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Research article

Sufficient conditions for isolated tough graphs to have path-factors

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Abstract: Let G be a connected graph with n vertices, where n is a positive integer. The size of G is denoted by $e(G)$. The isolated toughness of G, denoted by $I(G)$, is defined by

$$I(G) = \min\left\{\frac{|S|}{i(G-S)} : S \subseteq V(G) \text{ and } i(G-S) \geq 2\right\}$$

or $I(G) = +\infty$ if G is complete. A graph G is called isolated r-tough if $I(G) \geq r$. The distance signless Laplacian matrix $\mathcal{Q}(G)$ of G is defined by $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$, where $\mathcal{D}(G)$ denotes the distance matrix of G and $Tr(G)$ is the diagonal matrix of the vertex transmissions in G. The largest eigenvalue of $\mathcal{Q}(G)$, denoted by $\eta(G)$, is called the distance signless Laplacian spectral radius of G. A $P_{\geq k}$-factor means a path factor with every component containing at least k vertices, where k is an integer with $k \geq 2$. In this paper, we aim to establish two tight sufficient conditions based on $e(G)$ and $\eta(G)$ to guarantee that a graph G contains a $P_{\geq 2}$-factor. Let G be a connected isolated $\frac{t}{2t+1}$-tough graph of order n, where $t \geq 1$ is an integer. Then the following two results hold.
(i) If $n \geq 6t+2$ and $e(G) \geq e(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1))$, then G contains a $P_{\geq 2}$-factor unless $G = K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$.
(ii) If $n \geq 9t+2$ and $\eta(G) \leq \eta(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1))$, then G contains a $P_{\geq 2}$-factor unless $G = K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$.

Keywords: isolated toughness; size; distance signless Laplacian spectral radius; $P_{\geq 2}$-factor
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1. Introduction

A simple graph is a graph without loops or multiple edges. In this paper, we deal only with finite, undirected and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is called its order, and the number of edges in G is called its size. We write $|V(G)| = n$ and $|E(G)| = e(G)$. For any $v \in V(G)$, we let $N_G(v) = \{u : u \in V(G) \text{ and } vu \in E(G)\}$. The degree of v

in G, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. A vertex v with $d_G(v) = 0$ in G is called an isolated vertex of G. We denote by $i(G)$ the number of isolated vertices in G. For any subset $S \subseteq V(G)$, we use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G) \setminus S$, respectively. For any subset $E' \subseteq E(G)$, we denote by $G - E'$ the graph formed from G by removing all edges in E'. The complete graph and the path with n vertices are denoted by K_n and P_n, respectively. Given two vertex-disjoint graphs G_1 and G_2, let $G_1 \cup G_2$ and $G_1 \vee G_2$ denote their union and join, respectively.

Yang, Ma and Liu [20] first introduced the concept of isolated toughness. The isolated toughness of G, denoted by $I(G)$, is defined by

$$I(G) = \min\left\{\frac{|S|}{i(G-S)} : S \subseteq V(G) \text{ and } i(G-S) \geq 2\right\}$$

if G is not complete, or $I(G) = +\infty$ if G is complete. A graph G is said to be isolated t-tough if its isolated toughness $I(G) \geq t$. In fact, a graph G is isolated t-tough if and only if $t \cdot i(G - S) \leq |S|$ for each subset $S \subseteq V(G)$ satisfying $i(G - S) \geq 2$.

A spanning subgraph H of a graph G is called a perfect matching or a 1-factor of G if $d_H(v) = 1$ for every $v \in V(G)$. Let $\mathcal{H}$ denote a set of connected graphs. A spanning subgraph H of G is called an $\mathcal{H}$-factor if each component of H is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred to as a component factor. Let k be an integer with $k \geq 2$. An $\mathcal{H}$-factor is called a $\{K_{1,j} : 1 \leq j \leq k\}$-factor if $\mathcal{H} = \{K_{1,j} : 1 \leq j \leq k\}$. An $\mathcal{H}$-factor is called a $\{P_i : i \geq k\}$-factor if $\mathcal{H} = \{P_i : i \geq k\}$. In particular, a $\{P_i : i \geq k\}$-factor is simply written as a $P_{\geq k}$-factor, which is a path-factor. In fact, a perfect matching is also called a $\{P_2\}$-factor.

Tutte [15] provided a criterion for a graph to possess a $\{P_2\}$-factor. Niessen [13] showed a neighborhood union condition for a graph to contain a $\{P_2\}$-factor. Enomoto [4] gave a toughness condition for the existence of a $\{P_2\}$-factor in a graph. Anderson [2] proposed a binding number condition for a graph to have a $\{P_2\}$-factor. Aharoni, Georgakopoulos and Sprüssel [1] presented a sufficient condition for the existence of a $\{P_2\}$-factor in a graph. Las Vergnas [7] obtained a characterization for a graph with a $P_{\geq 2}$-factor. Kaneko [6] posed a necessary and sufficient condition for a graph to have a $P_{\geq 3}$-factor. Zhang and Zhou [22] investigated the existence of a $P_{\geq 2}$-factor and a $P_{\geq 3}$-factor in a graph, respectively. Zhou and Sun [30] obtained two results for a graph to possess a $P_{\geq 2}$-factor and a $P_{\geq 3}$-factor. Liu and Pan [10] gave two independence number and minimum degree conditions for a graph to contain a $P_{\geq 2}$-factor and a $P_{\geq 3}$-factor, respectively. Dai [3] got some results on the existence of a $P_{\geq 2}$-factor and a $P_{\geq 3}$-factor in a graph. Zhang [23] showed some binding number conditions for a graph to have a $P_{\geq 3}$-factor.

Given a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the distance between v_i and v_j, denoted by $d_G(v_i, v_j)$ or d_{ij}, is the length of a shortest path connecting them in G. The sum of distances from a vertex v_i to all other vertices of G is called the transmission of v_i in G, which is denoted by $Tr(v_i)$. That is to say, $Tr(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j)$. The distance matrix of G, denoted by $\mathcal{D}(G)$, is defined by $\mathcal{D}(G) = (d_{ij})_{n\times n}$. Let $Tr(G) = \operatorname{diag}(Tr(v_1), Tr(v_2), \ldots, Tr(v_n))$ be the diagonal matrix of the vertex transmissions in G. Then the distance signless Laplacian matrix of G is defined by $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$. The largest eigenvalue of $\mathcal{Q}(G)$, denoted by $\eta(G)$, is called the distance signless Laplacian spectral radius of G.

Many scholars characterized a graph with a path-factor by using the spectral radius. O [14] established a lower bound for the adjacency spectral radius in a graph G to ensure that G contains a $\{P_2\}$-factor. Zhang and Lin [24] created two connections between distance spectral radius and $\{P_2\}$-factors in graphs or bipartite graphs. Li and Miao [8] established an adjacency spectral condition for the existence of a $P_{\geq 2}$-factor in a graph. Zhou [26] showed two spectral conditions for a bipartite graph to contain a $\{P_3\}$-factor. Zhou, Zhang and Sun [33] obtained a result on the existence of a $P_{\geq 2}$-fator in a graph by utilizing A_α-spectral radius. Zhou, Sun and Liu [31] characterized a graph with a $P_{\geq 2}$-factor by using distance signless Laplacian spectral radius. Hao and Li [5] proposed two spectral conditions for a graph to possess a $P_{\geq 2}$-factor. Zhou, Bian and Sun [28] proposed some spectral radius conditions for the existence of $P_{\geq 2}$-factors in isolated tough graphs. More results on the relationships between spectral radii and graph factors can be referred to [9, 11, 16–18, 25, 27, 29, 32].

In this paper, we put forward two sufficient conditions for isolated tough graphs to contain $P_{\geq 2}$-factors with respect to the size and the distance signless Laplacian spectral radius.

Theorem 1.1. Let t be a positive integer, and let G be a connected isolated $\frac{t}{2t+1}$-tough graph of order n with $n \geq 6t + 2$. If

$$e(G) \geq e(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)),$$

then G contains a $P_{\geq 2}$-factor, unless $G = K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$.

Theorem 1.2. Let t be a positive integer, and let G be a connected isolated $\frac{t}{2t+1}$-tough graph of order n with $n \geq 9t + 2$. If

$$\eta(G) \leq \eta(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)),$$

then G contains a $P_{\geq 2}$-factor, unless $G = K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$.

2. Some preliminaries

Las Vergnas [7] proposed a necessary and sufficient condition for the existence of a $P_{\geq 2}$-factor in a graph.

Lemma 2.1. (Las Vergnas [7]) A graph G has a $P_{\geq 2}$-factor if and only if

$$i(G - S) \leq 2|S|$$

for any vertex subset S of G.

Lemma 2.2. (Minc [12]) Let G be a graph and $e \in E(G)$. If $G - e$ is connected, then

$$\eta(G - e) > \eta(G).$$

The Wiener index of a connected graph G with n vertices is defined by $W(G) = \sum_{i<j} d_{ij}$. Xing, Zhou and Li [19] provided a lower bound on the distance signless Laplacian spectral radius of a graph.

Lemma 2.3. (Xing, Zhou and Li [19]) Let G be a connected graph of order n. Then

$$\eta(G) = \max_{X \in \mathbb{R}^n} \frac{X^T Q(G) X}{X^T X} \geq \frac{4W(G)}{n},$$

where the second equality occurs if and only if G is transmission regular.

Let M be an $n \times n$ real matrix described in the following block form

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \cdots, X_r$ of $\{1, 2, \cdots, n\}$, where the diagonal blocks M_{ii} are $n_i \times n_i$ matrices for $1 \le i \le r$ and $n = \sum_{i=1}^{r} n_i$. Let M_{ij} denote the block of M by deleting the rows in $\{1, 2, \cdots, n\} - X_i$ and deleting the columns in $\{1, 2, \cdots, n\} - X_j$. For any $i, j \in \{1, 2, \cdots, r\}$, we denote by b_{ij} the average row sum of M_{ij}. Then $B(M) = (b_{ij})_{r\times r}$ (simply by B) is called the quotient matrix of M. If for every pair i, j, the matrix M_{ij} has constant row sum, then the partition is called equitable.

Lemma 2.4. (You, Yang, So and Xi [21]) Let M denote an $n\times n$ real matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then the eigenvalues of M_π are eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then the largest eigenvalues of M and M_π are equal.

Lemma 2.5. Graph $K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$ contains no $P_{\ge 2}$-factor.

Proof. Let $G = K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$. Set $S = V(K_t)$. Then we have $|S| = t$ and $i(G - S) \ge 2t + 1 = 2|S| + 1 > 2|S|$. In terms of Lemma 2.1, G contains no $P_{\ge 2}$-factor. This completes the proof of Lemma 2.5.

3. The proof of Theorem 1.1

Proof of Theorem 1.1. Suppose that a connected isolated $\frac{t}{2t+1}$-tough graph G contains no $P_{\ge 2}$-factor, where t is a positive integer. In view of Lemma 2.1, we conclude

$$i(G - S) \ge 2|S| + 1 \tag{3.1}$$

for some nonempty subset $S \subseteq V(G)$. In terms of the definition of isolated $\frac{t}{2t+1}$-tough graphs, we get

$$\frac{t}{2t + 1} \le I(G) \le \frac{|S|}{i(G - S)}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$(2t + 1)|S| \ge t \cdot i(G - S) \ge t(2|S| + 1),$$

and so $|S| \ge t$. Let $|S| = s$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n-3s-1} \cup (2s + 1)K_1)$, where $n \ge 3s + 1$. Thus, we have

$$e(G) \le e(G_1), \tag{3.3}$$

where the equality occurs if and only if $G = G_1$. If $s = t$, then we obtain $G_1 = K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$. Combining this with (3.3), we get

$$e(G) \le e(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)),$$

with equality if and only if $G = K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$. According to Lemma 2.5, $K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$ contains no $P_{\ge 2}$-factor. Thus, we obtain a contradiction. Next, we consider $s \ge t + 1$.

Let $G_* = K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)$. By a direct calculation, we obtain

$$e(G_1) = \binom{n - 2s - 1}{2} + s(2s + 1)$$

$$
\begin{aligned}
&=\frac{(n-2s-1)(n-2s-2)}{2}+s(2s+1)\\
&=\frac{1}{2}(n^2-(4s+3)n+8s^2+8s+2)
\end{aligned}
$$

and

$$e(G_*)=\frac{1}{2}(n^2-(4t+3)n+8t^2+8t+2). \tag{3.4}$$

The following proof will be divided into three cases based on the value of n.

Case 1. $n \geq 3s+3$.

In terms of (3.4), $n \geq 3s+3$, $n \geq 6t+2$ and $s \geq t+1$, we have

$$
\begin{aligned}
e(G_1)-e(G_*)=&\frac{1}{2}(n^2-(4s+3)n+8s^2+8s+2)-\frac{1}{2}(n^2-(4t+3)n+8t^2+8t+2)\\
=&(s-t)\left(-\frac{4}{3}n-\frac{2}{3}n+4s+4t+4\right)\\
\leq&(s-t)\left(-\frac{4}{3}(3s+3)-\frac{2}{3}(6t+2)+4s+4t+4\right)\\
=&-\frac{4}{3}(s-t)\\
<&0,
\end{aligned}
$$

which leads to

$$e(G_1)<e(G_*). \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$e(G)\leq e(G_1)<e(K_t\vee(K_{n-3t-1}\cup(2t+1)K_1)),$$

which contradicts $e(G)\geq e(K_t\vee(K_{n-3t-1}\cup(2t+1)K_1))$.

Case 2. $n=3s+2$.

In this case, $G_1=K_s\vee(2s+2)K_1$. Since $n=3s+2$ and $n\geq 6t+2$, we deduce $s\geq 2t$. According to (3.4), $t\geq 1$, $s\geq 2t$ and $n=3s+2$, we have

$$
\begin{aligned}
e(G_1)-e(G_*)=&s(2s+2)-\frac{1}{2}(n^2-(4t+3)n+8t^2+8t+2)\\
=&s(2s+2)-\frac{1}{2}((3s+2)^2-(4t+3)(3s+2)+8t^2+8t+2)\\
=&\frac{1}{2}(-5s^2+(12t+1)s-8t^2)\\
\leq&\frac{1}{2}(-5(2t)^2+2t(12t+1)-8t^2)\\
=&-2t^2+t\\
<&0,
\end{aligned}
$$

which yields that

$$e(G_1) < e(G_*).$$

Combining this with (3.3), we conclude

$$e(G) \le e(G_1) < e(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)),$$

which contradicts $e(G) \ge e(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1))$.

Case 3. $n = 3s + 1$.

In this case, $G_1 = K_s \vee (2s+1)K_1$. Note that $n = 3s+1$ and $n \ge 6t+2$. Then we see $s \ge 2t+1$. By virtue of (3.4), $t \ge 1$, $s \ge 2t+1$ and $n = 3s+1$, we get

$$\begin{aligned} e(G_1) - e(G_*) =& s(2s+1) - \frac{1}{2}(n^2 - (4t+3)n + 8t^2 + 8t + 2) \\ =& s(2s+1) - \frac{1}{2}((3s+1)^2 - (4t+3)(3s+1) + 8t^2 + 8t + 2) \\ =& \frac{1}{2}(-5s^2 + (12t+5)s - 8t^2 - 4t) \\ \le& \frac{1}{2}(-5(2t+1)^2 + (12t+5)(2t+1) - 8t^2 - 4t) \\ =& -2t^2 - t \\ <& 0, \end{aligned}$$

which implies

$$e(G_1) < e(G_*).$$

Together with (3.3), we obtain

$$e(G) \le e(G_1) < e(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)),$$

which contradicts $e(G) \ge e(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1))$. Theorem 1.1 is proved. □

4. The proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that a connected isolated $\frac{t}{2t+1}$-tough graph G contains no $P_{\ge 2}$-factor, where t is a positive integer. By Lemma 2.1, we obtain

$$i(G-S) \ge 2|S| + 1 \tag{4.1}$$

for some nonempty subset $S \subseteq V(G)$. According to the definition of isolated $\frac{t}{2t+1}$-tough graph, we have

$$\frac{t}{2t+1} \le I(G) \le \frac{|S|}{i(G-S)}. \tag{4.2}$$

From (4.1) and (4.2), we get

$$(2t+1)|S| \ge t \cdot i(G-S) \ge t(2|S|+1),$$

and so $|S| \geq t$. Let $|S| = s$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n-3s-1} \cup (2s+1)K_1)$, where $n \geq 3s+1$. In terms of Lemma 2.2, we deduce

$$\eta(G) \geq \eta(G_1), \tag{4.3}$$

where the equality occurs if and only if $G = G_1$. If $s = t$, then $G_1 = K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$. Together with (4.3), we have

$$\eta(G) \geq \eta(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)),$$

with equality if and only if $G = K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$. In view of Lemma 2.5, $K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$ contains no $P_{\geq 2}$-factor. Thus, we get a contradiction. Next, we consider $s \geq t+1$.

Recall that $G_1 = K_s \vee (K_{n-3s-1} \cup (2s+1)K_1)$. The proof will be divided into three cases.

Case 1. $n \geq 3s+3$.

The quotient matrix of $Q(G_1)$ according to the partition $V(G_1) = V((2s+1)K_1) \cup V(K_{n-3s-1}) \cup V(K_s)$ can be written as

$$B_1 = \begin{pmatrix} 2n+3s-2 & 2n-6s-2 & s \\ 4s+2 & 2n-s-2 & s \\ 2s+1 & n-3s-1 & n+s-2 \end{pmatrix}.$$

The characteristic polynomial of B_1 is denoted by $\varphi_{B_1}(x)$. By a simple calculation, we have

$$\begin{aligned}\varphi_{B_1}(x) =& x^3 - (5n+3s-6)x^2 + (8n^2 + sn - 24n + 24s^2 + 8s + 16)x \\ &- 4n^3 + 2sn^2 + 20n^2 - 24s^2n - 18sn - 32n - 8s^3 + 44s^2 + 28s + 16.\end{aligned}$$

According to Lemma 2.4 and the equitable partition $V(G_1) = V((2s+1)K_1) \cup V(K_{n-3s-1}) \cup V(K_s)$, the largest root, say η_1, of $\varphi_{B_1}(x) = 0$ is equal to the distance signless Laplacian spectral radius of G_1. Namely, $\varphi_{B_1}(\eta_1) = 0$ and $\eta(G_1) = \eta_1$.

Let $G_* = K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)$. Then the quotient matrix of $Q(G_*)$ with respect to the partition $V(G_*) = V((2t+1)K_1) \cup V(K_{n-3t-1}) \cup V(K_t)$ is given by

$$B_* = \begin{pmatrix} 2n+3t-2 & 2n-6t-2 & t \\ 4t+2 & 2n-t-2 & t \\ 2t+1 & n-3t-1 & n+t-2 \end{pmatrix}.$$

The characteristic polynomial of B_* is denoted by $\varphi_{B_*}(x)$. By a simple computation, we obtain

$$\begin{aligned}\varphi_{B_*}(x) =& x^3 - (5n+3t-6)x^2 + (8n^2 + tn - 24n + 24t^2 + 8t + 16)x \\ &- 4n^3 + 2tn^2 + 20n^2 - 24t^2n - 18tn - 32n - 8t^3 + 44t^2 + 28t + 16.\end{aligned}$$

In view of Lemma 2.4 and the equitable partition $V(G_*) = V((2t+1)K_1) \cup V(K_{n-3t-1}) \cup V(K_t)$, the largest root, say η_*, of $\varphi_{B_*}(x) = 0$ equals the distance signless Laplacian spectral radius of G_*. Obviously, $\varphi_{B_*}(\eta_*) = 0$ and $\eta(G_*) = \eta_*$.

By $t \geq 1$ and $n \geq 9t+2$, we get

$$\begin{aligned}\varphi_{B_*}(3n-2) =& (3n-2)^3 - (5n+3t-6)(3n-2)^2 + (8n^2 + tn - 24n + 24t^2 + 8t + 16)(3n-2) \\ &- 4n^3 + 2tn^2 + 20n^2 - 24t^2n - 18tn - 32n - 8t^3 + 44t^2 + 28t + 16\end{aligned}$$

$$
\begin{aligned}
&=n(2n^2-(22t+8)n+48t^2+40t+8)-8t^3-4t^2\\
&\geq n(2(9t+2)^2-(22t+8)(9t+2)+48t^2+40t+8)-8t^3-4t^2\\
&=n(12t^2-4t)-8t^3-4t^2\\
&\geq(9t+2)(12t^2-4t)-8t^3-4t^2\\
&=100t^3-16t^2-8t\\
&>0. \qquad (4.4)
\end{aligned}
$$

The derivative function of $\varphi_{B_*}(x)$ is

$$\varphi'_{B_*}(x)=3x^2-2(5n+3t-6)x+8n^2+tn-24n+24t^2+8t+16.$$

Notice that

$$\frac{2(5n+3t-6)}{2\times 3}=\frac{5n+3t-6}{3}<2n+6t.$$

So when $x\geq 2n+6t$, we obtain

$$
\begin{aligned}
\varphi'_{B_*}(x)\geq&\varphi'_{B_*}(2n+6t)\\
=&3(2n+6t)^2-2(5n+3t-6)(2n+6t)+8n^2+tn-24n+24t^2+8t+16\\
=&tn+96t^2+80t+16\\
>&0
\end{aligned}
$$

by $t\geq 1$ and $n\geq 9t+2$, which implies that $\varphi_{B_*}(x)$ is increasing for $x\geq 2n+6t$. Combining this with (4.4) and $3n-2>2n+6t$, we infer

$$\eta(G_*)=\eta_*<3n-2. \qquad (4.5)$$

Recall that $G_*=K_t\vee(K_{n-3t-1}\cup(2t+1)K_1)$. It follows from Lemma 2.3, $t\geq 1$ and $n\geq 9t+2$ that

$$
\begin{aligned}
\eta_*=&\eta(G_*)\\
\geq&\frac{4W(G_*)}{n}\\
=&\frac{2n^2+(8t+2)n-16t^2-16t-4}{n}\\
=&\frac{2n^2+6tn+(2t+2)n-16t^2-16t-4}{n}\\
\geq&\frac{2n^2+6tn+(2t+2)(9t+2)-16t^2-16t-4}{n}\\
=&\frac{2n^2+6tn+2t^2+6t}{n}\\
>&2n+6t. \qquad (4.6)
\end{aligned}
$$

By plugging the value η_* into x of $\varphi_{B_1}(x)-\varphi_{B_*}(x)$, it follows from $\varphi_{B_*}(\eta_*)=0$ that

$$\varphi_{B_1}(\eta_*)=\varphi_{B_1}(\eta_*)-\varphi_{B_*}(\eta_*)=(s-t)f(\eta_*), \qquad (4.7)$$

where $f(\eta_*) = -3\eta_*^2 + (n+24s+24t+8)\eta_* + 2n^2 - 24sn - 24tn - 18n - 8s^2 - 8ts + 44s - 8t^2 + 44t + 28$. Notice that

$$\frac{n+24s+24t+8}{6} < 2n+6t < \eta_*$$

by (4.6), $s \geq t+1$ and $n \geq 3s+3$. Thus, we obtain

$$\begin{aligned} f(\eta_*) <& f(2n+6t) \\ =& -8s^2 + (24n+136t+44)s - 8n^2 - 42tn - 2n + 28t^2 + 92t + 28. \end{aligned} \tag{4.8}$$

Let $g(s) = -8s^2 + (24n+136t+44)s - 8n^2 - 42tn - 2n + 28t^2 + 92t + 28$. Notice that

$$\frac{24n+136t+44}{16} > \frac{n-3}{3} \geq s \geq t+1.$$

Then we deduce

$$\begin{aligned} g(s) \leq& g\left(\frac{n-3}{3}\right) \\ =& -8\left(\frac{n-3}{3}\right)^2 + (24n+136t+44)\left(\frac{n-3}{3}\right) - 8n^2 - 42tn - 2n + 28t^2 + 92t + 28 \\ =& \frac{1}{9}(-8n^2 + (30t-54)n + 252t^2 - 396t - 216) \\ \leq& \frac{1}{9}(-8(9t+2)^2 + (30t-54)(9t+2) + 252t^2 - 396t - 216) \quad (\text{since } n \geq 9t+2) \\ =& \frac{1}{9}(-126t^2 - 1110t - 356) \\ <& 0 \quad (\text{since } t \geq 1). \end{aligned} \tag{4.9}$$

Using (4.7)–(4.9) and $s \geq t+1$, we conclude

$$\varphi_{B_1}(\eta_*) = (s-t)f(\eta_*) < (s-t)g(s) < 0,$$

which implies

$$\eta(G_1) = \eta_1 > \eta_* = \eta(G_*) = \eta(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)). \tag{4.10}$$

From (4.3) and (4.10), we deduce

$$\eta(G) \geq \eta(G_1) > \eta(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1)),$$

which contradicts $\eta(G) \leq \eta(K_t \vee (K_{n-3t-1} \cup (2t+1)K_1))$.

Case 2. $n = 3s+2$.

In this case, $G_1 = K_s \vee (2s+2)K_1$. The quotient matrix of $Q(G_1)$ with respective to the partition $V(G_1) = V(K_s) \cup V((2s+2)K_1)$ is

$$B_2 = \begin{pmatrix} n+s-2 & 2s+2 \\ s & 9s+4 \end{pmatrix},$$

and so its characteristic polynomial equals

$$\varphi_{B_2}(x) = x^2 - (n + 10s + 2)x + 9sn + 4n + 7s^2 - 16s - 8.$$

Using Lemma 2.4 and the equitable partition $V(G_1) = V(K_s) \cup V((2s + 2)K_1)$, the largest root of $\varphi_{B_2}(x) = 0$ is equal to $\eta(G_1)$. Thus, we get

$$\begin{aligned}\eta(G_1) &= \frac{n + 10s + 2 + \sqrt{(n + 10s + 2)^2 - 4(9sn + 4n + 7s^2 - 16s - 8)}}{2} \\ &= \frac{13n - 14 + \sqrt{33n^2 + 12n - 12}}{6}.\end{aligned} \tag{4.11}$$

According to (4.11), $t \geq 1$ and $n \geq 9t + 2$, we easily prove that

$$\eta(G_1) = \frac{13n - 14 + \sqrt{33n^2 + 12n - 12}}{6} > 3n - 2. \tag{4.12}$$

In terms of (4.3), (4.5) and (4.12), we conclude

$$\eta(G) \geq \eta(G_1) > 3n - 2 > \eta(G_*) = \eta(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)),$$

which contradicts $\eta(G) \leq \eta(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1))$.

Case 3. $n = 3s + 1$.

In this case, $G_1 = K_s \vee (2s + 1)K_1$. The quotient matrix of $Q(G_1)$ in view of the partition $V(G_1) = V(K_s) \cup V((2s + 1)K_1)$ equals

$$B_3 = \begin{pmatrix} n + s - 2 & 2s + 1 \\ s & 9s \end{pmatrix}.$$

The characteristic polynomial of B_3 is given by

$$\varphi_{B_3}(x) = x^2 - (n + 10s - 2)x + 9sn + 7s^2 - 19s.$$

From Lemma 2.4 and the equitable partition $V(G_1) = V(K_s) \cup V((2s + 1)K_1)$, the largest root of $\varphi_{B_3}(x) = 0$ equals $\eta(G_1)$. Hence, we have

$$\begin{aligned}\eta(G_1) &= \frac{n + 10s - 2 + \sqrt{(n + 10s - 2)^2 - 4(9sn + 7s^2 - 19s)}}{2} \\ &= \frac{13n - 16 + \sqrt{33n^2 - 24n}}{6}.\end{aligned} \tag{4.13}$$

By means of (4.13), $t \geq 1$ and $n \geq 9t + 2$, we easily deduce that

$$\eta(G_1) = \frac{13n - 16 + \sqrt{33n^2 - 24n}}{6} > 3n - 2.$$

Combining this with (4.3) and (4.5), we have

$$\eta(G) \geq \eta(G_1) > 3n - 2 > \eta(G_*) = \eta(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1)),$$

which contradicts $\eta(G) \leq \eta(K_t \vee (K_{n-3t-1} \cup (2t + 1)K_1))$. Theorem 1.2 is verified. □

5. Concluding remarks

In this paper, we establish two sufficient conditions to guarantee that a connected isolated tough graph G contains a $P_{\geq 2}$-factor with respect to its size and distance signless Laplacian spectral radius. It is natural and interesting to propose some other sufficient conditions to ensure that a connected isolated tough graph G contains a $P_{\geq 2}$-factor. It is also natural and interesting to put forward some sufficient conditions to guarantee that a connected isolated tough graph G has some other substructure based on the adjacency spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance signless Laplacian spectral radius, the A_α-spectral radius, and so on.

Author contributions

Quanru Pan: Writing-original draft preparation, review and editing; Sizhong Zhou: Writing-original draft preparation, review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to this work.

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