alpha,\lambda)$ with true values of parameters chosen as $(\alpha,\lambda) = (1.3, 0.7)$, while $S(t)$ and $H(t)$ are calculated at $t = 1.5$.

In the simulation study, the hyper-parameters for the Informative Priors (IP) are determined using the Empirical Bayes (EB) approach, also referred to here as elective hyper-parameters. In this framework, the hyper-parameters are estimated based on the Maximum Likelihood Estimates (MLEs) of the simulated data. While this involves a data-driven prior, it is utilized in this simulation context to evaluate the performance and sensitivity of the Bayesian estimators under controlled scenarios where prior information is aligned with the underlying model parameters. This empirical Bayes construction is used solely for simulation purposes and does not affect the real data analysis, where NIPs are adopted. For the BEs, estimates are obtained based on different loss functions, such as the SEL, LINEX with hyperparameter $h = 0.5$, GEL with hyperparameter $q = 0.5$, and ALB with hyperparameter $c = 0.5$. For the prior distributions of the parameters, we use informative Gamma priors with hyper-parameters (a_1, b_1) for α and (a_2, b_2) for λ; the values of hyper parameters can be obtained as follows:

1. Obtain l number of samples generated from the GLD under the PF-FC.
2. Calculate the corresponding MLEs $(\hat{\alpha}^j, \hat{\lambda}^j),\ j = 1,2,...,l$.

3. Obtain the mean and variance of $(\hat{\alpha}^j, \hat{\lambda}^j)$, $j = 1, 2, ..., l.$ as

$$\frac{1}{l}\sum_{j=1}^{l}\hat{\Theta}^j, \quad \frac{1}{l-1}\sum_{j=1}^{l}\left(\hat{\Theta}^j - \frac{1}{l}\sum_{j=1}^{l}\hat{\Theta}^j\right)^2,$$

where Θ refers to α, λ.

4. For the parameter Θ that follows the prior Gamma distribution $\pi(\Theta) \propto \Theta^{a-1}\exp -b\Theta$ with mean a/b and variance a/b^2, the hyperparameters can be obtained by solving the following two equations:

$$\frac{1}{l}\sum_{j=1}^{l}\hat{\Theta}^j = \frac{a}{b}, \text{ and } \frac{1}{l-1}\sum_{j=1}^{l}\left(\hat{\Theta}^j - \frac{1}{l}\sum_{j=1}^{l}\hat{\Theta}^j\right)^2 = \frac{a}{b^2}. \tag{5.1}$$

To enhance reproducibility, the actual values of the hyperparameters used in the simulation study are reported. For example, in the case $(n = 20, m = 15)$, the informative prior parameters are obtained from the MLEs using Eq (5.1), yielding $a_1 = 33.75$, $b_1 = 26.03$, $a_2 = 9.78$, and $b_2 = 14.13$. Similar calculations are performed for other sample configurations.

Additionally, the NIPs are used with putting $(a_1, b_1) = (0, 0)$ and $(a_2, b_2) = (0, 0)$. Moreover, the BEs are derived using the MCMC method in Algorithm 1, where the number of MCMC iterations is 12000 with a burn-in period of 2000. The comparative analysis between Informative and NIP serves as a sensitivity analysis, illustrating the stability of the Bayesian estimators. It was observed that while Informative priors improve precision, the model remains robust and provides consistent results even when NIPs are used, reflecting its ability to handle potential variability in the data.

For each estimate, the mean squared error (MSE) and the absolute bias (Bias) are calculated to compare the estimation methods. Regarding the interval estimation for the parameters and their reliability functions $\alpha, \lambda, S(t = 1.5)$ and $H(t = 1.5)$, we compute three kinds of 95% intervals: One frequentist CI and two Bayesian CRIs (informative and non-informative). For each interval, the average length (AL) and the coverage probability (CP) are calculated under PF-FC with $k = 1, 2$. Algorithm 2 summarizes the simulation process with all steps:

Algorithm 2 Simulation procedure under PF-FC scheme

Input: True parameters (α, λ), sample sizes (n, m, k), censoring schemes, number of replications $N = 1000$.

for $r = 1$ to N **do**

Generate a sample of size n from $GLD(\alpha, \lambda)$.

Apply the PF-FC censoring scheme (Scheme 1, 2, or 3) to obtain censored data.

Compute the MLEs $(\hat{\alpha}, \hat{\lambda})$.

Estimate hyperparameters (a_1, b_1, a_2, b_2) using the Empirical Bayes approach based on MLEs.

Obtain Bayesian estimates using MCMC under:

- Informative priors
- Non-informative priors

Compute estimates under different loss functions (SEL, LINEX, GEL, ALB).

Calculate reliability measures $S(t)$ and $H(t)$ at $t = 1.5$.

Construct 95% confidence/credible intervals.

Record Bias, MSE, Average Length (AL), and Coverage Probability (CP).

end for

Output: Average performance measures over all replications.

Tables 1 and 2 show the MLEs and BEs of α and λ, respectively, while Tables 3 and 4 show the MLEs and BEs of $S(t = 1.5)$ and $H(t = 1.5)$, respectively. The AL and CP for the 95% CIs and CRIs for parameters α, λ are presented in Tables 5 and 6. Moreover, Tables 7 and 8 show the AL and CP for the 95% CIs and CRIs for $S(t = 1.5)$ and $H(t = 1.5)$, respectively.
Based on the simulation Tables 1–9, we can conclude the following notes:

1. In most cases, the BEs based on informative priors perform better than the methods of ML and the BEs of NIPs.
2. We discovered that the GEL outperforms the SEL, LINEX, and ALB loss functions in BEs.
3. Generally, the MSE decreases as n and m increase.
4. There is no more difference between the estimates of the parameters under $k = 2$ and $k = 1$, which means that the estimation methods under the PF-FC is efficient like the classical estimation under PT-II censoring.
5. The numerical study shows that the CS II performs better than the other CSs.
6. The performance of the approximate confidence intervals is evaluated for sample sizes $n = 20, 30$, and 50. The results show that even for the smaller sample size ($n = 20$), the coverage probabilities remain close to the nominal levels and are comparable to those obtained for larger samples. This indicates that the normal approximation remains practically adequate under the considered censoring schemes.
7. In most cases, the asymptotic CIs and the CRIs have similar performance according to the CP measure.
8. The sensitivity analysis presented in Table 9 demonstrates that the BEs remain stable across a wide range of hyperparameter values governing the asymmetry. Although slight systematic variations are observed as the hyperparameters change, the resulting estimates and associated

performance measures (e.g., bias and MSE) exhibit no significant deterioration. This indicates that the proposed method is robust to prior specification and does not rely on a narrowly tuned choice of hyperparameters. Furthermore, the gradual and interpretable effect of the asymmetry parameters confirms that they provide controlled penalization for over- and under-estimation, enhancing the practical interpretability of the model in reliability applications. Additionally, ositive h penalizes overestimation (critical for safety in hazard rate estimation); $q = 0.5$ in GEL provides robustness against outliers in heterogeneous failure data; $c = 0.5$ in ALB balances absolute and relative errors for maintenance planning.

9. Based on these findings, we caution researchers against relying on conventional asymptotic intervals for $S(t)$ early censoring schemes (especially Schemes 2 and 3) without conducting a sensitivity analysis for initial values. We recommend using bootstrap or penalized Bayesian methods whenever possible.

Table 1. MLEs and BEs for α under PF-FC with $k = 1, 2$.

(n, m)	scheme		ML	IP BS	BL	BG	BB	NIP BS	BL	BG	BB
			k=1								
(20,15)	I	MSE	1.5705	0.0036	0.0032	0.0033	0.0045	5.1378	1.5765	1.9865	9.1909
		Bias	0.6414	0.0181	0.0071	0.0070	0.0349	1.2950	0.6520	0.5636	1.8873
	II	MSE	1.9762	0.0033	0.0031	0.0032	0.0041	10.1833	2.4747	3.7539	18.3619
		Bias	0.7868	0.0149	0.0043	0.0095	0.0312	1.9785	0.9830	0.9749	2.8135
	III	MSE	1.3682	0.0035	0.0032	0.0032	0.0043	3.7063	1.3339	1.5974	6.4549
		Bias	0.5499	0.0164	0.0056	0.0084	0.0330	1.0787	0.5496	0.4281	1.5931
(30,20)	I	MSE	1.1813	0.0040	0.0036	0.0036	0.0050	2.6481	1.1170	1.2887	4.2977
		Bias	0.5067	0.0203	0.0096	0.0041	0.0366	0.8690	0.4773	0.3624	1.2605
	II	MSE	1.6166	0.0039	0.0035	0.0035	0.0049	6.2215	2.2608	2.8376	10.3192
		Bias	0.6791	0.0184	0.0081	0.0053	0.0343	1.6452	0.9373	0.8942	2.2519
	III	MSE	1.4287	0.0040	0.0036	0.0035	0.0051	2.8821	1.3161	1.5139	4.3860
		Bias	0.5471	0.0204	0.0098	0.0039	0.0367	0.8925	0.5102	0.4189	1.2515
(50,30)	I	MSE	0.7289	0.0044	0.0041	0.0041	0.0052	1.2670	0.7313	0.7801	1.8385
		Bias	0.3268	0.0152	0.0049	0.0083	0.0309	0.4966	0.2862	0.1814	0.7321
	II	MSE	1.1334	0.0042	0.0038	0.0037	0.0052	2.9990	1.5733	1.7156	4.4588
		Bias	0.2157	0.0211	0.0110	0.0018	0.0364	0.8465	0.4863	0.4330	1.1752
	III	MSE	0.9316	0.0049	0.0045	0.0045	0.0057	1.4818	0.9003	0.9790	2.0278
		Bias	0.3680	0.0183	0.0083	0.0046	0.0336	0.5225	0.3223	0.2355	0.7362
			k=2								
(20,15)	I	MSE	1.5101	0.0132	0.0110	0.0089	0.0169	5.0600	1.8537	2.3012	8.4584
		Bias	0.5110	0.1012	0.0899	0.0769	0.1174	1.3574	0.7299	0.6542	1.9259
	II	MSE	2.4591	0.0073	0.0061	0.0050	0.0096	14.2346	4.2973	5.9153	25.8160
		Bias	0.9426	0.0595	0.0487	0.0356	0.0755	2.9231	1.5537	1.6798	4.0029
	III	MSE	1.2861	0.0113	0.0092	0.0073	0.0147	4.3685	1.4302	1.7966	8.0000
		Bias	0.3548	0.0935	0.0821	0.0690	0.1099	1.1735	0.5985	0.4899	1.7389
(30,20)	I	MSE	1.1338	0.0148	0.0123	0.0099	0.0187	2.8108	1.2538	1.4441	4.5197
		Bias	0.3644	0.1097	0.0982	0.0849	0.1263	0.9283	0.5254	0.4154	1.3341
	II	MSE	3.9033	0.0156	0.0136	0.0117	0.0189	19.8311	6.9080	10.2346	30.5780
		Bias	1.4650	0.0926	0.0818	0.0694	0.1081	3.4197	2.0020	2.2795	4.3385
	III	MSE	0.9826	0.0130	0.0108	0.0087	0.0166	2.2080	1.0742	1.1734	3.4592
		Bias	0.2465	0.1026	0.0916	0.0788	0.1185	0.8226	0.4731	0.3580	1.1873
(50,30)	I	MSE	0.9298	0.0177	0.0150	0.0124	0.0219	1.7870	1.0206	1.1313	2.4892
		Bias	0.2975	0.1189	0.1076	0.0948	0.1351	0.6122	0.3745	0.2869	0.8588
	II	MSE	3.2210	0.0276	0.0244	0.0214	0.0324	13.3674	6.2776	8.1278	18.8485
		Bias	1.4612	0.1496	0.1388	0.1272	0.1646	2.9019	1.9518	2.1282	3.5187
	III	MSE	0.7343	0.0131	0.0110	0.0090	0.0165	1.1426	0.7259	0.7899	1.5384
		Bias	0.0420	0.1007	0.0899	0.0775	0.1162	0.3510	0.1776	0.0799	0.5543

Table 2. MLEs and BEs for λ under PF-FC with $k = 1, 2$.

(n, m)	scheme		ML	IP BS	IP BL	IP BG	IP BB	NIP BS	NIP BL	NIP BG	NIP BB
			k=1								
(20,15)	I	MSE	0.2145	0.0104	0.0101	0.0102	0.0112	0.2805	0.2400	0.2237	0.0112
		Bias	0.1779	0.0009	0.0046	0.0224	0.0165	0.2137	0.1766	0.0857	0.2968
	II	MSE	0.3318	0.0098	0.0094	0.0093	0.0108	0.5189	0.4309	0.3812	0.0108
		Bias	0.2769	0.0099	0.0042	0.0138	0.0257	0.4230	0.3615	0.2489	0.5386
	III	MSE	0.2279	0.0099	0.0096	0.0095	0.0109	0.2451	0.2084	0.1908	0.0109
		Bias	0.1754	0.0071	0.0016	0.0164	0.0228	0.1971	0.1586	0.0594	0.2845
(30,20)	I	MSE	0.1473	0.0078	0.0076	0.0077	0.0084	0.1806	0.1614	0.1576	0.0084
		Bias	0.1367	0.0022	0.0026	0.0181	0.0158	0.1461	0.1187	0.0393	0.2148
	II	MSE	0.2630	0.0077	0.0073	0.0072	0.0085	0.4328	0.3677	0.3240	0.0085
		Bias	0.2465	0.0125	0.0074	0.0089	0.0268	0.4034	0.3494	0.2457	0.5095
	III	MSE	0.1966	0.0083	0.0081	0.0081	0.0089	0.2257	0.2007	0.1916	0.0089
		Bias	0.1566	0.0008	0.0041	0.0199	0.0147	0.1653	0.1361	0.0594	0.2344
(50,30)	I	MSE	0.0893	0.0075	0.0073	0.0075	0.0078	0.1027	0.0960	0.0988	0.0078
		Bias	0.0819	0.0020	0.0059	0.0186	0.0090	0.0701	0.0518	0.0081	0.1209
	II	MSE	0.1929	0.0059	0.0058	0.0060	0.0061	0.2983	0.2665	0.2402	0.0061
		Bias	0.0256	0.0031	0.0072	0.0208	0.0088	0.1759	0.1420	0.0743	0.2460
	III	MSE	0.1025	0.0061	0.0060	0.0062	0.0063	0.1186	0.1107	0.1116	0.0063
		Bias	0.0920	0.0046	0.0085	0.0216	0.0068	0.0838	0.0644	0.0050	0.1362
			k=2								
(20,15)	I	MSE	0.1241	0.0425	0.0434	0.0488	0.0386	0.1239	0.1189	0.1396	0.0386
		Bias	0.1811	0.1863	0.1894	0.2038	0.1747	0.0947	0.1149	0.1933	0.0284
	II	MSE	0.2490	0.0097	0.0103	0.0132	0.0078	0.5491	0.4393	0.3621	0.0078
		Bias	0.1735	0.0752	0.0798	0.0972	0.0606	0.5061	0.4280	0.3054	0.6436
	III	MSE	0.1439	0.0407	0.0416	0.0470	0.0368	0.1227	0.1178	0.1425	0.0368
		Bias	0.2040	0.1839	0.1870	0.2016	0.1721	0.1055	0.1266	0.2104	0.0349
(30,20)	I	MSE	0.1246	0.0535	0.0544	0.0597	0.0496	0.1060	0.1064	0.1349	0.0496
		Bias	0.2374	0.2185	0.2209	0.2331	0.2088	0.1762	0.1902	0.2555	0.1222
	II	MSE	0.2257	0.0278	0.0285	0.0322	0.0252	0.6347	0.5327	0.4681	0.0252
		Bias	0.2127	0.1276	0.1312	0.1459	0.1152	0.5140	0.4480	0.3475	0.6271
	III	MSE	0.1365	0.0448	0.0457	0.0508	0.0410	0.1062	0.1062	0.1320	0.0410
		Bias	0.2328	0.1993	0.2019	0.2146	0.1891	0.1504	0.1657	0.2331	0.0939
(50,30)	I	MSE	0.1249	0.0647	0.0655	0.0703	0.0611	0.1155	0.1171	0.1408	0.0611
		Bias	0.2659	0.2462	0.2480	0.2577	0.2385	0.2321	0.2406	0.2858	0.1955
	II	MSE	0.0909	0.0528	0.0536	0.0584	0.0492	0.2530	0.2234	0.1986	0.0492
		Bias	0.0728	0.2220	0.2240	0.2348	0.2135	0.2613	0.2267	0.1563	0.3338
	III	MSE	0.1461	0.0484	0.0492	0.0537	0.0451	0.1211	0.1224	0.1474	0.0451
		Bias	0.2765	0.2090	0.2111	0.2218	0.2005	0.2213	0.2319	0.2837	0.1781

Table 3. MLEs and BEs for $S(t = 1.5)$ under PF-FC with $k = 1, 2$.

(n, m)	scheme		ML	IP BS	IP BL	IP BG	IP BB	NIP BS	NIP BL	NIP BG	NIP BB
			k=1								
(20,15)	I	MSE	0.0099	0.0034	0.0033	0.0033	0.0036	0.0106	0.0103	0.0104	0.0116
		Bias	0.0122	0.0139	0.0124	0.0005	0.0233	0.0291	0.0262	0.0012	0.0482
	II	MSE	0.0094	0.0031	0.0030	0.0032	0.0032	0.0098	0.0097	0.0111	0.0097
		Bias	0.0069	0.0083	0.0070	0.0053	0.0173	0.0033	0.0007	0.0249	0.0214
	III	MSE	0.0088	0.0030	0.0030	0.0031	0.0032	0.0091	0.0089	0.0094	0.0098
		Bias	0.0022	0.0101	0.0087	0.0041	0.0194	0.0202	0.0172	0.0102	0.0394
(30,20)	I	MSE	0.0062	0.0026	0.0025	0.0025	0.0028	0.0065	0.0063	0.0062	0.0071
		Bias	0.0088	0.0112	0.0100	0.0010	0.0192	0.0224	0.0201	0.0013	0.0375
	II	MSE	0.0065	0.0025	0.0025	0.0026	0.0026	0.0069	0.0069	0.0085	0.0064
		Bias	0.0126	0.0056	0.0044	0.0057	0.0130	0.0092	0.0111	0.0312	0.0051
	III	MSE	0.0062	0.0026	0.0026	0.0026	0.0028	0.0065	0.0064	0.0066	0.0070
		Bias	0.0036	0.0122	0.0110	0.0001	0.0202	0.0174	0.0152	0.0052	0.0320
(50,30)	I	MSE	0.0044	0.0024	0.0024	0.0024	0.0026	0.0046	0.0045	0.0045	0.0050
		Bias	0.0071	0.0106	0.0096	0.0010	0.0169	0.0167	0.0150	0.0004	0.0271
	II	MSE	0.0037	0.0021	0.0021	0.0020	0.0023	0.0045	0.0045	0.0050	0.0043
		Bias	0.0054	0.0118	0.0109	0.0033	0.0173	0.0028	0.0015	0.0113	0.0119
	III	MSE	0.0035	0.0020	0.0020	0.0020	0.0022	0.0037	0.0036	0.0036	0.0040
		Bias	0.0052	0.0120	0.0110	0.0027	0.0181	0.0143	0.0128	0.0007	0.0240
			k=2								
(20,15)	I	MSE	0.0471	0.0254	0.0250	0.0229	0.0271	0.0522	0.0512	0.0461	0.0563
		Bias	0.1964	0.1443	0.1429	0.1344	0.1509	0.2082	0.2056	0.1906	0.2191
	II	MSE	0.0094	0.0054	0.0052	0.0041	0.0063	0.0084	0.0082	0.0085	0.0094
		Bias	0.0558	0.0615	0.0601	0.0497	0.0692	0.0406	0.0377	0.0103	0.0596
	III	MSE	0.0426	0.0235	0.0231	0.0210	0.0253	0.0476	0.0466	0.0415	0.0517
		Bias	0.1882	0.1405	0.1390	0.1302	0.1471	0.2000	0.1974	0.1815	0.2114
(30,20)	I	MSE	0.0507	0.0316	0.0312	0.0292	0.0333	0.0551	0.0543	0.0501	0.0585
		Bias	0.2127	0.1676	0.1665	0.1597	0.1728	0.2224	0.2204	0.2096	0.2303
	II	MSE	0.0239	0.0176	0.0173	0.0161	0.0186	0.0246	0.0244	0.0247	0.0252
		Bias	0.0957	0.1042	0.1031	0.0954	0.1100	0.0761	0.0738	0.0529	0.0907
	III	MSE	0.0411	0.0260	0.0256	0.0236	0.0275	0.0449	0.0441	0.0400	0.0481
		Bias	0.1907	0.1518	0.1506	0.1435	0.1572	0.1994	0.1974	0.1858	0.2079
(50,30)	I	MSE	0.0530	0.0385	0.0381	0.0364	0.0398	0.0558	0.0552	0.0523	0.0581
		Bias	0.2223	0.1891	0.1882	0.1833	0.1928	0.2282	0.2268	0.2200	0.2335
	II	MSE	0.0370	0.0335	0.0332	0.0318	0.0347	0.0337	0.0332	0.0308	0.0357
		Bias	0.1791	0.1773	0.1765	0.1721	0.1807	0.1653	0.1638	0.1538	0.1727
	III	MSE	0.0371	0.0276	0.0273	0.0257	0.0288	0.0387	0.0382	0.0357	0.0407
		Bias	0.1827	0.1575	0.1565	0.1512	0.1616	0.1866	0.1853	0.1777	0.1924

Table 4. MLEs and BEs for $H(t = 1.5)$ under PF-FC with $k = 1, 2$.

(n, m)	scheme		ML	IP BS	IP BL	IP BG	IP BB	NIP BS	NIP BL	NIP BG	NIP BB
			k=1								
(20,15)	I	MSE	0.1132	0.0134	0.0131	0.0135	0.0138	0.1497	0.1253	0.1113	0.1819
		Bias	0.0937	0.0029	0.0088	0.0258	0.0122	0.1188	0.0957	0.0535	0.1619
	II	MSE	0.1842	0.0129	0.0125	0.0126	0.0137	0.3063	0.2506	0.2191	0.3774
		Bias	0.1774	0.0075	0.0017	0.0146	0.0221	0.2805	0.2414	0.1868	0.3442
	III	MSE	0.1257	0.0128	0.0124	0.0126	0.0134	0.1313	0.1095	0.0950	0.1630
		Bias	0.1084	0.0042	0.0017	0.0185	0.0191	0.1214	0.0979	0.0542	0.1659
(30,20)	I	MSE	0.0664	0.0101	0.0099	0.0102	0.0104	0.0838	0.0740	0.0671	0.0981
		Bias	0.0683	0.0016	0.0066	0.0208	0.0112	0.0785	0.0634	0.0312	0.1098
	II	MSE	0.1279	0.0102	0.0099	0.0099	0.0108	0.2295	0.1923	0.1630	0.2849
		Bias	0.1602	0.0098	0.0048	0.0090	0.0222	0.2739	0.2410	0.1929	0.3296
	III	MSE	0.0923	0.0105	0.0103	0.0106	0.0109	0.1068	0.0929	0.0830	0.1267
		Bias	0.0907	0.0032	0.0083	0.0227	0.0096	0.1011	0.0839	0.0500	0.1350
(50,30)	I	MSE	0.0336	0.0094	0.0093	0.0095	0.0095	0.0368	0.0342	0.0315	0.0416
		Bias	0.0388	0.0044	0.0083	0.0194	0.0055	0.0378	0.0289	0.0074	0.0580
	II	MSE	0.0538	0.0077	0.0077	0.0080	0.0078	0.1143	0.0996	0.0844	0.1398
		Bias	0.0364	0.0073	0.0110	0.0215	0.0022	0.1378	0.1186	0.0865	0.1736
	III	MSE	0.0334	0.0076	0.0076	0.0079	0.0077	0.0383	0.0352	0.0317	0.0442
		Bias	0.0461	0.0080	0.0118	0.0229	0.0019	0.0486	0.0388	0.0160	0.0706
			k=2								
(20,15)	I	MSE	0.1076	0.0625	0.0638	0.0703	0.0578	0.1046	0.1051	0.1175	0.0985
		Bias	0.2761	0.2277	0.2310	0.2452	0.2162	0.2474	0.2560	0.2879	0.2202
	II	MSE	0.1042	0.0145	0.0153	0.0186	0.0123	0.2399	0.1816	0.1400	0.3366
		Bias	0.0363	0.0960	0.1006	0.1161	0.0826	0.2693	0.2165	0.1399	0.3608
	III	MSE	0.1082	0.0589	0.0603	0.0668	0.0541	0.1012	0.1015	0.1136	0.0957
		Bias	0.2719	0.2233	0.2266	0.2410	0.2115	0.2402	0.2495	0.2828	0.2117
(30,20)	I	MSE	0.1175	0.0778	0.0790	0.0852	0.0732	0.1119	0.1134	0.1251	0.1043
		Bias	0.3142	0.2651	0.2676	0.2792	0.2558	0.2977	0.3029	0.3254	0.2792
	II	MSE	0.1027	0.0423	0.0432	0.0471	0.0395	0.3155	0.2567	0.2200	0.4004
		Bias	0.0018	0.1613	0.1647	0.1772	0.1508	0.2324	0.1869	0.1244	0.3080
	III	MSE	0.1063	0.0654	0.0666	0.0724	0.0610	0.0956	0.0971	0.1082	0.0886
		Bias	0.2871	0.2426	0.2452	0.2572	0.2328	0.2622	0.2682	0.2927	0.2416
(50,30)	I	MSE	0.1238	0.0940	0.0950	0.1001	0.0901	0.1195	0.1209	0.1294	0.1134
		Bias	0.3350	0.2979	0.2996	0.3083	0.2910	0.3255	0.3283	0.3425	0.3142
	II	MSE	0.0647	0.0822	0.0831	0.0878	0.0786	0.0786	0.0731	0.0728	0.0879
		Bias	0.1797	0.2792	0.2809	0.2893	0.2725	0.0590	0.0758	0.1152	0.0190
	III	MSE	0.1026	0.0700	0.0710	0.0756	0.0665	0.0942	0.0955	0.1031	0.0889
		Bias	0.2929	0.2530	0.2550	0.2642	0.2456	0.2734	0.2773	0.2939	0.2596

Table 5. The asymptotic CIs and Bayesian CRI for α under PF-FC with $k = 1, 2$.

k	(n, m)	scheme	Asymp. CI		CRI (IP)		CRI (NIP)	
			AL	CP	AL	CP	AL	CP
1	(20,15)	I	6.021	1.000	0.811	1.000	6.498	0.884
		II	8.487	1.000	0.796	1.000	8.514	0.834
		III	5.687	1.000	0.812	1.000	5.814	0.892
	(30,20)	I	4.869	0.998	0.796	1.000	4.820	0.910
		II	7.879	1.000	0.788	1.000	6.895	0.780
		III	4.894	1.000	0.800	1.000	4.638	0.870
	(50,30)	I	3.627	0.988	0.789	1.000	3.419	0.891
		II	6.649	0.990	0.765	1.000	4.261	0.645
		III	3.704	0.982	0.777	1.000	3.251	0.855
2	(20,15)	I	7.218	1.000	0.820	1.000	6.398	0.814
		II	12.347	1.000	0.779	1.000	10.869	0.648
		III	6.862	1.000	0.823	1.000	6.039	0.878
	(30,20)	I	5.904	1.000	0.836	1.000	4.878	0.858
		II	13.102	1.000	0.781	0.994	10.655	0.464
		III	5.699	1.000	0.809	1.000	4.535	0.868
	(50,30)	I	4.692	1.000	0.830	1.000	3.541	0.810
		II	11.390	1.000	0.776	0.982	8.089	0.380
		III	4.544	1.000	0.798	1.000	2.956	0.856

Table 6. The asymptotic CIs and Bayesian CRI for λ under PF-FC with $k = 1, 2$.

k	(n, m)	scheme	Asymp. CI		CRI (IP)		CRI (NIP)	
			AL	CP	AL	CP	AL	CP
1	(20,15)	I	1.812	0.998	0.573	0.988	1.329	0.854
		II	2.976	1.000	0.580	0.992	1.720	0.800
		III	1.865	0.994	0.578	0.994	1.357	0.868
	(30,20)	I	1.550	0.992	0.539	0.992	1.172	0.890
		II	3.031	1.000	0.548	0.996	1.606	0.798
		III	1.634	0.988	0.540	0.992	1.189	0.848
	(50,30)	I	1.247	0.985	0.483	0.981	0.968	0.883
		II	2.856	0.984	0.493	0.991	1.173	0.661
		III	1.361	0.988	0.487	0.994	0.988	0.853
2	(20,15)	I	1.707	0.970	0.423	0.602	0.960	0.824
		II	4.175	1.000	0.518	0.988	1.913	0.786
		III	1.743	0.982	0.429	0.640	0.976	0.800
	(30,20)	I	1.388	0.952	0.376	0.442	0.801	0.768
		II	3.862	1.000	0.451	0.740	1.739	0.716
		III	1.510	0.964	0.392	0.562	0.845	0.790
	(50,30)	I	1.094	0.846	0.325	0.226	0.637	0.608
		II	2.738	1.000	0.339	0.322	1.244	0.734
		III	1.327	0.926	0.354	0.428	0.694	0.656

Table 7. The asymptotic CIs and Bayesian CRI for $S(t = 1.5)$ under PF-FC with $k = 1, 2$.

k	(n, m)	scheme	Asymp. CI		CRI (IP)		CRI (NIP)	
			AL	CP	AL	CP	AL	CP
1	(20,15)	I	0.388	0.910	0.293	0.988	0.411	0.948
		II	0.368	0.926	0.280	0.986	0.385	0.946
		III	0.379	0.930	0.289	0.988	0.410	0.966
	(30,20)	I	0.337	0.940	0.269	0.986	0.372	0.968
		II	0.340	0.956	0.256	0.992	0.337	0.950
		III	0.326	0.946	0.268	0.986	0.363	0.968
	(50,30)	I	0.275	0.961	0.240	0.977	0.312	0.977
		II	0.298	0.982	0.225	0.977	0.273	0.945
		III	0.265	0.973	0.237	0.990	0.299	0.986
2	(20,15)	I	0.363	0.402	0.287	0.502	0.390	0.442
		II	0.500	0.974	0.282	0.960	0.411	0.976
		III	0.363	0.432	0.289	0.528	0.395	0.480
	(30,20)	I	0.312	0.232	0.265	0.300	0.344	0.256
		II	0.477	0.742	0.255	0.668	0.357	0.716
		III	0.311	0.274	0.266	0.362	0.347	0.344
	(50,30)	I	0.362	0.264	0.354	0.279	0.395	0.233
		II	0.355	0.486	0.215	0.064	0.301	0.468
		III	0.254	0.196	0.234	0.228	0.285	0.254

Table 8. The asymptotic CIs and Bayesian CRI for $H(t = 1.5)$ under PF-FC with $k = 1, 2$.

k	(n, m)	scheme	Asymp. CI		CRI (IP)		CRI (NIP)	
			AL	CP	AL	CP	AL	CP
1	(20,15)	I	1.105	0.962	0.587	0.992	1.050	0.916
		II	1.739	0.984	0.582	0.984	1.374	0.892
		III	1.149	0.974	0.590	0.984	1.074	0.934
	(30,20)	I	0.913	0.964	0.542	0.980	0.890	0.942
		II	1.743	0.992	0.536	0.992	1.263	0.898
		III	0.991	0.970	0.541	0.990	0.928	0.932
	(50,30)	I	0.707	0.968	0.477	0.969	0.699	0.941
		II	1.362	0.899	0.465	0.977	0.923	0.767
		III	0.786	0.968	0.476	0.988	0.728	0.946
2	(20,15)	I	0.683	0.440	0.434	0.470	0.651	0.526
		II	2.293	1.000	0.515	0.960	1.594	0.898
		III	0.698	0.466	0.440	0.504	0.670	0.556
	(30,20)	I	0.517	0.308	0.379	0.272	0.514	0.388
		II	2.140	0.924	0.435	0.676	1.411	0.806
		III	0.592	0.394	0.393	0.356	0.560	0.490
	(50,30)	I	0.389	0.164	0.314	0.108	0.391	0.208
		II	1.297	0.908	0.311	0.092	0.871	0.794
		III	0.506	0.328	0.340	0.244	0.453	0.366

Table 9. Sensitivity analysis of hyperparameters for different loss functions under PF-FC with $(n, m, k) = (50, 30, 3)$, Scheme II, for α and λ.

Loss Function	Hyperparameter	MSE(IP)		MSE(NIP)	
		α	λ	α	λ
LINEX	$h = -1.0$	0.06034	0.11125	3.3654	0.15853
	$h = -0.5$	0.05417	0.11214	3.5742	0.12872
	$h = 0.5$ (Original)	0.04333	0.11392	3.8982	0.09676
	$h = 1.0$	0.03859	0.1148	2.9759	0.09785
GEL	$q = 0.5$ (Original)	0.03861	0.12037	4.53261	0.1218
	$q = 1.0$	0.03559	0.12288	3.56104	0.13515
	$q = 2.0$	0.0300	0.12798	1.8905	0.16534
ALB	$c = 0.5$ (Original)	0.05587	0.10827	4.357	0.14654
	$c = 1.0$	0.06384	0.10361	4.9231	0.2137
	$c = 2.0$	0.08158	0.09466	5.0889	0.32380
MLE (for reference)		3.438	0.0876		

6. Applications

In this section, we examine two real-world datasets to illustrate the applicability of the proposed PF-FC technique using the GLD. Under the PF-FC framework, MLE and Bayesian approaches are used to find the model parameters. For Bayesian estimation in real data applications, different prior specifications are adopted depending on the availability of prior information. For Dataset I, due to the absence of reliable historical information or expert knowledge, NIP are employed to avoid the issue of double use of data (data snooping), ensuring that the posterior inference is driven solely by the observed data. In contrast, for Dataset II, informative priors are incorporated to improve estimation stability and reduce variability, particularly under the PF-FC scheme.

We use the Akaike information criterion (AIC), the corrected Akaike information criterion (AIC_C), the consistent Akaike information criterion (CAIC), the Bayesian information criterion (BIC), and the Hannan–Quinn information criterion (HQIC) to check how well each dataset fits the model, as well as the Anderson–Darling (AD), Cramér–von Mises (CvM), and Kolmogorov–Smirnov (KS) tests. Here, $\log l$ denotes the maximized log-likelihood, k is the number of parameters, and n is the sample size. The corresponding formulas are given by:

$$\text{AIC} = 2k - 2\log l, \quad \text{AIC}_C = \text{AIC} + \frac{2k(k+1)}{n-k-1}, \quad \text{BIC} = k\log n - 2\log l,$$

$$\text{HQIC} = 2k\log(\log n) - 2\log l, \quad \text{CAIC} = -2\log l + k(\log n + 1).$$

Using probability-probability (P-P) plots, quantile-quantile (Q-Q) plots, and fitted probability density function (PDF) and cumulative distribution function (CDF) curves, we may check the accuracy of the graphs.

The GLD consistently demonstrates superior fitting performance across both datasets when compared to Weibull (W) [32], Perks [33], Power Rayleigh (PRay) [34], Epanechnikov-Weibull (EpW) [35], Power Lindley (PLin), Inverse Exponentiated Pareto (IEP) [36], Gompertz (Gom) [37], Power Muth (PMuth) [38, 39], and Chen [40].

6.1. Dataset I: Kevlar fatigue fracture

The dataset represents fatigue fracture lifetimes of 76 identical Kevlar 373/epoxy specimens under constant pressure at a 90% stress level [31]. All specimens fail, providing complete failure times. Specimens are divided into $n = 19$ groups with $k = 4$ units each. Within each group, only the first failure time is recorded, and remaining units are progressively censored per the PF-FC scheme. Descriptive statistics for Dataset I are provided in Table 10. The failure times are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

The MLE estimates of Dataset I are $\hat{\alpha} = 4.0508$ (SE = 1.3985) and $\hat{\lambda} = 0.8966$ (SE = 0.1571). Table 11 compares GLD against competing distributions, showing that GLD achieves the lowest AIC (248.67) and highest KS p-value (0.3713).

Table 10. Descriptive statistics for both datasets.

Dataset	n	Mean	Variance	Median	SD	Min	Max	Skewness	Kurtosis
I	76.00	1.96	2.48	1.74	1.57	0.03	9.10	1.98	8.16
II	66	2.76	0.79	2.83	0.89	0.39	4.90	-0.13	3.22

Table 11. Distribution comparison for Dataset I (Kevlar fatigue data).

Dist.	AIC	AICC	BIC	HQIC	CAIC	AD stat	AD p	CvM stat	CvM p	KS Stat	KS p
GLD	248.6709	248.8353	253.3324	250.5339	255.3324	0.7605	0.5099	0.1215	0.4905	0.1029	0.3713
Perk	248.8422	249.0066	253.5037	250.7051	255.5037	0.7670	0.5050	0.1224	0.4868	0.1043	0.3555
W	249.0494	249.2138	253.7108	250.9123	255.7108	0.7889	0.4886	0.1354	0.4382	0.1099	0.2953
PRay	249.0494	249.2138	253.7108	250.9123	255.7108	0.7889	0.4886	0.1354	0.4382	0.1099	0.2953
EpW	249.2336	249.3980	253.8950	251.0965	255.8950	0.8191	0.4670	0.1401	0.4218	0.1109	0.2859
PLin	248.8001	248.9645	253.4616	250.6631	255.4616	0.7863	0.4906	0.1349	0.4397	0.1123	0.2723
IEP	257.5220	257.6864	262.1835	259.3850	264.1835	1.3089	0.2294	0.2221	0.2286	0.1198	0.2084
Gom	254.7489	254.9133	259.4103	256.6118	261.4103	1.8040	0.1182	0.3217	0.1174	0.1268	0.1593
PMuth	252.0135	252.1778	256.6749	253.8764	258.6749	1.1576	0.2841	0.2036	0.2609	0.1277	0.1536
Chen	263.4625	263.6269	268.1240	265.3255	270.1240	2.4476	0.0529	0.4190	0.0641	0.1475	0.0660

Figure 2 displays the fitted PDF, CDF, and P-P plots, confirming excellent agreement between empirical and theoretical distributions. Figure 3 compares the fitted PDF and CDF of GLD against other distributions, visually confirming GLD's superior fit. The profile likelihood plots in Figure 4 demonstrate that the MLE estimates are well-identified, with clear peaks in the log-likelihood surface. The Q-Q and P-P plots in Figure 5 show points closely following the diagonal line, further validating the GLD assumption.

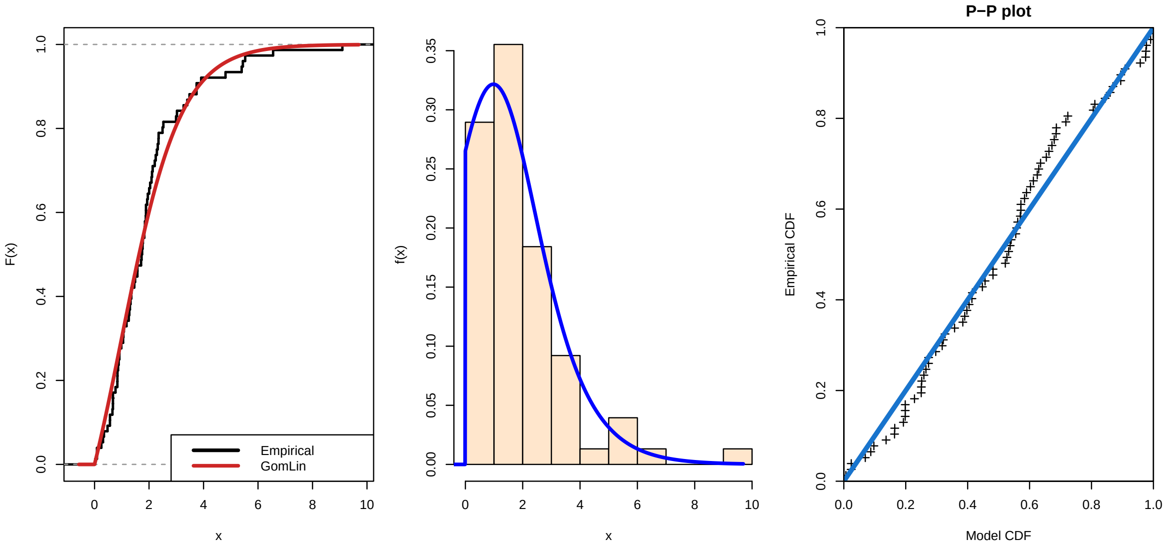


Figure 2. Fitted PDF, CDF, and P-P plots for Dataset I showing close alignment between empirical and theoretical distributions.

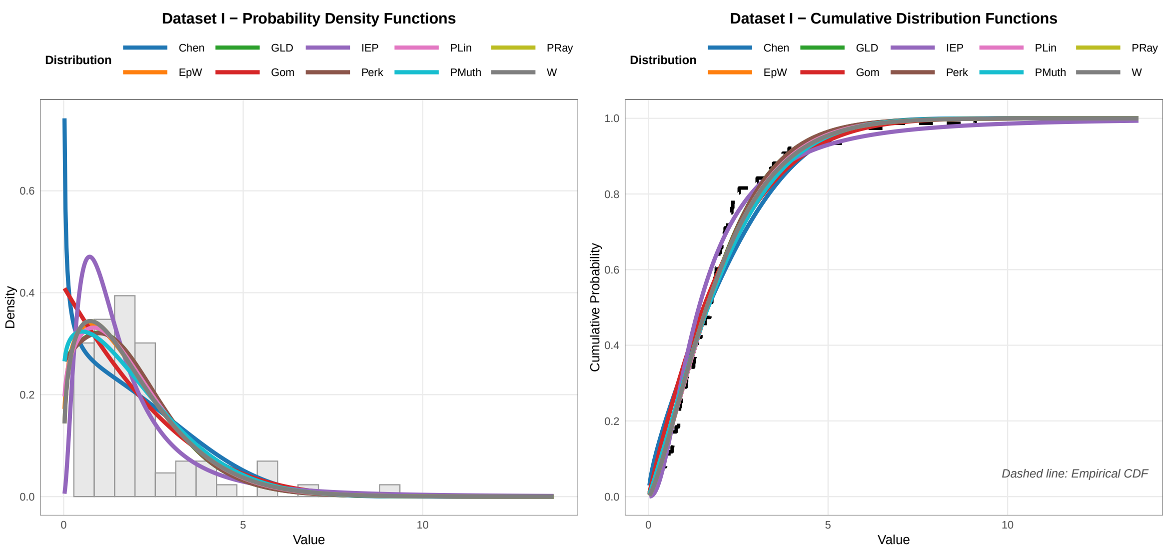


Figure 3. Fitted PDF and CDF for Dataset I comparing GLD with competing distributions.

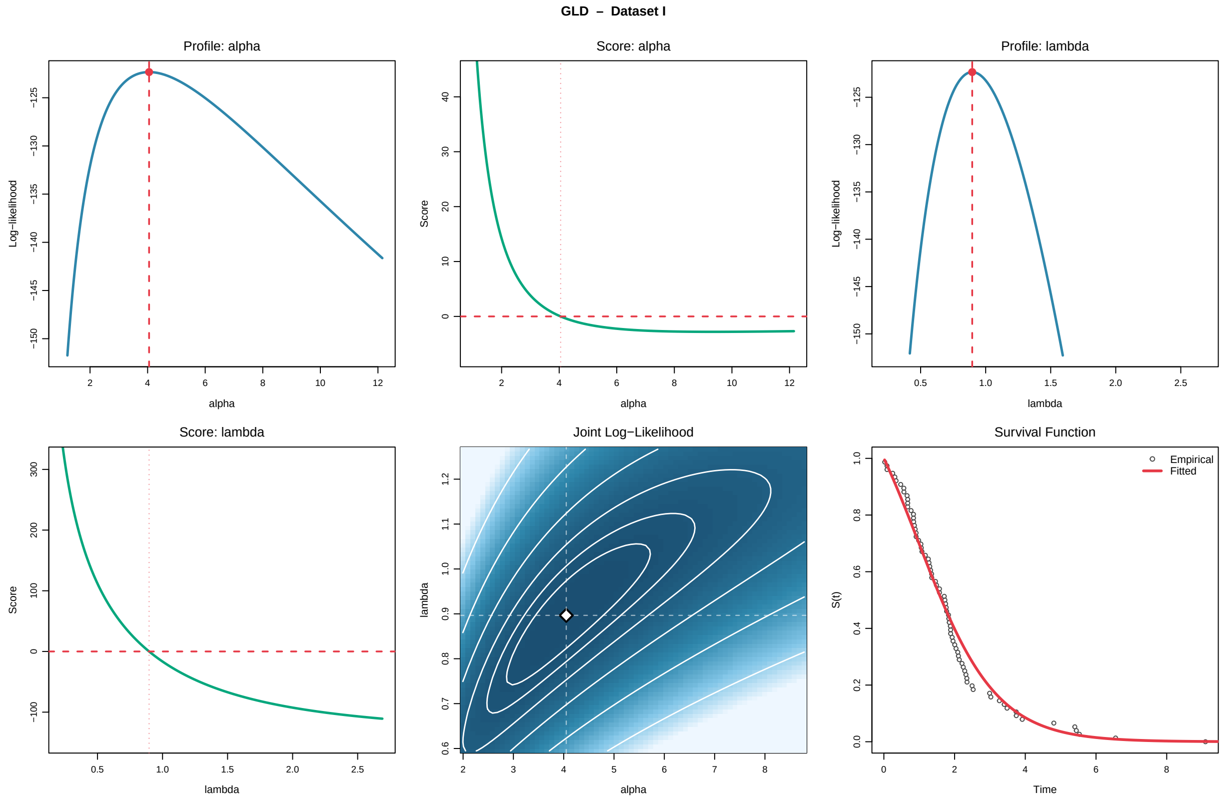
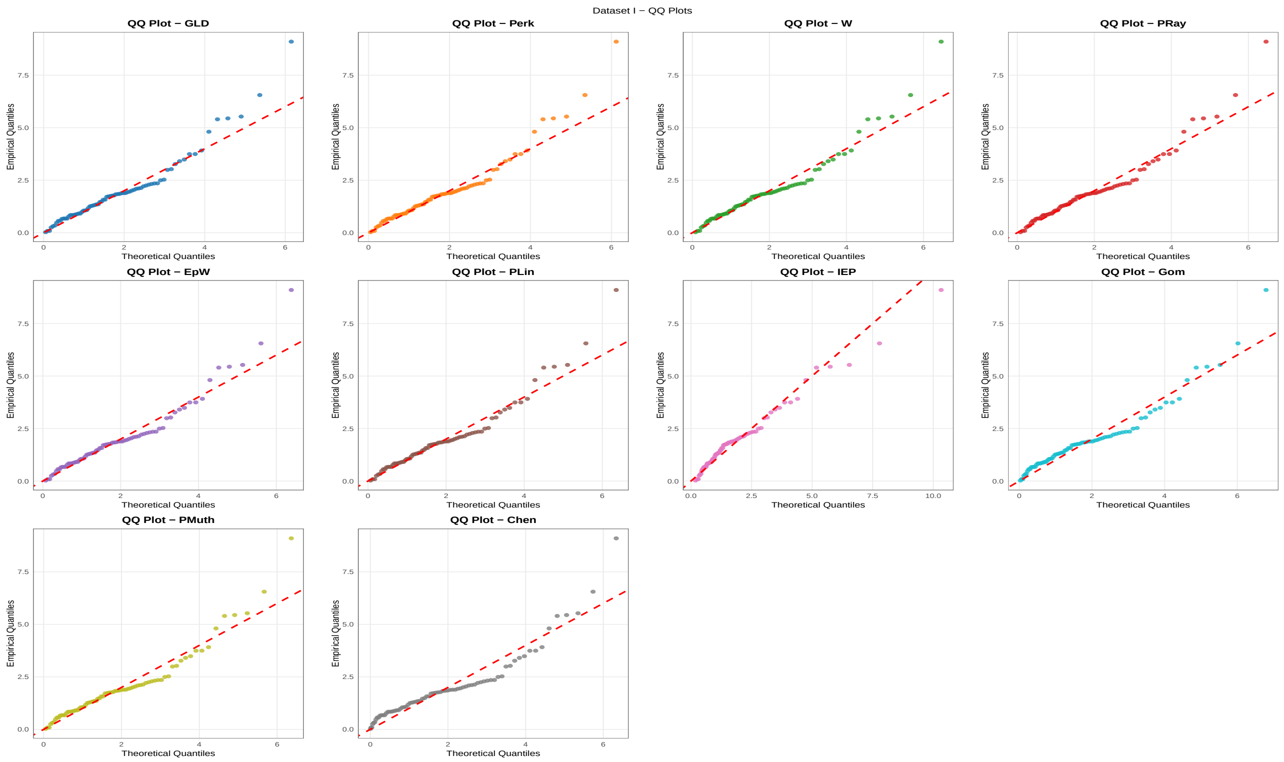
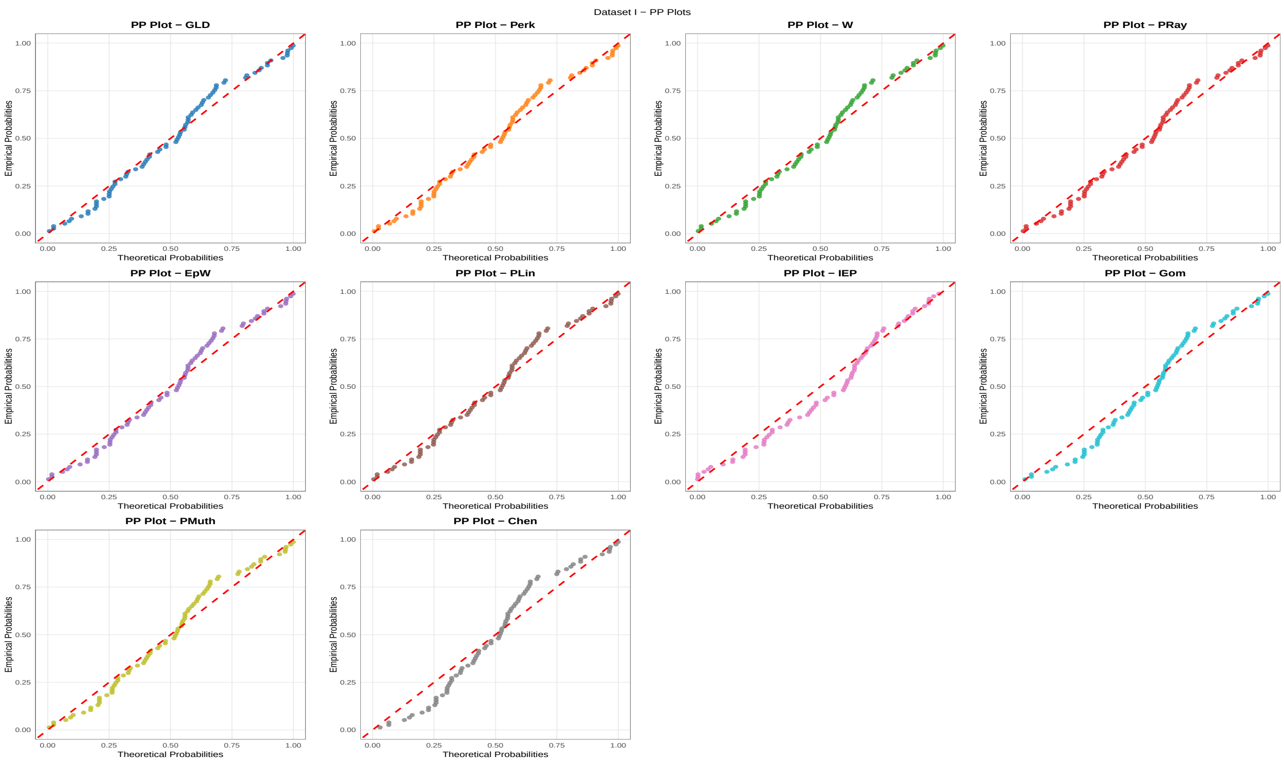


Figure 4. Profile log-likelihood, score function, and contour plots for Dataset I showing stable MLE estimates.



(a) Q-Q plot



(b) P-P plot

Figure 5. Q-Q and P-P plots for Dataset I confirming distributional adequacy.

Table 12 presents PF-FC observed failure times (x_f) and removals (R) under four censoring schemes with $m = 10$, $k = 4$, $n = 19$. Table 13 compares MLE and BEs.

Note: The BEs in Table 13 are obtained using **NIP**. Bayesian standard errors are generally smaller than MLE standard errors, indicating improved precision.

Table 12. PF-FC data for Dataset I with $m = 10$, $k = 4$, and $n = 19$ under four schemes.

scheme	1		2		3		4	
	x_f	R	x_f	R	x_f	R	x_f	R
1	0.0251	0	0.0251	1	0.0251	0	0.0251	3
2	0.3113	0	0.3113	1	0.3113	0	0.5671	0
3	0.5671	0	0.5671	1	0.5671	0	0.6753	0
4	0.6753	0	0.8425	1	0.6753	0	0.8425	0
5	0.8425	0	0.9120	1	0.8425	0	1.0773	0
6	0.9120	0	1.2985	1	0.9120	0	1.2985	0
7	1.0773	0	1.4595	1	1.0773	0	1.4595	0
8	1.2985	0	1.7083	1	1.2985	0	1.7083	0
9	1.4595	0	1.9558	1	1.4595	9	1.7746	0
10	1.7083	9	3.2678	0	1.8808	0	1.8808	6

Table 13. MLE and Bayesian (NIP) estimates for Dataset I under four censoring schemes.

scheme		MLE estimates	StEr	Bayesian (NIP) estimates	StEr
1	α	11.290	16.830	9.434	14.910
	λ	0.6692	0.6571	0.4087	0.3940
2	α	16.165	15.766	14.174	14.844
	λ	0.8445	0.4213	0.6214	0.3460
3	α	29.157	36.197	40.125	47.220
	λ	1.2587	0.7316	1.1093	0.5900
4	α	34.395	42.353	21.657	29.831
	λ	1.2169	0.6273	0.7338	0.4581

As depicted in Figure 6, the MCMC diagnostics for Dataset I confirm the validity of the posterior inference. The trace plots exhibit a high degree of mixing, suggesting that the chains have traversed the parameter space efficiently. Furthermore, the convergence of the ergodic means confirms that the number of iterations is sufficient for stable parameter estimation. The posterior histograms for both α and λ show a near-normal distribution, justifying the use of the posterior mean as a reliable point estimator.

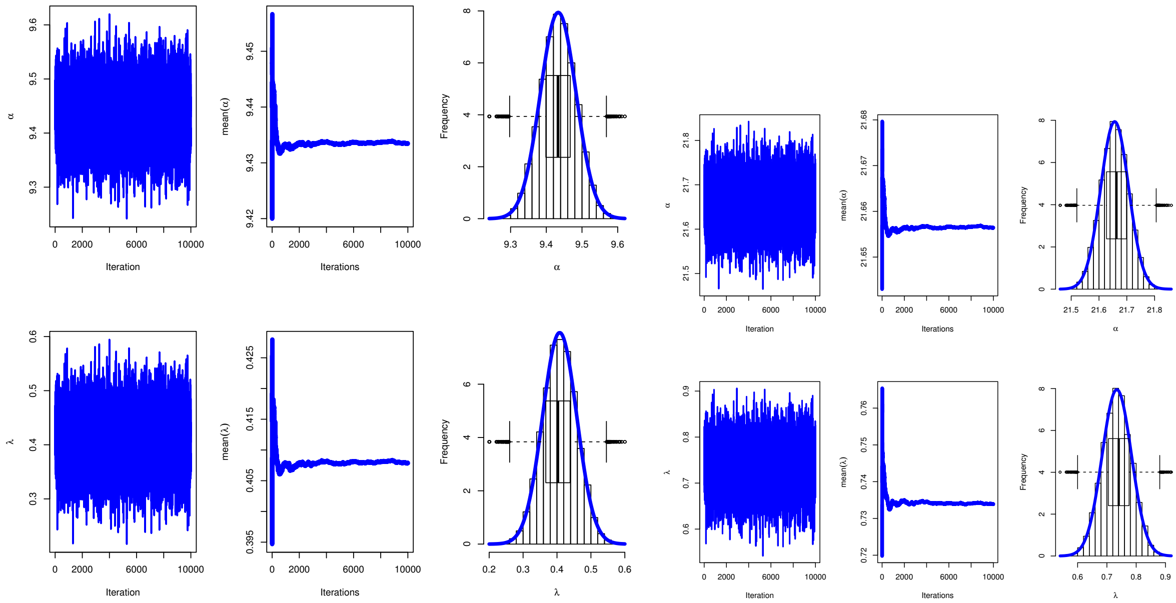


Figure 6. MCMC plots for Dataset I under censoring schemes 1 and 4.

6.2. Dataset II: Carbon fiber tensile strength

The dataset represents breaking stress of carbon fibers (in GPa) from Cordeiro and Lemonte [41], previously used for the Power Muth distribution [38]. The GLD provides superior fit based on AIC and KS measures. Table 10 reports the descriptive statistics for Dataset II. Data (n = 66) are:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69,
3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31,
2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05,
3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

MLE estimates are $\hat{\alpha}$ = 1.3623 (SE = 0.3934) and $\hat{\lambda}$ = 0.0998 (SE = 0.0220). AD (0.3270, p = 0.9163), CVM (0.0523, p = 0.8641), and KS (0.0608, p = 0.9677) confirm an excellent fit. Table 14 shows that GLD achieves the lowest AIC (174.35) and highest KS p-value (0.9677) among all competitors.

Table 14. Distribution comparison for Dataset II (carbon fiber).

Dist.	AIC	AICC	BIC	HQIC	CAIC	AD stat	AD p	CvM stat	CvM p	KS Stat	KS p
GLD	174.3457	174.5362	178.7250	176.0762	180.7250	0.3270	0.9163	0.0523	0.8641	0.0608	0.9677
Perk	174.4402	174.6306	178.8195	176.1706	180.8195	0.3378	0.9069	0.0595	0.8190	0.0632	0.9549
PLin	175.6111	175.8015	179.9904	177.3415	181.9904	0.4651	0.7820	0.0819	0.6823	0.0790	0.8049
EpW	175.7538	175.9443	180.1331	177.4843	182.1331	0.4604	0.7869	0.0780	0.7053	0.0797	0.7958
PRay	176.1352	176.3256	180.5145	177.8656	182.5145	0.4859	0.7606	0.0837	0.6725	0.0823	0.7625
W	176.1356	176.3261	180.5149	177.8661	182.5149	0.4901	0.7564	0.0850	0.6651	0.0832	0.7512
PMuth	176.1113	176.3018	180.4906	177.8418	182.4906	0.5305	0.7153	0.0872	0.6530	0.0886	0.6782
Chen	180.3060	180.4965	184.6853	182.0365	186.6853	0.9853	0.3647	0.1460	0.4027	0.1115	0.3850
Gom	180.1767	180.3672	184.5560	181.9072	186.5560	0.9488	0.3849	0.1397	0.4233	0.1118	0.3809
IEP	187.8878	188.0782	192.2671	189.6182	194.2671	1.2549	0.2474	0.2321	0.2132	0.1269	0.2382

Figure 7 visually confirms that GLD provides the closest fit to the empirical distribution of carbon fiber tensile strength data. The profile likelihood plots in Figure 8 show a well-defined peak, confirming parameter identifiability. The Q-Q and P-P plots in Figure 9 demonstrate that the GLD adequately captures the distribution of the carbon fiber data, with points closely following the diagonal reference line.

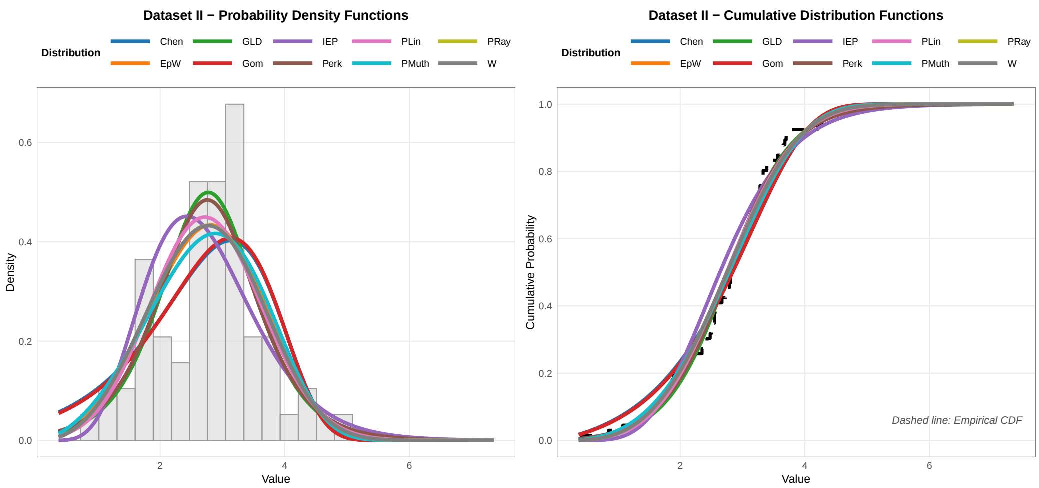


Figure 7. Fitted PDF and CDF for Dataset II comparing GLD with competing distributions.

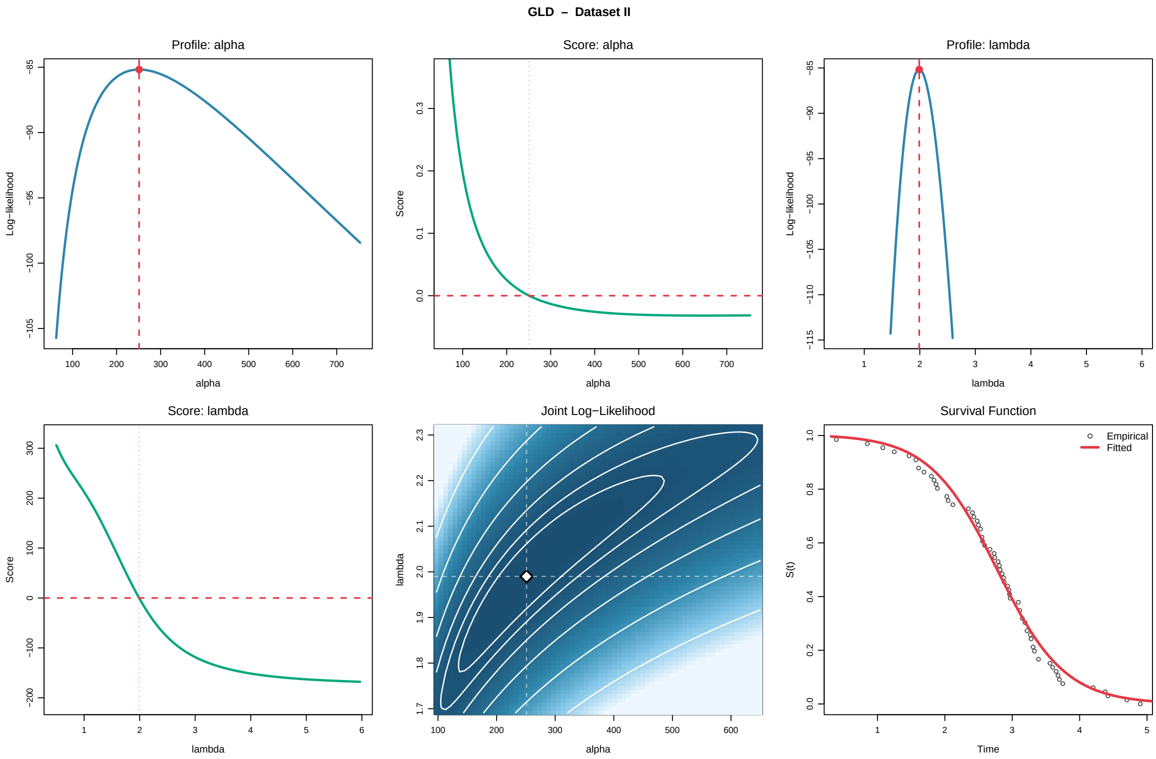
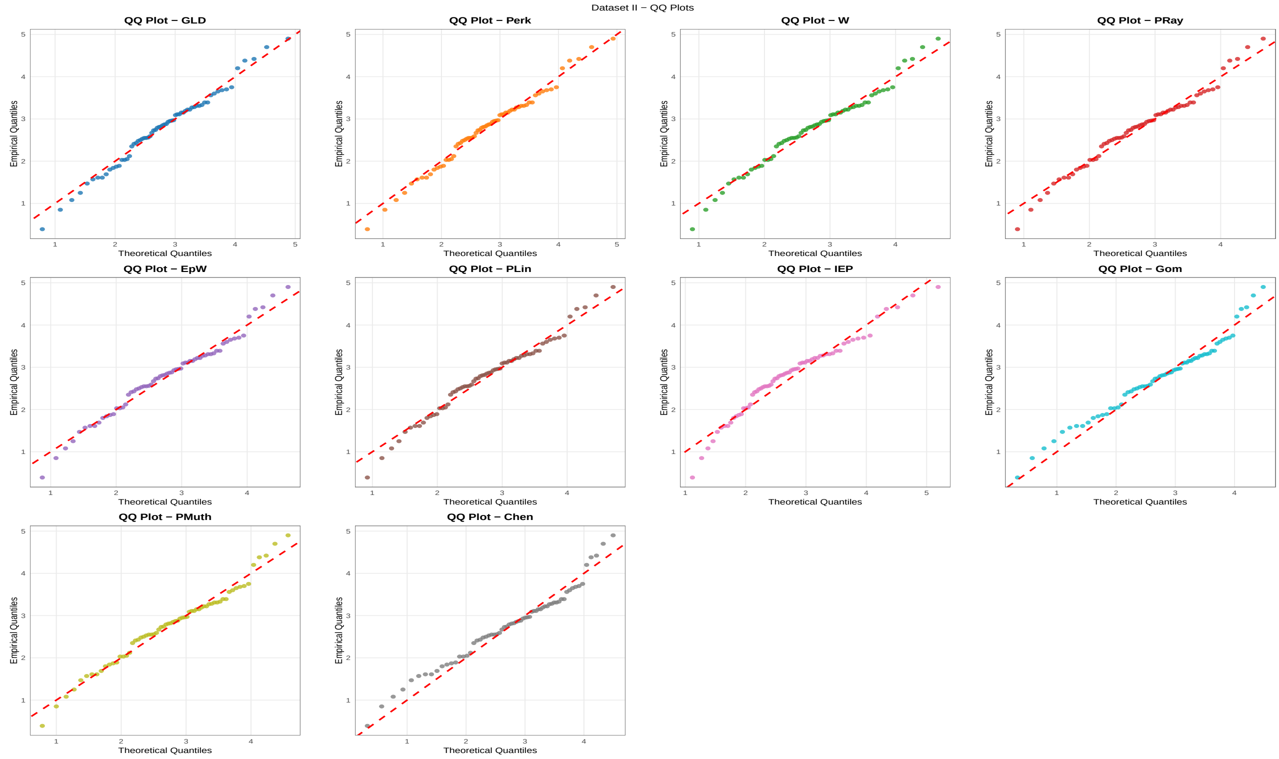


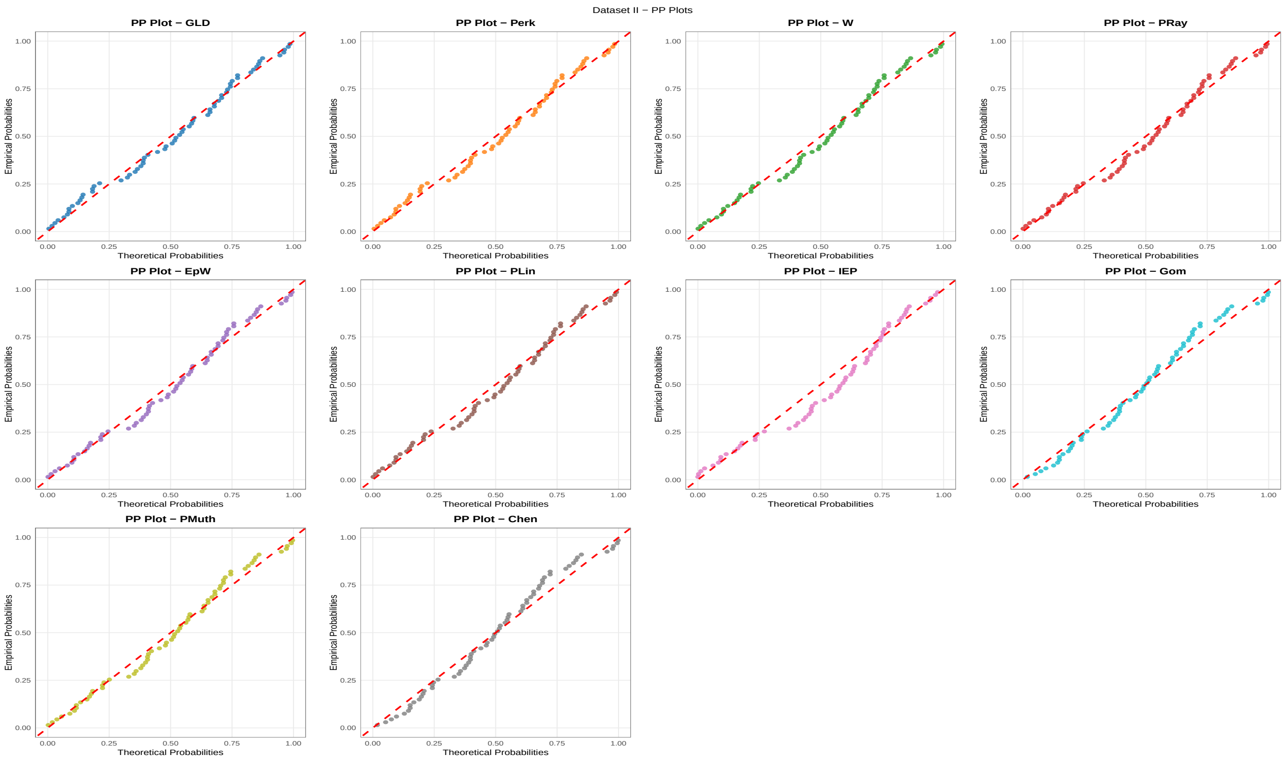
Figure 8. Profile log-likelihood, score function, and contour plots for Dataset II under the GLD model.

Figure 10 shows MCMC plots for Dataset II under censoring schemes 2 and 3, and these the Bayesian estimators have been confirmed convergences.

Tables 15 and 16 summarize the PF-FC data and the corresponding estimation results for Dataset II under four censoring schemes with $n = 11$, $k = 6$, and $m = 8$. The removal vectors are: Scheme 1: $R = (0, 0, 0, 0, 0, 0, 0, 3)$; Scheme 2: $R = (1, 1, 1, 0, 0, 0, 0, 0)$; Scheme 3: $R = (3, 0, 0, 0, 0, 0, 0, 0)$, and Scheme 4: $R = (0, 0, 0, 1, 1, 1, 0, 0)$. The MLE estimates of α are large (216–494) with high standard errors, indicating poor precision. In contrast, the informative Bayesian (IP-BS) approach yields more stable estimates with substantially smaller standard errors, particularly for λ, demonstrating the stabilizing effect of informative priors.



(a) Q-Q plot



(b) P-P plot

Figure 9. Q-Q and P-P plots for Dataset II confirming GLD adequacy.

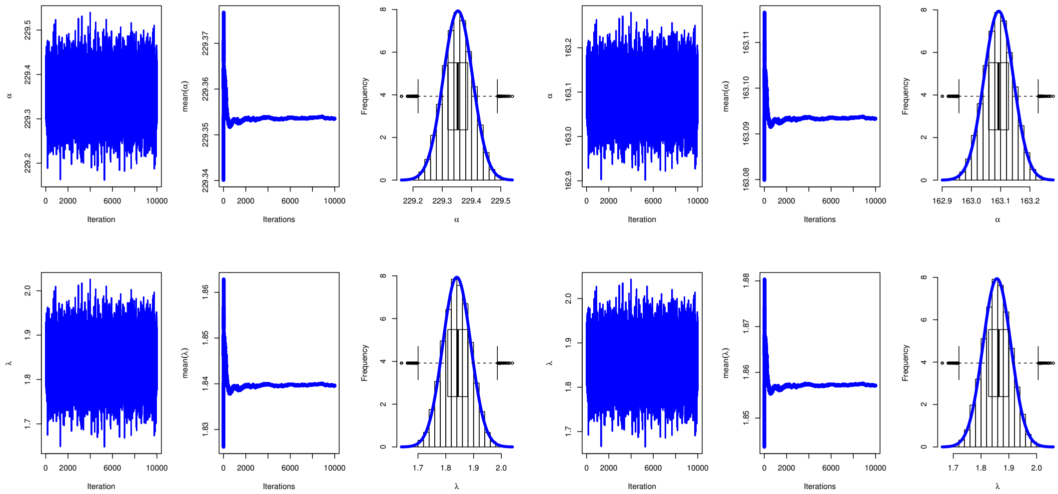


Figure 10. MCMC plots for Dataset II under censoring schemes 2 and 3.

Table 15. PF-FC data for Dataset II (Carbon fiber) with $n = 11$, $k = 6$,and $m = 8$. x_f = observed failure time.

scheme	1		2		3		4	
	x_f	R	x_f	R	x_f	R	x_f	R
1	0.39	0	0.39	1	0.39	3	0.39	0
2	1.08	0	1.08	1	1.25	0	1.08	0
3	1.25	0	1.47	1	1.47	0	1.25	0
4	1.47	0	1.57	0	1.57	0	1.47	1
5	1.57	0	1.61	0	1.61	0	1.57	1
6	1.61	0	1.89	0	1.69	0	1.69	1
7	1.69	0	2.43	0	1.89	0	1.89	0
8	1.89	3	2.55	0	2.43	0	2.43	0

Table 16. Summary of MLE and informative Bayesian (IP-BS) estimates for Dataset II under PF-FC with $n = 11$, $k = 6$, and $m = 8$.

		MLE		IP-BS	
Scheme	Parameter	Estimate	Std. Error	Estimate	Std. Error
1	α	494.2637	798.7342	537.5794	492.6679
	λ	2.5155	0.8969	2.3216	0.5179
2	α	216.1951	278.6830	229.3541	112.6340
	λ	1.8778	0.5907	1.8402	0.1900
3	α	305.0451	413.0909	163.0939	51.2826
	λ	2.2778	0.6866	1.8577	0.1438
4	α	303.4736	403.7363	301.9153	199.5587
	λ	2.1730	0.6850	2.0389	0.2981

7. Conclusions

In this study, we developed comprehensive inference methods for the GLD under PF-FC. Key contributions include the derivation of MLE with asymptotic CIs and BE frameworks under multiple loss functions (SEL, LINEX, GEL, ALB), implemented via efficient MCMC algorithms.

Simulation studies demonstrated that Bayesian methods with informative priors consistently outperformed classical approaches, particularly when using the generalized entropy loss function. Estimation accuracy improved with larger sample sizes, and Scheme II (removals at final failure) yielded optimal performance. The methodology proved robust across different group sizes ($k = 1, 2$), showing comparable efficiency to standard progressive censoring.

Practical applications to microcircuit failure data and cancer remission times confirmed the framework's utility in reliability engineering and survival analysis. The GLD effectively modeled diverse failure patterns, while the PF-FC scheme balanced experimental efficiency with statistical precision.

Researchers should extend this work to accelerated life testing, competing risks models, and optimal CS design. The proposed methods provide practitioners with robust tools for reliability analysis with censored lifetime data.

Author contributions

Hisham Mohamed Almongy: Conceptualization, data curation, formal analysis, investigation, methodology, resources, visualization, validation, writing–original draft; Ehab M. Almetwally: Conceptualization, data curation, formal analysis, investigation, methodology, resources, software, visualization, validation, writing– original draft; Eslam Hussam: Conceptualization, data curation, formal analysis, investigation, methodology, resources, software, visualization, validation, writing–original draft, proofreading; M. H. Abu-Moussa: Formal analysis, investigation, methodology, resources, software, visualization, validation, writing–original draft; T.S. Taher: Formal analysis, methodology, supervision, visualization, software, writing–review & editing; A. M. Sharawy: Formal analysis, methodology, supervision, visualization, software, writing–review & editing, proofreading. All authors have read and approved the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare there is no conflict of interest.

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