Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 13304–13338.
DOI: 10.3934/math.2026549
Received: 12 December 2025
Revised: 24 February 2026
Accepted: 10 March 2026
Published: 13 May 2026

Research article

Key results obtained during the search for contractive mappings that can generate fractal interpolation functions

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Abstract: In this paper, we show that, through two counterexamples, iterated function systems consisting of Bryant contraction mappings cannot generate fractals in general. Also, we present two new families of Rakotch contraction mappings that can generate new fractal interpolation functions with a wide range of applications in practice.

Keywords: fixed-point theorem; Bryant contraction mapping; Rakotch contraction mapping; iterated function systems (IFSs); fractal interpolation functions (FIFs)
Mathematics Subject Classification: 27A80, 37L30

1. Introduction

In 1986, Barnsley proposed the concept of a fractal interpolation function (FIF) as an attractor of an iterated function system (IFS) with certain constraints related to a given data point [1].

An IFS is a family of functions consisting of a finite number of Banach contraction mappings, and some typical fractals are generated by the attractors of these IFSes.

The properties of the invariant set (i.e., existence, uniqueness, compactness, attractiveness) of an IFS, a fixed point of the Hutchinson-Barnsley operator constructed by finite Banach contraction mappings of the IFS, have been studied [2], where the attractor (i.e., fractal) of an IFS is the attractive invariant set of an IFS, that is, it is the attractive fixed point of the Hutchinson-Barnsley operator.

The Hutchinson-Barnsley operator constructed by a finite number of Banach contraction mappings is a natural generalization of Banach contraction mapping, and the characterization of a fixed point (i.e., fractal) of the Hutchinson-Barnsley operator is revealed by the Banach fixed-point theorem.

Many researchers have constructed new IFSes using well-known fixed-point theorems and studied their analytical properties.

In [3–5], the authors constructed IFSes using Matkowski's contraction mappings [6] and generalizations of the Banach contraction mappings, and they showed that the attractors of these IFSes can generate fractals with wider coverage.

The Kannan contraction mapping [7], which emerged in 1968, is comparably important as the Banach contraction mapping, and in 1971, a contraction mapping formde by a simultaneous generalization of the Banach contraction mapping and the Kannan contraction mapping, was proposed by Reich [8].

The question of whether an IFS constructed by Kannan contraction mappings (or Reich contraction mappings) can generate an attractor (fractal) was raised in 2009 [9].

The authors of [10] presented notable counterexamples showing that fractals cannot be generated by IFSes constructed by Kannan contraction mappings (or Reich contraction mappings) in general.

Until now, a number of fixed-point theorems have been proposed by many authors [11–13], but the results of [10] show that fixed-point theorems which are applicable to attractor generation of IFSes are extremely limited.

Because Banach contraction mappings and Kannan contraction mappings are mutually independent, the theory of IFSes based on the Banach contraction principle may not hold for Kannan contraction mappings (or Reich contraction mappings).

The results of [10] show that the problem of finding IFSes which are capable of generating fractals is one of the main problems in fractal theory.

Therefore, the question naturally arises of whether contractive mappings which are not independent of the Banach contraction mappings and which are generalizations of the Banach contraction mappings always generate fractals.

As far as we know, Bryant contraction mappings, although generally discontinuous, have fixed-point properties that are too similar to Banach contraction mappings, which are of great theoretical and practical importance [11].

Because Bryant contraction mappings are the simplest and most obvious generalization of Banach contraction mappings, one might think that an IFS consisting of a finite number of Bryant contraction mappings would also produce a fractal.

Based on the above discussion, the first question that we set up and solve in this paper is as follows.

Can one always generate attractors (fractals) by IFSes consisting of Bryant contraction mappings, which are the simplest and most obvious generalization of Banach contraction mappings?

Fractal interpolation theory holds an important place in fractal theory.

Unlike the graphs of traditional interpolation functions, the graph of an FIF is an attractor of some IFS passing through a given data point.

FIFs are widely used in various applied sciences because they can represent a large number of geometric objects in nature [14].

Fractal interpolation theory uses IFSes for the generation, modeling, and structural analysis of highly elegant and complex natural images [15, 16].

Because an FIF is generated by the attractor of an IFS, the IFS is a powerful tool for generating and analyzing the FIF. Fractal interpolation has a stronger advantage than traditional polynomial interpolation in modeling the self-similarity of natural geometric objects.

Due to the limited range of FIFs studied so far, it is necessary to generate new FIFs and analyze their complexity in order to vividly describe the various and complex natural geometries.

As far as we know, the first significant generalization of Banach's principle was obtained by Rakotch in 1962 [17].

In 2017, Ri [18] exhibited that by using Rakotch fixed-point theorem, a generalization of Banach fixed-point theorem, one can generate an FIF with a wider range of coverage than Barnsley's FIF; however, only two Rakotch contraction mappings that are not Banach contraction mappings were used for the generation of FIFs.

Hence, the problem of finding a new family of Rakotch contraction mappings which are capable of generating fractal interpolation curves (FICs) is one of the most important problems in fractal interpolation theory [19–22].

This leads to the following problem addressed in this paper: How can Rakotch contraction mappings be constructed which fail to satisfy the Banach contraction condition?

The first question that we set up in this paper arises in the course of discussing whether or not generalized Banach contraction mappings can be used for the generation of FIFs with a wide range of applications in practice, and the second question arises in the process of finding Rakotch contraction mappings as generalized Banach contraction mappings that can generate new FIFs because in general, Bryant contraction mappings, the simplest and most obvious generalization of Banach contraction mappings, do not have attractors.

This paper is organized as follows.

In Section 2, we recall the typical contractive mappings and fractal interpolation principle by using Rakotch contraction mappings, and on this basis, we present the questions to be solved in fractal interpolation theory.

In Section 3, we present two counterexamples showing that IFSes consisting of two discontinuous (or continuous) Bryant contraction mappings, which are the simplest and most obvious generalization of Banach contraction mappings which have no attractors, that is, we show that, in general, one cannot generate fractals (in particular, FIFs) even using continuous Bryant contraction mappings as well as discontinuous Bryant contraction mappings.

Also, based on these two counterexamples, we give a sufficient condition for an IFS consisting of two Bryant contraction mappings to have a fractal, and we give an illustrative example.

In Section 4, we present a new Rakotch contraction mapping family that does not satisfy the Banach contraction condition and give one variable FIF (that is, FIC) generated by a new Rakotch contraction mapping family.

In Section 5, we condense the particular significance of results obtained in our paper.

The new families of Rakotch contraction mappings proposed in this paper will make some contribution to the development of fractal interpolation theory.

2. Preparatory facts and questions to be solved

In this section, we recall typical contractive mappings and the fractal interpolation principle by using Rakotch contraction mappings, and on this basis, we present questions to be solved in fractal interpolation theory.

2.1. Typical contractive mappings and fixed-point theorems

In this subsection, we recall the well-known typical contractive mappings and fixed-point theorems, which are the key object of discussion in this paper.

Definition 2.1. *(See [12]) Let (X, d) be a complete metric space with distance d, and let f be a mapping X into itself.*

(1) *(Banach [23]) There exists a number α, $0 \le \alpha < 1$ such that, for each $x, y \in X$,*

$$d(f(x), f(y)) \le \alpha d(x, y).$$

(2) *(Rakotch [17]) There exists a monotone decreasing function $\alpha : (0, +\infty) \to [0, 1)$ such that, for each $x, y \in X$,*

$$d(f(x), f(y)) \le \alpha(d(x, y))d(x, y).$$

(3) *(Bryant [24]) There exists some positive integer p such that f^p is Banach contraction on X, that is, there exist some $p \in \mathbb{N}$ and a number $\alpha \in [0, 1)$ such that, for each $x, y \in X$,*

$$d(f^p(x), f^p(y)) \le \alpha d(x, y).$$

Note: By the definitions of Banach, Rakotch, and Bryant, it is obvious that (1) ⇒ (2) and (1) ⇒ (3). Now, we give well-known fixed-point theorems.

Theorem 2.2. *(See [17, 23, 24]) Let X be a complete metric space.*

(1) Let $f : X \to X$, $0 \le \alpha < 1$, and

$$d(f(x), f(y)) \le \alpha d(x, y), \ \forall x, y \in X.$$

Then, f has a unique fixed point in X.

In particular, for any fixed $x_0 \in X$,

$$\lim_{n\to+\infty} f^n(x_0) = x_f,$$

where x_f is a unique fixed point of a mapping f, and $f^0(x) := x$, $f^n(x) := f^{n-1}(f(x))$, $n \in \mathbb{N}$. That is, a mapping f has a globally attracting fixed point.

(2) Let $f : X \to X$, and

$$d(f(x), f(y)) \le \alpha(d(x, y))d(x, y), \ \forall x, y \in X,$$

where α is a monotone decreasing function such that $\alpha : (0, +\infty) \to [0, 1)$. Then, f has a unique fixed point in X.

In particular, for any fixed $x_0 \in X$,

$$\lim_{n\to+\infty} f^n(x_0) = x_f,$$

where x_f is a unique fixed point of a mapping f, and $f^0(x) := x$, $f^n(x) := f^{n-1}(f(x))$, $n \in \mathbb{N}$. That is, a mapping f has a globally attracting fixed point.

(3) Let $f: X \to X$, $p \in \mathbb{N}$, $0 \le \alpha < 1$, and

$$d(f^p(x), f^p(y)) \le \alpha d(x, y), \ \forall x, y \in X.$$

Then, f has a unique fixed point in X.

In particular, for any fixed $x_0 \in X$,

$$\lim_{n\to+\infty} f^n(x_0) = x_f,$$

where x_f is a unique fixed point of a mapping f, and $f^0(x) := x$, $f^n(x) := f^{n-1}(f(x))$, $n \in \mathbb{N}$. That is, a mapping f has a globally attracting fixed point.

Remark 2.3. *(1) If f is a Banach contraction mapping on a complete metric space (X, d) with Banach contraction constant h, then for any $n \in \mathbb{N}$, f^n is also a Banach contraction mapping on (X, d) with Banach contraction constant h^n, and the unique fixed point of f is also the unique fixed point of any f^n (see [11, 24]).*

(2) The Bryant fixed-point theorem is of great theoretical and practical importance in the sense that the Bryant mapping f has the same properties as those of special contractive continuous mappings, such as the existence and uniqueness of a fixed point, even if f is not continuous (see [11]).

(3) There is another convenient definition for Rakotch contraction mappings (see [17], p. 463, Corollary; see [12], p. 257; see [3, 4, 18]; see [5], p. 100).

Let (X, d) be a complete metric space, and let $f: X \to X$.

Let $\varphi: (0, +\infty) \to (0, +\infty)$ be a function satisfying for all $x, y \in X$,

$$d(f(x), f(y)) \le \varphi(d(x, y)),$$

where φ is a monotone increasing function, and for every $t > 0$, $\varphi(t) < t$, and the function $\varphi(t)/t$ is monotone decreasing. Then, f is called Rakotch contraction mapping.

2.2. Mutual relations between typical contractive mappings

The statement $(a) \Rightarrow (b)$ means that any function which satisfies condition (a) also satisfies condition (b), and the statement $(a) \not\Rightarrow (b)$ means that any function which satisfies condition (a) does not satisfy condition (b) in general.

Obviously, any mapping that satisfies the Banach contraction condition also satisfies the Rakotch (or Bryant) contraction condition, but any mapping that satisfies the Rakotch (or Bryant) contraction condition does not satisfy the Banach contraction condition in general.

2.2.1. Rakotch contraction mapping$\not\Rightarrow$ Banach contraction mapping

To show that $(2) \not\Rightarrow (1)$, define $\alpha: (0, +\infty) \to [0, 1)$ by $\alpha(t) = 1/(t + 1)$, $t \in (0, +\infty)$ (see the example on p. 262 of [12]).

Let $f(x) = 1/(x + 1)$, $0 \le x \le 1$. Then, $f: [0, 1] \to [0, 1]$ and has a fixed point at $(\sqrt{5} - 1)\big/2$.

For any fixed $\beta \in (0, 1)$, choose $y < -1 + (1/\beta)$ and $y \in (0, (\sqrt{5} - 1)\big/2)$. Then,

$$d(f(0), f(y)) = \frac{y}{y+1} > \beta y = \beta d(0, y),$$

and it does not satisfy (1).

On the other hand, for $0 \le x < y \le 1$,

$$d(f(x), f(y)) = \frac{y-x}{(x+1)(y+1)} \le \frac{y-x}{y-x+1} = \alpha(d(x,y))d(x,y),$$

and f satisfies (2).

2.2.2. Continuous Bryant contraction mapping $\not\Rightarrow$ Banach contraction mapping

The Example 2.2 on p. 34 of [11] shows that (3) $\not\Rightarrow$ (1).

(a) Define $f(x) = \cos x$ on $\mathbb{R}$. Then, f is not a Banach contraction mapping on $\mathbb{R}$. Indeed, suppose there exists $h \in (0,1)$ such that

$$\left|\frac{\cos x - \cos y}{x-y}\right| \le h, \ for\ all\ x \ne y.$$

Letting $y \to x$, we get $|\sin x| \le h$ for all $x \in \mathbb{R}$, which is false. Note that the iterated function $f^2(x) = \cos(\cos x)$ satisfies

$$\left|\frac{d(\cos(\cos x))}{dx}\right| = |\sin(\cos x)\sin x| \le |\sin(\cos x)| < \sin 1 < 1,$$

and thus, by the mean-value theorem, f^2 is a Banach contraction mapping on $\mathbb{R}$.

(b) Define $f(x) = e^{-x}$ on $\mathbb{R}$. Then, f is not a Banach contraction mapping on $\mathbb{R}$. Indeed, suppose there exists $h \in (0,1)$ such that

$$\left|\frac{e^{-x} - e^{-y}}{x-y}\right| \le h, \quad for\ all\ x \ne y.$$

Letting $y \to x$, we get $|e^{-x}| \le h$ for all $x \in \mathbb{R}$, which is false.

However, $f^2(x) = e^{-e^{-x}}$ is a Banach contraction mapping on $\mathbb{R}$. Indeed,

$$\left|\frac{d(e^{-e^{-x}})}{dx}\right| = |e^{-x-e^{-x}}| \le e^{-1} < 1,$$

and thus, the conclusion follows by the mean-value theorem.

2.2.3. Discontinuous Bryant contraction mapping $\not\Rightarrow$ Rakotch (or Banach) contraction mapping

Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then, $f^2(x) = 0$ for all $x \in [0,1]$, and so f^2 is a Banach contraction mapping on $[0,1]$.

So, f is a Bryant contraction mapping, but f is not continuous, and thus, it is not a Rakotch mapping (or Banach) (cf. Example 2.2 on p. 34 of [11]).

2.2.4. Rakotch contraction mapping $\nRightarrow$ Bryant contraction mapping

Let $f(x) = x/(x+1), 0 \leq x \leq 1$. Then, $f : [0,1] \to [0,1]$ is a Rakotch contraction mapping, which does not satisfy Banach contraction condition (see [18]), but it is not a Bryant contraction mapping. In fact, for every fixed $n \in \mathbb{N}$, a function

$$f^n(x) = f(f^{n-1}(x)) = \frac{x}{1+nx}$$

is also a Rakotch contraction mapping, which does not satisfy the Banach contraction condition (see [18]).

2.3. Contractive mappings that can generate fractals

In this subsection, we introduce contractive mappings that can generate fractals among contractive mappings that have unique attracting fixed points.

2.3.1. Definition of fractal generated by contractive mappings

Based on the definitions of a Hutchinson-Barnsley operator, IFSes and attractor in [2,9,25,26], we give the concepts of Hutchinson-Barnsley operator, IFSes and attractor in the case of finite contractive mappings, where the continuity of finite contractive mappings is not assumed.

Definition 2.4. *(See [2], cf. Definition 2.2 on p. 305 of [25]; cf. [5,9,10,26–29].)*

Let (X, d) be a complete metric space.

Let $(H(X), h_d)$ be a fractal space generated by (X, d), where $H(X) := \{A \subset X | A$ is a nonempty compact subset of $X\}$, and h_d expresses the Hausdorff metric on $H(X)$, that is, for all $A, B \in H(X)$,

$$h_d(A, B) := \max\{\max_{x\in A} \min_{y\in B} d(x, y), \ \max_{y\in B} \min_{x\in A} d(x, y)\}.$$

Let $f_n : X \to X, n = 1, ..., N$.

Let for every $n = 1, 2, \ldots, N$, $f_n : X \to X$ be a contractive mapping that maps any nonempty compact subset of (X, d) to a nonempty compact subset of (X, d), where the continuity of f_n is not assumed.

Consider a mapping $F : H(X) \to H(X)$ defined by

$$F(A) := f_1(A) \bigcup f_2(A) \bigcup \cdots \bigcup f_N(A) = \bigcup_{n=1}^{N} f_n(A), \ A \in H(X).$$

Then, $F : H(X) \to H(X)$ is called a Hutchinson-Barnsley operator constructed by finite contractive mappings $f_1, f_2, \cdots, f_N$.

Also, $\{X; f_1, f_2, \ldots, f_N\}$ constructed by a complete metric space (X, d) and finite contractive mappings f_1, f_2, $\ldots$, f_N is called an iterated function system (IFS).

Definition 2.5. *(cf. Definition 2.2 on p. 305 of [25]; cf. [27].)*

Let (X, d) be a complete metric space.

Let $f_n : X \to X, n = 1, ..., N$.

Let for every $n = 1, 2, \ldots, N$, $f_n : X \to X$ *be a contractive mapping that maps any nonempty compact subset of* (X, d) *to a nonempty compact subset of* (X, d), *where the continuity of* f_n *is not assumed.*

Let $F : H(X) \to H(X)$ *be a Hutchinson-Barnsley operator consisting of* $f_n : X \to X, n = 1, 2, \cdots, N$.

A set $A \in H(X)$ *is called an attractive fixed point of the given Hutchinson-Barnsley operator, that is, an attractor of the given IFS* $\{X; f_1, f_2, \cdots, f_N\}$, *if it satisfies the following conditions:*

(1) (*invariance*)

$$F(A) = \bigcup_{i=1}^{N} f_i(A) = A,$$

(2) (*attractiveness*)

$$\lim_{\substack{k \to +\infty \\ h_d}} F^k(S) = A$$

for all $S \in H(X)$, *where* $\lim\limits_{\substack{k \to +\infty \\ h_d}}$ *denotes the limit with respect to the Hausdorff distance* h_d *on* $H(X)$.

In particular, if a compact set $A \in H(X)$ *is an attracting and invariant set of a complete metric space* (X, d), *then we call* $A \in H(X)$ *a fractal generated by the IFS* $\{X; f_1, f_2, \cdots, f_N\}$.

Remark 2.6. *(1) Hutchinson proposed that, in the case when* $f_n : X \to X$, $n = 1, ..., N$ *are Banach contraction mappings, the IFS consisting of them generates a unique fractal (see [26]).*

(2) There exists an example of a discontinuous typical contractive mapping that does not preserve the compactness of the set.

For example, consider a mapping $f : [0, 1] \to [0, 1]$, *defined by*

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ x + \frac{1}{2} & \text{if } x \in (0, \frac{1}{2}), \\ \frac{3}{4} & \text{if } x = \frac{1}{2}, \\ 1 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Let d be the Euclidean distance of $\mathbb{R}$. *Then,* $([0, 1], d)$ *is a complete metric space.*

Also, $x = 1$ *is the unique fixed point of* $f : [0, 1] \to [0, 1]$.

Because f is discontinuous, f is not a Banach contraction mapping.

However, because $f^2(x) = f(f(x)) = 1$ *for all* $x \in [0, 1]$, *we can see that f is a Bryant contraction mapping.*

On the other hand, because

$$f : \left[\frac{1}{3}, \frac{1}{2}\right] \to \left\{\frac{3}{4}\right\} \bigcup \left[\frac{5}{6}, 1\right),$$

the mapping $f : [0, 1] \to [0, 1]$ *does not always map a compact set of the complete metric space* $([0, 1], d)$ *to a compact set of the complete metric space* $([0, 1], d)$.

That is, in general, the Bryant contraction mapping, a typical contractive mapping, does not always preserve the compactness of the set.

Based on the above example of a discontinuous Bryant contraction that does not preserve the compactness of a set, we give an essential and indispensable prerequisite that, in Definition 2.4 and

Definition 2.5, contractive mappings should map any compact set of a given complete metric space to a compact set of that complete metric space.

(3) Because even if mappings $f_n : X \to X, n = 1, ..., N$ are discontinuous, the mappings $f_n : X \to X, n = 1, ..., N$ are contractive mappings with globally attracting fixed points in a complete metric space (X, d) as Banach contraction mappings, we can easily see that, if an attractor of the given IFS $\{X; f_1, f_2, \cdots, f_N\}$ exists, then it is unique.

2.3.2. Contractive mappings that can generate fractals

In this subsubsection, we recall F-contraction mappings, Matkowski contraction mappings, and Meir-Keeler-type mappings that can always generate fractals.

Definition 2.7. *(See [3, 6, 28]) Let (X, d) be a metric space, and let $f : X \to X$.*

(1) *(F-contraction mapping)*

Let $F : (0, +\infty) \to \mathbb{R}$ be a mapping such that

(a) for all $t_1, t_2 \in (0, +\infty)$ $(t_1 < t_2)$, $F(t_1) < F(t_2)$,

(b)

$$\lim_{t \to +0} F(t) = -\infty,$$

(c) for some $\lambda \in (0, 1)$,

$$\lim_{t \to +0} t^{\lambda} F(t) = 0,$$

(d) for some $\mu > 0$ and for all $x, y \in X$ such that $f(x) \neq f(y)$,

$$\mu + F(d(f(x), f(y))) \leq F(d(x, y)).$$

Then, $f : X \to X$ is called an F-contraction mapping (see [28]).

(2) *(Matkowski contraction mapping)*

Let $\alpha : (0, +\infty) \to (0, +\infty)$ be a function satisfying for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha(d(x, y)),$$

where α is an increasing function, and for fixed every $t > 0$,

$$\lim_{n \to +\infty} \alpha^n(t) = 0.$$

Then, f is called a Matkowski contraction mapping (see [6]).

(3) *(Meir-Keeler-type mapping)*

If for any $\varepsilon > 0$, there is $\delta > 0$ satisfying

$$x, y \in X,\ \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon,$$

then f is called a Meir-Keeler-type mapping (see [3]).

Remark 2.8. *(1) All Banach contraction mappings are F-contraction mappings (or Rakotch contraction mappings, or Matkowski contraction mappings, or Meir-Keeler-type mappings) (see [3, 28]), but any mapping which satisfies the F-contraction (or Rakotch contraction, or Matkowski contraction, or Meir-Keeler contraction) condition does not always satisfy the Banach contraction condition in general.*

(2) F-contraction mappings (or Rakotch contraction mappings, or Matkowski contraction mappings, or Meir-Keeler-type mappings) are continuous (see [3, 6, 28]).

Theorem 2.9. *(Existence and uniqueness of fractals; see Theorem 2 on p. 218 of [2], [3, 28].)*

Let (X, d) *be a complete metric space, and for all* $i = 1, 2, \ldots, n$, $f_i : X \to X$.

Let $\{X; f_1, f_2, \ldots, f_n\}$ *be an IFS consisting of Banach contraction mappings (or* F*-contraction mappings, or Rakotch contraction mappings, or Matkowski contraction mappings, or Meir-Keeler-type mappings).*

Then, there exists a unique fractal generated by this IFS.

2.4. Contractive mappings that cannot generate fractals in general

In this subsection, we closely investigate Example 2.4 on p. 2274 and Example 2.7 on p. 2277 in [10] and give a particular remark.

2.4.1. Kannan (or Reich) contraction mappings

Definition 2.10. *(See [12].)*

Let (X, d) *be a complete metric space with distance* d*, and let* f *be a mapping* X *into itself.*

(1) *(Kannan [7, 30]) There exists a number* β, $0 < \beta < 1/2$ *such that for each* $x, y \in X$,

$$d(f(x), f(y)) \le \beta[d(x, f(x)) + d(y, f(y))].$$

(2) *(Reich [8]) There exist non-negative numbers* α, β, γ *satisfying* $\alpha + \beta + \gamma < 1$ *such that, for each* $x, y \in X$,

$$d(f(x), f(y)) \le \alpha d(x, f(x)) + \beta d(y, f(y)) + \gamma d(x, y).$$

Now, we give the well-known fixed-point theorems.

Theorem 2.11. *(See [8, 17, 23, 24, 30].)*

Let X *be a complete metric space.*

(1) Let $f : X \to X$, $0 < \beta < 1/2$, *and let*

$$d(f(x), f(y)) \le \beta[d(x, f(x)) + d(y, f(y))], \ \forall x, y \in X.$$

Then, f *has a unique fixed point in* X.

In particular, for any fixed $x_0 \in X$,

$$\lim_{n \to +\infty} f^n(x_0) = x_f,$$

where x_f *is a unique fixed point of a mapping* f*, and* $f^0(x) := x$, $f^n(x) := f^{n-1}(f(x))$, $n \in \mathbb{N}$. *That is, a mapping* f *has a globally attracting fixed point.*

(2) Let $f : X \to X$, $0 \le \alpha + \beta + \gamma < 1$, *and*

$$d(f(x), f(y)) \le \alpha d(x, f(x)) + \beta d(y, f(y)) + \gamma d(x, y), \ \forall x, y \in X.$$

Then, f *has a unique fixed point in* X.

In particular, for any fixed $x_0 \in X$,

$$\lim_{n \to +\infty} f^n(x_0) = x_f,$$

where x_f *is a unique fixed point of a mapping* f*, and* $f^0(x) := x$, $f^n(x) := f^{n-1}(f(x))$, $n \in \mathbb{N}$. *That is, a mapping* f *has a globally attracting fixed point.*

In 1976, the author of [31] gave a simple relationship between Banach contraction mapping and Kannan contraction mapping.

Proposition 2.12. *(See Lemma 2.2 on p. 173 of [31].)*

Let (X, d) be a metric space, and let $f : X \to X$ be a Banach contraction mapping with Banach contraction factor α, where $0 \leq \alpha < 1/3$. Then, f is a Kannan contraction mapping with respect to metric d.

Remark 2.13. *(1) There is a Banach contraction (or Rakotch contraction, or Bryant contraction) mapping that is not a Kannan contraction mapping.*

For example, $f(x) = x/3$ (see the example on p. 263 of [12]).

(2) There is a Kannan contraction mapping that is not a Banach contraction (or Rakotch contraction, or Bryant contraction) mapping.

For example,

$$f(x) := \begin{cases} \frac{1}{2} & \text{if } x \in [0, 1), \\ \frac{1}{4} & \text{if } x = 1 \end{cases}$$

(see the example on p. 262 of [12]).

(3) There is a Reich contraction mapping that is not a Kannan contraction mapping.

For example,

$$f(x) := \begin{cases} \frac{x}{3} & \text{if } x \in [0, 1), \\ \frac{1}{6} & \text{if } x = 1 \end{cases}$$

(see the example on p. 122 of [8]).

2.4.2. Counterexamples for Kannan (or Reich) contraction mappings

The properties of a Kannan contraction mapping f are not hereditary to a mapping F in general, where $F(A) := \{f(x) : x \in A\}$ for all $A \in H(X)$.

Counterexample 2.14. *(See Example 2.4 on p. 2274 in [10].)*

There are a complete metric space (X, d) and a Kannan (or Reich) contraction mapping satisfying the following conditions:

(1) *For any $\alpha \in [2/5, 1/2)$, $f : X \to X$ is a Kannan contraction mapping on a complete metric space (X, d) with Kannan contraction factor α;*

(2) *For all $A \in H(X)$,*

$$F(A) := \bigcup_{x \in A} f(x) \in H(X);$$

(3) *The map $F : H(X) \to H(X)$ is not a Kannan map on a complete metric space $(H(X), h_d)$ for any Kannan contraction factor $\alpha \in (0, 1/2)$.*

The properties of two Kannan contraction mappings, f_1 and f_2, are not hereditary to a mapping F, where for all $A \in H(X)$,

$$F(A) := f_1(A) \bigcup f_2(A).$$

Counterexample 2.15. *(See Example 2.7 on p. 2277 in [10].)*

There are a complete metric space (X, d) and Kannan (or Reich) contraction mappings satisfying the following conditions:

(1) *For any $\alpha \in [1/3, 1/2)$, $f_1, f_2 : X \to X$ are Kannan contraction mappings on a complete metric space (X, d) with Kannan contraction factor α;*

(2) *For all $A \in H(X)$,*

$$F(A) := F_1(A) \bigcup F_2(A) \in H(X);$$

(3) *The map $F : H(X) \to H(X)$ is not a Kannan map on a complete metric space $(H(X), h_d)$ for any Kannan contraction factor $\alpha \in (0, 1/2)$;*

(4) *$F(A) \neq A$ for all $A \in H(X)$.*

2.5. Contractive mappings that can generate continuous FIFs

In this subsection, we introduce the contractive mappings that can generate continuous FIFs (FICs) among mappings that can generate fractals.

Let $\{(x_i, y_i) \in \mathbb{R}^2 : i \in \{0, 1, 2, \cdots, N\}\}$ be a data set, where $N \geq 2$, $-\infty < x_0 < x_1 < x_2 < \cdots < x_N < +\infty$, and $y_i \in \mathbb{R}$.

Let for $(x, y) \in [x_0, x_N] \times \mathbb{R}$,

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} u_i x + v_i \\ F_i(x, y) \end{pmatrix},$$

where

$$u_i := \frac{x_i - x_{i-1}}{x_N - x_0},$$

$$v_i := x_{i-1} - \frac{x_i - x_{i-1}}{x_N - x_0} x_0,$$

F_i is a continuous function such that $F_i : [x_0, x_N] \times \mathbb{R} \to \mathbb{R}$, and every w_i satisfies the following conditions:

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix},$$

$$w_i \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

The following theorem gives the generative principle of one variable FIFs by using Banach contraction (or Rakotch contraction) mappings.

Theorem 2.16. *(See [1, 2, 18, 32].)*

Let N be a positive integer greater than 1. Let $\{\mathbb{R}^2; \omega_i, i = 1, 2, \cdots, N\}$ denote the IFS defined above, associated with the data set $\{(x_i, y_i) \in \mathbb{R}^2 : i = 1, 2, \cdots, N\}$.

If for every $i \in \{1, 2, \cdots, N\}$, a mapping $F_i(\cdot, y)$ satisfies the Banach contraction (or Rakotch contraction) condition, and $F_i(x, \cdot)$ satisfies the Lipschitz condition [18] (or Hölder condition [32]), then there is a continuous function that passes through the given data, and the graph of this continuous function is an attractor of the IFS defined above.

Remark 2.17. *(1) Although we can generate fractals by using F-contraction mappings (or Matkowski contraction mappings, or Meir-Keeler-type mappings), so far, there are no studies that can generate FIFs by using F-contraction mappings (or Matkowski contraction mappings, or Meir-Keeler-type mappings).*

(2) Moreover, there is no theoretical guarantee that the mappings which can generate fractals can necessarily generate FIFs.

Remark 2.18. *(1) In [18], the author only used Rakotch contraction mappings $f(x) = 1/(1+x)$ and $g(x) = x/(1+x)$ $(x \geq 0)$, which do not satisfy the Banach contraction condition.*

By using these Rakotch contraction mappings, which do not satisfy Banach contraction condition, the author of [18] gave a novel idea and an original way for new FIF generation.

(2) By using the Rakotch contraction mappings of [18], we can obtain the Rakotch contraction mapping family

$$\left\{g_n(x) := \frac{x}{1+nx} | n \in \mathbb{N}\right\},$$

which does not satisfy the Banach contraction condition.

This Rakotch contraction family, which does not satisfy the Banach contraction condition, has an essential and important significance for the solution of new FIC generation problem. In fact, because

$$\frac{x}{1+x} \geq \frac{x}{1+2x} \geq \frac{x}{1+3x} \geq \frac{x}{1+4x} \geq \cdots \geq \frac{x}{1+nx} \geq \cdots,$$

and for all $n \in \mathbb{N}$ and for $\delta(t) := t/(1+t)$ $(t \geq 0)$,

$$|g_n(x) - g_n(x)| \leq \delta(|x-y|),$$

we can obtain a new Rakotch contraction family which does not satisfy the Banach contraction condition such that for all $m, n \in \mathbb{N}$,

$$g_{m,n}(x) := \begin{cases} \frac{x}{1+mx} & \text{if } x \in [0, \frac{n-1}{m}], \\ \frac{1}{n}x & \text{if } x > \frac{n-1}{m}. \end{cases}$$

(3) In [32], in order to present new FIFs, the authors used a function $\sin x$ $(x > 0)$ that is a Rakotch contraction mapping which does not satisfy Banach contraction condition.

Indeed, $\sin x$ is not a Banach contraction mapping on $(0, +\infty)$. In fact, if there is $0 < c < 1$ such that for all $x \neq y$ $(x, y \in (0, +\infty))$,

$$\left|\frac{\sin x - \sin y}{x - y}\right| \leq c,$$

then for all $x \in (0, +\infty)$,

$$\left|\lim_{y \to x} \frac{\sin x - \sin y}{x - y}\right| = |\cos x| \leq c.$$

It is false, and so a function $\sin x$ is not a Banach contraction mapping on $(0, +\infty)$.

On the other hand, for all $x, y \in (0, +\infty)$,

$$|\sin x - \sin y| = 2\left|\cos\frac{x+y}{2} \sin\frac{x-y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right| \leq 2\left|\sin\frac{|x-y|}{2}\right|.$$

Let for all $t \in (0, +\infty)$,

$$\alpha(t) := \begin{cases} 2\sin\frac{t}{2} & \text{if } t \in [0, \pi], \\ \frac{2}{\pi}t & \text{if } t > \pi. \end{cases}$$

Then, for all $t \in (0, +\infty)$, $\alpha(t) < t$ *and* $\alpha : [0, +\infty) \to [0, +\infty)$ *are increasing functions, and a mapping* $t \to \varphi(t)/t$ *is a decreasing function, and for all* $x, y \in (0, +\infty)$,

$$|\sin x - \sin y| \leq \alpha(|x - y|).$$

So, a function $\sin x$ *is a Rakotch contraction mapping that does not satisfy the Banach contraction condition.*

Difficulty 2.19. *Obviously, a Rakotch contraction is a generalized Banach contraction, but it is no easy matter to find a new Rakotch contraction mapping family that does not satisfy the Banach contraction condition because it is difficult to make a mapping* $f(x)$ *and a function* $\varphi(t)$ *such that for all* $x, y \in X$,

$$d(f(x), f(y)) \leq \varphi(d(x, y)),$$

where $\varphi : (0, +\infty) \to (0, +\infty)$ *is a function satisfying for all* $t > 0$, $\varphi(t) < t$, *a function* $\varphi(t)$ *is a monotone increasing function, and a function* $\varphi(t)/t$ *is a monotone decreasing function.*

Even if it is difficult to make a mapping $f(x)$ *and a function* $\varphi(t)$, *if one finds a new Rakotch contraction mapping family that does not satisfy the Banach contraction condition, then by Ri's nonlinear fractal interpolation principle, by using Rakotch contraction mappings [18], one can always generate a new FIF by using it.*

2.6. Questions to be solved

The results of [10, 18] show that the problem of finding IFSes which are capable of generating fractals, especially, the problem of finding contractive mappings which are capable of generating FIFs, is one of the main problems in fractal theory.

Although there exists a result of [18] that by using Rakotch contraction mappings which do not satisfy Banach contraction conditions, one can generate more generalized FIFs than Barnsley's FIFs, it is not easy to find such Rakotch contraction mappings which do not satisfy the Banach contraction condition.

Based on the above discussion, we are reminded of the questions that we set up and solve in this paper as follows.

Question 2.20. *Can one always generate FIFs by IFSes consisting of Bryant contraction mappings, which are the simplest and most obvious generalization of the Banach contraction mappings?*

Question 2.21. *How do we construct new Rakotch contraction mappings that do not satisfy the Banach contraction condition?*

3. Bryant contraction mappings that cannot generate fractals in general

Bryant contraction mappings are not necessarily continuous mappings (see 2.2.3 of Section 2); in particular, they are not necessarily Banach contraction mappings (see 2.2.2 of Section 2).

Therefore, Bryant contraction mappings do not preserve the compactness of sets in general, as shown in the example of 2) in Remark 2.6.

In particular, the notion of the Hutchinson-Barnsley operator in [2,26], which was defined for a finite number of Banach contraction mappings, cannot be applied to a finite number of Bryant contraction mappings in general.

In Definitions 2.4 and 2.5, we gave the new, exact, and valid notions of the Hutchinson-Barnsley operator and an attractor for a finite number of discontinuous (or continuous) contractive mappings.

However, even in the case of Bryant contraction mappings, preserving the compactness of sets, the IFS theory based on the Banach fixed-point theorem may not entirely hold for these Bryant contraction mappings in general.

In particular, in general, the contractive property of Bryant contraction mappings may not be inherited to the Hutchinson-Barnsley operator (that is, the induced set-valued mapping) consisting of these Bryant contraction mappings.

In this section, we first show that the property of the unique existence of a fixed point of a Bryant contraction mapping preserving the compactness of sets on a complete metric space (X, d) is inherited to the Hutchinson-Barnsley operator consisting of this Bryant contraction mapping in a fractal space $(H(X), h_d)$.

Next, we give two discontinuous Bryant contraction mappings and two continuous Bryant contraction mappings which, despite preserving the compactness of sets, cannot generate fractals.

That is, we present the inevitable counterexamples in fractal theory.

These two counterexamples show that IFSes consisting of two discontinuous (or even continuous) Bryant contraction mappings, the simplest and most obvious generalization of Banach contraction mappings, cannot always generate fractals in general.

Also, based on these two counterexamples, we give a sufficient condition for an IFS consisting of two Bryant contraction mappings to have a fractal, and we give an illustrative example.

3.1. Existence of a set fixed point by one Bryant contraction mapping

We present the following theorem based on the well-known theorem of the existence and uniqueness of a fixed point of a Bryant contraction mapping in a complete metric space (Theorem 2.4 in [11]) and the well-known theorem in fractal theory (Lemma 7.3 of [2]).

Theorem 3.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a Bryant contraction mapping that maps any compact subset of X to a compact subset of X, where the continuity of f is not assumed.*

Then, a mapping $F : A \to f(A)$, $A \in H(X)$ has a globally attracting unique fixed point in a complete metric space $(H(X), h_d)$.

Proof. Because $f : X \to X$ is a Bryant contraction mapping on (X, d), there exist $p \in \mathbb{N}$ such that f^p is a Banach contraction mapping.

Because f^p is a Banach contraction mapping on X, f^p is continuous on X.

Thus, f^p maps every compact subset of X to a compact subset of X.

That is, $f^p(A) \in H(X)$ for all $A \in H(X)$.

Consider a mapping

$$F : A \to f(A),\ A \in H(X).$$

From the assumption of the theorem, because a mapping $f : X \to X$ maps any compact subset of a complete metric space (X, d) to a compact subset of a complete metric space (X, d), we have

$$F : A \to f(A) \in H(X),\ A \in H(X),$$

that is, $F : H(X) \to H(X)$.

Now, let $F^p(A) := F(F^{p-1}(A))$ for all $A \in H(X)$.

Because $f^p(A) \in H(X)$ for all $A \in H(X)$, and

$$F^p(A) = F(F^{p-1}(A)) = \cdots = F^{p-1}(F(A)) = F^{p-1}(f(A)) = \cdots = F(f^{p-1}(A)) = \cdots = f^p(A),$$

we see that F^p maps $H(X)$ to itself.

Because f^p is a Banach contraction mapping in a complete metric space (X, d), and $F^p : H(X) \to H(X)$, by Lemma 7.3 on p. 79 of [2], we obtain that F^p is a Banach contraction mapping in a complete metric space $(H(X), h_d)$.

So, we can see that $F : H(X) \to H(X)$ is a Bryant contraction mapping in a complete metric space $(H(X), h_d)$.

Hence, by the Bryant fixed-point theorem [24], a mapping $F : H(X) \to H(X)$ has a unique globally attracting fixed point in a complete metric space $(H(X), h_d)$. □

Remark 3.2. *(1) As shown in Example (2) in Remark 2.6, a discontinuous Bryant contraction mapping $f : X \to X$ does not map a compact subset of a complete metric space (X, d) to a compact subset of a complete metric space (X, d) in general, and so we cannot guarantee the compactness of a set $F(A) = f(A) \subset X$ in general.*

That is, the Hutchinson-Barnsley operator consisting of one Bryant contraction generally does not map a given fractal space $(H(X), h_d)$ to itself.

To discuss the unique existence of a fixed point of the Hutchinson-Barnsley operator $F(A) := f(A) \subset X,\ A \in H(X)$ defined in the complete metric space $(H(X), h_d)$, basically, this Hutchinson-Barnsley operator must map a complete metric space $(H(X), h_d)$ to a complete metric space $(H(X), h_d)$.

Hence, the assumption of Theorem 3.1 that a Bryant contraction mapping $f : X \to X$ must map any compact subset of a complete metric space (X, d) to a compact subset of a complete metric space (X, d) is an essential prerequisite.

(2) Theorem 3.1 shows that the property of the unique existence of a fixed point of a Bryant contraction mapping f preserving the compactness of set in a complete metric space (X, d) is inherited to the Hutchinson-Barnsley operator $F(A) := f(A),\ A \in H(X)$ consisting of this Bryant contraction mapping f in a complete metric space $(H(X).h_d)$.

Especially in Theorem 3.1, we can see that the Hutchinson-Barnsley operator consisting of one Bryant contraction mapping preserving the compactness of set in a complete metric space (X, d) is a Bryant contraction mapping in a fractal space $(H(X).h_d)$ that has a unique globally attracting fixed point.

However, in the case of Kannan contraction (or Reich contraction) mapping, even if this Kannan contraction (or Reich contraction) mapping is continuous, the Hutchinson-Barnsley operator consisting of this Kannan contraction (or Reich contraction) mapping is not always a Kannan contraction (or Reich contraction) mapping on a fractal space in general (see Example 2.4 on p. 2274 and Example 2.7 on p. 2277 of [10]).

(3) Theorem 3.1 naturally raises the question of whether the property of the unique existence of fixed points of two Bryant contraction mappings, f_1 and f_2, preserving compactness of sets on a complete metric space (X, d) is inherited by the Hutchinson-Barnsley operator defined by $F(A) := f_1(A) \bigcup f_2(A)$, $A \in X$ consisting of these Bryant contraction mappings, where the continuity of mappings f_1 and f_2 is not assumed.

This problem is discussed in Subsections 3.2 and 3.3.

Below, we give an example of a discontinuous Bryant contraction mapping which satisfies the sufficient condition of Theorem 3.1.

This example shows that the assumption of Theorem 3.1 above is not an impossible assumption.

Example 3.3. *(A simple example)*

Consider a function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, $([0,\ 1],\ d)$ is a complete metric space, and f maps any compact subset of a complete metric space $([0, 1], d)$ to a compact subset of a complete metric space $([0, 1], d)$, where d is the Euclidean metric in $\mathbb{R}$.

Then, f is not a Banach contraction mapping because f is not continuous.

However, f^2 is a Banach contraction mapping because $f^2(x) = 0$ for all $x \in [0, 1]$.

So, f is a discontinuous Bryant contraction mapping but is not a Banach contraction, satisfying the sufficient condition of Theorem 3.1.

3.2. Necessary condition for generating fractals by finite contractive mappings

In this subsection, we give a necessary condition for the Hutchinson-Barnsley operator consisting of finite contractive mappings to have an attractive fixed point. That is, we give a necessary condition for an IFS consisting of a finite number of contractive mappings to generate a fractal, where continuity of contractive mappings are not assumed.

Theorem 3.4. *Let (X, d) be a complete metric space.*

Let $f_n : X \to X, n = 1, ..., N$.

Let for every $n = 1, 2, \ldots, N$, $f_n : X \to X$ be a contractive mapping that maps any nonempty compact subset of (X, d) to a nonempty compact subset of (X, d), where the continuity of f_n is not assumed.

If the Hutchinson-Barnsley operator $F : H(X) \to H(X)$ defined by

$$F(A) := f_1(A) \bigcup f_2(A) \bigcup \cdots \bigcup f_N(A) = \bigcup_{n=1}^{N} f_n(A),\ A \in H(X)$$

has an attractive fixed point, that is, if an attractor of the given IFS $\{X : f_1, f_2, \ldots, f_N\}$ exists, then for every infinite word $\omega \in \Omega$, the limit of $f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}(x)$ exists and is independent of $x \in X$, where

$$\Omega := \{1,\ 2,\ \cdots,\ N\}^{\mathbb{N}} = \{\omega = \omega_1\omega_2\cdots\omega_n\cdots : \omega_n \in \{1,\ 2,\ \cdots,\ N\}, n \in \mathbb{N}\}.$$

Proof. Even if mappings $f_n : X \to X, n = 1, ..., N$ are discontinuous, from the assumption of the theorem, the mappings $f_n : X \to X, n = 1, ..., N$ preserve the compactness of the set as continuous mappings, especially as Banach contraction mappings.

Also, the mappings $f_n : X \to X, n = 1, ..., N$ are contractive mappings with globally attracting fixed points in a complete metric space (X, d) as Banach contraction mappings.

Therefore, the proof of Theorem 3.1 is similar to that of "(4) Proof of Uniqueness" in "3. invariant sets" of [26]. We therefore omit this proof here. □

3.3. Bryant contraction mappings that cannot generate fractals

In this subsection, we give the motivation to discuss the counterexamples of Bryant contraction mappings that cannot generate fractals.

Motivation 3.5. *Let (X, d) be a complete metric space and $f_1 : X \to X$ and $f_2 : X \to X$ be Bryant contraction mappings that map any compact subset of X to a compact subset of X, where the continuity of f_1 and f_2 are not assumed.*

Then, in general, one cannot always guarantee the fact that a mapping $F : A \to f_1(A) \bigcup f_2(A)$, $A \in H(X)$ has a unique globally attracting fixed point in a complete metric space $(H(X), h_d)$.

In fact, because $f_1 : X \to X$ and $f_2 : X \to X$ are the Bryant contraction mappings on (X, d), there exist $p_1, p_2 \in \mathbb{N}$ such that $f_1^{p_1}$ and $f_2^{p_2}$ are Banach contraction mappings.

Without loss of generality, let $p := p_1 = p_2$.

Because f_1^p and f_2^p are Banach contraction mappings on X, f_1^p and f_2^p are continuous on X.

Thus, f_1^p and f_2^p map every compact subset of X to a compact subset of X.

That is, $f_1^p(A) \bigcup f_2^p(A) \in H(X)$ for all $A \in H(X)$.

In the case of $p = 2$, the consider the mappings

$$F : A \to f_1(A) \bigcup f_2(A),\ A \in H(X)$$

and

$$T : A \to f_1^2(A) \bigcup f_2^2(A),\ A \in H(X).$$

Because mappings $f_1 : X \to X$ and $f_1 : X \to X$ map any compact subset of a complete metric space (X, d) to a compact subset of a complete metric space (X, d), we have

$$F : A \to f_1(A) \bigcup f_2(A) \in H(X),\ A \in H(X),$$

that is, $F : H(X) \to H(X)$.

Also, because f_1^2 and f_2^2 are Banach contraction mappings, by [2], a mapping T has a unique attracting fixed point in a complete metric space $(H(X), h_d)$.

On the other hand, in general, for all $A \in H(X)$,

$$\begin{aligned} T(A) &= f_1^2(A) \cup f_2^2(A) \neq f_1^2(A) \cup f_2^2(A) \cup f_1(f_2(A)) \cup f_2(f_1(A)) \\ &= f_1(f_1(A) \cup f_2(A)) \cup f_2(f_1(A) \cup f_2(A)) = F(f_1(A) \cup f_2(A)) \\ &= F(F(A)) = F^2(A). \end{aligned}$$

Because $T(A) \neq F^2(A)$ for all $A \in H(X)$ in general, as in Theorem 3.1, one cannot always guarantee that the unique attracting fixed point of a mapping T in a complete metric space $(H(X), h_d)$ is the unique attracting fixed point of a mapping F in general.

3.3.1. Discontinuous Bryant contraction mappings that cannot generate fractals

Below, we give a counterexample that for two discontinuous Bryant contraction mappings, the well-known IFS theory (in particular, the theorem of existence and uniqueness of fractals) based on Banach contraction mappings doesn't hold.

Counterexample 3.6. *Let* $X := [0, 2]$ *and* d *be the Euclidean metric. Then,* (X, d) *is a complete metric space.*

Let

$$f_1(x) := \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x \in (1,2], \end{cases}$$

and

$$f_2(x) := \begin{cases} 1 & \text{if } x \in [0,1), \\ 2 & \text{if } x \in [1,2]. \end{cases}$$

Because $f_1(A) \in \{\{0\}, \{1\}, \{0, 1\}\}$ *and* $f_2(A) \in \{\{1\}, \{2\}, \{1, 2\}\}$ *for all* $A \in H([0, 2])$*,* f_1 *and* f_2 *map any nonempty compact set of a complete metric space* $([0, 2], d)$ *to a nonempty compact set of a complete metric space* $([0, 2], d)$*.*

Obviously,

$$(f_1 \circ f_2)(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,2], \end{cases}$$

and

$$(f_2 \circ f_1)(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 2 & \text{if } x \in (1,2]. \end{cases}$$

Also, f_1 *and* f_2 *are not Banach contraction mappings, but* f_1 *and* f_2 *are Bryant contraction mappings because for all* $x \in [0, 2]$*,* $f_1^2(x) = 0$ *(see Example 2.2 of [11]) and* $f_2^2(x) = 2$*. Then,* $x_1 = 0$ *is a unique fixed point of* $f_1 : [0, 2] \to [0, 2]$*, and* $x_2 = 2$ *is a unique fixed point of* $f_2 : [0, 2] \to [0, 2]$*.*

On the other hand,

$$(f_2 \circ f_1 \circ f_2)(x) = \begin{cases} 1 & \text{if } x \in [0,1), \\ 2 & \text{if } x \in [1,2], \end{cases}$$

and

$$(f_1 \circ f_2 \circ f_1 \circ f_2)(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,2]. \end{cases}$$

Hence,

$$f_1 \circ f_2 = f_1 \circ f_2 \circ f_1 \circ f_2.$$

Thus, by induction, for all $m \in \mathbb{N}$*,*

$$f_1 \circ f_2 = (f_1 \circ f_2)^m.$$

Also,

$$(f_1 \circ f_2 \circ f_1)(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x \in (1,2], \end{cases}$$

and

$$(f_2 \circ f_1 \circ f_2 \circ f_1)(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 2 & \text{if } x \in (1,2]. \end{cases}$$

Hence,

$$f_2 \circ f_1 = f_2 \circ f_1 \circ f_2 \circ f_1.$$

Thus, by induction, for all $m \in \mathbb{N}$,

$$f_2 \circ f_1 = (f_2 \circ f_1)^m.$$

Finally, for any $m \in \mathbb{N}$,

$$(f_1 \circ f_2)^m(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,2], \end{cases}$$

and

$$(f_1 \circ (f_2 \circ f_1)^m)(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

If $x_0 \in [0,1)$, then

$$f_1(x_0) = 0,\ f_1(f_2(x_0)) = 0,\ f_1(f_2(f_1(x_0))) = 0,\ f_1(f_2(f_1(f_2(x_0)))) = 0,$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

and if $x_0 \in (1,2]$, then

$$f_1(x_0) = 1,\ f_1(f_2(x_0)) = 1,\ f_1(f_2(f_1(x_0))) = 1,\ f_1(f_2(f_1(f_2(x_0)))) = 1,$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ .$$

In particular, if $x_0 = 1$, then

$$f_1(x_0) = 0,\ (f_1 \circ f_2)(x_0) = 1,\ (f_1 \circ f_2 \circ f_1)(x_0) = 0,\ (f_1 \circ f_2 \circ f_1 \circ f_2)(x_0) = 1, \cdots .$$

Hence, for some special symbol

$$(\omega_1, \omega_2, , \omega_3, \omega_4, \cdots) := (1,2,1,2,\cdots) \in \{1,2\}^{\mathbb{N}},$$

the limit

$$\lim_{n\to+\infty} f_{\omega_1} \circ f_{\omega_2} \circ f_{\omega_3} \circ f_{\omega_4} \cdots \circ f_{\omega_n}(x_0),$$

that does not depend on $x_0 \in [0,2]$ does not exist.

Therefore, for all symbols

$$(\sigma_1, \sigma_2, , \sigma_3, \sigma_4, \ldots) \in \{1,2\}^{\mathbb{N}},$$

the limits

$$\lim_{n\to+\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ f_{\sigma_3} \circ f_{\sigma_4} \cdots \circ f_{\sigma_n}(x)$$

that do not depend on $x \in [0,2]$ *do not exist.*

Therefore, by Theorem 3.1, there is not $A \in H([0,2])$ *which satisfies*

$$A = f_1(A) \bigcup f_2(A),$$

that is,

$$f_1(A) \bigcup f_2(A) \neq A,$$

for all $A \in H([0,2])$*, where A refers to an attracting invariant subset.*

Note that because

$$f_1(\{0,1,2\}) \bigcup f_2(\{0,1,2\}) = \{0,1,2\},$$

a compact set $\{0,1,2\} \in H([0,2])$ is an invariant set but not an attracting invariant set.

3.3.2. Continuous Bryant contraction mappings that cannot generate fractals

Below, we give a counterexample showing that even for two continuous Bryant contraction mappings, the well-known IFS theory based on Banach contraction mappings does not hold.

Counterexample 3.7. *Let* $X := [0,1]$ *and d be the Euclidean metric. Then,* (X,d) *is a complete metric space.*

Let

$$f_1(x) := \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$f_2(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}), \\ x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Because f_1 *and* f_2 *are continuous,* f_1 *and* f_2 *map any nonempty compact set of a complete metric space* $([0,2],d)$ *to a nonempty compact set of a complete metric space* $([0,2],d)$*.*

Obviously,

$$(f_1 \circ f_2)(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ x & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$(f_2 \circ f_1)(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}), \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and so $f_1 \circ f_2 \neq f_2 \circ f_1$*.*

Also, f_1 *and* f_2 *are not Banach contraction mappings, but* f_1 *and* f_2 *are continuous Bryant contraction mappings because for all* $x \in [0,1]$*,* $f_1^2(x) = 1$ *and* $f_2^2(x) = 0$*,* $x_1 = 1$ *is a unique fixed point of* $f_1 : [0,1] \to [0,1]$*, and* $x_2 = 0$ *is a unique fixed point of* $f_2 : [0,1] \to [0,1]$*.*

On the other hand,

$$(f_2 \circ f_1 \circ f_2)(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}), \\ x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$(f_1 \circ f_2 \circ f_1 \circ f_2)(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Hence,

$$f_1 \circ f_2 = f_1 \circ f_2 \circ f_1 \circ f_2.$$

Thus, by induction, for all $m \in \mathbb{N}$*,*

$$f_1 \circ f_2 = (f_1 \circ f_2)^m.$$

Also,

$$(f_1 \circ f_2 \circ f_1)(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$(f_2 \circ f_1 \circ f_2 \circ f_1)(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}), \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Hence,

$$f_2 \circ f_1 = f_2 \circ f_1 \circ f_2 \circ f_1.$$

Thus, by induction, for all $m \in \mathbb{N}$*,*

$$f_2 \circ f_1 = (f_2 \circ f_1)^m.$$

Finally, any $m \in \mathbb{N}$*,*

$$(f_1 \circ f_2)^m(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ x & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$(f_1 \circ (f_2 \circ f_1)^m)(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Now, for a fixed $x_0 \in [0, 1]$*, consider an infinite sequence*

$$\{a_1(x_0) := f_1(x_0),\ a_2(x_0) := (f_1 \circ f_2)(x_0),\ a_3(x_0) := (f_1 \circ f_2 \circ f_1)(x_0),\ a_4(x_0) := (f_1 \circ f_2 \circ f_1 \circ f_2)(x_0), \cdots\}.$$

If $x_0 \in [0, 1/2)$*, then*

$$a_1(x_0) = f_1(x_0) = x_0 + \frac{1}{2},\ a_2(x_0) = f_1(f_2(x_0)) = \frac{1}{2},\ a_3(x_0) = f_1(f_2(f_1(x_0))) = x_0 + \frac{1}{2},$$
$$a_4(x_0) = f_1(f_2(f_1(f_2(x_0)))) = \frac{1}{2}, \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and if $x_0 \in [1/2, 1]$*, then*

$$a_1(x_0) = f_1(x_0) = 1,\ a_2(x_0) = f_1(f_2(x_0)) = x_0,\ a_3(x_0) = f_1(f_2(f_1(x_0))) = 1,$$
$$a_4(x_0) = f_1(f_2(f_1(f_2(x_0)))) = x_0, \cdots\ \cdots\ \cdots\ \cdots\ \cdots .$$

Hence, for every fixed $x_0 \in (0, 1)$ *and for some special symbol*

$$(1, 2, 1, 2, \dots) \in \{1, 2\}^{\mathbb{N}},$$

an infinite sequence $\{a_n(x_0)\}_{n=1}^{+\infty}$ *is not convergent in* $[0, 1]$*, and only in two points,* $x_0 = 0$ *and* $x_0 = 1$*,*

$$\lim_{n\to+\infty} a_n(0) = \frac{1}{2},$$

and

$$\lim_{n\to+\infty} a_n(1) = 1.$$

So, for all symbols

$$(\sigma_1, \sigma_2, , \sigma_3, \sigma_4, \dots) \in \{1, 2\}^{\mathbb{N}},$$

the limits

$$\lim_{n\to+\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ f_{\sigma_3} \circ f_{\sigma_4} \cdots \circ f_{\sigma_n}(x)$$

that do not depend on $x \in [0, 1]$ *do not exist.*

Therefore, by Theorem 3.1, there is not $A \in H([0, 1])$ *which satisfies*

$$A = f_1(A) \bigcup f_2(A),$$

that is,

$$f_1(A) \bigcup f_2(A) \neq A$$

for all $A \in H([0, 1])$*, where A refers to an attracting invariant subset.*

Note that because

$$f_1(\{0, \frac{1}{2}, 1\}) \bigcup f_2(\{0, \frac{1}{2}, 1\}) = \{0, \frac{1}{2}, 1\},$$

a compact set $\{0, \frac{1}{2}, 1\} \in H([0, 1])$ is an invariant set but not an attracting invariant set.

Remark 3.8. *The above two counterexamples require finding sufficient conditions for the IFS consisting of a finite number (especially two) of Bryant contraction mappings to generate a fractal.*

3.4. A sufficient condition for two Bryant contraction mappings to generate fractals

In this subsection, based on the two counterexamples above, we give a sufficient condition for the Hutchinson-Barnsley operator consisting of two Bryant contraction mappings to have a unique attracting fixed point in a fractal space, and we give an illustrative example.

Theorem 3.9. *Let (X, d) be a complete metric space and $f_1, f_2 : X \to X$ be Bryant contraction mappings that map any compact subset of X to a compact subset of X, where the continuity of $f_1, f_2 : X \to X$ is not assumed.*

Without loss of generality, let $f_1^2 : X \to X$ and $f_2^2 : X \to X$ be Banach contraction mappings.

Let $f_1 \circ f_2 : X \to X$ and $f_2 \circ f_1 : X \to X$ be Banach contraction mappings.

Then, for any $A \in H(X)$, the mapping $F : H(X) \to H(X)$ defined by

$$F(A) := \{f_1(x)|x \in A\} \bigcup \{f_2(x)|x \in A\} = f_1(A) \bigcup f_2(A)$$

has a unique globally attracting fixed point in a complete metric space $(H(X), h_d)$.

Proof. From the assumption of the theorem, $f_1^2(A), f_2^2(A), f_1 \circ f_2(A), f_2 \circ f_1(A) \in H(X)$ for all $A \in H(X)$.

Now, let $F^2(A) := F(F(A))$ for all $A \in H(X)$.

Because $f_1^2 : X \to X$, $f_2^2 : X \to X$, $f_1 \circ f_2 : X \to X$, and $f_2 \circ f_1 : X \to X$ are Banach contraction mappings and because for all $A \in H(X)$,

$$F^2(A) = f_1^2(A) \bigcup f_2^2(A) \bigcup f_1 \circ f_2(A) \bigcup f_2 \circ f_1(A) \in H(X),$$

by Theorem 7.1 on p. 81 of [2], we can see that $F^2 : H(X) \to H(X)$ is a Banach contraction mapping in a complete metric space $(H(X), h_d)$, and so $F : H(X) \to H(X)$ is a Bryant contraction mapping in a complete metric space $(H(X), h_d)$.

So, by the Bryant fixed-point theorem [24], a mapping $F : H(X) \to H(X)$ has a unique globally attracting fixed point in a complete metric space $(H(X), h_d)$. □

Below we give an illustrative example of two discontinuous Bryant contraction mappings satisfying a sufficient condition of the above theorem.

Example 3.10. *(A simple example)*

Let $X := [0, 1]$ and d be the Euclidean metric. Then, (X, d) is a complete metric space.

Let

$$f_1(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

and

$$f_2(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{3} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Because f_1 and f_2 are not continuous, they are not Banach contraction mappings.

However, because $f_1^2(x) = 0$, and $f_2^2(x) = 0$ for all $x \in [0, 1]$, f_1^2 and f_2^2 are Banach contraction mappings in the interval $[0, 1]$.

Thus, f_1 and f_2 are Bryant contraction mappings that are not Banach contraction mappings.

On the other hand, because

$$(f_1 \circ f_2)(x) = (f_2 \circ f_1)(x) = 0$$

for all $x \in [0, 1]$, f_1 and f_2 are discontinuous Bryant contraction mappings which satisfy the assumptions of the above theorem.

Because obviously, $x_1 = 0$ *is a unique fixed point of* $f_1 : [0,1] \to [0,1]$, *and* $x_2 = 0$ *is a unique fixed point of* $f_2 : [0,1] \to [0,1]$ *and because*

$$f_1(\{0\}) \cup f_2(\{0\}) = \{0\},$$

we can see that a set $\{0\}$ *is an attracting invariant set of the given IFS.*

Remark 3.11. *The following problems arise naturally to be solved in the future.*

Let the IFS consisting of $N(> 2)$ *discontinuous Bryant contraction mappings satisfy the sufficient condition of the above theorem.*

(1) Does the result of the above theorem hold?

(2) If the result of the above theorem does not hold, what is the sufficient condition for the result of the above theorem to hold?

4. FICs generated by two new Rakotch contraction mapping families

In this section, we present two new Rakotch contraction mapping families that do not satisfy the Banach contraction condition and give one-variable FIFs (that is, FICs) generated by two new Rakotch contraction mapping families.

4.1. Two new Rakotch contraction mapping families

In this subsection, in order to obtain novelty fractal interpolation curves with wide, essential, and deep meaning, we give two new Rakotch contraction mapping families that do not satisfy the Banach contraction condition.

4.1.1. An irrational-type Rakotch contraction mapping family

Let for all $t \geq 0$, $\alpha(t) := t\Big/\sqrt{1+t^2}$. Then, for all $n \in \mathbb{N}$ and for all $t \geq 0$, $\alpha^n(t) = t\Big/\sqrt{1+nt^2}$. So, for all $t \geq 0$, $\lim_{n\to+\infty} \alpha^n(t) = 0$. Also, if $t > 0$, then $\alpha(t) = t\Big/\sqrt{1+t^2} < t$ because if $t > 0$, then $\sqrt{1+t^2} > 1$. Moreover, α is an increasing function because for all $t \geq 0$, $d\alpha(t)/dt = 1\Big/\left[(1+t^2)\sqrt{1+t^2}\right] > 0$. Hence, for all $t > 0$, $\alpha(t) < t$, and for all $t > 0$, $\lim_{n\to+\infty} \alpha^n(x) = 0$, and a function $\alpha(t)$ is an increasing function, and a function $\alpha(t)/t = 1\Big/\sqrt{1+t^2}$ is a decreasing function.

Let for all $x > 0$, $f(x) := x\Big/\sqrt{1+x^2}$. Then, for all $x, y > 0$,

$$\begin{aligned}
|f(x) - f(y)| = |\frac{x}{\sqrt{1+x^2}} - \frac{y}{\sqrt{1+y^2}}| &= \frac{|\sqrt{x^2+x^2y^2} - \sqrt{y^2+x^2y^2}|}{\sqrt{1+x^2}\sqrt{1+y^2}} \\
&= \frac{|x^2-y^2|}{\sqrt{1+x^2}\sqrt{1+y^2}(x\sqrt{1+y^2}+y\sqrt{1+x^2})} \\
&\leq \frac{|x^2-y^2|}{\sqrt{1+x^2}\sqrt{1+y^2}(x+y)} \leq \frac{|x-y|}{\sqrt{1+x^2}\sqrt{1+y^2}} \\
&\leq \frac{|x-y|}{\sqrt{1+|x-y|^2}} = \alpha(|x-y|).
\end{aligned}$$

So, for all $x > 0$, $f(x)$ is a new Rakotch contraction mapping.

Then, obviously, $f(x)$ does not satisfy the Banach contraction condition. In fact, a mapping $f(x) = x\Big/\sqrt{1+x^2}$ $(x \geq 0)$ has a unique fixed point $x_0 = 0$.

Fix a real number $c \in (0, 1)$. Then, for all $x \in (0, \sqrt{\left(1\big/c^2\right) - 1})$,

$$d(f(0), f(x)) = \frac{x}{\sqrt{1+x^2}} > \frac{x}{\sqrt{1 + (\sqrt{\frac{1}{c^2} - 1})^2}} = cx = cd(0, x).$$

So, a mapping $f(x) = x\Big/\sqrt{1+x^2}$ $(x \geq 0)$ does not satisfy the Banach contraction condition.

Now, we give another Rakotch contraction mapping.

Let for all $t \geq 0, \beta(t) := t\Big/\sqrt[3]{1+t^3}$. Then, for all $n \in \mathbb{N}$ and for all $t \geq 0, \beta^n(t) = t\Big/\sqrt[3]{1+nt^3}$. So, for all $t \geq 0$, $\lim_{n\to+\infty} \beta^n(t) = 0$. Also, if $t > 0$, then $\beta(t) = t\Big/\sqrt[3]{1+t^3} < t$ because if $t > 0$, then $\sqrt[3]{1+t^3} > 1$.

Moreover, β is an increasing function because for all $t \geq 0$, $d\beta(t)/dt = 1\Big/\left(1+t^3\right)^{\frac{4}{3}} > 0$. Hence, for all $t > 0, \beta(t) < t$, and for all $t > 0$, $\lim_{n\to+\infty} \beta^n(x) = 0$, and a function $\beta(t)$ is an increasing function, and a function $\beta(t)/t = 1\Big/\sqrt[3]{1+t^3}$ is a decreasing function.

Let for all $x > 0$, $g(x) := x\Big/\sqrt[3]{1+x^3}$. Then, obviously, for all $x \in \mathbb{R}$,

$$f(x) = \frac{x}{\sqrt[2]{1+x^2}} \leq \frac{x}{\sqrt[3]{1+x^3}} = g(x).$$

Moreover, for all $x, y > 0$,

$$\begin{aligned}
|g(x) - g(y)| &= |\frac{x}{\sqrt[3]{1+x^3}} - \frac{y}{\sqrt[3]{1+y^3}}| = \frac{|\sqrt[3]{x^3 + x^3y^3} - \sqrt[3]{y^3 + x^3y^3}|}{\sqrt[3]{1+x^3}\sqrt[3]{1+y^3}} \\
&= \frac{|x^3 - y^3|}{\sqrt[3]{1+x^3}\sqrt[3]{1+y^3}\{(x^3 + x^3y^3)^{\frac{2}{3}} + [(x^3 + x^3y^3)(y^3 + x^3y^3)]^{\frac{1}{3}} + (y^3 + x^3y^3)^{\frac{2}{3}}\}} \\
&= \frac{|x - y|(x^2 + xy + y^2)}{\sqrt[3]{1+x^3}\sqrt[3]{1+y^3}\{x^2(1+y^3)^{\frac{2}{3}} + xy[(1+y^3)(1+x^3)]^{\frac{1}{3}} + y^2(1+x^3)^{\frac{2}{3}}\}} \\
&\leq \frac{|x - y|}{\sqrt[3]{1+x^3}\sqrt[3]{1+y^3}} \leq \frac{|x - y|}{\sqrt[3]{1+|x-y|^3}} = \beta(|x - y|).
\end{aligned}$$

So, for all $x > 0$, $g(x)$ is a new Rakotch contraction mapping that does not satisfy the Banach contraction condition.

By using the inductive method, we can see that for every $n \in \mathbb{N}$, a mapping $x\Big/\sqrt[n]{1+x^n}$ is a Rakotch contraction mapping that does not satisfy the Banach contraction condition, and

$$\frac{x}{1+x} \leq \frac{x}{\sqrt[2]{1+x^2}} \leq \frac{x}{\sqrt[3]{(1+x^3)}} \leq \frac{x}{\sqrt[4]{(1+x^4)}} \leq \cdots \leq \frac{x}{\sqrt[n]{(1+x^n)}} \leq \cdots.$$

In the case when $n = 1$, our function is equal to the function in [17] (or [18]).

In this manner, finally, we present a new Rakotch contraction mapping family that does not satisfy the Banach contraction condition such that

$$\left\{f_n(x) := \frac{x}{\sqrt[n]{1+x^n}} | n \in \mathbb{N}\right\}.$$

4.1.2. A rational-type Rakotch contraction mapping family

We give another new Rakotch family,

$$\left\{h_n(x) := \frac{nx}{nx+n+1} | n \in \mathbb{N}\right\},$$

which does not satisfy the Banach contraction condition, where $x \in [0, +\infty)$.

Obviously, for all $n \in \mathbb{N}$, every $h_n(x)$ is not the Banach contraction mapping.

However, for all $n \in \mathbb{N}$, every $h_n(x)$ is a Rakotch contraction mapping because for all $x \in [0, +\infty)$,

$$\frac{dh_n(x)}{dx} = \frac{n^2+n}{(nx+n+1)^2} > 0,$$

$h_n(x)$ is an increasing mapping on $x \in [0, +\infty)$, and for all $x, y \in [0, +\infty)$, if $x > y$, then

$$\begin{aligned}
|h_n(x) - h_n(y)| &= |\frac{nx}{nx+n+1} - \frac{ny}{ny+n+1}| \\
&= \frac{n(n+1)(x-y)}{(nx+n+1)(ny+n+1)} \\
&\leq \frac{n(n+1)(x-y)}{(nx+n)[(n+1)(\frac{n}{n+1}y+1)]} \\
&\leq \frac{x-y}{(x+1)(\frac{n}{n+1}y+1)} \\
&\leq \frac{x-y}{1+x-y} = \alpha(x-y),
\end{aligned}$$

and if $y > x$, then

$$|h_n(x) - h_n(y)| \leq \frac{y-x}{1+y-x} = \alpha(y-x),$$

where for all $t \geq 0$, $\alpha(t) := t/(1+t)$, and so, for all $x, y \in [0, +\infty)$,

$$|h_n(x) - h_n(y)| \leq \frac{|x-y|}{1+|x-y|} = \alpha(|x-y|).$$

Also, obviously, for every fixed $n \in \mathbb{N}$, a function $h_n(x)/x$ is decreasing.

Hence, for all $n \in \mathbb{N}$, every $h_n(x)$ is a Rakotch contraction mapping which does not satisfy the Banach contraction condition.

Also, obviously, for all $n \in \mathbb{N}$ and for all $x \in [0, +\infty)$,

$$h_n(x) \leq h_{n+1}(x),$$

and for all $x \in [0, +\infty)$,

$$\lim_{n \to +\infty} h_n(x) = \frac{x}{1+x}.$$

Remark 4.1. *In [18], Ri exhibited that by using Rakotch contraction mappings, one can always generate FIFs.*

However, in [18], the author only proposed Rakotch contraction mappings $f(x) = 1/(1+x)$ and $g(x) = x/(1+x)$ that do not satisfy the Banach contraction condition. Also, in [32], the authors only *presented the Rakotch contraction mapping* $h(x) = \sin x$*, which does not satisfy the Banach contraction condition.*

Because the irrational-type Rakotch contraction mapping family and the rational type Rakotch contraction mapping family presented above differ qualitatively or quantitatively from previously known examples of [18, 32], these two new Rakotch contraction mapping families can be used to generate new FIFs.

In the next subsection, we give new FIFs in comparison with previous examples of [2, 18, 32].

4.2. FICs generated by two new Rakotch contraction mapping families

By using two new Rakotch contraction mappings, we give the new one-variable FIFs (that is, FICs).

Example 4.2. *Let* $x_0 = 0$, $0 < x_1 < 1$, $x_2 = 1$.

Let (x_0, y_0), (x_1, y_1), *and* (x_2, y_2) *be a data set, where* $y_0, y_1, y_2 \in \mathbb{R}$.

Let

$$\alpha_1(y) := \alpha y,$$

where $0 \le \alpha < 1$. *Then,* $\alpha_1(y)$ *is a Banach contraction (see [2]).*

Let $\alpha_2(y) := y\Big/\sqrt{1 + y^2}$.

Then, $\alpha_2(y)$ *is a new Rakotch contraction which does not satisfy the Banach contraction condition (see Subsection 4.1).*

Let for all $x \in [0, 1]$, $y \in \mathbb{R}$ *and for* $i \in \{1, 2\}$,

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_i x + v_i \\ F_i(x, y) \end{pmatrix} := \begin{pmatrix} u_i x + v_i \\ \alpha_i(y) + g_i(x) \end{pmatrix},$$

where we assume that

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix},$$

$$w_i \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix},$$

and $g_i(x)$ *is a Lipschitz continuous (or Hölder continuous) function.*

Let for all $t \ge 0$,

$$\beta(t) := \max\left\{\alpha t, \frac{t}{\sqrt{1 + t^2}}\right\}.$$

Then, for all $y', y'' \in \mathbb{R}$ *and for* $i \in \{1, 2\}$,

$$|\alpha_i(y') - \alpha_i(y'')| \le \beta(|y' - y''|).$$

So, by Theorem 2.2, there exists one-variable FIF which passes through the data points (x_0, y_0), (x_1, y_1), *and* (x_2, y_2).

Below, we give the figures of a Barnsley affine FIF [2], a nonlinear FIF of [18], and a new nonlinear FIF with respect to a Banach contraction, a new irrational and a rational Rakotch contraction and two Lipschitz continuous functions. Then, we present the figures of a Barnsley affine FIF [2], a nonlinear

FIF of [32], and a new nonlinear FIF with respect to a Banach contraction, a new irrational and a rational Rakotch contraction, and two Hölder continuous functions that are not Lipschitz continuous.

Let $(x_0, y_0) := (0, 0.25)$, $(x_1, y_1) := (0.5, 1)$ and $(x_2, y_2) := (1, 0.5)$.

Let

$$\alpha_1(y) := 0.5y.$$

(1) A Barnsley affine FIF [2]

Let $\alpha_2(y) := 0.75y$, and for $i = 1, 2$, $g_i(x) := c_i x + d_i$.

Figure 1 shows a figure of a Barnsley affine FIC generated by two Banach contraction mappings and by two Lipschitz continuous functions.

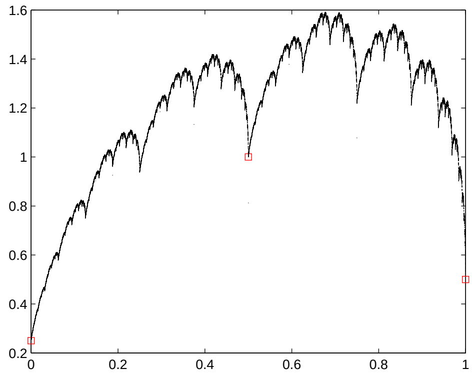

Figure 1. A Barnsley affine FIC.

(2) A nonlinear FIF of [18]

Let $\alpha_2(y) := y/(1 + y)$, and for $i = 1, 2$, $g_i(x) := c_i x + d_i$.

Figure 2 shows a figure of a nonlinear FIC generated by one Banach contraction mapping, by one Rakotch contraction mapping of [18] which does not satisfy the Banach contraction condition and by two Lipschitz continuous functions.

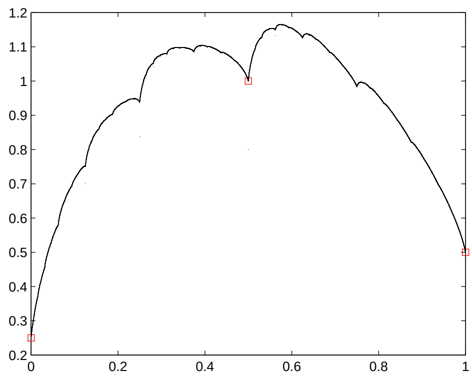

Figure 2. A nonlinear FIC of [18].

(3) A new nonlinear FIF generated by a Banach contraction mapping, an irrational-type Rakotch contraction mapping, and Lipschitz continuous functions

Let $\alpha_2(y) := y \Big/ \sqrt{1+y^2}$, and for $i = 1, 2$, $g_i(x) := c_i x + d_i$.

Figure 3 shows a figure of a new nonlinear FIC generated by one Banach contraction mapping, a new irrational-type Rakotch contraction mapping that is not a Rakotch contraction mapping of [18], and by two Lipschitz continuous functions.

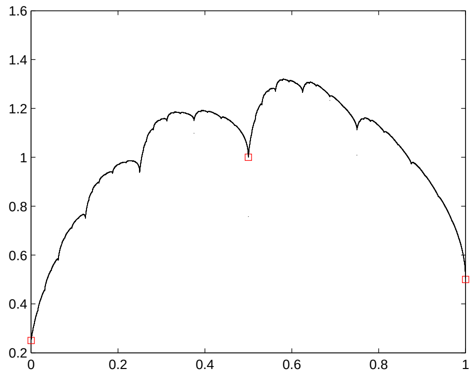

Figure 3. A new nonlinear FIC generated by an irrational-type Rakotch contraction mapping and the Lipschitz continuous functions.

(4) A new nonlinear FIF generated by a rational-type Rakotch contraction mapping and Lipschitz continuous functions

Let $\alpha_2(y) := 2y/(2y + 3)$, and for $i = 1, 2$, $g_i(x) := c_i x + d_i$.

Figure 4 shows a figure of a new nonlinear FIC generated by one Banach contraction mapping, a new rational-type Rakotch contraction mapping that is not a Rakotch contraction mapping, and two Lipschitz continuous functions.

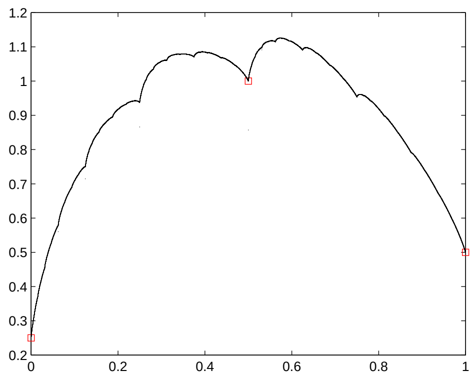

Figure 4. A new nonlinear FIC generated by a rational-type Rakotch contraction mapping and Lipschitz continuous functions.

(5) A Barnsley affine FIF of [2] with respect to Hölder continuous functions

Let $\alpha_2(y) := 0.75y$, and for $i = 1, 2$, $g_i(x) := c_i \sqrt{x} + d_i$.

Figure 5 shows a figure of a Barnsley affine FIC generated by two Banach contraction mappings and by two Hölder continuous functions that are not Lipschitz continuous.

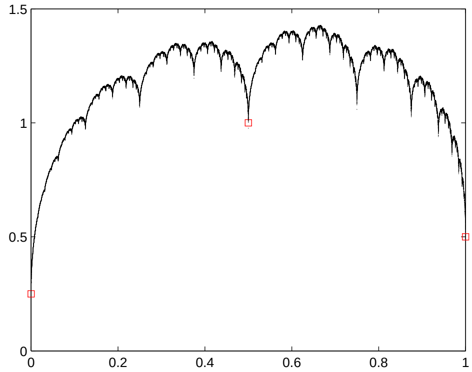

Figure 5. A Barnsley affine FIC with respect to Hölder continuous functions.

(6) A nonlinear FIF of [32] with respect to Hölder continuous functions

Let $\alpha_2(y) := y/(1 + y)$, and for $i = 1, 2$, $g_i(x) := c_i \sqrt{x} + d_i$.

Figure 6 shows a figure of a nonlinear FIC generated by one Banach contraction mapping, one Rakotch contraction mapping of [32] (or [18]) which does not satisfy the Banach contraction condition, and two Hölder continuous functions that are not Lipschitz continuous.

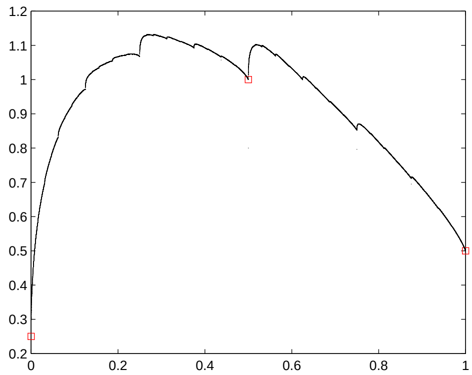

Figure 6. A nonlinear FIC with respect to Hölder continuous functions.

(7) A new nonlinear FIF generated by an irrational-type Rakotch contraction mapping and Hölder continuous functions

Let $\alpha_2(y) := y\Big/\sqrt{1 + y^2}$, and for $i = 1, 2$, $g_i(x) := c_i \sqrt{x} + d_i$.

Figure 7 shows a figure of a new nonlinear FIC generated by a Banach contraction mapping, a new irrational Rakotch contraction mapping that is not a Rakotch contraction mapping of [18] (or [32]), and two Hölder continuous functions that are not Lipschitz continuous.

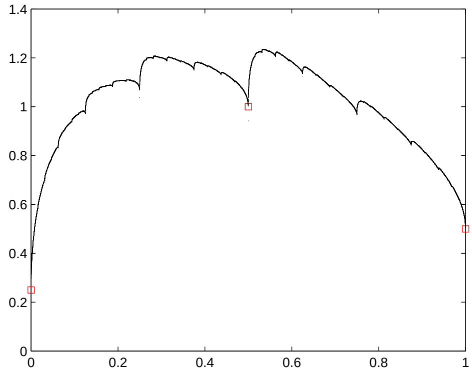

Figure 7. A new nonlinear FIC generated by an irrational-type Rakotch contraction mapping and Hölder continuous functions.

(8) A new nonlinear FIF generated by a rational-type Rakotch contraction mapping and Hölder continuous functions that are not Lipschitz continuous

Let $\alpha_2(y) := 2y/(2y + 3)$, and for $i = 1, 2$, $g_i(x) := c_i \sqrt{x} + d_i$.

Figure 8 shows a figure of a new nonlinear FIC generated by a Banach contraction mapping, a new rational-type Rakotch contraction mapping that is not a Rakotch contraction mapping of [18] (or [32]), and two Hölder continuous functions that are not Lipschitz continuous.

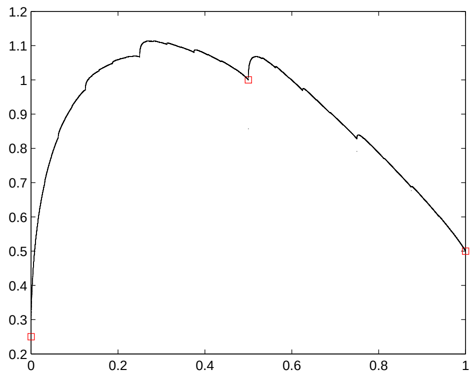

Figure 8. A new nonlinear FIC generated by a rational-type Rakotch contraction mapping and Hölder continuous functions.

5. Conclusions

One of the most important problems to be solved in fractal interpolation theory is to find a new contractive mapping family that can generate fractals (in particular, FIFs).

We presented Counterexample 3.6 and Counterexample 3.7 showing that IFSes consisting of two discontinuous (or continuous) Bryant contraction mappings, which are the simplest and most obvious generalization of the Banach contraction mappings, have no attractors. Although these IFSes have invariant sets, by using these IFSes, one cannot generate fractals because these invariant sets are not attractors (see Counterexample 3.6 and Counterexample 3.7), and so, in particular, one cannot generate FIFs by using Bryant contraction mappings in general.

Our Counterexample 3.6 and Counterexample 3.7 show that there exist only a few generalized Banach contraction mappings that can generate fractals, such as F-contraction mappings, Rakotch contraction mappings, Matkowski contraction mappings, and Meir-Keeler-type mappings.

That is, it is of particular significance in this paper that we show through two counterexamples using discontinuous (or continuous) Bryant contraction mappings, the simplest and most obvious generalization of Banach contraction mappings, that there exist generalized Banach contraction mappings which cannot generate fractals (in particular, FIFs).

Also, the new families of Rakotch contraction mappings proposed in this paper will make some contribution to the development of fractal interpolation theory.

Author contributions

Song-Il Ri: Project administration, methodology, writing; Gwang-Jin O: Investigation, editing, review; Gyong-Jin Jo: Formal analysis, investigation; Chol-U Pak: Methodology, review; In-Son Ri: Formal analysis, investigation, review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence(AI) tools in the creation of this article.

Acknowledgments

The authors greatly thank the anonymous referees and editor for many valuable suggestions and comments which helped to improve the original paper. The authors would like to thank Guest Editor Professor Vasileios Drakopoulos for his great help during the whole period of writing and publishing this paper.

Conflict of interest

Prof. Vasileios Drakopoulos is the Guest Editor of special issue “Advances in Fractal Geometry: Theory and Applications” for AIMS Mathematics. Vasileios Drakopoulos was not involved in the editorial review and the decision to publish this article.

The authors declare no conflicts of interest.

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