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Research article

Asymptotic behavior and numerical simulation of a stochastic multi-group SEIR epidemic model with infinite distributed delays

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Abstract: In this paper, we studied the asymptotic behavior of a stochastic multi-group Susceptible-Exposed-Infectious-Recovered (SEIR) epidemic model with infinite distributed delays and Lévy jumps. First, by the methods of Lyapunov functions, Itô's formula, and the theory of stopping times, we proved the existence and uniqueness of the global positive solution to the stochastic delayed system. Furthermore, by using appropriate Lyapunov functions, graph theory and stochastic analysis, we established the asymptotic dynamical behaviors around the disease-free equilibrium P_0 and the endemic equilibrium P^* of the deterministic system, respectively. It was shown that if the threshold $R_0 < 1$, the solution of the stochastic delayed system oscillates around the disease-free equilibrium P_0; while if $R_0 > 1$, the solution fluctuates around the endemic equilibrium P^*. Finally, numerical simulations were performed to intuitively analyze the impact of Lévy noise on the dynamical behavior of the stochastic delayed system.

Keywords: stochastic multi-group epidemic model; infinite distributed delays; Lévy jumps; asymptotic behavior; numerical simulation
Mathematics Subject Classification: 34K20, 60G51, 60H10, 92D30

1. Introduction

With the increasing complexity of disease transmission across heterogeneous populations, the mechanisms of inter-group transmission and cross-infection have become a critical research focus in infectious disease dynamics. Lajmanovich and Yorke [1] pioneered the development of a multi-group Susceptible-Infectious-Susceptible (SIS) model for gonorrhea transmission. By dividing the population into groups based on sexual activity, they captured contact heterogeneity and proved the global stability of the unique endemic equilibrium, thereby laying the foundation for subsequent multi-group models. Building on this work, Beretta and Capasso [2] analyzed a non-migrating multi-group SIR model. They employed methods including Lyapunov functionals, matrix analysis, the fixed-point theorem, Perron-Frobenius spectral radius calculation, and numerical simulations. Their research explored the existence, uniqueness, and global stability of the model's non-trivial endemic equilibrium, while verifying its applicability to human diseases. Guo et al. [3, 4] investigated the multi-group SIR and SEIR epidemic models, respectively. Through the integration of global Lyapunov functional methods and graph-theoretic approaches, they clarified that the basic reproduction number R_0 plays an important role in governing the existence, uniqueness, and global stability of disease equilibria. Building on these foundational works, a variety of multi-group epidemic models have been developed, which are widely applied to the investigation of diverse infectious diseases (see [5–7]).

However, time delays naturally arise in infectious disease transmission due to biological processes such as pathogen incubation, immune responses, and infectious periods. For example, the incubation period of COVID-19 and the time from infection to infectivity in influenza vary significantly among individuals. A large number of studies employ discrete time delays to model the delay effects; see [8]. Such idealized hypotheses remain valid only for systems with distinct time-varying behaviors, whereas complex pathological dynamics commonly display considerable individual variability. Consequently, distributed time delays defined by probability distributions have emerged as a key area for model improvement. This integral kernel-based approach can address dynamically changing transmission intensities. It can clearly describe the sustained effects of pathogens. It also accurately represents the non-uniform nature of infection processes. As a result, the explanatory power of biomathematical models is greatly enhanced. Busenberg and Cooke [9] pioneered the use of distributed delays in epidemic models and demonstrated their superior fitting performance. Thieme [10], investigating population-related delays, provided valuable methodological insights for subsequent research. Further advancing the field, Beretta and Takeuchi [11] developed a distributed delay model for vector-borne diseases. Following this work, Beretta et al. [12] investigated the permanence of an SIR epidemic model with distributed time delays. On this basis, Li [13] analyzed the influence of delay distribution functions and refined the analytical framework for stability, facilitating the transition of multi-group models from discrete to distributed delays. Subsequently, Safi and Gumel [14, 15] and Shu et al. [16] extended this framework to quarantine, age-structured, and multi-group SEIR models, respectively. Collectively, these studies have provided a more comprehensive theoretical foundation for accurately characterizing complex infectious disease dynamics. De la Sen et al. [17] extended distributed-delay SEIR models with vaccination and proved that delays affect stability and convergence, supporting the use of infinite distributed delays in this work. Meanwhile, multiple infection stages and delays are common in real transmission. Zhang et al. [18] built a reaction-diffusion model with nonlocal delays and studied traveling wave fronts between the two equilibria.

In addition, infectious diseases are often subject to external random disturbances, such as variations in light intensity, temperature, humidity, precipitation, wind speed, and direction. These environmental fluctuations can influence key epidemic parameters, including the transmission rate. Continuous stochastic perturbations are commonly modeled using Brownian motion (see [19–21]). Liu et al. [22] constructed a two-group stochastic SEIR model with infinite delays. They proved the stochastic asymptotic stability of the endemic equilibrium and demonstrated that sufficiently small environmental noise can preserve the stability of the model. Liu et al. [23] further constructed an n-group stochastic epidemic model incorporating the distributed time delay term $\int_{r=0}^{\infty} f_j(r)E_j(t-r)dr$ and general kernel functions $f_j(\cdot)$, whose specific form was given as follows:

$$
\begin{cases}
\mathrm{d}S_k = \left[\Lambda_k - \sum_{j=1}^{n}\beta_{kj}S_k\int_{r=0}^{\infty}f_j(r)E_j(t-r)\mathrm{d}r - d_k^S S_k\right]\mathrm{d}t + \alpha_k S_k \mathrm{d}B_{1k}(t), \\
\mathrm{d}E_k = \left[\sum_{j=1}^{n}\beta_{kj}S_k\int_{r=0}^{\infty}f_j(r)E_j(t-r)\mathrm{d}r - \left(d_k^E + \epsilon_k\right)E_k\right]\mathrm{d}t + \beta_k E_k \mathrm{d}B_{2k}(t), \quad k = 1, 2, \ldots, n,
\end{cases} \tag{1.1}
$$

where S_k and E_k denote the size of the susceptible compartment and the exposed (infected but non-infectious) compartment of the kth subgroup at time t, respectively. The parameters in system (1.1) have the following biological meanings: Λ_k is the influx of individuals into the kth group; β_{kj} represents the transmission coefficient between S_k and E_j; d_k^S and d_k^E are the death rates of S_k and E_k, respectively; ϵ_k is the rate at which exposed individuals become infectious after a latent period. All parameters are non-negative, with $\Lambda_k, d_k^S, d_k^E > 0$ for all k. In addition, the kernel function $f_j(r)$ is non-negative and continuous, with $\int_{r=0}^{\infty} f_j(r)\mathrm{d}r = a_j > 0$ ($j = 1, 2, \ldots, n$). $B_{ik}(t)$ ($i = 1, 2$) are standard Brownian motions with independent components, defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (increasing, right-continuous, and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). α_k, β_k denote the intensities of the white noise. For stochastic system (1.1), the basic reproduction number $R_0 = \rho(M_0)$ (spectral radius of M_0) determines disease occurrence, in which $M_0 = \left(\frac{\beta_{kj}S_k^0 a_j}{d_k^E + \epsilon_k}\right)_{n\times n}$ and $S_k^0 = \frac{\Lambda_k}{d_k^S}$. It has been shown that if $R_0 < 1$, the solutions oscillate around the disease-free equilibrium $E_0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$, and the disease tends to die out almost surely; if $R_0 > 1$, E_0 becomes unstable, and the solutions fluctuate around the endemic equilibrium $E^* = (S_1^*, E_1^*, \ldots, S_n^*, E_n^*)$, which indicates that the disease would persist in the population.

In reality, populations are exposed not only to continuous environmental noise but also to sudden disturbances, such as natural disasters (e.g., earthquakes, floods, or toxic spills) or abrupt climate shifts. These events can induce drastic, discontinuous fluctuations in population sizes, which traditional Brownian motion fails to capture accurately. Therefore, stochastic models with jumps have been introduced as a more realistic framework for modeling such perturbations, as evidenced by recent studies [24–26]. Motivated by these concerns, this paper improves the existing stochastic delayed epidemic model (1.1) and develops a complete stochastic multi-group SEIR model with infinite distributed delays, Lévy jumps, multi-group heterogeneity, and disease-induced mortality. The model can characterize disease transmission dynamics more realistically. By constructing integral-type Lyapunov functionals with graph-theoretic weights, combined with the Itô formula for jump-diffusion processes, stopping-time techniques, and compensated Poisson random measure estimates, we establish the existence and uniqueness of the global positive solution. We then investigate the asymptotic behavior of solutions around the disease-free equilibrium and the endemic equilibrium, analyze the combined effects of noise and time delays on the dynamical behavior and transmission trends of the system, and further enrich the theoretical framework of stochastic delayed epidemic models with Lévy perturbations.

The organization of this paper is as follows: In Section 2, we formulate the stochastic delayed system, which serves as the main model for our analysis. In Section 3, we introduce the preliminary concepts and lemmas required for subsequent analyses. In Section 4, we prove the existence and uniqueness of the global positive solution for the stochastic delayed system (2.1). In Sections 5 and 6, we discuss the dynamical behavior of stochastic delayed system (2.1): when $R_0 < 1$, the solutions will oscillate around the disease-free equilibrium P_0 under certain conditions; when $R_0 > 1$, the solutions will fluctuate around the endemic equilibrium P^* under sufficient conditions, respectively. In Section 7, we perform numerical simulations to support the theoretical findings. In the last section, we provide a brief discussion and summary of the main results.

2. Stochastic multi-group SEIR epidemic model with an infinite distributed delay

This paper focuses on a class of infectious diseases predominantly driven by latent infection. In such diseases, latently infected individuals account for a large proportion of all cases. Once symptoms appear, disease transmissibility declines sharply as symptomatic individuals receive timely isolation or mount effective immune responses. Typical examples include chickenpox and hand, foot, and mouth disease (HFMD). The transmission of these pathogens is jointly shaped by multiple factors, including population heterogeneity, random environmental noise, and disease-induced mortality—factors that conventional epidemic models fail to fully incorporate. Accordingly, we construct a multi-group SEIR model with distributed time delays and stochastic jump processes to better describe the transmission dynamics of this type of infectious disease. The model is formulated as follows:

$$\begin{cases} \mathrm{d}S_k(t) = \left[\Lambda_k - \sum_{j=1}^{n}\beta_{kj}S_k(t)\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r - d_k^S S_k(t)\right]\mathrm{d}t + \sigma_{1k}S_k(t)\mathrm{d}B_{1k}(t) \\ \qquad\qquad + \int_Y S_k(t)D_{1k}(y)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \\ \mathrm{d}E_k(t) = \left[\sum_{j=1}^{n}\beta_{kj}S_k(t)\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r - \left(d_k^E + \epsilon_k\right)E_k(t)\right]\mathrm{d}t + \sigma_{2k}E_k(t)\mathrm{d}B_{2k}(t) \\ \qquad\qquad + \int_Y E_k(t)D_{2k}(y)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \\ \mathrm{d}I_k(t) = \left[\epsilon_k E_k(t) - \left(d_k^I + \gamma_k + \theta_k\right)I_k(t)\right]\mathrm{d}t + \sigma_{3k}I_k(t)\mathrm{d}B_{3k}(t) + \int_Y I_k(t)D_{3k}(y)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \\ \mathrm{d}R_k(t) = \left[\gamma_k I_k(t) - d_k^R R_k(t)\right]\mathrm{d}t + \sigma_{4k}R_k(t)\mathrm{d}B_{4k}(t) + \int_Y R_k(t)D_{4k}(y)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \quad k = 1,2,\ldots,n, \end{cases} \tag{2.1}$$

where $B_{ik}(t)$ $(i = 1,2,3,4)$ are independent standard Brownian motions, defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions (i.e., it is increasing and right-continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). The constants $\sigma_{ik} > 0$ $(i = 1,2,3,4)$ denote the intensities of the white noise. $N(\mathrm{d}t,\mathrm{d}y)$ is a Poisson random measure with compensator $\tilde{N}$, where the compensated Poisson random measure is defined by $\tilde{N}(\mathrm{d}t,\mathrm{d}y) = N(\mathrm{d}t,\mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}t$. Here, ν is a Lévy measure defined on a Borel set $Y \subset (0,\infty)$ satisfying $\nu(Y) < \infty$. Furthermore, $D_{ik}(y)$ $(i = 1,2,3,4)$ represents the jump intensity function for the i-th noise term in the k-th group, which is assumed to be bounded and continuously differentiable in y. The other parameters in system (2.1) and their biological meanings are summarized in Table 1. All parameters are non-negative.

Here, the matrix $B = (\beta_{kj})_{n\times n}$ denotes the contact matrix, where $\beta_{kj} \geq 0$ for all $k, j \in \{1,2,\ldots,n\}$. The corresponding Laplacian matrix can be defined as follows:

$$L_B = \begin{bmatrix} \sum\limits_{k\neq 1} \beta_{1k} & -\beta_{12} & \dots & -\beta_{1n} \\ -\beta_{21} & \sum\limits_{k\neq 2} \beta_{2k} & \dots & -\beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{n1} & -\beta_{n2} & \dots & \sum\limits_{k\neq n} \beta_{nk} \end{bmatrix}.$$

Table 1. Definitions of the parameters in system (2.1).

Symbol	**Biological significance**
Λ_k	Constant recruitment rate of the k-th subgroup.
β_{kj}	Transmission coefficient between S_k and E_j.
ϵ_k	Infection rate of exposed individuals in the k-th subgroup.
γ_k	Recovery rate of infected individuals in the k-th subgroup.
θ_k	Disease-induced mortality rate of infected individuals in the k-th subgroup.
r	Time delay variable.
$d_k^S, d_k^E, d_k^I, d_k^R$	Natural mortality rate of S_k, E_k, I_k, R_k in the k-th subgroup.

3. Preliminaries

First, we give some fundamental results of stochastic differential equations.

Theorem 3.1. *([27, 28]) Let $0 \leq t_0 \leq T < \infty$. Assume that the mappings $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ (the space of $d \times m$ real matrices), $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are all Borel measurable. Consider the d-dimensional stochastic differential equation:*

$$\mathrm{d}x(t) = f(x(t-))\mathrm{d}t + g(x(t-))\mathrm{d}B(t) + \int_{|y|<c} H\left(x(t-), y\right) \tilde{N}\left(\mathrm{d}t, \mathrm{d}y\right) + \int_{|y|\geq c} K\left(x(t-), y\right) N\left(\mathrm{d}t, \mathrm{d}y\right),$$

for $t_0 \leq t \leq T$ with initial value $x(t_0) = x_0 \in \mathbb{R}^d$, where f and g are the drift and diffusion coefficients, respectively, and H, K are the jump coefficients. The equation has a unique solution if the following conditions hold:

(1) Lipschitz condition: There exists a positive constant L such that for all $x_1, x_2 \in \mathbb{R}^d$,

$$|f(x_1) - f(x_2)|^2 \leq L\,|x_1 - x_2|^2\,, \quad \|g(x_1) - g(x_2)\|^2 \leq L\,|x_1 - x_2|^2\,,$$

$$\int_{|y|<c} |H(x_1, y) - H(x_2, y)|^2\, \nu(\mathrm{d}y) \leq L\,|x_1 - x_2|^2\,.$$

(2) Growth condition: There exists a positive constant K such that for all $x \in \mathbb{R}^d$,

$$|f(x)|^2 \leq K(1 + |x|^2), \quad \|g(x)\|^2 \leq K(1 + |x|^2), \quad \int_{|y|<c} |H(x, y)|^2\, \nu(\mathrm{d}y) \leq K(1 + |x|^2).$$

(3) The mapping $x \mapsto K(x, y)$ is continuous for all $|y| \geq c$.

Theorem 3.2. *([27, 28]) Let $Y(t)$ be a Lévy-type stochastic process satisfying:*

$$\mathrm{d}Y(t) = G(t)\mathrm{d}t + F(t)\mathrm{d}B(t) + \int_{|y|<c} H(t, y)\,\tilde{N}(\mathrm{d}t, \mathrm{d}y) + \int_{|y|\geq c} K(t, y)\,N(\mathrm{d}t, \mathrm{d}y),$$

where Y is a Lévy-type stochastic process. For any $f \in C^2(\mathbb{R}^d)$ and $t \geq 0$, the following identity holds:

$$\begin{aligned} f(Y(t)) - f(Y(0)) = &\int_0^t \partial_i f(Y(s-))\mathrm{d}Y_c^i(s) + \frac{1}{2}\int_0^t \partial_i\partial_j f(Y(s-))\mathrm{d}[Y_c^i, Y_c^j](s) \\ &+ \int_0^t \int_{|y|\geq c} [f(Y(s-) + K(s, y)) - f(Y(s-))]N(\mathrm{d}s, \mathrm{d}y) \\ &+ \int_0^t \int_{|y|<c} [f(Y(s-) + H(s, y)) - f(Y(s-))]\tilde{N}(\mathrm{d}s, \mathrm{d}y) \\ &+ \int_0^t \int_{|y|<c} [f(Y(s-) + H(s, y)) - f(Y(s-)) - H^i(s, y)\partial_i f(Y(s-))]\nu(\mathrm{d}y)\mathrm{d}s. \end{aligned}$$

Next, we introduce some notations and results of graph theory [29, 30]. A directed graph $\mathcal{G} = (V, E)$ consists of a vertex set $V = \{1, 2, \cdots, n\}$ and an edge set E of directed arcs (j, k). The graph is weighted if each arc (j, k) carries a positive weight a_{kj}. For a weighted directed graph $(\mathcal{G}, A)$ with n vertices, where $A = (a_{kj})_{n\times n}$ is the weight matrix of the directed graph, the element a_{kj} equals the weight of the arc (j, k) if the arc exists; otherwise, it is 0. The graph is strongly connected if and only if its weight matrix A is irreducible. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L_A = \begin{pmatrix} \sum\limits_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum\limits_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum\limits_{k\neq n} a_{nk} \end{pmatrix}.$$

Let c_k be the cofactor of the k-th diagonal entry of L_A.

Lemma 3.1. *If A is nonnegative and irreducible, then the spectral radius $\rho(A)$ is a simple eigenvalue, and A has a positive eigenvector $\mathbf{e} = (e_1, \ldots, e_n)$ corresponding to $\rho(A)$.*

Lemma 3.2. *Suppose $n \geq 2$.*

(1) If $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$ for all $1 \leq k \leq n$.

(2) The identity $\sum_{k=1}^n \sum_{j=1}^n c_k a_{kj} G_k(\chi_k) = \sum_{k=1}^n \sum_{j=1}^n c_k a_{kj} G_j(\chi_j)$ holds for arbitrary functions G_k.

4. The existence and uniqueness of the global positive solution

In order to study the dynamic behavior of stochastic delayed system (2.1), it is necessary to determine the existence and uniqueness of a global positive solution. Before presenting our main theorems, we introduce two assumptions on the jump diffusion coefficients that will be used throughout the paper.

(**A1**) For each $N > 0$, there exists $L_N > 0$ such that $\int_Y |H_{ik}(x_1, y) - H_{ik}(x_2, y)|^2 \nu(\mathrm{d}y) \leq L_N|x_1 - x_2|^2$ (i = 1, 2, 3, 4), where $H_{1k}(x, y) = D_{1k}(y)S_k$, $H_{2k}(x, y) = D_{2k}(y)E_k$, $H_{3k}(x, y) = D_{3k}(y)I_k$, $H_{4k}(x, y) = D_{4k}(y)R_k$, $|x_1| \vee |x_2| \leqslant N$.

(**A2**) $1 + D_{ik}(y) > 0$ and $\int_Y [D_{ik}(y) - \ln(1 + D_{ik}(y))] \nu(\mathrm{d}y) < \infty$ for $i = 1, 2, 3, 4$, $y \in Y$.

The local Lipschitz condition (A1) guarantees the existence and uniqueness of the local solution. It constrains the intensity of Lévy jumps within a reasonable range and avoids unrealistic extreme fluctuations. Moreover, it conforms to the natural variation of group sizes across different stages of real disease transmission. Assumption (A2) shows that under the condition $D_{ik} > -1$, the mean level remains bounded even if extreme jump events occur.

Let $X(t) = (S_1(t), E_1(t), I_1(t), R_1(t), \cdots, S_n(t), E_n(t), I_n(t), R_n(t))$ be the solution to stochastic delayed system (2.1), and define $\mathbb{R}^{4n}_+ = \{x \in \mathbb{R}^{4n} \mid x_i > 0, i = 1, 2, \ldots, 4n\}$.

Theorem 4.1. *Let assumptions (A1) and (A2) hold. If the matrix $B = (\beta_{ij})_{n\times n}$ is irreducible, then for any initial values $S_k(0) = S_{k,0}$, $E_k(0) = \phi_k(s)$ ($s \in (-\infty, 0]$), $I_k(0) = I_{k,0}$, $R_k(0) = R_{k,0}$ ($k = 1, 2, \ldots, n$), the stochastic delayed system* (2.1) *admits a unique positive solution $X(t)$ for all $t \geq 0$. Moreover, this solution will remain in $\mathbb{R}^{4n}_+$ with probability one almost surely.*

Proof. By assumption (A1), the coefficients of the stochastic delayed system (2.1) are locally Lipschitz continuous. Thus there exists a unique local solution $X(t)$ on $t \in (-\infty, \tau_e]$, where τ_e is the explosion time. To show the existence and uniqueness of the global solution to system (2.1), it suffices to show that $\tau_e = \infty$ almost surely. Let m_0 be a sufficiently large constant such that $S_{k,0}$, $\phi_k(s)$, $I_{k,0}$, and $R_{k,0}$ ($1 \leq k \leq n$) all lie in the interval $\left[\frac{1}{m_0}, m_0\right]$. For each integer $m \geq m_0$, we define the stopping time

$$\tau_m = \inf\left\{t \in (-\infty, \tau_e] : \min_{1\leq k\leq n} \{S_k(t), E_k(t), I_k(t), R_k(t)\} \leq \frac{1}{m} \text{ or } \max_{1\leq k\leq n} \{S_k(t), E_k(t), I_k(t), R_k(t)\} \geq m\right\}.$$

As usual, we set $\inf \emptyset = \infty$. Clearly, τ_m is increasing. Let $\tau_\infty = \lim_{m\to\infty} \tau_m$, where $\tau_\infty \leq \tau_e$ a.s. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and the solution stays in $\mathbb{R}^{4n}_+$ for all $t \in \mathbb{R}$ a.s. If this claim is untrue, then there exist a constant $T > 0$ and a constant $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} \geq \epsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$\mathbb{P}\{\tau_m \leq T\} \geq \epsilon, \quad \forall m \geq m_1. \tag{4.1}$$

Let c_k be an algebraic cofactor corresponding to the k-th diagonal entry of L_B (the Laplacian matrix of the graph $(\mathcal{G}, B)$). Since $B = (\beta_{ij})_{n\times n}$ is irreducible, the directed graph $(\mathcal{G}, B)$ is strongly connected. By Lemma 3.2(1), it follows that $c_k > 0$ for all $1 \leq k \leq n$. Define a C^2-function $V : \mathbb{R}^{4n}_+ \to \mathbb{R}_+$ as

$$\begin{aligned}
&V(S_1, E_1, I_1, R_1, \cdots, S_n, E_n, I_n, R_n)\\
&= \sum_{k=1}^{n} \left[\left(S_k - ac_k - ac_k \ln \frac{S_k}{ac_k}\right) + (E_k - 1 - \ln E_k) + (I_k - 1 - \ln I_k) + (R_k - 1 - \ln R_k)\right.\\
&\left. + ac_k \sum_{j=1}^{n} \beta_{kj} \int_t^{t+r} \int_0^{\infty} f_j(r) E_j(u - r)\, \mathrm{d}r \mathrm{d}u\right],
\end{aligned} \tag{4.2}$$

where a is a positive constant to be determined later. Obviously, this function is non-negative, which can be seen from $x-1-\ln x\geq 0, \forall x>0$. Making use of Itô's formula for V, we have

$$
\begin{aligned}
\mathrm{d}V(X) =& LV(X)\mathrm{d}t + \sum_{k=1}^{n}\left[\sigma_{1k}\left(S_k - ac_k\right)\mathrm{d}B_{1k}(t) + \sigma_{2k}\left(E_k-1\right)\mathrm{d}B_{2k}(t) + \sigma_{3k}\left(I_k-1\right)\mathrm{d}B_{3k}(t)\right. \\
&\left. + \sigma_{4k}\left(R_k-1\right)\mathrm{d}B_{4k}(t)\right] + \sum_{k=1}^{n}\int_Y\left[S_kD_{1k}(y) - ac_k\ln(1+D_{1k}(y)) + E_kD_{2k}(y)\right. \\
&\left. - \ln\left(1+D_{2k}(y)\right) + I_kD_{3k}(y) - \ln\left(1+D_{3k}(y)\right) + R_kD_{4k}(y) - \ln\left(1+D_{4k}(y)\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
LV(X) =& \sum_{k=1}^{n}\left(1-\frac{ac_k}{S_k}\right)\left(\Lambda_k - \sum_{j=1}^{n}\beta_{kj}S_k\int_0^{\infty}f_j(r)E_j(t-r)\,\mathrm{d}r - d_k^SS_k\right) + \sum_{k=1}^{n}\left(1-\frac{1}{R_k}\right)\left(\gamma_kI_k - d_k^RR_k\right) \\
&+ \sum_{k=1}^{n}\sum_{j=1}^{n}ac_k\beta_{kj}a_jE_j + \sum_{k=1}^{n}\left(1-\frac{1}{E_k}\right)\left(\sum_{j=1}^{n}\beta_{kj}S_k\int_0^{\infty}f_j(r)E_j(t-r)\,\mathrm{d}r - (d_k^E+\epsilon_k)E_k\right) \\
&+ \sum_{k=1}^{n}\left(1-\frac{1}{I_k}\right)\left[\epsilon_kE_k - \left(d_k^I+\gamma_k+\theta_k\right)I_k\right] - \sum_{k=1}^{n}\sum_{j=1}^{n}ac_k\beta_{kj}\int_0^{\infty}f_j(r)E_j(t-r)\,\mathrm{d}r \\
&+ \frac{1}{2}\sum_{k=1}^{n}\left[\frac{ac_k}{S_k^2}\left(\sigma_{1k}S_k\right)^2 + \frac{1}{E_k^2}\left(\sigma_{2k}E_k\right)^2 + \frac{1}{I_k^2}\left(\sigma_{3k}I_k\right)^2 + \frac{1}{R_k^2}\left(\sigma_{4k}R_k\right)^2\right] \\
&+ \sum_{k=1}^{n}ac_k\int_Y\left[D_{1k}(y) - \ln(1+D_{1k}(y))\right]\nu(\mathrm{d}y) + \sum_{k=1}^{n}\int_Y\left[D_{2k}(y) - \ln(1+D_{2k}(y))\right]\nu(\mathrm{d}y) \\
&+ \sum_{k=1}^{n}\int_Y\left[D_{3k}(y) - \ln(1+D_{3k}(y))\right]\nu(\mathrm{d}y) + \sum_{k=1}^{n}\int_Y\left[D_{4k}(y) - \ln(1+D_{4k}(y))\right]\nu(\mathrm{d}y) \\
=& \sum_{k=1}^{n}\left[\Lambda_k + ac_kd_k^S + d_k^E + \epsilon_k + d_k^I + \gamma_k + \theta_k + d_k^R + ac_k\sum_{j=1}^{n}\beta_{kj}a_jE_j - d_k^SS_k - \frac{ac_k\Lambda_k}{S_k} - d_k^EE_k\right. \\
&- \left(d_k^I+\theta_k\right)I_k - d_k^RR_k + ac_k\sum_{j=1}^{n}\beta_{kj}\int_0^{\infty}f_j(r)E_j(t-r)\mathrm{d}r - ac_k\sum_{j=1}^{n}\beta_{kj}\int_0^{\infty}f_j(r)E_j(t-r)\mathrm{d}r \\
&\left. - \frac{S_k}{E_k}\sum_{j=1}^{n}\beta_{kj}\int_0^{\infty}f_j(r)E_j(t-r)\,\mathrm{d}r - \frac{\epsilon_kE_k}{I_k} - \frac{\gamma_kI_k}{R_k}\right] + \frac{1}{2}\sum_{k=1}^{n}\left(ac_k\sigma_{1k}^2 + \sigma_{2k}^2 + \sigma_{3k}^2 + \sigma_{4k}^2\right) \\
&+ \sum_{k=1}^{n}ac_k\int_Y\left[D_{1k}(y) - \ln(1+D_{1k}(y))\right]\nu(\mathrm{d}y) + \sum_{k=1}^{n}\int_Y\left[D_{2k}(y) - \ln(1+D_{2k}(y))\right]\nu(\mathrm{d}y) \\
&+ \sum_{k=1}^{n}\int_Y\left[D_{3k}(y) - \ln(1+D_{3k}(y))\right]\nu(\mathrm{d}y) + \sum_{k=1}^{n}\int_Y\left[D_{4k}(y) - \ln(1+D_{4k}(y))\right]\nu(\mathrm{d}y) \\
\leq& \sum_{k=1}^{n}\left[\Lambda_k + ac_kd_k^S + d_k^E + \epsilon_k + d_k^I + \gamma_k + \theta_k + d_k^R - \left(d_k^EE_k - ac_k\sum_{j=1}^{n}\beta_{kj}a_jE_j\right)\right] \\
&+ \frac{1}{2}\sum_{k=1}^{n}\left(ac_k\sigma_{1k}^2 + \sigma_{2k}^2 + \sigma_{3k}^2 + \sigma_{4k}^2\right) + \sum_{k=1}^{n}ac_k\int_Y\left[D_{1k}(y) - \ln(1+D_{1k}(y))\right]\nu(\mathrm{d}y)
\end{aligned}
$$

$$+\sum_{k=1}^{n}\int_Y [D_{2k}(y)-\ln(1+D_{2k}(y))]\,\nu(\mathrm{d}y)+\sum_{k=1}^{n}\int_Y [D_{3k}(y)-\ln(1+D_{3k}(y))]\,\nu(\mathrm{d}y)$$
$$+\sum_{k=1}^{n}\int_Y [D_{4k}(y)-\ln(1+D_{4k}(y))]\,\nu(\mathrm{d}y).$$

According to Lemma 3.2, we know that

$$\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}E_j=\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}E_k.$$

Choose $a=\min\left\{\frac{d_k^E}{c_k\sum_{j=1}^{n}\beta_{kj}a_j}, k=1,2,\cdots,n\right\}$, such that for each $k=1,2,\cdots,n$,

$$ac_k\sum_{j=1}^{n}\beta_{kj}a_j-d_k^E\le 0.$$

Therefore,

$$LV(X)\le\sum_{k=1}^{n}\left[\Lambda_k+ac_kd_k^S+d_k^E+\epsilon_k+d_k^I+\gamma_k+\theta_k+d_k^R+\frac{1}{2}\left(ac_k\sigma_{1k}^2+\sigma_{2k}^2+\sigma_{3k}^2+\sigma_{4k}^2\right)+4F_k\right]:=\tilde{F}, \tag{4.4}$$

where

$$F_k=\max\left\{ac_k\int_Y [D_{1k}(y)-\ln(1+D_{1k}(y))]\,\nu(\mathrm{d}y), \int_Y [D_{2k}(y)-\ln(1+D_{2k}(y))]\,\nu(\mathrm{d}y),\right.$$
$$\left.\int_Y [D_{3k}(y)-\ln(1+D_{3k}(y))]\,\nu(\mathrm{d}y), \int_Y [D_{4k}(y)-\ln(1+D_{4k}(y))]\,\nu(\mathrm{d}y)\right\}.$$

Since $x-\ln(1+x)\ge 0, \forall x>-1$, and hypothesis (A2) is satisfied, we conclude that F_k is non-negative. Therefore, $\tilde{F}$ is a positive constant, and combined with inequality (4.4), we get

$$\begin{aligned}\mathrm{d}V(X)\le&\tilde{F}\mathrm{d}t+\sum_{k=1}^{n}[\sigma_{1k}(S_k-ac_k)\,\mathrm{d}B_{1k}(t)+\sigma_{2k}(E_k-1)\,\mathrm{d}B_{2k}(t)+\sigma_{3k}(I_k-1)\,\mathrm{d}B_{3k}(t)\\&+\sigma_{4k}(R_k-1)\,\mathrm{d}B_{4k}(t)]+\sum_{k=1}^{n}\int_Y [S_kD_{1k}(y)-ac_k\ln(1+D_{1k}(y))+E_kD_{2k}(y)\\&-\ln(1+D_{2k}(y))+I_kD_{3k}(y)-\ln(1+D_{3k}(y))+R_kD_{4k}(y)-\ln(1+D_{4k}(y))]\,\tilde{N}(\mathrm{d}t,\mathrm{d}y).\end{aligned} \tag{4.5}$$

Integrating (4.5) from 0 to $\tau_m\wedge T$ and taking the expectation on both sides yields

$$\mathbb{E}V(X(\tau_m\wedge T))\le V(X(0))+\mathbb{E}\int_0^{\tau_m\wedge T}\tilde{F}\mathrm{d}t\le V(X(0))+\tilde{F}T. \tag{4.6}$$

Let $\Omega_m=\{\tau_m\le T\}$, for $m\ge m_1$, and by (4.1), we have $\mathbb{P}(\Omega_m)\ge\epsilon$. Note that, for every $\omega\in\Omega_m$, there is at least one of $S_k(\tau_m\wedge T), E_k(\tau_m\wedge T), I_k(\tau_m\wedge T), R_k(\tau_m\wedge T)$ ($1\le k\le n$) that equals either m or $\frac{1}{m}$. Consequently,

$$V(X(\tau_m\wedge T))\ge\min_{0\le k\le n}(m-ac_k-ac_k\ln\frac{m}{ac_k})\wedge\min_{0\le k\le n}(\frac{1}{m}-ac_k+ac_k\ln ac_km),$$

where we define c_0 such that $ac_0 = 1$. It is easy to observe that

$$\begin{aligned}
&V(X(0)) + \tilde{F}T \\
\geq &\mathbb{E}\left[\mathbf{1}_{\Omega_m} V\left(S_k(\tau_m \wedge T), E_k(\tau_m \wedge T), I_k(\tau_m \wedge T), R_k(\tau_m \wedge T)\right), 1 \leq k \leq n\right] \\
\geq &\epsilon\left[\min_{0\leq k\leq n}\left(m - ac_k - ac_k \ln\frac{m}{ac_k}\right) \wedge \min_{0\leq k\leq n}\left(\frac{1}{m} - ac_k + ac_k \ln ac_k m\right)\right],
\end{aligned}$$

where $\mathbf{1}_{\Omega_m}$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction that $V(X(0)) + \tilde{F}T = \infty$. So $\tau_\infty = \infty$ a.s. □

5. Asymptotic behavior of the solution around the disease-free equilibrium point P_0

It is not difficult to show that $P_0\left(\frac{\Lambda_1}{d_1^S}, 0, 0, 0, \cdots, \frac{\Lambda_n}{d_n^S}, 0, 0, 0\right)$ is the disease-free equilibrium of the deterministic model. For the deterministic system, if the basic reproduction number $R_0 < 1$, then P_0 is globally asymptotically stable, which implies that the disease will die out. In this section, we discuss the asymptotic behavior of the solution for the stochastic multi-group delayed SEIR epidemic system (2.1) around P_0.

The basic reproduction number R_0 of system (2.1) is defined as the spectral radius of the next-generation matrix M_0, i.e.,

$$R_0 = \rho(M_0),$$

and

$$M_0 = \left(\frac{\beta_{kj} S_k^0 a_j}{d_k^E + \epsilon_k}\right)_{n\times n},$$

where $S_k^0 = \frac{\Lambda_k}{d_k^S}$ denotes the number of susceptible individuals in the k-th group at the disease-free equilibrium P_0, and $a_j = \int_0^\infty f_j(r)\mathrm{d}r$ is the integral of the delay kernel function. Since $S_k^0 > 0$, $a_j > 0$, and $d_k^E + \epsilon_k > 0$, the matrix M_0 is a non-negative matrix obtained by multiplying each entry of the irreducible matrix $B = (\beta_{kj})_{n\times n}$ by a positive scalar. Therefore, M_0 is also irreducible.

Theorem 5.1. *Suppose that $B = (\beta_{kj})_{n\times n}$ is irreducible. If $R_0 < 1$ and the following conditions hold:*

$$\begin{gathered}
\sigma_{1k}^2 + 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y) < d_k^S, \ \frac{\sigma_{2k}^2}{2} + 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y) < d_k^E + \epsilon_k, \\
\frac{\sigma_{3k}^2}{2} + \int_Y D_{3k}^2(y)\nu(\mathrm{d}y) < d_k^I + \gamma_k + \theta_k, \ \frac{\sigma_{4k}^2}{2} + \int_Y D_{4k}^2(y)\nu(\mathrm{d}y) < d_k^R, \quad k = 1, 2, \cdots, n,
\end{gathered}$$

then for any given initial value $(S_k(0), E_k(0), I_k(0), R_k(0),\ 1 \leq k \leq n) \in \mathbb{R}_+^{4n}$, the solution $(S_k(t), E_k(t), I_k(t), R_k(t),\ 1 \leq k \leq n) \in \mathbb{R}_+^{4n}$ of stochastic delayed system (2.1) *possesses the following property:*

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^n \mathbb{E}\int_0^t \left[\left(S_k(s) - \frac{\Lambda_k}{d_k^S}\right)^2 + E_k^2(s) + I_k^2(s) + R_k^2(s)\right]\mathrm{d}s \leq \frac{\sum_{k=1}^n J_k}{M},$$

where

$$J_k = (a_k b+1)\left(\frac{\Lambda_k}{d_k^S}\right)^2\left(\sigma_{1k}^2 + 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right),\ M_{1k} = d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y),$$
$$M_{2k} = \frac{1}{4}\left(d_k^E + \epsilon_k - \frac{1}{2}\sigma_{2k}^2 - 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right),\ M_{3k} = \frac{f_k}{4}\left(d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2 - \int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right),$$
$$M_{4k} = \frac{g_k}{2}\left(d_k^R - \frac{1}{2}\sigma_{4k}^2 - \int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right),\ M = \min\{M_{1k}, M_{2k}, M_{3k}, M_{4k}\},$$

and $b,\ a_k,\ f_k,\ g_k\ (k = 1, 2, \cdots, n)$ *are positive constants to be determined later.*

Proof. Let $u_k = S_k - \frac{\Lambda_k}{d_k^S}$, $v_k = E_k$, $w_k = I_k$, and $x_k = R_k$. Then we have $u_k \in \mathbb{R}$, $v_k \geqslant 0$, $w_k \geqslant 0$, $x_k \geqslant 0$, and the stochastic delayed system (2.1) can be rewritten as

$$\begin{cases} \mathrm{d}u_k(t) = \left[-d_k^S u_k - \sum_{j=1}^n \beta_{kj}\left(u_k + \frac{\Lambda_k}{d_k^S}\right)\int_0^\infty f_j(r)v_j(t-r)\mathrm{d}r\right]\mathrm{d}t + \sigma_{1k}\left(u_k + \frac{\Lambda_k}{d_k^S}\right)\mathrm{d}B_{1k}(t) \\ \qquad\qquad + \int_Y \left(u_k + \frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)\tilde{N}(\mathrm{d}t, \mathrm{d}y), \\ \mathrm{d}v_k(t) = \left[\sum_{j=1}^n \beta_{kj}\left(u_k + \frac{\Lambda_k}{d_k^S}\right)\int_0^\infty f_j(r)v_j(t-r)\mathrm{d}r - (d_k^E + \epsilon_k)v_k\right]\mathrm{d}t + \sigma_{2k}v_k\mathrm{d}B_{2k}(t) \\ \qquad\qquad + \int_Y v_k D_{2k}(y)\tilde{N}(\mathrm{d}t, \mathrm{d}y), \\ \mathrm{d}w_k(t) = \left[\epsilon_k v_k - (d_k^I + \gamma_k + \theta_k)w_k\right]\mathrm{d}t + \sigma_{3k}w_k\mathrm{d}B_{3k}(t) + \int_Y w_k D_{3k}(y)\tilde{N}(\mathrm{d}t, \mathrm{d}y), \\ \mathrm{d}x_k(t) = \left(\gamma_k w_k - d_k^R x_k\right)\mathrm{d}t + \sigma_{4k}x_k\mathrm{d}B_{4k}(t) + \int_Y x_k D_{4k}(y)\tilde{N}(\mathrm{d}t, \mathrm{d}y), \quad k = 1, 2, \ldots, n. \end{cases}$$

Since M_0 is non-negative and irreducible, by Lemma 3.1, there exists a strictly positive left eigenvector $e = (e_1, e_2, ..., e_n) \in \mathbb{R}^{1\times n}$ with $e_k > 0$ for all $k = 1, 2, \cdots, n$ corresponding to $\rho(M_0)$, that is,

$$(e_1, e_2, \cdots, e_n)\,\rho(M_0) = (e_1, e_2, \cdots, e_n)\,M_0.$$

Right-multiplying both sides by the vector $v = (v_1, v_2, \ldots, v_n)^T$, we obtain the key identity

$$(e_1, e_2, \cdots, e_n)\,\rho(M_0)(v_1, v_2, \ldots, v_n)^T = (e_1, e_2, \cdots, e_n)\,M_0(v_1, v_2, \ldots, v_n)^T. \tag{5.1}$$

Define the function V_i $(i = 1, 2, 3, 4, 5)$ as follows:

$$V_1 = \frac{1}{2}\sum_{k=1}^n (u_k + v_k)^2,\ V_2 = \frac{1}{2}\sum_{k=1}^n a_k u_k^2,\ V_3 = \sum_{k=1}^n \frac{e_k}{d_k^E + \epsilon_k}v_k,\ V_4 = \frac{1}{2}\sum_{k=1}^n f_k w_k^2,\ V_5 = \frac{1}{2}\sum_{k=1}^n g_k x_k^2.$$

By applying the Itô formula, the computation of $\mathrm{d}V_i$ $(i = 1, 2, 3, 4, 5)$ can be carried out:

$$\mathrm{d}V_1 = LV_1\mathrm{d}t + \sum_{k=1}^n (u_k + v_k)\left[\sigma_{1k}\left(u_k + \frac{\Lambda_k}{d_k^S}\right)\mathrm{d}B_{1k}(t) + \sigma_{2k}v_k\mathrm{d}B_{2k}(t)\right]$$

$$+\frac{1}{2}\sum_{k=1}^{n}\int_Y\left[\left(\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)+v_kD_{2k}(y)\right)^2+2\left(u_k+v_k\right)\left(\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)+v_kD_{2k}(y)\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),$$

$$\mathrm{d}V_2=LV_2\mathrm{d}t+\sum_{k=1}^{n}a_k\sigma_{1k}u_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)\mathrm{d}B_{1k}(t)+\frac{1}{2}\sum_{k=1}^{n}a_k\int_Y\left[\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2D_{1k}^2(y)+2u_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),$$

$$\mathrm{d}V_3=LV_3\mathrm{d}t+\sum_{k=1}^{n}\frac{e_k}{d_k^E+\epsilon_k}\sigma_{2k}v_k\mathrm{d}B_{2k}(t)+\sum_{k=1}^{n}\frac{e_k}{d_k^E+\epsilon_k}\int_Y v_kD_{2k}(y)\tilde{N}(\mathrm{d}t,\mathrm{d}y),$$

$$\mathrm{d}V_4=LV_4\mathrm{d}t+\sum_{k=1}^{n}f_k\sigma_{3k}w_k^2\mathrm{d}B_{3k}(t)+\frac{1}{2}\sum_{k=1}^{n}f_k\int_Y w_k^2\left(D_{3k}^2(y)+2D_{3k}(y)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}y),$$

$$\mathrm{d}V_5=LV_5\mathrm{d}t+\sum_{k=1}^{n}g_k\sigma_{4k}x_k^2\mathrm{d}B_{4k}(t)+\frac{1}{2}\sum_{k=1}^{n}g_k\int_Y x_k^2\left(D_{4k}^2(y)+2D_{4k}(y)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}y).$$

In addition:

$$\begin{aligned}LV_1=&\sum_{k=1}^{n}(u_k+v_k)\left[-d_k^Su_k-\left(d_k^E+\epsilon_k\right)v_k\right]+\frac{1}{2}\sum_{k=1}^{n}\left[\sigma_{1k}^2\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2+\sigma_{2k}^2v_k^2\right]\\&+\frac{1}{2}\sum_{k=1}^{n}\int_Y\left[\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)+v_kD_{2k}(y)\right]^2\nu(\mathrm{d}y)\\\le&-\sum_{k=1}^{n}\left[\left(d_k^S-\sigma_{1k}^2\right)u_k^2+\left(d_k^E+\epsilon_k-\frac{\sigma_{2k}^2}{2}\right)v_k^2+\left(d_k^S+d_k^E+\epsilon_k\right)u_kv_k-\left(\frac{\sigma_{1k}\Lambda_k}{d_k^S}\right)^2\right]\\&+\sum_{k=1}^{n}\int_Y\left[\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2D_{1k}^2(y)+v_k^2D_{2k}^2(y)\right]\nu(\mathrm{d}y),\\LV_2=&\sum_{k=1}^{n}a_ku_k\left[-d_k^Su_k-\sum_{j=1}^{n}\beta_{kj}\left(u_k+\frac{\Lambda_k}{d_k^S}\right)\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r\right]+\frac{1}{2}\sum_{k=1}^{n}a_k\sigma_{1k}^2\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2\\&+\frac{1}{2}\sum_{k=1}^{n}a_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\\\le&-\sum_{k=1}^{n}a_k\left[\left(d_k^S-\sigma_{1k}^2\right)u_k^2+\sum_{j=1}^{n}\beta_{kj}u_k^2\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r+\sum_{j=1}^{n}\beta_{kj}\frac{\Lambda_k}{d_k^S}u_k\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r\right.\\&\left.-\left(\frac{\sigma_{1k}\Lambda_k}{d_k^S}\right)^2\right]+\frac{1}{2}\sum_{k=1}^{n}a_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y).\end{aligned}$$

Due to $\sum_{j=1}^{n}\beta_{kj}u_k^2\int_0^\infty f_j(r)v_j(t-r)\,dr\ge 0, k,j=1,2,\ldots,n$, the above equation can be reduced to

$$\begin{aligned}LV_2\le&-\sum_{k=1}^{n}a_k\left[\left(d_k^S-\sigma_{1k}^2\right)u_k^2+\sum_{j=1}^{n}\beta_{kj}\frac{\Lambda_k}{d_k^S}u_k\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r-(\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2\right]\\&+\frac{1}{2}\sum_{k=1}^{n}a_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y),\end{aligned}\tag{5.2}$$

$$
\begin{aligned}
LV_3 &= \sum_{k=1}^{n} \frac{e_k}{d_k^E + \epsilon_k} \left[\sum_{j=1}^{n} \beta_{kj} \left(u_k + \frac{\Lambda_k}{d_k^S} \right) \int_0^{\infty} f_j(r) v_j(t-r)\,\mathrm{d}r - (d_k^E + \epsilon_k) v_k \right] \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} u_k \int_0^{\infty} f_j(r) v_j(t-r)\,\mathrm{d}r - \sum_{k=1}^{n} e_k v_k + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} \int_0^{\infty} f_j(r) v_j(t-r)\,\mathrm{d}r, \\
LV_4 &= \sum_{k=1}^{n} f_k \left[\epsilon_k v_k w_k - \left(d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2 \right) w_k^2 \right] + \frac{1}{2} \sum_{k=1}^{n} f_k w_k^2 \int_Y D_{3k}^2(y) \nu(\mathrm{d}y) \\
&\leq -\frac{1}{2} \sum_{k=1}^{n} f_k \left[\left(d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2 \right) w_k^2 - \frac{\epsilon_k^2}{d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2} v_k^2 \right] + \frac{1}{2} \sum_{k=1}^{n} f_k w_k^2 \int_Y D_{3k}^2(y) \nu(\mathrm{d}y), \\
LV_5 &= \sum_{k=1}^{n} g_k \left(\gamma_k w_k x_k - d_k^R x_k^2 \right) + \frac{1}{2} \sum_{k=1}^{n} g_k \sigma_{4k}^2 x_k^2 + \frac{1}{2} \sum_{k=1}^{n} g_k x_k^2 \int_Y D_{4k}^2(y) \nu(\mathrm{d}y) \\
&\leq -\frac{1}{2} \sum_{k=1}^{n} g_k \left[\left(d_k^R - \frac{1}{2}\sigma_{4k}^2 \right) x_k^2 - \frac{\gamma_k^2}{d_k^R - \frac{1}{2}\sigma_{4k}^2} w_k^2 \right] + \frac{1}{2} \sum_{k=1}^{n} g_k x_k^2 \int_Y D_{4k}^2(y) \nu(\mathrm{d}y).
\end{aligned}
$$

Let

$$
\tilde{V}_3 = V_3 + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} \int_t^{t+r} \int_0^{\infty} f_j(r) v_j(u-r)\,\mathrm{d}r\mathrm{d}u,
$$

and then

$$
\begin{aligned}
\mathrm{d}\tilde{V}_3 &= \mathrm{d}V_3 + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} \left[\int_0^{\infty} f_j(r) v_j(t+r-r)\,\mathrm{d}r - \int_0^{\infty} f_j(r) v_j(t-r)\,\mathrm{d}r \right] \mathrm{d}t \\
&:= L\tilde{V}_3 \mathrm{d}t + \sum_{k=1}^{n} \frac{e_k \sigma_{2k}}{d_k^E + \epsilon_k} v_k \mathrm{d}B_{2k}(t) + \sum_{k=1}^{n} \frac{e_k}{d_k^E + \epsilon_k} \int_Y v_k D_{2k}(y) \tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{aligned}
$$

where

$$
\begin{aligned}
L\tilde{V}_3 &= LV_3 + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} a_j v_j - \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} \int_0^{\infty} f_j(r) v_j(t-r)\,\mathrm{d}r \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} u_k \int_0^{\infty} f_j(r) v_j(t-r)\,\mathrm{d}r - \sum_{k=1}^{n} e_k v_k + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} a_j v_j.
\end{aligned}
$$

Combining with formula (5.1), it is noted that

$$
\begin{aligned}
&- \sum_{k=1}^{n} e_k v_k + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj}}{d_k^E + \epsilon_k} \frac{\Lambda_k}{d_k^S} a_j v_j \\
&= -(e_1, e_2, \cdots, e_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + (e_1, e_2, \cdots, e_n) \times \begin{pmatrix}
\frac{\beta_{11} \frac{\Lambda_1}{d_1^S} a_1}{d_1^E + \epsilon_1} & \frac{\beta_{12} \frac{\Lambda_1}{d_1^S} a_2}{d_1^E + \epsilon_1} & \cdots & \frac{\beta_{1n} \frac{\Lambda_1}{d_1^S} a_n}{d_1^E + \epsilon_1} \\
\frac{\beta_{21} \frac{\Lambda_2}{d_2^S} a_1}{d_2^E + \epsilon_2} & \frac{\beta_{22} \frac{\Lambda_2}{d_2^S} a_2}{d_2^E + \epsilon_2} & \cdots & \frac{\beta_{2n} \frac{\Lambda_2}{d_2^S} a_n}{d_2^E + \epsilon_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\beta_{n1} \frac{\Lambda_n}{d_n^S} a_1}{d_n^E + \epsilon_n} & \frac{\beta_{n2} \frac{\Lambda_n}{d_n^S} a_2}{d_n^E + \epsilon_n} & \cdots & \frac{\beta_{nn} \frac{\Lambda_n}{d_n^S} a_n}{d_n^E + \epsilon_n}
\end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
:= &-\mathbf{e}v + \mathbf{e}M_0 v = -\mathbf{e}v + \rho(M_0)\mathbf{e}v = (\rho(M_0) - 1)\mathbf{e}v \\
= &(R_0 - 1)\sum_{k=1}^{n} e_k v_k.
\end{aligned}
$$

If $R_0 < 1$, then $(R_0 - 1)e^T v < 0$, and thus

$$
L\tilde{V}_3 \leq \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{e_k\beta_{kj}}{d_k^E + \epsilon_k} u_k \int_0^\infty f_j(r) v_j(t - r)\,\mathrm{d}r. \tag{5.3}
$$

Applying (5.2) and (5.3), yields

$$
\begin{aligned}
LV_2 + L\tilde{V}_3 \leq &-\sum_{k=1}^{n} a_k \left[\left(d_k^S - \sigma_{1k}^2\right)u_k^2 + \sum_{j=1}^{n}\beta_{kj}\frac{\Lambda_k}{d_k^S}u_k\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r - \left(\frac{\sigma_{1k}\Lambda_k}{d_k^S}\right)^2\right] \\
&+ \frac{1}{2}\sum_{k=1}^{n} a_k\left(u_k + \frac{\Lambda_k}{d_k^S}\right)^2 \int_Y D_{1k}^2(y)\nu(\mathrm{d}y) + \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{e_k\beta_{kj}}{d_k^E + \epsilon_k}u_k\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r \\
= &-\sum_{k=1}^{n} a_k\left[\left(d_k^S - \sigma_{1k}^2\right)u_k^2 - \left(\frac{\sigma_{1k}\Lambda_k}{d_k^S}\right)^2\right] - \sum_{k=1}^{n}\sum_{j=1}^{n}\beta_{kj}\left(\frac{a_k\Lambda_k}{d_k^S} - \frac{e_k}{d_k^E + \epsilon_k}\right) \\
&\times u_k\int_0^\infty f_j(r)v_j(t-r)\,\mathrm{d}r + \frac{1}{2}\sum_{k=1}^{n} a_k\left(u_k + \frac{\Lambda_k}{d_k^S}\right)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y).
\end{aligned}
$$

Choosing $a_k = \frac{e_k d_k^S}{(d_k^E + \epsilon_k)\Lambda_k}$ such that

$$
\frac{a_k\Lambda_k}{d_k^S} - \frac{e_k}{d_k^E + \epsilon_k} = 0
$$

leads to

$$
LV_2 + L\tilde{V}_3 \leq -\sum_{k=1}^{n} a_k\left[\left(d_k^S - \sigma_{1k}^2\right)u_k^2 - \left(\frac{\sigma_{1k}\Lambda_k}{d_k^S}\right)^2\right] + \frac{1}{2}\sum_{k=1}^{n} a_k\left(u_k + \frac{\Lambda_k}{d_k^S}\right)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y).
$$

We have

$$
\begin{aligned}
&LV_1 + b\left(LV_2 + L\tilde{V}_3\right) \\
\leq &-\sum_{k=1}^{n}\left[(a_k b + 1)\left(d_k^S - \sigma_{1k}^2\right)u_k^2 + \left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2}\right)v_k^2 + \left(d_k^S + d_k^E + \epsilon_k\right)u_k v_k - (a_k b + 1)\left(\frac{\sigma_{1k}\Lambda_k}{d_k^S}\right)^2\right] \\
&+ \sum_{k=1}^{n}\left(\frac{1}{2}a_k b + 1\right)\left(u_k + \frac{\Lambda_k}{d_k^S}\right)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y) + \sum_{k=1}^{n} v_k^2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y) \\
\leq &-\sum_{k=1}^{n}\left[(a_k b + 1)\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right) - \frac{\left(d_k^S + d_k^E + \epsilon_k\right)^2}{2\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2}\right)}\right]u_k^2 \\
&-\frac{1}{2}\sum_{k=1}^{n}\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2} - 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right)v_k^2 + \sum_{k=1}^{n}(a_k b + 1)\left(\frac{\Lambda_k}{d_k^S}\right)^2\left(\sigma_{1k}^2 + 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right),
\end{aligned}
$$

where b is a positive constant defined by

$$b = \max\left\{\frac{\left(d_k^S + d_k^E + \epsilon_k\right)^2}{2a_k\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2}\right)\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)},\ k = 1, 2, \cdots, n\right\},$$

such that $a_k b\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right) - \frac{\left(d_k^S + d_k^E + \epsilon_k\right)^2}{2\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2}\right)} \geq 0$ for each k. Accordingly, we obtain

$$\begin{aligned} LV_1 + b(LV_2 + L\tilde{V}_3) \leq & -\sum_{k=1}^{n}\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)u_k^2 - \frac{1}{2}\sum_{k=1}^{n}\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2}\right. \\ & \left. -2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right)v_k^2 + \sum_{k=1}^{n}(a_k b + 1)\left(\frac{\Lambda_k}{d_k^S}\right)^2\left(\sigma_{1k}^2 + 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} & LV_1 + b(LV_2 + L\tilde{V}_3) + LV_4 + LV_5 \\ \leq & -\sum_{k=1}^{n}\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)u_k^2 - \frac{1}{2}\sum_{k=1}^{n}\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2} - 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right. \\ & \left. - \frac{f_k\epsilon_k^2}{d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2}\right)v_k^2 - \frac{1}{2}\sum_{k=1}^{n} f_k\left(d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2 - \int_Y D_{3k}^2(y)\nu(\mathrm{d}y) - \frac{g_k\gamma_k^2}{f_k\left(d_k^R - \frac{1}{2}\sigma_{4k}^2\right)}\right)w_k^2 \\ & - \frac{1}{2}\sum_{k=1}^{n} g_k\left(d_k^R - \frac{1}{2}\sigma_{4k}^2 - \int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right)x_k^2 + \sum_{k=1}^{n}(a_k b + 1)\left(\frac{\Lambda_k}{d_k^S}\right)^2\left(\sigma_{1k}^2 + 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right). \end{aligned}$$

Let $f_k = \frac{1}{2\epsilon_k^2}\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2} - 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right)\left(d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2\right)$, which leads to

$$\frac{f_k\epsilon_k^2}{d_k^I + \gamma_k + \theta_k - \frac{1}{2}\sigma_{3k}^2} = \frac{1}{2}\left(d_k^E + \epsilon_k - \frac{\sigma_{2k}^2}{2} - 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right). \tag{5.4}$$

Let $g_k = \frac{f_k}{2\gamma_k^2}\left(d_k^I + \gamma_k + \theta_k - \frac{\sigma_{3k}^2}{2} - \int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right)\left(d_k^R - \frac{1}{2}\sigma_{4k}^2\right)$, and then

$$\frac{g_k\gamma_k^2}{f_k\left(d_k^R - \frac{1}{2}\sigma_{4k}^2\right)} = \frac{1}{2}\left(d_k^I + \gamma_k + \theta_k - \frac{\sigma_{3k}^2}{2} - \int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right). \tag{5.5}$$

According to the conditions of Theorem 5.1, it can be inferred that f_k, g_k are non-negative constants. Therefore,

$$\begin{aligned} LV(X) = & LV_1 + b\left(LV_2 + L\tilde{V}_3\right) + LV_4 + LV_5 \\ \leq & -\sum_{k=1}^{n}\left[\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)u_k^2 + \frac{1}{4}\left(d_k^E + \epsilon_k - \frac{1}{2}\sigma_{2k}^2 - 2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right)v_k^2\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{f_k}{4}\left(d_k^I+\gamma_k+\theta_k-\frac{\sigma_{3k}^2}{2}-\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right)w_k^2+\frac{g_k}{2}\left(d_k^R-\frac{1}{2}\sigma_{4k}^2-\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right)x_k^2\Bigg]\\
&+\sum_{k=1}^n(a_kb+1)\left(\frac{\Lambda_k}{d_k^S}\right)^2\left(\sigma_{1k}^2+2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)\\
=&-\sum_{k=1}^n\left(M_{1k}u_k^2+M_{2k}v_k^2+M_{3k}w_k^2+M_{4k}x_k^2\right)+\sum_{k=1}^n J_k,
\end{aligned}\tag{5.6}
$$

where

$$
\begin{aligned}
&M_{1k}=d_k^S-\sigma_{1k}^2-2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y),\ M_{2k}=\frac{1}{4}\left(d_k^E+\epsilon_k-\frac{1}{2}\sigma_{2k}^2-2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right),\\
&M_{3k}=\frac{f_k}{4}\left(d_k^I+\gamma_k+\theta_k-\frac{1}{2}\sigma_{3k}^2-\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right),\ M_{4k}=\frac{g_k}{2}\left(d_k^R-\frac{1}{2}\sigma_{4k}^2-\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right),\\
&J_k=(a_kb+1)\left(\frac{\Lambda_k}{d_k^S}\right)^2\left(\sigma_{1k}^2+2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathrm{d}V(X)=&\mathrm{d}V_1+b(\mathrm{d}V_2+\mathrm{d}\tilde{V}_3)+\mathrm{d}V_4+\mathrm{d}V_5 \qquad (5.7)\\
\le&-\sum_{k=1}^n\left(M_{1k}u_k^2+M_{2k}v_k^2+M_{3k}w_k^2+M_{4k}x_k^2\right)+\sum_{k=1}^n J_k+\sum_{k=1}^n(u_k+v_k)\left[\sigma_{1k}\left(u_k+\frac{\Lambda_k}{d_k^S}\right)\mathrm{d}B_{1k}(t)\right.\\
&+\sigma_{2k}v_k\mathrm{d}B_{2k}(t)]+b\sum_{k=1}^n a_k\sigma_{1k}u_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)\mathrm{d}B_{1k}(t)+b\sum_{k=1}^n\frac{e_k\sigma_{2k}}{d_k^E+\epsilon_k}v_k\mathrm{d}B_{2k}(t)\\
&+\sum_{k=1}^n f_k\sigma_{3k}w_k^2\mathrm{d}B_{3k}(t)+\sum_{k=1}^n g_k\sigma_{4k}x_k^2\mathrm{d}B_{4k}(t)+\frac{1}{2}\sum_{k=1}^n\int_Y\left[\left(\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)+v_kD_{2k}(y)\right)^2\right.\\
&\left.+2\left(u_k+v_k\right)\left(\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)+v_kD_{2k}(y)\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)+\frac{b}{2}\sum_{k=1}^n a_k\int_Y\left[\left(u_k+\frac{\Lambda_k}{d_k^S}\right)^2D_{1k}^2(y)\right.\\
&\left.+2u_k\left(u_k+\frac{\Lambda_k}{d_k^S}\right)D_{1k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)+b\sum_{k=1}^n\frac{e_k}{d_k^E+\epsilon_k}\int_Y v_kD_{2k}(y)\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&+\frac{1}{2}\sum_{k=1}^n f_k\int_Y w_k^2\left(D_{3k}^2(y)+2D_{3k}(y)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}y)+\frac{1}{2}\sum_{k=1}^n g_k\int_Y x_k^2\left(D_{4k}^2(y)+2D_{4k}(y)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}y).
\end{aligned}
$$

Integrating both sides of inequality (5.7) from 0 to t and taking the expectation, we get

$$
\mathbb{E}V(X(t))=V(X(0))+\mathbb{E}\int_0^t LV(X)\mathrm{d}s.
$$

Then by (5.6), we find

$$
0\le\mathbb{E}V(X(t))\le V(X(0))-\sum_{k=1}^n\mathbb{E}\int_0^t\left(M_{1k}u_k^2(s)+M_{2k}v_k^2(s)+M_{3k}w_k^2(s)+M_{4k}x_k^2(s)\right)\mathrm{d}s+t\sum_{k=1}^n J_k.
$$

Since $M_{1k} > 0, M_{2k} > 0, M_{3k} > 0, M_{4k} > 0$, we obtain

$$\sum_{k=1}^{n} \mathbb{E} \int_0^t \left(M_{1k} u_k^2(s) + M_{2k} v_k^2(s) + M_{3k} w_k^2(s) + M_{4k} x_k^2(s) \right) \mathrm{d}s \leq V(X(0)) + t \sum_{k=1}^{n} J_k.$$

Dividing both sides of the above inequality by t and taking the limit superior as $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=1}^{n} \mathbb{E} \int_0^t \left(u_k^2(s) + v_k^2(s) + w_k^2(s) + x_k^2(s) \right) \mathrm{d}s \leq \frac{\sum_{k=1}^{n} J_k}{M},$$

in which $M = \min\{M_{1k}, M_{2k}, M_{3k}, M_{4k}\}$. Hence

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=1}^{n} \mathbb{E} \int_0^t \left[\left(S_k(s) - \frac{\Lambda_k}{d_k^S} \right)^2 + E_k^2(s) + I_k^2(s) + R_k^2(s) \right] \mathrm{d}s \leq \frac{\sum_{k=1}^{n} J_k}{M}.$$

This completes the proof. □

Remark 1. *Theorem 5.1 shows that when $R_0 < 1$ and certain assumptions are satisfied, the solution of the stochastic delayed SEIR system* (2.1) *fluctuates randomly around the disease-free equilibrium P_0 of the deterministic system. The oscillation amplitude is affected by perturbation intensity: weaker perturbations result in smaller oscillations near P_0, while stronger perturbations generate larger amplitudes. When $\sigma_{ik} = 0$ $(i = 1, 2, 3, 4)$ and $D_{ik} = 0$ $(i = 1, 2, 3, 4)$, the stochastic delayed SEIR system* (2.1) *reduces to the deterministic delay system, where P_0 is globally asymptotically stable and the solution converges to P_0.*

6. Asymptotic behavior of the solution around the endemic equilibrium point P^*

When $R_0 = \rho(M_0) > 1$, the deterministic multi-group SEIR model admits a positive equilibrium, namely, the endemic equilibrium $P^*(S_1^*, E_1^*, I_1^*, R_1^*, \cdots, S_n^*, E_n^*, I_n^*, R_n^*)$. Obviously, P^* is no longer an endemic equilibrium of the stochastic multi-group SEIR model (2.1). In this section, we show that the solution of stochastic delayed system (2.1) oscillates around P^* under certain conditions.

We introduce the following notations:

$$\bar{\beta}_{kj} = \beta_{kj} S_k^* a_j E_j^*, \bar{B} = (\beta_{kj} S_k^* a_j E_j^*)_{n\times n},\ 1 \leq k, j \leq n,\ n \geq 2,$$

$$L_{\bar{B}} = \begin{pmatrix} \sum\limits_{k\neq 1} \bar{\beta}_{1k} & -\bar{\beta}_{12} & \cdots & -\bar{\beta}_{1n} \\ -\bar{\beta}_{21} & \sum\limits_{k\neq 2} \bar{\beta}_{2k} & \cdots & -\bar{\beta}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{n1} & -\bar{\beta}_{n2} & \cdots & \sum\limits_{k\neq n} \bar{\beta}_{nk} \end{pmatrix}.$$

Note that $L_{\bar{B}}$ is the Laplacian matrix of matrix $\bar{B}$. If $(\beta_{kj})_{n\times n}$ is irreducible, $\bar{B}$ and $L_{\bar{B}}$ are also irreducible. Let $\bar{c}_k$ denote the cofactor of the k-th diagonal entry of $L_{\bar{B}}$.

Theorem 6.1. *Suppose that $B = (\beta_{kj})_{n\times n}$ is irreducible. If $R_0 > 1$ and the following conditions hold:*

$$\sigma_{1k}^2 + 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y) < d_k^S,\ \sigma_{2k}^2 + 4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y) < d_k^E + \epsilon_k,$$
$$\sigma_{3k}^2 + 2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y) < d_k^I + \gamma_k + \theta_k,\ \sigma_{4k}^2 + 2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y) < d_k^R, \quad k = 1, 2, \cdots, n,$$

then for any given initial value $(S_1(0), E_1(0), I_1(0), R_1(0), \cdots, S_n(0), E_n(0), I_n(0), R_n(0)) \in \mathbb{R}_+^{4n}$, the solution $(S_1(t), E_1(t), I_1(t), R_1(t), \cdots, S_n(t), E_n(t), I_n(t), R_n(t)) \in \mathbb{R}_+^{4n}$ to stochastic model (2.1) *has the following property:*

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[(S_k(s) - S_k^*)^2 + (E_k(s) - E_k^*)^2 + (I_k(s) - I_k^*)^2 + (R_k(s) - R_k^*)^2\right]\mathrm{d}s \le \frac{\sum_{k=1}^n H_k}{\eta},$$

where

$$\begin{aligned}
H_k = &\left(\frac{a+2}{2}\bar{c}_k + b_k S_k^*\right) S_k^* \sigma_{1k}^2 + \left(\frac{a+1}{2}\bar{c}_k + b_k E_k^*\right) E_k^* \sigma_{2k}^2 + A_k (I_k^*)^2 \sigma_{3k}^2 + B_k (R_k^*)^2 \sigma_{4k}^2 \\
&+ \left(2b_k + \frac{\bar{c}_k}{S_k^*}\right)(S_k^*)^2 \int_Y D_{1k}^2(y)\nu(\mathrm{d}y) + 2b_k (E_k^*)^2 \int_Y D_{2k}^2(y)\nu(\mathrm{d}y) + A_k (I_k^*)^2 \int_Y D_{3k}^2(y)\nu(\mathrm{d}y) \\
&+ B_k (R_k^*)^2 \int_Y D_{4k}^2(y)\nu(\mathrm{d}y) + 3G_k,
\end{aligned}$$
$$\eta_{1k} = \frac{\bar{c}_k}{2S_k^*}\left[d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right],\ \eta_{2k} = \frac{b_k}{4}\left[d_k^E + \epsilon_k - \sigma_{2k}^2 - 4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right],$$
$$\eta_{3k} = \frac{A_k}{4}\left[d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2 - 2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right],\ \eta_{4k} = \frac{B_k}{2}\left[d_k^R - \sigma_{4k}^2 - 2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right],$$
$$\eta = \min\{\eta_{1k}, \eta_{2k}, \eta_{3k}, \eta_{4k}\},$$

in which $a,\ b_k,\ l_k,\ A_k,\ B_k\ (k = 1, \cdots, n)$ are positive constants to be determined later.

Proof. Since P^* is the deterministic endemic equilibrium of the system, we have

$$\begin{cases}
\Lambda_k = d_k^S S_k^* + \sum_{j=1}^n \beta_{kj} S_k^* a_j E_j^*, \\
(d_k^S + \epsilon_k) E_k^* = \sum_{j=1}^n \beta_{kj} S_k^* a_j E_j^*, \\
\epsilon_k E_k^* = (d_k^I + \gamma_k + \theta_k) I_k^*, \\
\gamma_k I_k^* = d_k^R R_k^*.
\end{cases}$$

Define C^2-functions V_i $(i = 1, \cdots, 6)$:

$$V_1 = \sum_{k=1}^n \bar{c}_k \left(S_k - S_k^* - S_k^* \ln\frac{S_k}{S_k^*} + E_k - E_k^* - E_k^* \ln\frac{E_k}{E_k^*}\right),\ V_2 = \sum_{k=1}^n \bar{c}_k \left(E_k - E_k^* - \ln\frac{E_k}{E_k^*}\right),$$

$$V_3=\frac{1}{2}\sum_{k=1}^{n}b_k\left(S_k-S_k^*+E_k-E_k^*\right)^2,\ V_4=\frac{1}{2}\sum_{k=1}^{n}l_k\left(S_k-S_k^*\right)^2,\ V_5=\frac{1}{2}\sum_{k=1}^{n}A_k\left(I_k-I_k^*\right)^2,$$
$$V_6=\frac{1}{2}\sum_{k=1}^{n}B_k\left(R_k-R_k^*\right)^2,$$

where $\bar{c}_k>0\ (k=1,\cdots,n)$. Thus $V_i\ (i=1,\cdots,6)$ is positive definite. According to Itô's formula, we compute that

$$\begin{aligned}
\mathrm{d}V_1=&LV_1\mathrm{d}t+\sum_{k=1}^{n}\bar{c}_k\left[\sigma_{1k}\left(S_k-S_k^*\right)\mathrm{d}B_{1k}(t)+\sigma_{2k}\left(E_k-E_k^*\right)\mathrm{d}B_{2k}(t)\right]\\
&+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_kD_{1k}(y)-S_k^*\ln(1+D_{1k}(y))+E_kD_{2k}(y)-E_k^*\ln(1+D_{2k}(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_2=&LV_2\mathrm{d}t+\sum_{k=1}^{n}\bar{c}_k\sigma_{2k}\left(E_k-E_k^*\right)\mathrm{d}B_{2k}(t)+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[E_kD_{2k}(y)-E_k^*\ln(1+D_{2k}(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_3=&LV_3\mathrm{d}t+\sum_{k=1}^{n}b_k\left(S_k-S_k^*+E_k-E_k^*\right)\left(\sigma_{1k}S_k\mathrm{d}B_{1k}(t)+\sigma_{2k}E_k\mathrm{d}B_{2k}(t)\right)\\
&+\frac{1}{2}\sum_{k=1}^{n}b_k\int_Y\left[\left(S_kD_{1k}(y)+E_kD_{2k}(y)\right)^2+2\left(S_k-S_k^*+E_k-E_k^*\right)\left(S_kD_{1k}(y)+E_kD_{2k}(y)\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_4=&LV_4\mathrm{d}t+\sum_{k=1}^{n}l_k\left(S_k-S_k^*\right)\sigma_{1k}S_k\mathrm{d}B_{1k}(t)+\frac{1}{2}\sum_{k=1}^{n}l_k\int_Y\left[S_k^2D_{1k}^2(y)+2S_k\left(S_k-S_k^*\right)D_{1k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_5=&LV_5\mathrm{d}t+\sum_{k=1}^{n}A_k\left(I_k-I_k^*\right)\sigma_{3k}I_k\mathrm{d}B_{3k}(t)+\frac{1}{2}\sum_{k=1}^{n}A_k\int_Y\left[I_k^2D_{3k}^2(y)+2I_k\left(I_k-I_k^*\right)D_{3k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_6=&LV_6\mathrm{d}t+\sum_{k=1}^{n}B_k\left(R_k-R_k^*\right)\sigma_{4k}R_k\mathrm{d}B_{4k}(t)+\frac{1}{2}\sum_{k=1}^{n}B_k\int_Y\left[R_k^2D_{4k}^2(y)+2R_k\left(R_k-R_k^*\right)D_{4k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}$$

which implies

$$\begin{aligned}
LV_1=&\sum_{k=1}^{n}\bar{c}_k\left[-d_k^S\frac{\left(S_k-S_k^*\right)^2}{S_k}+\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}+2\sum_{j=1}^{n}\bar{\beta}_{kj}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k^*}{S_k}\right.\\
&\left.-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}rE_k^*}{S_k^*a_jE_j^*E_k}\right]+\frac{1}{2}\sum_{k=1}^{n}\bar{c}_k\left(S_k^*\sigma_{1k}^2+E_k^*\sigma_{2k}^2\right)\\
&+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y),
\end{aligned}$$

$$
\begin{aligned}
LV_2 &= \sum_{k=1}^{n} \bar{c}_k \left(1 - \frac{E_k^*}{E_k}\right)\left[\sum_{j=1}^{n} \bar{\beta}_{kj} \frac{S_k \int_0^\infty f_j(r) E_j(t-r)\,\mathrm{d}r}{S_k^* a_j E_j^*} - \sum_{j=1}^{n} \bar{\beta}_{kj} \frac{E_k}{E_k^*}\right] + \frac{1}{2}\sum_{k=1}^{n} \bar{c}_k E_k^* \sigma_{2k}^2 \\
&\quad + \sum_{k=1}^{n} \bar{c}_k E_k^* \int_Y \left(D_{2k}(y) - \ln(1 + D_{2k}(y))\right) \nu(\mathrm{d}y) \\
&= \sum_{k=1}^{n} \bar{c}_k \left[\sum_{j=1}^{n} \bar{\beta}_{kj} \frac{S_k \int_0^\infty f_j(r) E_j(t-r)\,\mathrm{d}r}{S_k^* a_j E_j^*} - \sum_{j=1}^{n} \bar{\beta}_{kj} \frac{E_k}{E_k^*} - \sum_{j=1}^{n} \bar{\beta}_{kj} \frac{S_k \int_0^\infty f_j(r) E_j(t-r)\,\mathrm{d}r E_k^*}{S_k^* a_j E_j^* E_k} \right. \\
&\quad \left. + \sum_{j=1}^{n} \bar{\beta}_{kj} + \frac{E_k^* \sigma_{2k}^2}{2}\right] + \sum_{k=1}^{n} \bar{c}_k E_k^* \int_Y \left(D_{2k}(y) - \ln(1 + D_{2k}(y))\right) \nu(\mathrm{d}y),
\end{aligned} \tag{6.1}
$$

$$
\begin{aligned}
LV_3 &= \sum_{k=1}^{n} b_k (S_k - S_k^* + E_k - E_k^*)\left[-d_k^S (S_k - S_k^*) - \left(d_k^E + \epsilon_k\right)(E_k - E_k^*)\right] + \sum_{k=1}^{n} \frac{b_k}{2}\left(S_k^2 \sigma_{1k}^2 + E_k^2 \sigma_{2k}^2\right) \\
&\quad + \frac{1}{2}\sum_{k=1}^{n} b_k \int_Y \left[S_k D_{1k}(y) + E_k D_{2k}(y)\right]^2 \nu(\mathrm{d}y) \\
&= -\sum_{k=1}^{n} b_k \left[d_k^S (S_k - S_k^*)^2 + \left(d_k^E + \epsilon_k\right)(E_k - E_k^*)^2 + \left(d_k^S + d_k^E + \epsilon_k\right)(S_k - S_k^*)(E_k - E_k^*)\right] \\
&\quad + \frac{1}{2}\sum_{k=1}^{n} b_k S_k^2 \sigma_{1k}^2 + \frac{1}{2}\sum_{k=1}^{n} b_k E_k^2 \sigma_{2k}^2 + \frac{1}{2}\sum_{k=1}^{n} b_k \int_Y \left[S_k D_{1k}(y) + E_k D_{2k}(y)\right]^2 \nu(\mathrm{d}y) \\
&\le -\sum_{k=1}^{n} b_k \left(d_k^S - \sigma_{1k}^2 - \frac{\left(d_k^S + d_k^E + \epsilon_k\right)^2}{2\left(d_k^E + \epsilon_k - \sigma_{2k}^2\right)}\right)(S_k - S_k^*)^2 + \sum_{k=1}^{n} b_k \left(\sigma_{1k}^2 (S_k^*)^2 + \sigma_{2k}^2 b_k (E_k^*)^2\right) \\
&\quad - \frac{1}{2}\sum_{k=1}^{n} b_k \left(d_k^E + \epsilon_k - \sigma_{2k}^2\right)(E_k - E_k^*)^2 + \frac{1}{2}\sum_{k=1}^{n} b_k \int_Y \left[S_k D_{1k}(y) + E_k D_{2k}(y)\right]^2 \nu(\mathrm{d}y),
\end{aligned}
$$

$$
\begin{aligned}
LV_4 &= \sum_{k=1}^{n} l_k (S_k - S_k^*)\left[\sum_{j=1}^{n} \beta_{kj}\left(S_k^* a_j E_j^* - S_k \int_0^\infty f_j(r) E_j(t-r)\,\mathrm{d}r\right) - d_k^S (S_k - S_k^*)\right] \\
&\quad + \frac{1}{2}\sum_{k=1}^{n} l_k S_k^2 \sigma_{1k}^2 + \frac{1}{2}\sum_{k=1}^{n} l_k S_k^2 \int_Y D_{1k}^2(y) \nu(\mathrm{d}y) \\
&\le -\sum_{k=1}^{n}\sum_{j=1}^{n} l_k \beta_{kj} S_k^* (S_k - S_k^*)\left[\int_0^\infty f_j(r)(E_j(t-r) - E_j^*)\,\mathrm{d}r\right] - \sum_{k=1}^{n} l_k \left(d_k^S - \sigma_{1k}^2\right)(S_k - S_k^*)^2 \\
&\quad + \sum_{k=1}^{n} l_k \sigma_{1k}^2 (S_k^*)^2 + \frac{1}{2}\sum_{k=1}^{n} l_k S_k^2 \int_Y D_{1k}^2(y) \nu(\mathrm{d}y),
\end{aligned}
$$

$$
\begin{aligned}
LV_5 &= \sum_{k=1}^{n} A_k (I_k - I_k^*)\left[\epsilon_k (E_k - E_k^*) - \left(d_k^I + \gamma_k + \theta_k\right)(I_k - I_k^*)\right] + \frac{1}{2}\sum_{k=1}^{n} A_k \sigma_{3k}^2 I_k^2 \\
&\quad + \frac{1}{2}\sum_{k=1}^{n} A_k I_k^2 \int_Y D_{3k}^2(y) \nu(\mathrm{d}y)
\end{aligned}
$$

$$\leq -\frac{1}{2}\sum_{k=1}^{n} A_k\left(d_k^I+\gamma_k+\theta_k-\sigma_{3k}^2\right)(I_k-I_k^*)^2+\sum_{k=1}^{n}\frac{A_k\epsilon_k^2}{2\left(d_k^I+\gamma_k+\theta_k-\sigma_{3k}^2\right)}(E_k-E_k^*)^2$$

$$+\sum_{k=1}^{n} A_k\sigma_{3k}^2(I_k^*)^2+\frac{1}{2}\sum_{k=1}^{n} A_k I_k^2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y),$$

$$LV_6=\sum_{k=1}^{n} B_k(R_k-R_k^*)\left[\gamma_k(I_k-I_k^*)-d_k^R(R_k-R_k^*)\right]+\frac{1}{2}\sum_{k=1}^{n} B_k\sigma_{4k}^2(R_k-R_k^*+R_k^*)^2$$

$$+\frac{1}{2}\sum_{k=1}^{n} B_k R_k^2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)$$

$$\leq -\frac{1}{2}\sum_{k=1}^{n} B_k\left(d_k^R-\sigma_{4k}^2\right)(R_k-R_k^*)^2+\sum_{k=1}^{n}\frac{B_k\gamma_k^2}{2\left(d_k^R-\sigma_{4k}^2\right)}(I_k-I_k^*)^2+\sum_{k=1}^{n} B_k\sigma_{4k}^2(R_k^*)^2$$

$$+\frac{1}{2}\sum_{k=1}^{n} B_k R_k^2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y).$$

Define

$$\tilde{V}_1=V_1+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\int_t^{t+r}\left[\frac{\int_0^\infty f_j(r)E_j(u-r)\,\mathrm{d}r}{a_jE_j^*}-1-\ln\frac{\int_0^\infty f_j(r)E_j(u-r)\,\mathrm{d}r}{a_jE_j^*}\right]\mathrm{d}u.$$

Therefore

$$\mathrm{d}\tilde{V}_1=\mathrm{d}V_1+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\left[\left(\frac{\int_0^\infty f_j(r)E_j(t)\mathrm{d}r}{a_jE_j^*}-1-\ln\frac{\int_0^\infty f_j(r)E_j(t)\mathrm{d}r}{a_jE_j^*}\right)\right.$$

$$\left.-\left(\frac{\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r}{a_jE_j^*}-1-\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r}{a_jE_j^*}\right)\right]\mathrm{d}t$$

$$=\mathrm{d}V_1+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\left[\left(\frac{E_j}{E_j^*}-1-\ln\frac{E_j}{E_j^*}\right)-\left(\frac{\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r}{a_jE_j^*}-1-\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r}{a_jE_j^*}\right)\right]\mathrm{d}t$$

$$:=L\tilde{V}_1\mathrm{d}t+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_kD_{1k}(y)-S_k^*\ln(1+D_{1k}(y))+E_kD_{2k}(y)-E_k^*\ln(1+D_{2k}(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)$$

$$+\sum_{k=1}^{n}\bar{c}_k\left[\sigma_{1k}(S_k-S_k^*)\,\mathrm{d}B_{1k}(t)+\sigma_{2k}(E_k-E_k^*)\,\mathrm{d}B_{2k}(t)\right],$$

where

$$L\tilde{V}_1=LV_1+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\left[\left(\frac{E_j}{E_j^*}-1-\ln\frac{E_j}{E_j^*}\right)-\left(\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}-1-\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right)\right]$$

$$=\sum_{k=1}^{n}\bar{c}_k\left[-d_k^S\frac{\left(S_k-S_k^*\right)^2}{S_k}+\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}+2\sum_{j=1}^{n}\bar{\beta}_{kj}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k^*}{S_k}\right.$$

$$
\begin{aligned}
&-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r\,E_k^*}{S_k^*a_jE_j^*E_k}\Bigg]+\frac{1}{2}\sum_{k=1}^{n}\bar{c}_k\left(S_k^*\sigma_{1k}^2+E_k^*\sigma_{2k}^2\right)\\
&+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y)\\
&+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\left[\left(\frac{E_j}{E_j^*}-1-\ln\frac{E_j}{E_j^*}\right)-\left(\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}-1-\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right)\right]\\
=&\sum_{k=1}^{n}\bar{c}_k\Bigg[-d_k^S\frac{\left(S_k-S_k^*\right)^2}{S_k}-\left(\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{E_j}{E_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right)+\left(\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_j}{E_j^*}\right.\\
&\left.-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}\right)+\left(2\sum_{j=1}^{n}\bar{\beta}_{kj}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k^*}{S_k}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r\,E_k^*}{S_k^*a_jE_j^*E_k}\right)+\frac{1}{2}S_k^*\sigma_{1k}^2+\frac{1}{2}E_k^*\sigma_{2k}^2\Bigg]\\
&+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y). \qquad (6.2)
\end{aligned}
$$

Applying inequality $x \geq 1+\ln x$, $x>0$, we have

$$
\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k^*}{S_k}\geq\sum_{j=1}^{n}\bar{\beta}_{kj}\left(1+\ln\frac{S_k^*}{S_k}\right), \qquad (6.3)
$$

$$
\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}rE_k^*}{S_k^*a_jE_j^*E_k}=\sum_{j=1}^{n}\bar{\beta}_{kj}\left(1+\ln\frac{S_k}{S_k^*}+\ln\frac{E_k^*}{E_k}+\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right). \qquad (6.4)
$$

According to Lemma 3.2, we obtain

$$
\sum_{k=1}^{n}\bar{c}_k\left(\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_j}{E_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}\right)=0. \qquad (6.5)
$$

Similarly, we get

$$
\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{E_j}{E_j^*}-\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{E_k}{E_k^*}=0. \qquad (6.6)
$$

Substituting (6.3)–(6.6) into (6.2), then

$$
\begin{aligned}
L\tilde{V}_1\leq&\sum_{k=1}^{n}\bar{c}_k\Bigg[-d_k^S\frac{(S_k-S_k^*)^2}{S_k}-\left(\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{E_j}{E_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right)-\sum_{j=1}^{n}\bar{\beta}_{kj}\left(1+\ln\frac{S_k^*}{S_k}\right)\\
&+\left(\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_j}{E_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}\right)-\sum_{j=1}^{n}\bar{\beta}_{kj}\left(1+\ln\frac{S_k}{S_k^*}+\ln\frac{E_k^*}{E_k}+\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right)+2\sum_{j=1}^{n}\bar{\beta}_{kj}\\
&+\frac{1}{2}S_k^*\sigma_{1k}^2+\frac{1}{2}E_k^*\sigma_{2k}^2\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y)\\
=&\sum_{k=1}^{n}\bar{c}_k\left[-d_k^S\frac{(S_k-S_k^*)^2}{S_k}+\frac{1}{2}S_k^*\sigma_{1k}^2+\frac{1}{2}E_k^*\sigma_{2k}^2\right]\\
&+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y).
\end{aligned}
$$

By combining (6.1) with (6.4), the following result can be derived:

$$
\begin{aligned}
LV_2\leq&\sum_{k=1}^{n}\bar{c}_k\left[\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{S_k^*a_jE_j^*}-\sum_{k=1}^{n}\bar{\beta}_{kj}\left(1+\ln\frac{S_k}{S_k^*}+\ln\frac{E_k^*}{E_k}+\ln\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right)\right.\\
&\left.-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}+\sum_{j=1}^{n}\bar{\beta}_{kj}\right]+\frac{1}{2}\sum_{k=1}^{n}\bar{c}_kE_k^*\sigma_{2k}^2+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left[D_{2k}(y)-\ln(1+D_{2k}(y))\right]\nu(\mathrm{d}y)\\
=&\sum_{k=1}^{n}\bar{c}_k\left[\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{S_k^*a_jE_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}+\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{S_k^*}{S_k}-\sum_{j=1}^{n}\bar{\beta}_{kj}\left(\ln\frac{E_j}{E_j^*}-\ln\frac{E_k}{E_k^*}\right)\right.\\
&\left.+\frac{1}{2}E_k^*\sigma_{2k}^2\right]+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left[D_{2k}(y)-\ln(1+D_{2k}(y))\right]\nu(\mathrm{d}y)\\
=&\sum_{k=1}^{n}\bar{c}_k\left[\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{S_k^*a_jE_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}+\sum_{j=1}^{n}\bar{\beta}_{kj}\ln\frac{S_k^*}{S_k}+\frac{1}{2}E_k^*\sigma_{2k}^2\right]\\
&+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left[D_{2k}(y)-\ln(1+D_{2k}(y))\right]\nu(\mathrm{d}y)\\
\leq&\sum_{k=1}^{n}\bar{c}_k\left[\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{S_k^*a_jE_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_k}{E_k^*}+\sum_{k=1}^{n}\bar{\beta}_{kj}\left(\frac{S_k}{S_k^*}-1\right)+\frac{1}{2}E_k^*\sigma_{2k}^2\right]\\
&+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left[D_{2k}(y)-\ln(1+D_{2k}(y))\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{S_k^*a_jE_j^*}\\
=&\sum_{k=1}^{n}\bar{c}_k\left[\sum_{j=1}^{n}\bar{\beta}_{kj}\left(\frac{S_k}{S_k^*}-1\right)\left(\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}-1\right)+\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{S_k}{S_k^*}+\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{\int_0^\infty f_j(r)E_j(t-r)\mathrm{d}r}{a_jE_j^*}-\sum_{j=1}^{n}\bar{\beta}_{kj}\right].
\end{aligned}
\tag{6.7}
$$

Define

$$
\tilde{V}_2=V_2+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{\int_t^{t+r}\int_0^\infty f_j(r)E_j(u-r)\,\mathrm{d}r\mathrm{d}u}{a_jE_j^*},
$$

and then

$$\begin{aligned}
\mathrm{d}\tilde{V}_2 =&\mathrm{d}V_2 + \sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\left[\frac{E_j}{E_j^*}-\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\right]\mathrm{d}t\\
:=&L\tilde{V}_2\mathrm{d}t+\sum_{k=1}^{n}\bar{c}_k\sigma_{2k}\left(E_k-E_k^*\right)\mathrm{d}B_{2k}(t)+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[E_kD_{2k}(y)-E_k^*\ln(1+D_{2k}(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}$$

where

$$\begin{aligned}
L\tilde{V}_2 =&LV_2+\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{E_j}{E_j^*}-\sum_{k=1}^{n}\bar{c}_k\sum_{j=1}^{n}\bar{\beta}_{kj}\frac{\int_0^\infty f_j(r)E_j(t-r)\,\mathrm{d}r}{a_jE_j^*}\\
\le&\sum_{k=1}^{n}\bar{c}_k\left[\sum_{j=1}^{n}\beta_{kj}\left(S_k-S_k^*\right)\left(\int_0^\infty f_j(r)(E_j(t-r)-E_j^*)\,dr\right)+\sum_{j=1}^{n}\beta_{kj}a_jE_j^*\frac{\left(S_k-S_k^*\right)^2}{S_k}\right]\\
&+\frac{1}{2}\sum_{k=1}^{n}\bar{c}_kE_k^*\sigma_{2k}^2+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y).
\end{aligned}$$

Thus

$$\begin{aligned}
L\tilde{V}_2+LV_4\le&-\sum_{k=1}^{n}\sum_{j=1}^{n}\beta_{kj}\left(l_kS_k^*-\bar{c}_k\right)\left(S_k-S_k^*\right)\int_0^\infty f_j(r)\left(E_j(t-r)-E_j^*\right)\mathrm{d}r\\
&+\sum_{k=1}^{n}\sum_{j=1}^{n}\bar{c}_k\beta_{kj}a_jE_j^*\frac{\left(S_k-S_k^*\right)^2}{S_k}-\sum_{k=1}^{n}l_k\left(d_k^S-\sigma_{1k}^2\right)\left(S_k-S_k^*\right)^2+\frac{1}{2}\sum_{k=1}^{n}l_kS_k^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\\
&+\sum_{k=1}^{n}\left[\frac{1}{2}E_k^*\sigma_{2k}^2+l_k\sigma_{1k}^2(S_k^*)^2\right]+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y).
\end{aligned}$$

Choose $l_k=\frac{\bar{c}_k}{S_k^*}$ such that $l_kS_k^*-\bar{c}_k=0$. Hence

$$\begin{aligned}
L\tilde{V}_2+LV_4\le&\sum_{k=1}^{n}\sum_{j=1}^{n}\bar{c}_k\beta_{kj}a_jE_j^*\frac{\left(S_k-S_k^*\right)^2}{S_k}-\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}\left(d_k^S-\sigma_{1k}^2\right)\left(S_k-S_k^*\right)^2\\
&+\sum_{k=1}^{n}\bar{c}_k\left(\sigma_{1k}^2S_k^*+\frac{1}{2}E_k^*\sigma_{2k}^2\right)+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y)\\
&+\frac{1}{2}\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}S_k^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y),\\
aL\tilde{V}_1+L\tilde{V}_2+LV_4\le&-\sum_{k=1}^{n}\bar{c}_k\left(ad_k^S-\sum_{j=1}^{n}\beta_{kj}a_jE_j^*\right)\frac{\left(S_k-S_k^*\right)^2}{S_k}-\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}\left(d_k^S-\sigma_{1k}^2\right)\left(S_k-S_k^*\right)^2\\
&+\sum_{k=1}^{n}\bar{c}_k\left(\frac{a+2}{2}S_k^*\sigma_{1k}^2+\frac{a+1}{2}E_k^*\sigma_{2k}^2\right)+\frac{1}{2}\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}S_k^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)
\end{aligned}$$

$$+\sum_{k=1}^{n} a\bar{c}_k \int_Y \left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y)$$
$$+\sum_{k=1}^{n} \bar{c}_k E_k^* \int_Y \left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y).$$

Choose $a=\max\left\{\frac{1}{d_k^S}\sum_{j=1}^{n}\beta_{kj}a_jE_j^*, k=1,2,\cdots,n\right\}$ such that $ad_k^S-\sum_{j=1}^{n}\beta_{kj}a_jE_j^*\geq 0$, $k=1,2,\cdots,n$. We have

$$\begin{aligned}
aL\tilde{V}_1+L\tilde{V}_2+LV_4 \leq &-\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}\left(d_k^S-\sigma_{1k}^2\right)(S_k-S_k^*)^2+\sum_{k=1}^{n}\bar{c}_k\left(\frac{a+2}{2}S_k^*\sigma_{1k}^2+\frac{a+1}{2}E_k^*\sigma_{2k}^2\right)\\
&+\sum_{k=1}^{n} a\bar{c}_k \int_Y \left[S_k^*\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)+E_k^*\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\right]\nu(\mathrm{d}y)\\
&+\frac{1}{2}\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}S_k^{2}\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)+\sum_{k=1}^{n}\bar{c}_kE_k^*\int_Y\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y).
\end{aligned}$$

In addition, we get

$$\begin{aligned}
&aL\tilde{V}_1+L\tilde{V}_2+LV_3+LV_4+LV_5+LV_6\\
&\leq-\sum_{k=1}^{n}\left[\frac{\bar{c}_k}{S_k^*}\left(d_k^S-\sigma_{1k}^2-\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)+b_k\left(d_k^S-\sigma_{1k}^2-\frac{(d_k^S+d_k^E+\epsilon_k)^2}{2(d_k^E+\epsilon_k-\sigma_{2k}^2)}-2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)\right]\\
&\times(S_k-S_k^*)^2-\sum_{k=1}^{n}\frac{b_k}{2}\left[d_k^E+\epsilon_k-\sigma_{2k}^2-4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)-\frac{A_k\epsilon_k^2}{b_k\left(d_k^I+\gamma_k+\theta_k-\sigma_{3k}^2\right)}\right](E_k-E_k^*)^2\\
&-\sum_{k=1}^{n}\frac{A_k}{2}\left[d_k^I+\gamma_k+\theta_k-\sigma_{3k}^2-2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)-\frac{B_k\gamma_k^2}{A_k\left(d_k^R-\sigma_{4k}^2\right)}\right](I_k-I_k^*)^2-\sum_{k=1}^{n}\frac{B_k}{2}\left[d_k^R-\sigma_{4k}^2\right.\\
&\left.-2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right](R_k-R_k^*)^2+\sum_{k=1}^{n}\left[\left(\frac{a+2}{2}\bar{c}_k+b_kS_k^*\right)S_k^*\sigma_{1k}^2+\left(\frac{a+1}{2}\bar{c}_k+b_kE_k^*\right)E_k^*\sigma_{2k}^2+A_k(I_k^*)^2\sigma_{3k}^2\right.\\
&+B_k(R_k^*)^2\sigma_{4k}^2+\left(2b_k+\frac{\bar{c}_k}{S_k^*}\right)(S_k^*)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)+2b_k(E_k^*)^2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)+A_k(I_k^*)^2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\\
&\left.+B_k(R_k^*)^2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)+3G_k\right],
\end{aligned}$$

where

$$\begin{aligned}
G_k=\max\Bigg\{&a\bar{c}_kS_k^*\int_Y\left(D_{1k}(y)-\ln(1+D_{1k}(y))\right)\nu(\mathrm{d}y), a\bar{c}_kE_k^*\int_Y\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y),\\
&\bar{c}_kE_k^*\int_Y\left(D_{2k}(y)-\ln(1+D_{2k}(y))\right)\nu(\mathrm{d}y)\Bigg\}.
\end{aligned}$$

We choose $b_k>0$ such that

$$\frac{\bar{c}_k}{S_k^*}\left(d_k^S-\sigma_{1k}^2-\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)+b_k\left(d_k^S-\sigma_{1k}^2-\frac{\left(d_k^S+d_k^E+\epsilon_k\right)^2}{2\left(d_k^E+\epsilon_k-\sigma_{2k}^2\right)}-2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right)$$

$$\geq \frac{\bar{c}_k}{2S_k^*}\left(d_k^S - \sigma_{1k}^2 - 2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right),$$

and $A_k = \frac{b_k}{2\epsilon_k^2}\left(d_k^E + \epsilon_k - \sigma_{2k}^2 - 4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right)\left(d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2\right)$ such that

$$d_k^E + \epsilon_k - \sigma_{2k}^2 - 4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y) - \frac{A_k\epsilon_k^2}{b_k\left(d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2\right)} = \frac{1}{2}\left(d_k^E + \epsilon_k - \sigma_{2k}^2 - 4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right).$$

Set $B_k = \frac{A_k}{2\gamma_k^2}\left(d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2 - 2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right)\left(d_k^R - \sigma_{4k}^2\right)$ such that

$$d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2 - 2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y) - \frac{B_k\gamma_k^2}{A_k\left(d_k^R - \sigma_{4k}^2\right)} = \frac{1}{2}\left(d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2 - 2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right).$$

Therefore

$$\begin{aligned} &aL\tilde{V}_1 + L\tilde{V}_2 + LV_3 + LV_4 + LV_5 + LV_6 \\ \leq& -\sum_{k=1}^{n}\left[\eta_{1k}\left(S_k - S_k^*\right)^2 + \eta_{2k}\left(E_k - E_k^*\right)^2 + \eta_{3k}\left(I_k - I_k^*\right)^2 + \eta_{4k}\left(R_k - R_k^*\right)^2\right] + \sum_{k=1}^{n} H_k, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \eta_{1k} =& \frac{\bar{c}_k}{2S_k^*}\left[d_k^S - \sigma_{1k}^2 - \int_Y D_{1k}^2(y)\nu(\mathrm{d}y)\right], \\ \eta_{2k} =& \frac{b_k}{4}\left[d_k^E + \epsilon_k - \sigma_{2k}^2 - 4\int_Y D_{2k}^2(y)\nu(\mathrm{d}y)\right], \\ \eta_{3k} =& \frac{A_k}{4}\left[d_k^I + \gamma_k + \theta_k - \sigma_{3k}^2 - 2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y)\right], \\ \eta_{4k} =& \frac{B_k}{2}\left[d_k^R - \sigma_{4k}^2 - 2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y)\right], \\ H_k =& \left(\frac{a+2}{2}\bar{c}_k + b_kS_k^*\right)S_k^*\sigma_{1k}^2 + \left(\frac{a+1}{2}\bar{c}_k + b_kE_k^*\right)E_k^*\sigma_{2k}^2 + A_k(I_k^*)^2\sigma_{3k}^2 + B_k(R_k^*)^2\sigma_{4k}^2 \\ &+ \left(2b_k + \frac{\bar{c}_k}{S_k^*}\right)(S_k^*)^2\int_Y D_{1k}^2(y)\nu(\mathrm{d}y) + 2b_k(E_k^*)^2\int_Y D_{2k}^2(y)\nu(\mathrm{d}y) + A_k(I_k^*)^2\int_Y D_{3k}^2(y)\nu(\mathrm{d}y) \\ &+ B_k(R_k^*)^2\int_Y D_{4k}^2(y)\nu(\mathrm{d}y) + 3G_k. \end{aligned}$$

Consider a Lyapunov function:

$$V(X) = a\tilde{V}_1 + \tilde{V}_2 + V_3 + V_4 + V_5 + V_6.$$

Then we derive

$$\mathrm{d}V(X) = a\mathrm{d}\tilde{V}_1 + \mathrm{d}\tilde{V}_2 + \mathrm{d}V_3 + \mathrm{d}V_4 + \mathrm{d}V_5 + \mathrm{d}V_6 \tag{6.9}$$

$$
\begin{aligned}
&\leq\left\{-\sum_{k=1}^{n}\left[\eta_{1k}\left(S_k-S_k^*\right)^2+\eta_{2k}\left(E_k-E_k^*\right)^2+\eta_{3k}\left(I_k-I_k^*\right)^2+\eta_{4k}\left(R_k-R_k^*\right)^2\right]+\sum_{k=1}^{n}H_k\right\}\mathrm{d}t\\
&\quad+\sum_{k=1}^{n}a\bar{c}_k\left[\sigma_{1k}\left(S_k-S_k^*\right)\mathrm{d}B_{1k}(t)+\sigma_{2k}\left(E_k-E_k^*\right)\mathrm{d}B_{2k}(t)\right]\\
&\quad+\sum_{k=1}^{n}a\bar{c}_k\int_Y\left[S_kD_{1k}(y)-S_k^*\ln(1+D_{1k}(y))+E_kD_{2k}(y)-E_k^*\ln(1+D_{2k}(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&\quad+\sum_{k=1}^{n}\bar{c}_k\sigma_{2k}\left(E_k-E_k^*\right)\mathrm{d}B_{2k}(t)+\sum_{k=1}^{n}\bar{c}_k\int_Y\left[E_kD_{2k}(y)-E_k^*\ln(1+D_{2k}(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&\quad+\sum_{k=1}^{n}b_k\left(S_k-S_k^*+E_k-E_k^*\right)\left(\sigma_{1k}S_k\mathrm{d}B_{1k}(t)+\sigma_{2k}E_k\mathrm{d}B_{2k}(t)\right)\\
&\quad+\frac{1}{2}\sum_{k=1}^{n}b_k\int_Y\left[(S_kD_{1k}(y)+E_kD_{2k}(y))^2+2\left(S_kD_{1k}(y)+\left(S_k-S_k^*+E_k-E_k^*\right)E_kD_{2k}(y)\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&\quad+\sum_{k=1}^{n}l_k\left(S_k-S_k^*\right)\sigma_{1k}S_k\mathrm{d}B_{1k}(t)+\frac{1}{2}\sum_{k=1}^{n}l_k\int_Y\left[S_k^2D_{1k}^2(y)+2S_k\left(S_k-S_k^*\right)D_{1k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&\quad+\sum_{k=1}^{n}A_k\left(I_k-I_k^*\right)\sigma_{3k}I_k\mathrm{d}B_{3k}(t)+\frac{1}{2}\sum_{k=1}^{n}A_k\int_Y\left[I_k^2D_{3k}^2(y)+2I_k\left(I_k-I_k^*\right)D_{3k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&\quad+\sum_{k=1}^{n}B_k\left(R_k-R_k^*\right)\sigma_{4k}R_k\mathrm{d}B_{4k}(t)+\frac{1}{2}\sum_{k=1}^{n}B_k\int_Y\left[R_k^2D_{4k}^2(y)+2R_k\left(R_k-R_k^*\right)D_{4k}(y)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y).
\end{aligned}
$$

Integrating both sides of the inequality (6.9) from 0 to t and taking the expectation, we get

$$
\mathbb{E}V(X(t))=V(X(0))+\mathbb{E}\int_0^t LV(X)\mathrm{d}s.
$$

Then by (6.8), we find

$$
0\leq\mathbb{E}V(X(t))\leq V(X(0))-\sum_{k=1}^{n}\mathbb{E}\int_0^t\left[\eta_{1k}(S_k-S_k^*)^2+\eta_{2k}\left(E_k-E_k^*\right)^2+\eta_{3k}\left(I_k-I_k^*\right)^2+\eta_{4k}\left(R_k-R_k^*\right)^2\right]\mathrm{d}s+t\sum_{k=1}^{n}H_k.
$$

Since $\eta_{1k}>0,\eta_{2k}>0,\eta_{3k}>0,\eta_{4k}>0$, it follows that

$$
\sum_{k=1}^{n}\mathbb{E}\int_0^t\left[\eta_{1k}\left(S_k-S_k^*\right)^2+\eta_{2k}\left(E_k-E_k^*\right)^2+\eta_{3k}\left(I_k-I_k^*\right)^2+\eta_{4k}\left(R_k-R_k^*\right)^2\right]\mathrm{d}s\leq V(X(0))+t\sum_{k=1}^{n}H_k.
$$

Dividing both sides of the above inequality by t and taking the limit superior as $t\to\infty$, we have

$$
\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[\eta_{1k}\left(S_k-S_k^*\right)^2+\eta_{2k}\left(E_k-E_k^*\right)^2+\eta_{3k}\left(I_k-I_k^*\right)^2+\eta_{4k}\left(R_k-R_k^*\right)^2\right]\mathrm{d}s\leq\sum_{k=1}^{n}H_k.
$$

Therefore,

$$
\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[(S_k(s)-S_k^*)^2+(E_k(s)-E_k^*)^2+(I_k(s)-I_k^*)^2+(R_k(s)-R_k^*)^2\right]\mathrm{d}s\leq\frac{\sum_{k=1}^{n}H_k}{\eta},
$$

where $\eta = \min\{\eta_{1k}, \eta_{2k}, \eta_{3k}, \eta_{4k}\}$. This completes the proof. $\square$

Remark 2. *According to Theorem 6.1, when $R_0 > 1$ and the parameter constraints are satisfied, the solution of the stochastic delay system* (2.1) *fluctuates randomly around the endemic equilibrium P^* of the deterministic system. The oscillation amplitude is affected by both white noise and Lévy jumps: weaker perturbations result in smaller oscillations, while stronger perturbations generate larger amplitudes. If both the Brownian motion and Lévy jumps are equal to zero, the stochastic system* (2.1) *reduces to the deterministic delay system, which has a unique globally asymptotically stable endemic equilibrium P^* when $R_0 > 1$.*

7. Numerical simulations

In this section, we use the Milstein high-order algorithm mentioned in [31] and the Euler-Maruyama algorithm mentioned in [32] to discretize the model. Assume that the delay kernel function is taken as $f_j(s) = e^{-s}$ as in [22], which satisfies the normalization condition $\int_0^{+\infty} e^{-s}\mathrm{d}s = 1$. The initial condition is $E_j(\theta) = z_j e^{\theta}$, where $\theta < 0, z_j > 0$. When $n = 2$, system (2.1) becomes

$$
\begin{cases}
\mathrm{d}S_k = \left[\Lambda_k - \dfrac{e^{-t}}{2}\displaystyle\sum_{j=1}^{2} z_j\beta_{kj}S_k - e^{-t}\sum_{j=1}^{2}\beta_{kj}S_k\int_0^t e^sE_j(s)\,\mathrm{d}s - d_k^S S_k\right]\mathrm{d}t \\
\qquad + \sigma_{1k}S_k\mathrm{d}B_{1k}(t) + \displaystyle\int_Y D_{1k}(y)S_k(t)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \\
\mathrm{d}E_k = \left[\dfrac{e^{-t}}{2}\displaystyle\sum_{j=1}^{2} z_j\beta_{kj}S_k + e^{-t}\sum_{j=1}^{2}\beta_{kj}S_k\int_0^t e^sE_j(s)\,\mathrm{d}s - \left(d_k^E + \epsilon_k\right)E_k\right]\mathrm{d}t \\
\qquad + \sigma_{2k}E_k\mathrm{d}B_{2k}(t) + \displaystyle\int_Y D_{2k}(y)E_k(t)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \\
\mathrm{d}I_k = \left[\epsilon_kE_k - \left(d_k^I + \gamma_k + \theta_k\right)I_k\right]\mathrm{d}t + \sigma_{3k}I_k\mathrm{d}B_{3k}(t) + \displaystyle\int_Y D_{3k}(y)I_k(t)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \\
\mathrm{d}R_k = \left(\gamma_kI_k - d_k^RR_k\right)\mathrm{d}t + \sigma_{4k}R_k\mathrm{d}B_{4k}(t) + \displaystyle\int_Y D_{4k}(y)R_k(t)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \quad k = 1,2.
\end{cases}
\tag{7.1}
$$

For simplicity, we always assume that $d_k^S = d_k^E = d_k^I = d_k^R = d_k$ and $\int_0^t E_j(s)\mathrm{d}s = a_j = 1$ $(j = 1,2)$. The parameter values adopted in the numerical simulations of this section are presented in Table 2.

Example 1. *To verify Theorem 5.1, we choose the parameters for population 1 as $\Lambda_1 = 0.3$, $\beta_{11} = 0.02$, $\beta_{12} = 0.05$, $d_1 = 0.2$, $\epsilon_1 = 0.5$, $\gamma_1 = 0.1$, $\theta_1 = 0.1$, and the positive initial value $(S_1(0), E_1(0), I_1(0), R_1(0)) = (5, z_1, 2.3, 2.5)$, where $z_1 = 2.7$; and for population 2 as $\Lambda_2 = 0.4$, $\beta_{21} = 0.04$, $\beta_{22} = 0.02$, $d_2 = 0.15$, $\epsilon_2 = 0.6$, $\gamma_2 = 0.15$, $\theta_2 = 0.07$, and the positive initial value $(S_2(0), E_2(0), I_2(0), R_2(0)) = (2, z_2, 1, 3)$, where $z_2 = 2.1$.*

Using Example 1, we obtain the matrix $M_0 = \begin{pmatrix} 0.04 & 0.11 \\ 0.07 & 0.53 \end{pmatrix}$ and the basic reproduction number $R_0 = \rho(M_0) \approx 0.55 < 1$, which satisfies the conditions of Theorem 5.1. When $\sigma_{jk} = 0$ and $D_{jk} = 0$ $(j = 1,2,3,4; k = 1,2)$, stochastic system (7.1) reduces to a deterministic system, and the evolutionary trajectories of its solutions correspond to the results presented in Figure 1. In this case, population 1 and population 2 admit the disease-free equilibrium $P_1^0(1.5, 0, 0, 0)$ and $P_2^0(2.67, 0, 0, 0)$, indicating that the disease dies out.

To intuitively illustrate the impact of stochastic perturbation intensity on the dynamic behavior of the system, we use the data in Table 2 to carry out numerical simulations while keeping other parameters unchanged.

Figure 2 shows that the solutions for population 1 and population 2 fluctuate around the disease-free equilibrium under stochastic perturbations. Despite these fluctuations, the disease dies out eventually. This result is consistent with our theoretical analysis of the stochastic asymptotic behavior of the disease-free equilibrium. Furthermore, as shown in Figures 3 and 4, when the jump intensity is reduced or the white noise intensity is increased, the system dynamics still imply that the disease will gradually die out.

In summary, the numerical simulation results presented in the figures fully verify the validity of Theorem 5.1: the greater the intensity of random perturbations, i.e., the more significant the random interference, the larger the oscillation of the solutions of system (2.1) around the disease-free equilibrium point of the deterministic SEIR model, and the faster the disease dies out.

Example 2. *For the purpose of verifying Theorem 6.1, we specify the parameters for population 1 as* $\Lambda_1 = 4.6$, $\beta_{11} = 0.38$, $\beta_{12} = 0.35$, $d_1 = 0.8$, $\epsilon_1 = 0.9$, $\gamma_1 = 0.8$, $\theta_1 = 0.1$*; and for population 2 as* $\Lambda_2 = 5.75$, $\beta_{21} = 0.32$, $\beta_{22} = 0.51$, $d_2 = 0.9$, $\epsilon_2 = 0.9$, $\gamma_2 = 1.75$, $\theta_2 = 0.07$*. The initial values* $(S_1(0),\ S_2(0),\ E_1(0),\ E_2(0),\ I_1(0),\ I_2(0),\ R_1(0),\ R_2(0)) = (4, 2.5, z_1,\ z_2,\ 2.2,\ 2.2,\ 2.5,\ 3)$*, where* $z_1 = 2.5,\ z_2 = 3.5$.

In Example 2, we select the parameters (refer to Table 2) such that $R_0 > 1$ and the conditions in Theorem 6.1 are satisfied, and the results are presented in Figures 5–8. In this case, we can calculate the matrix $M_0 = \begin{pmatrix} 1.01 & 0.93 \\ 0.94 & 1.5 \end{pmatrix}$, and then the basic reproduction number is determined as $R_0=\rho(M_0)\approx 2.21$, which satisfies the threshold condition $R_0 > 1$. When the stochastic perturbation parameters are set to zero (i.e., $\sigma_{jk} = 0$ and $D_{jk} = 0$ for $j = 1, 2, 3, 4$ and $k = 1, 2$), the model reduces to a deterministic system. As illustrated in Figure 5, the deterministic system admits endemic equilibria for population 1 and population 2, which are P_1^* (2.06, 1.15, 0.61, 0.61) and P_2^* (2.36, 1.46, 0.76, 0.64), respectively. This indicates that the disease will persist in the populations.

With all other parameters fixed and random perturbations introduced, Figure 6 depicts the time evolution curves of $S_k(t), E_k(t), I_k(t), R_k(t)$ (k=1,2) in populations 1 and 2 for the stochastic system. As can be seen from the figure, the solution of the stochastic system oscillates persistently around the endemic equilibrium point of the deterministic system, indicating that the disease remains endemic in the population.

Figures 7 and 8 further illustrate the influence of noise intensity on the stochastic system. In Figure 7, when the Lévy jump intensity is reduced, the system solutions still oscillate around the endemic equilibrium, with a smaller fluctuation amplitude compared to Figure 6. In Figure 8, as the white noise intensity is increased, the fluctuation intensity is found to be larger than that in Figure 7. However, the solutions still remain around the endemic equilibrium, and the disease remains endemic.

The results validate Theorem 6.1: stochastic perturbations alter the system trajectories and disease transmission dynamics, with the oscillation intensity of stochastic system (2.1) around the endemic equilibrium P^* increasing significantly as the perturbation intensity rises. Nevertheless, the core trend of persistent disease endemicity remains unaffected.

Table 2. Parameter values used in the numerical simulations of system (7.1).

Population	Parameters	Numerical values					
	σ_{11}	0.2	0.2	0.35	0.15	0.15	0.31
	σ_{21}	0.3	0.3	0.8	0.1	0.1	0.22
	σ_{31}	0.2	0.2	0.6	0.2	0.2	0.4
	σ_{41}	0.05	0.05	0.4	0.1	0.1	0.25
Population 1	D_{11}	0.14	0.001	0.14	0.13	0.001	0.26
	D_{21}	0.2	0.002	0.2	0.11	0.003	0.22
	D_{31}	0.23	0.005	0.23	0.16	0.008	0.31
	D_{41}	0.21	0.003	0.21	0.15	0.005	0.3
	σ_{12}	0.18	0.18	0.3	0.15	0.15	0.35
	σ_{22}	0.25	0.25	0.8	0.1	0.1	0.2
	σ_{32}	0.15	0.15	0.5	0.25	0.25	0.45
	σ_{42}	0.1	0.1	0.4	0.15	0.15	0.32
Population 2	D_{12}	0.12	0.002	0.12	0.15	0.005	0.3
	D_{22}	0.21	0.003	0.21	0.12	0.002	0.23
	D_{32}	0.22	0.008	0.22	0.18	0.008	0.39
	D_{42}	0.17	0.006	0.17	0.16	0.005	0.32
		Figure 2	Figure 3	Figure 4	Figure 6	Figure 7	Figure 8

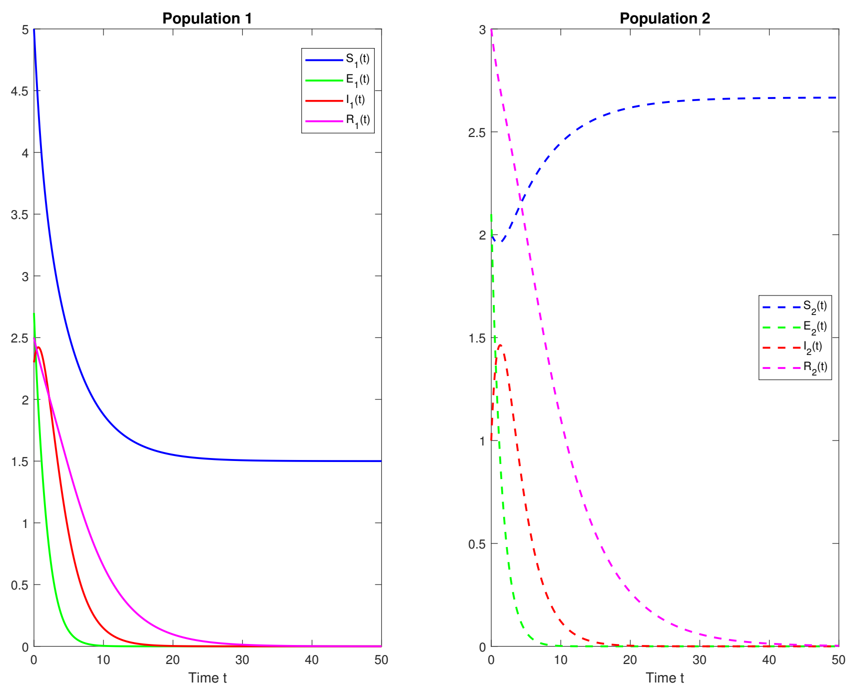

Figure 1. When $R_0 < 1$, the dynamic behavior of the solution of system (7.1) for $\sigma_{jk} = 0$, $D_{jk} = 0$, $j = 1, 2, 3, 4$, $k = 1, 2$, $\Delta t = 0.001$.

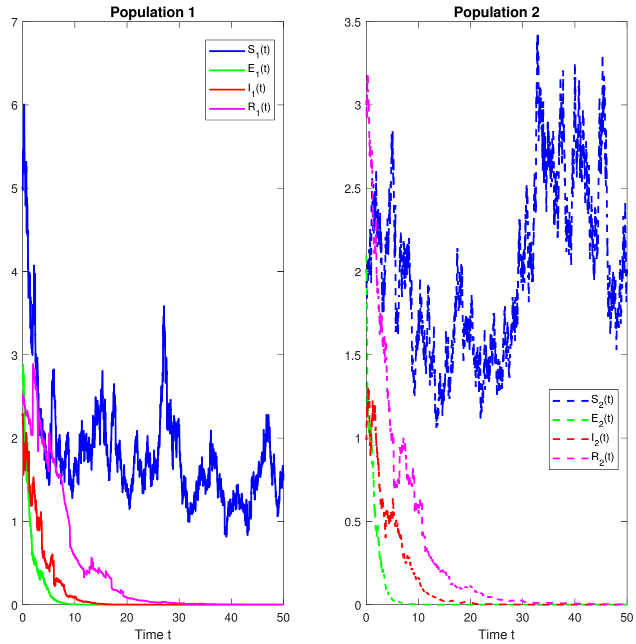

Figure 2. When $R_0 < 1$, the dynamic behavior of the solution of system (7.1) with Lévy noise. (Parameters are shown in Table 2, Column 1).

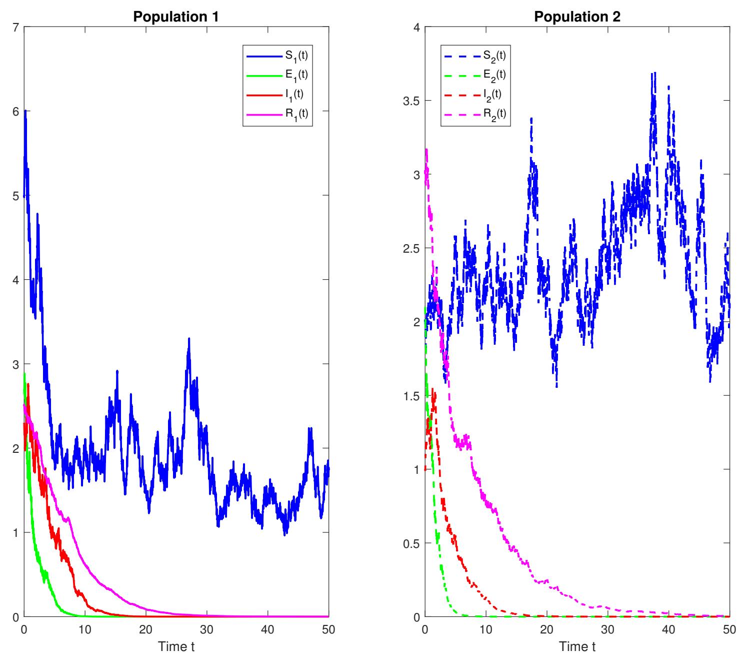

Figure 3. Under the condition of $R_0 < 1$, the dynamic behavior of system (7.1) when the intensity of the jumps decreases. (Parameters are shown in Table 2, Column 2).

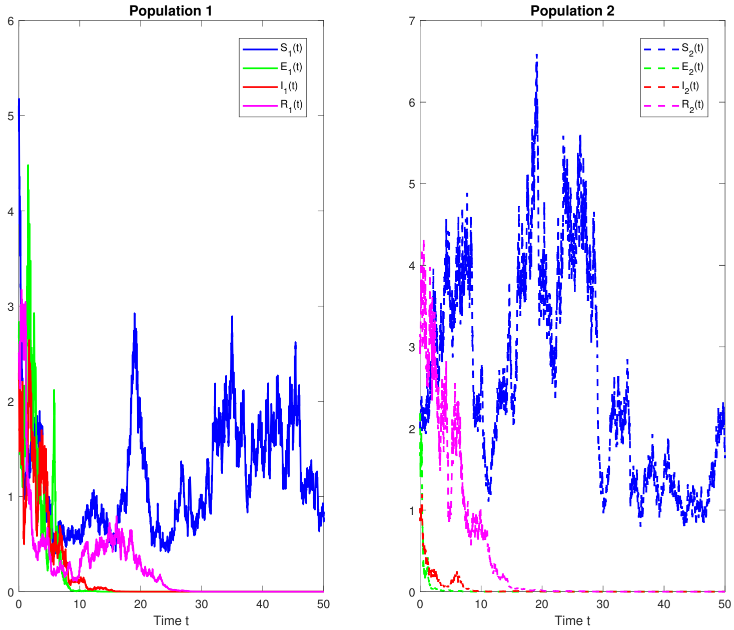

Figure 4. Under the condition of $R_0 < 1$, the dynamic behavior of the solution of system (7.1) when the intensity of white noise and Lévy jumps increases. (Parameters are shown in Table 2, Column 3).

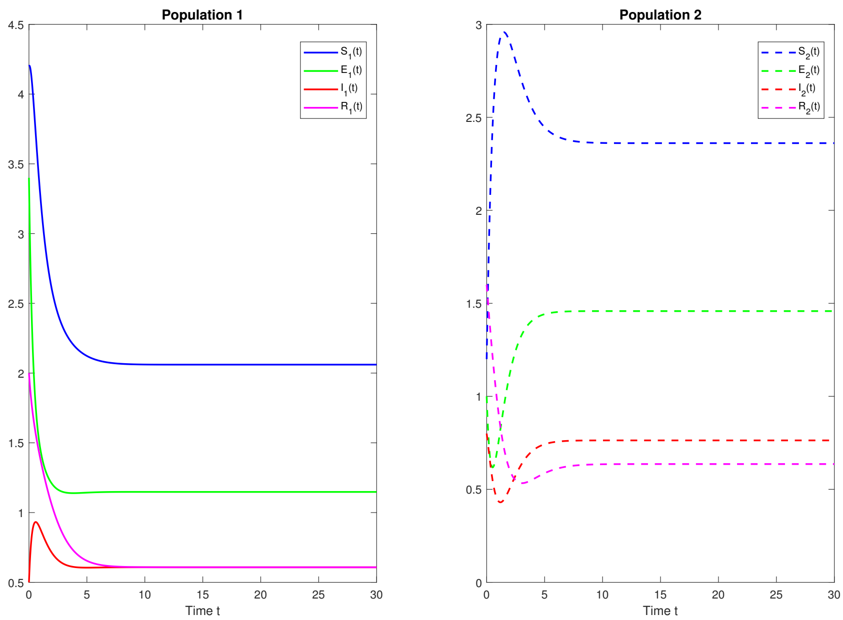

Figure 5. When $R_0 > 1$, the dynamic behavior of the solution of system (7.1) for $\sigma_{jk} = 0$, $D_{jk} = 0,\ j = 1, 2, 3, 4,\ k = 1, 2,\ \Delta t = 0.001$.

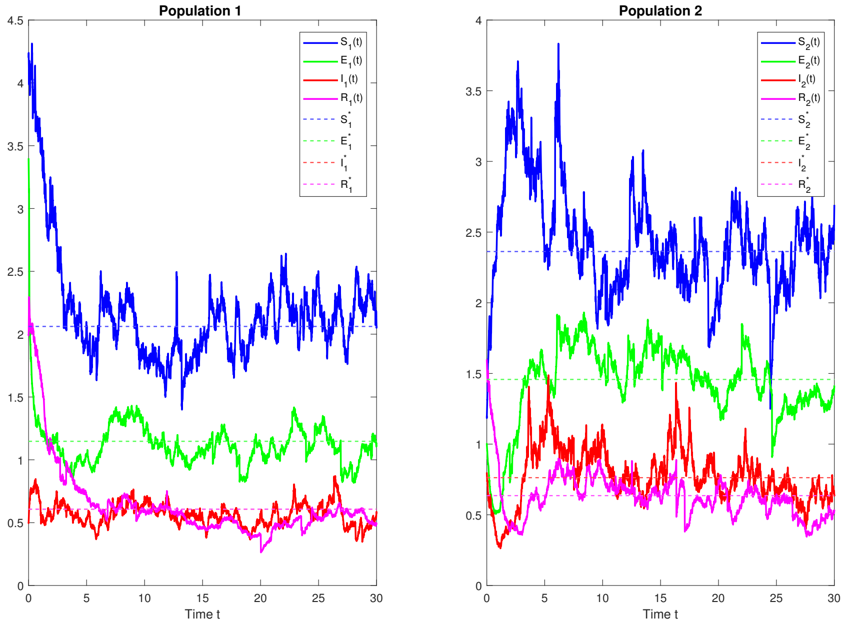

Figure 6. When $R_0 > 1$, the dynamic behavior of the solution of system (7.1) with Lévy noise. (Parameters are shown in Table 2, Column 4).

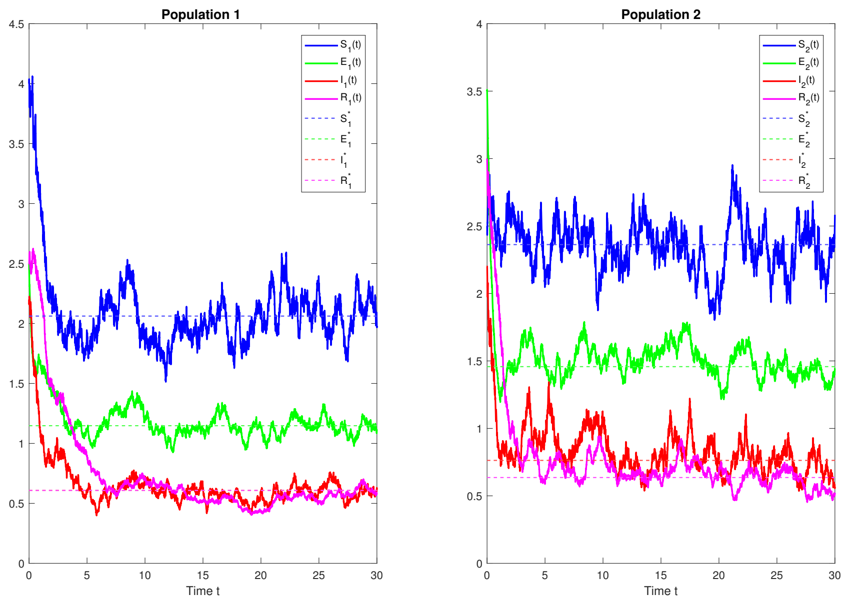

Figure 7. Under the condition of $R_0 > 1$, the dynamic behavior of the solution of system (7.1) when the intensity of the jumps decreases. (Parameters are shown in Table 2, Column 5).

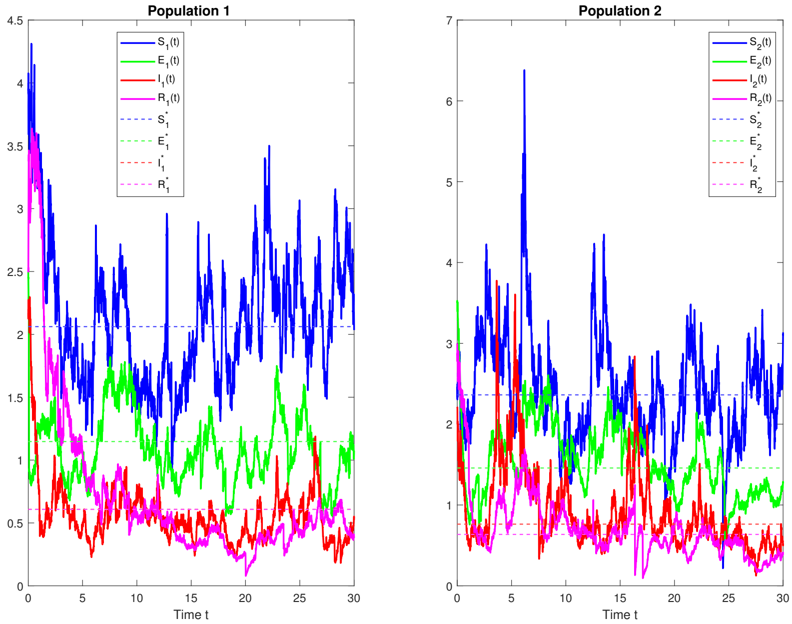

Figure 8. Under the condition of $R_0 > 1$, the dynamic behavior of the solution of system (7.1) when the intensity of white noise and Lévy jumps increases. (Parameters are shown in Table 2, Column 6).

8. Conclusions

A multi-group stochastic SEIR epidemic system with infinite distributed delays driven by Lévy jumps is proposed and analyzed in this paper. The dynamical behaviors of the system are systematically studied. Numerical simulations indicate that random noise causes the model solution to fluctuate around the two equilibrium points, and the intensity of the fluctuations is related to the strength of the noise. The results contribute to expanding the investigation and analysis for stochastic epidemic models. This work investigates the impact of Lévy noise exclusively on the transmission rate; in contrast, the scenario where noise perturbations are incorporated across all parameters would offer greater generality and better consistency with empirical epidemiological observations. If the threshold $R_0 < 1$, the solutions of the stochastic delayed system oscillate around the disease-free equilibrium P_0, and the disease will disappear; while if $R_0 > 1$, the solutions fluctuate around the endemic equilibrium P^*, and the disease will persist. An important direction for future investigation is to establish a stochastic multi-group SEIR model with infinite distributed delays under the combined influence of Markovian switching and Lévy noise. It would also be intriguing to examine how media noise modulates the dynamical properties of the solutions. Addressing these aspects is crucial for a deeper understanding of the combined effects of non-Gaussian fluctuations and memory on the persistence and extinction of infectious diseases.

Author contributions

Die Sun: Conceptualization, Formal analysis, Writing–original draft preparation, Software; Yingjia Guo: Methodology, Validation, Writing–review and editing. Both authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors confirm that no artificial intelligence technologies or tools were utilized in the writing and creation of this work.

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Conflict of interest

The authors declare no conflict of interest.

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