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Research article

Continuous orbit equivalence of groupoid actions

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Abstract: In this paper, I studied continuous actions of étale groupoids on compact Hausdorff spaces and the associated operator algebras. I introduced notions of conjugacy and continuous orbit equivalence for such groupoid actions and characterized them in terms of the corresponding transformation groupoids and their reduced C^*-algebras. In particular, I proved that two topologically free actions are continuously orbit equivalent if and only if their associated transformation groupoid C^*-algebras are isomorphic, and if and only if there exists a C^*-algebra isomorphism preserving the canonical Cartan subalgebras between the corresponding reduced C^*-algebras of these transformation groupoids.

Keywords: groupoid action; transformation groupoid; continuous orbit equivalence; conjugacy; groupoid C^*-algebra
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1. Introduction

The interplay between topological dynamics and operator algebras forms a profound and enduring branch of modern mathematics. A central bridging framework between these two fields is established on the basis of crossed product constructions and groupoid structures. Tracing back to their conceptual origins in the mid-twentieth century, the methodological strategy of encoding dynamical information into algebraic invariants has propelled substantial advancements in both disciplines. By constructing operator algebras from dynamical systems, researchers obtain a "noncommutative shadow"of the underlying dynamics; within this algebraic framework, fundamental dynamical notions such as equivalence, conjugacy, and entropy admit rigorous algebraic counterparts. This paradigmatic approach has furnished an indispensable theoretical scaffold for advancing the classification theories of dynamical systems and operator algebras [1, 2].

A central driving force in this field lies in the quest to elucidate how dynamical equivalence translates into algebraic isomorphism. A landmark achievement in this direction was the groundbreaking work of Giordano, Putnam, and Skau (GPS) on Cantor minimal systems [3]. They introduced pivotal notions, conjugacy, orbit equivalence, and strong orbit equivalence, and furnished a complete characterization of orbit equivalence by establishing a correspondence between this dynamical property and the K-theory as well as strong Morita equivalence of the associated transformation group C^*-algebras.

The GPS paradigm has since been profoundly generalized. A major extension was developed by Li, who investigated actions of countable discrete groups on locally compact spaces [4]. He demonstrated that continuous orbit equivalence for such systems is characterized by isomorphisms of the crossed product C^*-algebras that preserve the canonical Cartan subalgebra $C_0(X)$. This theoretical framework was further extended to the setting of partial group actions [5]. Subsequently, these results were generalized to the context of automorphism groups acting on étale equivalence relations [6]. For more interesting progress and applications on continuous orbit equivalence, see [7–9] and the references therein.

A parallel and immensely fruitful research strand originates in symbolic dynamics with the Cuntz-Krieger algebra $\mathcal{O}_A$. Its intrinsic structure is deeply intertwined with the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$. Cuntz established that flow equivalence and conjugacy of these shifts give rise to invariants for $\mathcal{O}_A$, and this correspondence was later proven to be complete by Carlsen and collaborators, who linked these dynamical equivalences to diagonal-preserving $*$-isomorphisms of the associated Cuntz-Krieger algebras [10–13].

This theory has been decisively extended to one-sided shifts (X_A, σ_A), which are regarded as semigroup ($\mathbb{N}$) actions. Matsumoto's introduction of the continuous orbit equivalence for such systems [14], coupled with Matui's subsequent identification of the Deaconu-Renault groupoid $\mathcal{G}_A$ satisfying $C_r^*(\mathcal{G}_A) \cong \mathcal{O}_A$ [15], laid a seminal bridge connecting one-sided dynamics, groupoid theory, and C^*-algebras. Matsumoto's follow-up work further refined this dynamical-algebraic correspondence [16, 17]. The framework was successfully extended to directed graphs [18, 19] and higher-rank graphs [20], where path groupoids and their associated C^*-algebras play a unifying role. Moreover, the core thesis, that semigroup dynamics is classified by its associated C^*-algebra, has been validated for broader classes of semigroups, including partial group actions [21] and actions of discrete Ore semigroups [22, 23].

In summary, the orbit equivalence classification of topological dynamical systems under group and semigroup actions, alongside its interplay with C^*-algebra classification, forms a mature yet dynamically evolving research field. However, the dynamical systems discussed above are all defined by the action of a classical algebraic structure (i.e., a group or a semigroup). A considerably broader theoretical framework is provided by groupoid actions: An étale groupoid acting continuously on a topological space defines a groupoid dynamical system. Moreover, Flores and Măntoiu have systematically investigated the fundamental dynamical properties of such systems, including transitivity, minimality, and the structure of their limit sets and periodic sets [24, 25]. Despite advances in clarifying their elementary dynamical behaviors, the core classification problem remains unresolved: How does the continuous orbit equivalence of étale groupoid actions manifest itself in the structure of their corresponding C^*-algebras?

In this paper, I study continuous actions of locally compact Hausdorff étale groupoids on compact spaces, with a focus on the relationship between their orbit structures and the associated algebraic invariants. I first introduce the notion of continuous orbit equivalence for such actions.

For topologically free actions, I then prove that continuous orbit equivalence is equivalent to the isomorphism of the corresponding transformation groupoids $\mathcal{G} \ltimes X$. Furthermore, I establish that this dynamical equivalence is, in turn, equivalent to the existence of a C^*-isomorphism between the reduced groupoid C^*-algebras that preserves the canonical Cartan subalgebras $C(X)$. Moreover, I prove that for a (discrete) groupoid action $\mathcal{G} \curvearrowright X$, reduced groupoid C^*-algebra $C^*_r(\mathcal{G} \ltimes X)$ is canonically isomorphic to the reduced crossed product $C(X) \rtimes^{\alpha}_r \mathcal{G}$.

The paper is organized as follows: In Section 2, I compile the necessary preliminary concepts regarding étale groupoids, their actions, and the corresponding transformation groupoids. Section 3 is devoted to C^*-algebraic constructions. I define the relevant algebras and prove the canonical isomorphism between the transformation groupoid algebra and the crossed product. In Section 4, I introduce the notion of continuous orbit equivalence, establish its equivalence to the isomorphism of transformation groupoids, and culminate in the proof of my main theorem. Finally, in Section 5, I present concluding remarks and propose directions for future research.

Throughout this paper, the following notions are used: For a groupoid $\mathcal{G}$, let $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(2)}$ be its unit space and the set of composable pairs, respectively. The range and source maps r, s from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ are defined by $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, respectively. A topological groupoid is called to be étale if its range and source maps are local homeomorphisms from $\mathcal{G}$ onto $\mathcal{G}^{(0)}$. In this case, $\mathcal{G}^{(0)}$ is necessarily a closed and open subset of $\mathcal{G}$. Given an étale groupoid $\mathcal{G}$, for $u \in \mathcal{G}^{(0)}$, write $\mathcal{G}^u = r^{-1}(u)$, $\mathcal{G}_u = s^{-1}(u)$, and $\mathcal{G}^u_u = \mathcal{G}^u \cap \mathcal{G}_u$. The isotropy bundle is defined as $\mathcal{G}' = \cup_{u \in \mathcal{G}^{(0)}} \mathcal{G}^u_u$. Clearly, $\mathcal{G}'$ contains $\mathcal{G}^{(0)}$ and is closed in $\mathcal{G}$. $\mathcal{G}$ is said to be *topologically principal* if the set $\{u \in \mathcal{G}^{(0)} : \mathcal{G}^u_u = \{u\}\}$ is dense in $\mathcal{G}^{(0)}$. The reader is referred to [26, 27] for more details regarding topological groupoids and their C^*-algebras.

2. The action groupoid of groupoid action

Let X be a set and $\mathcal{G}$ a groupoid. A left action of $\mathcal{G}$ on X is defined by a surjection $\rho_X : X \to \mathcal{G}^{(0)}$, called the moment map, and an action map defined on the set $\mathcal{G} * X := \{(\gamma, x) \in \mathcal{G} \times X : s(\gamma) = \rho_X(x)\}$ given by $(\gamma, x) \mapsto \gamma \cdot x \in X$, satisfying the following conditions:

(1) $\rho_X(x) \cdot x = x$ for all $x \in X$ (where $\rho_X(x) \in \mathcal{G}^{(0)}$ is viewed as a unit);
(2) If $(\gamma, \eta) \in \mathcal{G}^{(2)}$ and $(\eta, x) \in \mathcal{G} * X$, then $(\gamma\eta, x)$ and $(\gamma, \eta \cdot x) \in \mathcal{G} * X$, and $\gamma \cdot (\eta \cdot x) = \gamma\eta \cdot x$.

Denote such a groupoid action by $\mathcal{G} \curvearrowright X$. When it is not necessary to emphasize set X, simply write ρ for ρ_X. In this case, X is called a left $\mathcal{G}$-space.

Remark 2.1. *Given a groupoid action $\mathcal{G} \curvearrowright X$, for any $(\gamma, x) \in \mathcal{G} * X$,*

$$\rho(\gamma \cdot x) = r(\gamma), \quad \gamma^{-1} \cdot (\gamma \cdot x) = x.$$

Indeed, since $(r(\gamma), \gamma) \in \mathcal{G}^{(2)}$, condition (2) implies that $(r(\gamma), \gamma \cdot x) \in \mathcal{G} * X$. Therefore,

$$\rho(\gamma \cdot x) = s(r(\gamma)) = r(\gamma),$$

and

$$\gamma^{-1} \cdot (\gamma \cdot x) = (\gamma^{-1}\gamma) \cdot x = s(\gamma) \cdot x = \rho(x) \cdot x = x.$$

Definition 2.1. *Suppose $\mathcal{G}$ is a topological groupoid and X is a topological space equipped with a left $\mathcal{G}$-action. $\mathcal{G}$ is said to act continuously on X, or X is called a continuous left $\mathcal{G}$-space, if the moment* *map* $\rho_X : X \to \mathcal{G}^{(0)}$ *and the action map* $(\gamma, x) \in \mathcal{G} * X \mapsto \gamma \cdot x \in X$ *are both continuous. Here,* $\mathcal{G} * X$ *is endowed with the subspace topology from* $\mathcal{G} \times X$.

Remark that for an étale groupoid $\mathcal{G}$, a continuous left action has natural local topological properties. Since the source map of an étale groupoid is a local homeomorphism, the continuous action inherits local section features, and its local restrictions are local homeomorphisms. The following are some classical examples of continuous groupoid actions.

Example 2.1. *Let* $\mathcal{G}$ *be a locally compact Hausdorff groupoid.*

(1) *The multiplication operation on* $\mathcal{G}$ *defines a continuous action of* $\mathcal{G}$ *on itself.*

(2) *Let* ρ *be the identity map on* $\mathcal{G}^{(0)}$*, which may also be viewed as the restriction of range map* r *to* $\mathcal{G}^{(0)}$*. Define*
$$\gamma \cdot u := r(\gamma) = \gamma u \gamma^{-1}, \quad (\gamma, u) \in \mathcal{G} * \mathcal{G}^{(0)}.$$
Then $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ *is a continuous action. In fact, the left action of* $\mathcal{G}$ *on* $\mathcal{G}^{(0)}$ *can be expressed as* $\gamma \cdot s(\gamma) = r(\gamma)$.

(3) *Let* $\mathrm{Iso}(\mathcal{G}) := \{\gamma \in \mathcal{G} : \ s(\gamma) = r(\gamma)\}$ *be the isotropy subgroupoid of* $\mathcal{G}$*. Define* $\rho : \mathrm{Iso}(\mathcal{G}) \to \mathcal{G}^{(0)}$ *as the restriction of the range map* r *to* $\mathrm{Iso}(\mathcal{G})$*. For* $(\gamma, \eta) \in \mathcal{G} * \mathrm{Iso}(\mathcal{G})$*, the operation*
$$\gamma \cdot \eta := \gamma \eta \gamma^{-1}$$
defines a continuous left action of $\mathcal{G}$ *on* $\mathrm{Iso}(\mathcal{G})$.

Example 2.2. *Let* $G \curvearrowright X$ *and* $H \curvearrowright Y$ *be two continuous actions of discrete groups on compact spaces* X *and* Y*, respectively. Suppose these actions are conjugate, i.e., there exist a homeomorphism* $\phi : X \to Y$ *and a group isomorphism* $\pi : G \to H$ *such that* $\phi(g \cdot x) = \pi(g) \cdot \phi(x)$ *for all* $g \in G$ *and* $x \in X$*. For notational simplicity, the action of* $g \in G$ *on* $x \in X$ *is denoted simply by* gx*. Let* $\mathcal{H} = H \rtimes Y$ *be the transformation groupoid associated with the action* $H \curvearrowright Y$*. Define* $\rho_X : X \to \mathcal{H}^{(0)} = Y$ *by* $\rho_X(x) = \phi(x)$*, and the action on the space* $\mathcal{H} * X = \{((h, y), x) \in H \rtimes Y \times X \ : \ s(h, y) = y = \phi(x)\}$ *by* $(h, y) \cdot x = \pi^{-1}(h)x$*. Then* $\mathcal{H} \curvearrowright X$ *is a continuous groupoid action.*

Suppose that $\mathcal{G}$ is a locally compact Hausdorff groupoid and X is a compact Hausdorff space which is also a continuous left $\mathcal{G}$-space with moment map $\rho : X \to \mathcal{G}^{(0)}$. Analogous to the transformation groupoid associated with a group action, the action groupoid (or transformation groupoid) $\mathcal{G} \ltimes X$ is defined and equiped with the subspace topology inherited from the product topology on $\mathcal{G} \times X$. Its set of arrows is defined as
$$\mathcal{G} \ltimes X = \{(\gamma, x) \in \mathcal{G} \times X : \ r(\gamma) = \rho(x)\}.$$
Two elements $(\gamma, x), (\eta, y) \in \mathcal{G} \ltimes X$ are composable if and only if $y = \gamma^{-1} \cdot x$. Their product is then defined by $(\gamma, x)(\eta, y) = (\gamma\eta, x)$. The inverse is given by $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma^{-1} \cdot x)$.

Remark 2.2. *For* $(\gamma, x) \in \mathcal{G} \ltimes X$*, the source and range maps are given by*
$$s(\gamma, x) = (\gamma^{-1}, \gamma^{-1} \cdot x)(\gamma, x) = (s(\gamma), \gamma^{-1} \cdot x) = (\rho(\gamma^{-1} \cdot x), \gamma^{-1} \cdot x),$$
$$r(\gamma, x) = (\gamma, x)(\gamma^{-1}, \gamma^{-1} \cdot x) = (r(\gamma), x) = (\rho(x), x).$$
Therefore, the unit space is $(\mathcal{G} \ltimes X)^{(0)} = r(\mathcal{G} \ltimes X) = \{(\rho(x), x) : \ x \in X\}$*. If the unit space* $(\mathcal{G} \ltimes X)^{(0)}$ *is identified with* X *via the natural bijection* $(\rho(x), x) \mapsto x$*, then* $s(\gamma, x) = \gamma^{-1} \cdot x$*,* $r(\gamma, x) = x$*. Under this canonical identification, the topology induced on* X *from the unit space* $(\mathcal{G} \ltimes X)^{(0)}$ *is exactly the original topology of* X*.*

In fact, the set $\mathcal{G} * X = \{(\gamma, x) \in \mathcal{G} \times X : s(\gamma) = \rho(x)\}$ can also be endowed with a groupoid structure. Two elements $(\gamma, z), (\eta, x) \in \mathcal{G} * X$ are composable if and only if $z = \eta \cdot x$. Their product is then defined by $(\gamma, \eta \cdot x)(\eta, x) = (\gamma\eta, x)$. The inverse is defined by $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma \cdot x)$. If the unit $(\rho(x), x)$ of this groupoid is identified with the point $x \in X$, then the source and range maps take the following simple form, $s(\gamma, x) = (s(\gamma), x) = (\rho(x), x) \equiv x$, $r(\gamma, x) = (r(\gamma), \gamma \cdot x) = (\rho(\gamma \cdot x), \gamma \cdot x) \equiv \gamma \cdot x$. Moreover, the map $\partial : \mathcal{G} \ltimes X \to \mathcal{G} * X$, $(\gamma, x) \mapsto (\gamma, \gamma^{-1} \cdot x)$ is an isomorphism of topological groupoids.

Lemma 2.1. *The set* $\widetilde{\mathcal{G}}_x := \{\gamma \in \mathcal{G}_{\rho(x)} : \gamma \cdot x = x\}$ *is a closed subgroup of the isotropy group* $\mathcal{G}_{\rho(x)}^{\rho(x)} = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = \rho(x)\}$. *Moreover,* $\widetilde{\mathcal{G}}_x$ *is anti-isomorphic to the isotropy group at the unit x of the action groupoid* $\mathcal{G} \ltimes X$.

Proof. For any $\gamma \in \widetilde{\mathcal{G}}_x$, $s(\gamma) = \rho(x)$ by definition. Since $\gamma \cdot x = x$, it follows from the properties of the action that $\rho(\gamma \cdot x) = r(\gamma)$. Hence, $r(\gamma) = \rho(x)$. This shows that $\widetilde{\mathcal{G}}_x$ is contained in the isotropy group at $\rho(x)$. The closedness of $\widetilde{\mathcal{G}}_x$ follows from the continuity of the action map and the fact that $\{x\}$ is closed in X.

To show that $\widetilde{\mathcal{G}}_x$ is a subgroup, let $\gamma, \eta \in \widetilde{\mathcal{G}}_x$. Then $s(\gamma) = \rho(x) = r(\eta)$, so the product $\gamma\eta$ is defined. Moreover, $(\gamma\eta) \cdot x = \gamma \cdot (\eta \cdot x) = \gamma \cdot x = x$, which implies $\gamma\eta \in \widetilde{\mathcal{G}}_x$. If $\gamma \in \widetilde{\mathcal{G}}_x$, then $\gamma \cdot x = x$. Applying γ^{-1} to both sides yields $\gamma^{-1} \cdot x = \gamma^{-1} \cdot (\gamma \cdot x) = x$, so $\gamma^{-1} \in \widetilde{\mathcal{G}}_x$. The isotropy group of the action groupoid $\mathcal{G} \ltimes X$ at the unit x is given by $(\mathcal{G} \ltimes X)_x^x = \{(\gamma, x) \in \mathcal{G} \ltimes X : \gamma^{-1} \cdot x = x\}$. Consider the mapping $\phi : \widetilde{\mathcal{G}}_x \to (\mathcal{G} \ltimes X)_x^x$ defined by $\phi(\gamma) = (\gamma^{-1}, x)$. One can check that ϕ is a bijection. Furthermore, for $\gamma, \eta \in \widetilde{\mathcal{G}}_x$, one has

$$\phi(\gamma\eta) = ((\gamma\eta)^{-1}, x) = (\eta^{-1}\gamma^{-1}, x) = (\eta^{-1}, x)(\gamma^{-1}, x) = \phi(\eta)\phi(\gamma).$$

This shows that ϕ is a group anti-isomorphism. □

Proposition 2.1. *Suppose a locally compact Hausdorff groupoid $\mathcal{G}$ acts continuously on a compact Hausdorff topological space X. Then, under the relative topology inherited from the product topology on $\mathcal{G} \times X$, the action groupoid $\mathcal{G} \ltimes X$ is locally compact and Hausdorff. Furthermore, if $\mathcal{G}$ and X are second countable, then $\mathcal{G} \ltimes X$ is also second countable.*

Moreover, if $\mathcal{G}$ is an étale groupoid, then $\mathcal{G} \ltimes X$ is also étale.

Proof. By the continuity of the groupoid operations on $\mathcal{G}$ and the continuity of the action, $\mathcal{G} \ltimes X$ is a topological groupoid. To show that $\mathcal{G} \ltimes X$ is closed in $\mathcal{G} \times X$, consider the map $\Psi : \mathcal{G} \times X \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ defined by $\Psi(\gamma, x) = (r(\gamma), \rho(x))$. This map is continuous because r and ρ are continuous. Note that $\mathcal{G} \ltimes X = \Psi^{-1}(\Delta)$, where $\Delta = \{(u, u) : u \in \mathcal{G}^{(0)}\}$ is the diagonal in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is Hausdorff, Δ is closed. Therefore, $\mathcal{G} \ltimes X$ is closed. Thus, $\mathcal{G} \ltimes X$ is locally compact and Hausdorff. The second countability claim is straightforward.

Now, assume $\mathcal{G}$ is an étale groupoid. It is first proved that the range map $r : \mathcal{G} \ltimes X \to (\mathcal{G} \ltimes X)^{(0)}$ is open. Let $(\gamma, x) \in \mathcal{G} \ltimes X$, and let x_i be a net in X converging to x. Then $\rho(x_i) \to \rho(x) = r(\gamma)$. Since the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is open, by the net characterization of openness and Fell's Criterion, there exists a net γ_i in $\mathcal{G}$ such that $\gamma_i \to \gamma$ and $r(\gamma_i) = \rho(x_i)$ for all i. Consequently, $(\gamma_i, x_i) \in \mathcal{G} \ltimes X$ and $(\gamma_i, x_i) \to (\gamma, x)$, with $r(\gamma_i, x_i) = x_i$. This proves that r is open.

Next, it is shown that r is a local homeomorphism. Since $\mathcal{G}$ is étale, the map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism. Thus, for $(\gamma, x) \in \mathcal{G} \ltimes X$, there exists an open neighborhood $U \subseteq \mathcal{G}$ of γ such that $r|_U : U \to r(U)$ is a homeomorphism. Define $V = \rho^{-1}(r(U))$, which is open in X due to the continuity of ρ. Now, consider the set $W = (U \times V) \cap (\mathcal{G} \ltimes X)$, which is an open neighborhood of (γ, x) in $\mathcal{G} \ltimes X$. Claim: $r|_W : W \to V$ is a homeomorphism. To see this, for any $y \in V$, one has $\rho(y) \in r(U)$. Since $r|_U$ is a homeomorphism, there exists a unique $\eta \in U$ such that $r(\eta) = \rho(y)$. Then $(\eta, y) \in W$ and $r(\eta, y) = y$. Thus, $r|_W$ is surjective. Suppose $(\eta_1, y_1), (\eta_2, y_2) \in W$ with $r(\eta_1, y_1) = r(\eta_2, y_2)$, i.e., $y_1 = y_2 = y$. Then $r(\eta_1) = \rho(y) = r(\eta_2)$. Since η_1 and η_2 are in U and $r|_U$ is injective, it follows that $\eta_1 = \eta_2$. Hence, $(\eta_1, y) = (\eta_2, y)$. Therefore, r is a local homeomorphism, and $\mathcal{G} \ltimes X$ is étale. □

3. The C^*-algebra of groupoid action

Each groupoid action $\mathcal{G} \curvearrowright X$ induces a dual action, specifically, a groupoid dynamical system, of $\mathcal{G}$ on the C^*-algebra $C(X)$ of continuous functions. The researchers in [28] detail the construction of the universal crossed product C^*-algebra for groupoid dynamical systems, where the groupoid is equipped with a Haar system. In this section, I present the construction of this dual action and introduce the associated universal crossed product C^*-algebra and reduced crossed product C^*-algebra. For further details on the underlying concepts, the reader is referred to [28]. Some necessary preliminaries are recalled first.

Suppose X is a locally compact Hausdorff space and A is a C^*-algebra. If there exists a nondegenerate homomorphism Φ_A from $C_0(X)$ into $ZM(A)$ (the center of the multiplier algebra $M(A)$ of A), then A is called a $C_0(X)$-algebra. For $x \in X$, let J_x be the ideal of functions in $C_0(X)$ vanishing at x. Then $I_x = \overline{\Phi_A(J_x) \cdot A}$ is an ideal in A, where $\Phi_A(J_x) \cdot A = \operatorname{span}\{\Phi_A(f)a : f \in J_x, a \in A\}$. The quotient $A(x) = A/I_x$ is called the fiber of A over x. For every $a \in A$, the image of a in $A(x)$ is denoted by $a(x)$.

Let $\mathcal{A}$ be a topological space with a continuous and open surjection $p : \mathcal{A} \to X$. Suppose that for each $x \in X$, the fiber $\mathcal{A}_x = p^{-1}(x)$ is a C^*-algebra satisfying the following conditions: (1) The map $a \in \mathcal{A} \mapsto \|a\| \in \mathbb{R}^+$ is upper semicontinuous (i.e., for every $\varepsilon > 0$, the set $\{a \in \mathcal{A} : \|a\| < \varepsilon\}$ is open); (2) the operations of addition, multiplication, scalar multiplication, and involution in $\mathcal{A}$ are all continuous; and (3) if $\{a_i\} \subset \mathcal{A}$ is a net such that $\|a_i\| \to 0$ and $p(a_i) \to x$, then $a_i \to 0_x$, where 0_x is the zero element in $\mathcal{A}_x$. Then $\mathcal{A}$ is called an upper semicontinuous C^*-bundle over X. According to [28, Corollary 3.26], the disjoint union $\mathcal{A} = \bigsqcup_{x \in X} A(x)$ can be endowed with a unique topology that makes it an upper semicontinuous C^*-bundle; this $\mathcal{A}$ is then called the associated upper semicontinuous C^*-bundle to A.

Definition 3.1. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid, A be a $C_0(\mathcal{G}^{(0)})$-algebra, and $\mathcal{A}$ be the associated upper semicontinuous C^*-bundle. An action α of $\mathcal{G}$ on A by $*$-isomorphisms is a family $\{\alpha_\gamma\}_{\gamma \in \mathcal{G}}$ such that*

(a) *For all $\gamma \in \mathcal{G}$, $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$ is an isomorphism.*
(b) *For all $(\gamma, \eta) \in \mathcal{G}^{(2)}$. $\alpha_\gamma \circ \alpha_\eta = \alpha_{\gamma\eta}$.*
(c) *The map $(\gamma, a) \mapsto \alpha_\gamma(a)$ defines a continuous action of $\mathcal{G}$ on $\mathcal{A}$.*

The triple $(A, \mathcal{G}, \alpha)$ is called a (groupoid) C^*-dynamical system.

Suppose $\mathcal{G}$ is a locally compact Hausdorff étale groupoid acting continuously on a compact metric space X. Then $C(X)$ becomes a $C(\mathcal{G}^{(0)})$-algebra via the $*$-homomorphism $\Phi : C(\mathcal{G}^{(0)}) \to C(X)$ defined by $\Phi(g)(x) = g(\rho(x))$, where $\rho : X \to \mathcal{G}^{(0)}$ is the moment map. For $u \in \mathcal{G}^{(0)}$, let I_u be the ideal defining the fiber algebra $C(X)(u) = C(X)/I_u$, and let $I_{\rho^{-1}(u)}$ be the ideal of functions vanishing on $\rho^{-1}(u)$. One can verify that $I_u = I_{\rho^{-1}(u)}$, leading to the isomorphism $C(X)/I_u \cong C(\rho^{-1}(u))$. Denoting $X_u = \rho^{-1}(u)$, $C(X_u)$ can thus be viewed as the fiber over u of the upper semicontinuous bundle $\mathcal{A} = \bigsqcup_{u\in\mathcal{G}^{(0)}} C(X_u)$ over $\mathcal{G}^{(0)}$.

For $\gamma \in \mathcal{G}$, define the isomorphism $\alpha_\gamma : C(X_{s(\gamma)}) \to C(X_{r(\gamma)})$ by $\alpha_\gamma(f)(x) = f(\gamma^{-1} \cdot x)$. Then $(C(X), \mathcal{G}, \alpha)$ forms a groupoid dynamical system.

3.1. *Universal crossed product C^*-algebra*

As established, $\mathcal{A} = \bigsqcup_{u\in\mathcal{G}^{(0)}} C(X_u)$ is an upper semicontinuous C^*-bundle over $\mathcal{G}^{(0)}$. Its pullback along the range map r is the bundle $r^*(\mathcal{A}) = \{(\gamma, a) \in \mathcal{G} \times \mathcal{A} \mid r(\gamma) = p(a)\}$ over $\mathcal{G}$, where the bundle projection is given by the continuous, open, and surjective map $q(\gamma, a) = \gamma$. This structure makes $r^*(\mathcal{A})$ an upper semicontinuous C^*-bundle over $\mathcal{G}$. The space of compactly supported continuous sections of this pullback bundle is denoted by $\Gamma_c(\mathcal{G}, r^*(\mathcal{A}))$. These are continuous maps $f : \mathcal{G} \to r^*(\mathcal{A})$ satisfying $q(f(\gamma)) = \gamma$ for every $\gamma \in \mathcal{G}$.

Consider a net $\{f_i\}_{i\in I}$ and an element f in $\Gamma_c(\mathcal{G}, r^*(\mathcal{A}))$. The net f_i is said to converge to f in the inductive limit topology if f_i converges uniformly to f on $\mathcal{G}$, and there exists a compact set $K \subset \mathcal{G}$ such that eventually all f_i and f vanish off K. Furthermore, a map $F : \Gamma_c(\mathcal{G}, r^*(\mathcal{A})) \to Y$ is continuous in the inductive limit topology if whenever a net $f_i \to f$ in the inductive limit topology, then $F(f_i) \to F(f)$ in Y. It can be verified that $\Gamma_c(\mathcal{G}, r^*(\mathcal{A}))$ is a $*$-algebra with the convolution and involution defined by

$$(f * g)(\gamma) = \sum_{\eta\in\mathcal{G}^{r(\gamma)}} f(\eta)\alpha_\eta\left(g\left(\eta^{-1}\gamma\right)\right),$$

$$f^*(\gamma) = \alpha_\gamma\left(f\left(\gamma^{-1}\right)^*\right).$$

Furthermore, the convolution product and the involution are continuous with respect to the inductive limit topology.

The I-norm on $\Gamma_c(\mathcal{G}, r^*(\mathcal{A}))$ is defined as

$$\|f\|_I = \max\left\{\sup_{u\in\mathcal{G}^{(0)}} \sum_{r(\gamma)=u} \|f(\gamma)\|, \sup_{u\in\mathcal{G}^{(0)}} \sum_{s(\gamma)=u} \|f(\gamma)\|\right\}.$$

This norm is submultiplicative and involutive. Furthermore, convergence in the inductive limit topology implies convergence in the I-norm.

Consider a (nondegenerate) $*$-representation π of $\Gamma_c(\mathcal{G}, r^*(\mathcal{A}))$ on some Hilbert space $\mathcal{H}$. If π is either I-norm decreasing or continuous in the inductive limit topology, then π is equivalent to the integrated form $\pi \rtimes U$ of some covariant representation $(\mathcal{G}^{(0)} * \mathcal{H}, \pi, U)$ [28, Section 3.3]. The universal norm is given by

$$\|f\| = \sup\left\{\|\pi \rtimes U(f)\| : \ (\pi, U) \text{ is a covariant representation of } (C(X), \mathcal{G}, \alpha)\right\}.$$

The resulting completion is the groupoid crossed product C^*-algebra, denoted $C(X) \rtimes_\alpha \mathcal{G}$. From [28, Proposition 4.38], the following result is obtained:

Proposition 3.1. *The map $\Phi : C_c(\mathcal{G} \ltimes X) \to \Gamma_c(\mathcal{G}, r^*(\mathcal{A}))$ defined by $\Phi(f)(\gamma)(x) = f(\gamma, x)$ can be extended to a $*$-isomorphism from $C^*(\mathcal{G} \ltimes X)$ onto $C(X) \rtimes_\alpha G$.*

3.2. *Reduced crossed product C^*-algebra*

Let $\mathcal{G}$ be an étale groupoid acting continuously on a compact space X, and let $l^2(\mathcal{G} \ltimes X)$ be a Hilbert $C(X)$-module by the completion of $C_c(\mathcal{G} \ltimes X)$ under the following operations:

$$(\xi \cdot f)(\gamma, x) = \xi(\gamma, x)f(x), \quad \text{for } \xi \in C_c(\mathcal{G} \ltimes X), f \in C(X), (\gamma, x) \in \mathcal{G} \ltimes X,$$
$$\langle \xi, \zeta \rangle(x) = \sum_{\gamma \in \mathcal{G}^{\rho(x)}} \overline{\xi(\gamma, x)}\zeta(\gamma, x), \quad \text{for } \xi, \zeta \in C_c(\mathcal{G} \ltimes X), x \in X,$$
$$\|\xi\| = \sup_{x \in X} \left(\sum_{\gamma \in \mathcal{G}^{\rho(x)}} |\xi(\gamma, x)|^2 \right)^{1/2}, \quad \text{for } \xi \in C_c(\mathcal{G} \ltimes X).$$

Let $\mathcal{L}(l^2(\mathcal{G} \ltimes X))$ denote the C^*-algebra of all adjointable operators on $l^2(\mathcal{G} \ltimes X)$. Define a representation $\tilde{\pi} : C_c(\mathcal{G} \ltimes X) \to \mathcal{L}(l^2(\mathcal{G} \ltimes X))$ by

$$\tilde{\pi}(f)(\xi)(\gamma, x) = \sum_{\eta \in \mathcal{G}^{\rho(x)}} f(\gamma^{-1}\eta, \gamma^{-1} \cdot x)\xi(\eta, x), \quad \text{for } (\gamma, x) \in \mathcal{G} \ltimes X.$$

In particular, for any $f \in C(X)$, $\tilde{\pi}(f)(\xi)(\gamma, x) = f(\gamma^{-1} \cdot x)\xi(\gamma, x)$. The reduced groupoid C^*-algebra of the transformation groupoid $\mathcal{G} \ltimes X$, denoted $C_r^*(\mathcal{G} \ltimes X)$, is defined as the C^*-subalgebra of $\mathcal{L}(l^2(\mathcal{G} \ltimes X))$ generated by $\{\tilde{\pi}(f) : f \in C_c(\mathcal{G} \ltimes X)\}$ [29]. In the rest of this section, let $\mathcal{G}$ be a discrete groupoid.

Lemma 3.1. *Let $\mathcal{G} \curvearrowright X$ be a continuous groupoid action. For $\gamma \in \mathcal{G}$, set $U_\gamma = \{\gamma\} \times X_{r(\gamma)}$, where $X_{r(\gamma)} = \rho^{-1}(r(\gamma))$, and let u_γ be the characteristic function of U_γ. Then*

(1) *u_γ is a partial isometry in $C_c(\mathcal{G} \ltimes X)$, and $u_\gamma u_\eta = u_{\gamma\eta}$ for all $(\gamma, \eta) \in \mathcal{G}^{(2)}$.*
(2) *For all $f \in C(X)$ and $\gamma \in \mathcal{G}$, $u_\gamma f = (f \circ V_\gamma)u_\gamma$, where $V_\gamma(x) = \gamma^{-1} \cdot x$.*
(3) *$C_c(\mathcal{G} \ltimes X) = \operatorname{span}\{fu_\gamma : f \in C(X), \gamma \in \mathcal{G}\}$.*

Proof. Note that for $\gamma \in \mathcal{G}$, $X_{r(\gamma)}$ is an open and compact subset in $\mathcal{G} \ltimes X$, so $u_\gamma \in C_c(\mathcal{G} \ltimes X)$. Properties (1) and (2) follow from direct computations. The proof of (3) is as follows.

For $\xi \in C_c(\mathcal{G} \ltimes X)$, its support $\operatorname{supp} \xi$ is compact. Since the sets $\{U_\gamma\}_{\gamma \in \mathcal{G}}$ form an open cover of $\mathcal{G} \ltimes X$, there exist $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{G}$ such that $\operatorname{supp} \xi \subseteq \bigcup_{i=1}^n U_{\gamma_i}$. By a partition of unity argument, there exist functions $\xi_i \in C_c(\mathcal{G} \ltimes X)$ with $\operatorname{supp} \xi_i \subseteq U_{\gamma_i}$ and $\xi = \sum_{i=1}^n \xi_i$. For each i, define $f_i \in C(X)$ by

$$f_i(x) = \begin{cases} \xi_i(\gamma_i, x), & \text{if } x \in X_{r(\gamma_i)}, \\ 0, & \text{for otherwise.} \end{cases}$$

Then $\xi_i = f_i u_{\gamma_i}$ for each i. Thus, $C_c(\mathcal{G} \ltimes X) = \operatorname{span}\{fu_\gamma : f \in C(X), \gamma \in \mathcal{G}\}$. □

Define $l^2(\mathcal{G}, C(X)) = \{\xi : \mathcal{G} \to C(X) \mid \xi(\gamma) \in C(X_{r(\gamma)}) \text{ for all } \gamma \in \mathcal{G},\ \xi \text{ has finite support}\}$. Then $l^2(\mathcal{G}, C(X))$ becomes a pre-Hilbert $C(X)$-module under the following operations:

$$(\xi \cdot f)(\gamma)(x) = \begin{cases} \xi(\gamma)(x)f(x), & \text{if} x \in X_{r(\gamma)}, \\ 0, & \text{for otherwise,} \end{cases}$$
$$\langle \xi, \eta \rangle(x) = \sum_{\gamma \in \mathcal{G}^{\rho(x)}} \overline{\xi(\gamma)(x)}\, \eta(\gamma)(x), \qquad x \in X,$$

for $\xi, \eta \in l^2(\mathcal{G}, C(X))$ and $f \in C(X)$. The completion of $l^2(\mathcal{G}, C(X))$ with respect to the norm $\|\xi\| = \sup_{x \in X} \langle \xi, \xi \rangle (x)^{1/2}$ is a Hilbert $C(X)$-module, which is denoted by F. Let $\mathcal{L}(F)$ be the C^*-algebra of all adjointable operators on F.

Define a $*$-homomorphism $\pi : C(X) \to \mathcal{L}(F)$ by

$$\pi(f)\xi(\gamma)(x) = \begin{cases} f(\gamma^{-1} \cdot x)\, \xi(\gamma)(x), & \text{if } x \in X_{r(\gamma)}, \\ 0, & \text{for otherwise}, \end{cases} \qquad f \in C(X),\ \xi \in F,\ \gamma \in \mathcal{G}.$$

Note that for each $f \in C(X)$, $\pi(f)$ acts as a diagonal adjointable operator on the Hilbert C(X)-module F. For $\eta \in \mathcal{G}$, define an operator $v_\eta \in \mathcal{L}(F)$ by

$$v_\eta \xi(\gamma) = \begin{cases} \xi(\gamma\eta), & \text{if } \ s(\gamma) = r(\eta), \\ 0, & \text{otherwise}, \end{cases} \qquad \xi \in F,\ \gamma \in \mathcal{G}.$$

By direct calculation, the following conclusions can be drawn:

Lemma 3.2. *The operators v_η satisfy the following:*

(1) *v_η is a partial isometry, and $v_\eta^* = v_{\eta^{-1}}$ for every $\eta \in \mathcal{G}$.*
(2) *$v_\gamma v_\eta = v_{\gamma\eta}$ for every $(\gamma, \eta) \in \mathcal{G}^{(2)}$.*
(3) *For every $f \in C(X)$ and $\gamma \in \mathcal{G}$, $v_\gamma\, \pi(f)\, v_{\gamma^{-1}} = \pi(\alpha_\gamma(f))$, where $\alpha_\gamma(f)(x) = f(\gamma^{-1} \cdot x)$ (defined on $X_{r(\gamma)}$ and extended by zero to X).*

The *reduced crossed product* of the dynamical system $(C(X), \mathcal{G}, \alpha)$ is the C^*-subalgebra of $\mathcal{L}(F)$ generated by $\{\pi(f),\ v_\gamma : f \in C(X),\ \gamma \in \mathcal{G}\}$; it is denoted by

$$C(X) \rtimes_r^\alpha \mathcal{G} = \overline{\text{span}}^{\|\cdot\|}\{\pi(f)v_\gamma : f \in C(X),\ \gamma \in \mathcal{G}\} \subset \mathcal{L}(F).$$

Theorem 3.1. *$C_r^*(\mathcal{G} \ltimes X)$ is isomorphic to $C(X) \rtimes_r^\alpha \mathcal{G}$.*

Proof. Define

$$\Phi(f)(\gamma)(x) = \begin{cases} f(\gamma, x), & \text{if } x \in X_{r(\gamma)}, \\ 0, & \text{for otherwise}, \end{cases} \quad \text{for } f \in C_c(\mathcal{G} \ltimes X), \gamma \in \mathcal{G}.$$

Then Φ can be extended to an operator (still denoted by Φ) from $l^2(\mathcal{G} \ltimes X)$ onto F. Moreover, Φ is an adjointable unitary operator with $\Phi^*(\xi)(\gamma, x) = \xi(\gamma)(x)$ for $\xi \in F$, $(\gamma, x) \in \mathcal{G} \ltimes X$.

Define $\Psi : \mathcal{L}(l^2(\mathcal{G} \ltimes X)) \to \mathcal{L}(F)$ by $\Psi(T) = \Phi T \Phi^*$. Then Ψ is a $*$-isomorphism. Compute for $\xi \in F$,

$$\begin{aligned} \Psi(\tilde{\pi}(f))\xi(\gamma)(x) &= (\Phi\tilde{\pi}(f)\Phi^*\xi)(\gamma)(x) = \tilde{\pi}(f)(\Phi^*\xi)(\gamma, x) \\ &= f(\gamma^{-1} \cdot x)\, (\Phi^*\xi)(\gamma, x) = f(\gamma^{-1} \cdot x)\, \xi(\gamma)(x) \\ &= \pi(f)\xi(\gamma)(x). \end{aligned}$$

Hence, $\Psi(\tilde{\pi}(f)) = \pi(f)$.

For $\gamma \in \mathcal{G}$, let $u_\gamma \in C_c(\mathcal{G} \ltimes X)$ be the characteristic function of $\{\gamma\} \times X_{r(\gamma)}$. Its convolution action is

$$\tilde{\pi}(u_\gamma)\xi(\eta, x) = \begin{cases} \xi(\eta\gamma, x), & \text{if } s(\eta) = r(\gamma), \\ 0, & \text{for otherwise,} \end{cases} \qquad \xi \in l^2(\mathcal{G} \ltimes X).$$

Again compute for $\xi \in F$,

$$\begin{aligned} \Psi(\tilde{\pi}(u_\gamma))\xi(\eta)(x) &= (\Phi\tilde{\pi}(u_\gamma)\Phi^*\xi)(\eta)(x) = \tilde{\pi}(u_\gamma)(\Phi^*\xi)(\eta, x) \\ &= \begin{cases} (\Phi^*\xi)(\eta\gamma, x), & \text{if } s(\eta) = r(\gamma), \\ 0, & \text{for otherwise,} \end{cases} \\ &= \begin{cases} \xi(\eta\gamma)(x), & \text{if } s(\eta) = r(\gamma), \\ 0, & \text{for otherwise,} \end{cases} \\ &= v_\gamma\xi(\eta)(x). \end{aligned}$$

Thus, $\Psi(\tilde{\pi}(u_\gamma)) = v_\gamma$. Therefore, $C_r^*(\mathcal{G} \ltimes X)$ is isomorphic to $C(X) \rtimes_r^\alpha \mathcal{G}$. □

4. Continuous orbit equivalence

Let X be a continuous left $\mathcal{G}$-space. Then the groupoid action defines an equivalence relation on X as follows:

$$x \sim y \Leftrightarrow \exists\gamma \in \mathcal{G} \text{ such that } \gamma \cdot x = y.$$

The set $\mathcal{G} \cdot x = \{\gamma \cdot x : s(\gamma) = \rho(x)\} = \{\gamma \cdot x : \ \gamma \in \mathcal{G}_{\rho(x)}\}$ is called the orbit of $x \in X$.

Definition 4.1. *Two continuous groupoid actions $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are conjugate if there exists a homeomorphism $\varphi : X \to Y$ and an étale groupoid isomorphism $\Lambda : \mathcal{G} \to \mathcal{H}$ such that*

(1) $\rho_Y(\varphi(x)) = \Lambda(\rho_X(x))$ for all $x \in X$,
(2) $\varphi(\gamma \cdot x) = \Lambda(\gamma) \cdot \varphi(x)$ for all $x \in X$ and $\gamma \in \mathcal{G}_{\rho_X(x)}$.

Note that condition (1) ensures that for $\gamma \in \mathcal{G}_{\rho_X(x)}$, $s(\Lambda(\gamma)) = \Lambda(s(\gamma)) = \Lambda(\rho_X(x)) = \rho_Y(\varphi(x))$, so the action in condition (2) is well-defined.

Remark 4.1. *Given a continuous groupoid action $\mathcal{G} \curvearrowright X$, for $\gamma \in \mathcal{G}$ and $x \in X$, define the set $\rho_\gamma^{-1}(x) := \{y \in X : s(\gamma) = \rho(y),\ \gamma \cdot y = x\}$. Then the actions $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are conjugate (via φ and Λ) if and only if $\varphi(\rho_\gamma^{-1}(x)) = \rho_{\Lambda(\gamma)}^{-1}(\varphi(x))$ for all $\gamma \in \mathcal{G}$ and $x \in X$.*

Necessity is now proved. Assume the actions are conjugate. Let $\gamma \in \mathcal{G}$ and $x \in X$. The set equality is shown by proving two inclusions. If $z \in \rho_\gamma^{-1}(x)$, then $\gamma \cdot z = x$. By conjugacy, $\Lambda(\gamma) \cdot \varphi(z) = \varphi(\gamma \cdot z) = \varphi(x)$. Since $s(\Lambda(\gamma)) = \rho_Y(\varphi(z))$, it follows that $\varphi(z) \in \rho_{\Lambda(\gamma)}^{-1}(\varphi(x))$. If $w \in \rho_{\Lambda(\gamma)}^{-1}(\varphi(x))$, then $\Lambda(\gamma) \cdot w = \varphi(x)$. Applying the inverse conjugacy, gives $\gamma \cdot \varphi^{-1}(w) = x$. Thus, $\varphi^{-1}(w) \in \rho_\gamma^{-1}(x)$, so $w \in \varphi(\rho_\gamma^{-1}(x))$.

Conversely, assume the set condition holds. Consider the unit $\rho_X(x) \in \mathcal{G}^{(0)}$. Since $\rho_X(x) \cdot x = x$, it follows that $x \in \rho_{\rho_X(x)}^{-1}(x)$. By the assumed set condition with $\gamma = \rho_X(x)$, one has $\varphi(x) \in \varphi(\rho_{\rho_X(x)}^{-1}(x)) = \rho_{\Lambda(\rho_X(x))}^{-1}(\varphi(x))$. Then $\Lambda(\rho_X(x)) \cdot \varphi(x) = \varphi(x)$. However, for a unit $u \in \mathcal{H}^{(0)}$ to act on y with $\rho_Y(y) = u$, it must be that $u \cdot y = y$. Thus, $\rho_Y(\varphi(x)) = \Lambda(\rho_X(x))$. Let $z = \gamma \cdot x$ for $x \in X$ and $\gamma \in \mathcal{G}_{\rho_X(x)}$. Then $x \in \rho_\gamma^{-1}(z)$. By assumption, $\varphi(x) \in \varphi(\rho_\gamma^{-1}(z)) = \rho_{\Lambda(\gamma)}^{-1}(\varphi(z))$. This means $\Lambda(\gamma) \cdot \varphi(x) = \varphi(z) = \varphi(\gamma \cdot x)$.

Example 4.1. *Consider two conjugate group actions $G \curvearrowright X$ and $H \curvearrowright Y$ on compact spaces, where $G \ltimes X$ and $H \ltimes Y$ denote the corresponding transformation groupoids. Following the method in Example 2.2, I construct a continuous groupoid action of $H \ltimes Y$ on X. By symmetry, a similar continuous action of $G \ltimes X$ on Y exists. Consequently, the two groupoid actions are conjugate.*

Definition 4.2. *Two continuous groupoid actions $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are* orbit equivalent *if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi(\mathcal{G} \cdot x) = \mathcal{H} \cdot \varphi(x)$ for $x \in X$.*

In this case, for any $x, y \in X$ and $\gamma \in \mathcal{G}$ satisfying $y = \gamma \cdot x$, there exists an $\eta \in \mathcal{H}_{\rho(\varphi(x))}$ such that $\varphi(y) = \eta \cdot \varphi(x)$. Similarly, for any $u, v \in Y$ and $\eta \in \mathcal{H}$ with $v = \eta \cdot u$, there exists a $\gamma \in \mathcal{G}_{\rho(x)}$ such that $\varphi^{-1}(v) = \gamma \cdot \varphi^{-1}(u)$. This motivates the following notion:

Definition 4.3. *Two continuous groupoid actions $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are* continuously orbit equivalent *if there exist a homeomorphism $\varphi : X \to Y$ and continuous maps $a : \mathcal{G} * X \to \mathcal{H}$ and $b : \mathcal{H} * Y \to \mathcal{G}$ such that*

$$\varphi(\gamma \cdot x) = a(\gamma, x) \cdot \varphi(x), \quad \text{for } x \in X, \gamma \in \mathcal{G}_{\rho_X(x)}, \tag{4.1}$$

$$\varphi^{-1}(\eta \cdot y) = b(\eta, y) \cdot \varphi^{-1}(y), \quad \text{for } y \in Y, \eta \in \mathcal{H}_{\rho_Y(y)}. \tag{4.2}$$

Here, $a(\gamma, x) \in \mathcal{H}_{\rho_Y(\varphi(x))}$ and $b(\eta, y) \in \mathcal{G}_{\rho_X(\varphi^{-1}(y))}$, so that the groupoid actions on both sides are well-posed.

As such, continuous orbit equivalence implies orbit equivalence for the underlying groupoid actions.

For a groupoid action $\mathcal{G} \curvearrowright X$, consider the action groupoid $\mathcal{G} \ltimes X$ and the groupoid $\mathcal{G} * X$. Their isotropy bundles are given by

$$(\mathcal{G} \ltimes X)' = \{(\gamma, x) \in \mathcal{G} \ltimes X : \gamma^{-1} \cdot x = x\},$$

and

$$(\mathcal{G} * X)' = \{(\gamma, x) \in \mathcal{G} * X : \gamma \cdot x = x\}.$$

Moreover, under the isomorphism $\partial : \mathcal{G} \ltimes X \to \mathcal{G} * X$ defined by $\partial(\gamma, x) = (\gamma, \gamma^{-1} \cdot x)$, the relation $\partial((\mathcal{G} \ltimes X)') = (\mathcal{G} * X)'$ holds.

Motivated by [5, 23], topological freeness is defined as follows:

Definition 4.4. *A continuous groupoid action $\mathcal{G} \curvearrowright X$ is called* topologically free *if $\{x \in X : \widetilde{\mathcal{G}}_x = \{\rho(x)\}\}$ is dense in X. Here, $\widetilde{\mathcal{G}}_x = \{\gamma \in \mathcal{G}_{\rho(x)} : \gamma \cdot x = x\}$ is the stabilizer group at x.*

One can see that $\mathcal{G} \curvearrowright X$ is topologically free if and only if for a dense set of points x, the only groupoid element fixing x is the unit at $\rho(x)$. Moreover, a continuous groupoid action $\mathcal{G} \curvearrowright X$ is topologically free if and only if the groupoid $\mathcal{G} * X$ is topologically principal, which is also equivalent to the action groupoid $\mathcal{G} \ltimes X$ being topologically principal.

Lemma 4.1. *If $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are topologically free, then the mappings a and b in Definition 4.3 are uniquely determined by (4.1) and (4.2), respectively.*

Proof. Suppose that $a' : \mathcal{G} * X \to \mathcal{H}$ is another continuous map such that for all $(\gamma, x) \in \mathcal{G} * X$,

$$s(a'(\gamma, x)) = \rho_Y(\varphi(x)) \quad \text{and} \quad \varphi(\gamma \cdot x) = a'(\gamma, x) \cdot \varphi(x).$$

Fixing $(\gamma, x) \in \mathcal{G} * X$, one has $\varphi(\gamma \cdot x) = a(\gamma, x) \cdot \varphi(x) = a'(\gamma, x) \cdot \varphi(x)$. Let $\eta(\gamma, x) = a'(\gamma, x)^{-1} a(\gamma, x)$. Then $\eta(\gamma, x) \in \mathcal{H}$ and $\eta(\gamma, x) \cdot \varphi(x) = \varphi(x)$, so $\eta(\gamma, x) \in \widetilde{\mathcal{H}}_{\varphi(x)}$. Since a, a' are continuous, the map $\eta : \mathcal{G} * X \to \mathcal{H}$ is continuous.

By the topological freeness of $\mathcal{H} \curvearrowright Y$, there exists a dense subset $Y_0 \subseteq Y$ such that $\widetilde{\mathcal{H}}_y = \{\rho_Y(y)\}$ for every $y \in Y_0$. Since $\varphi : X \to Y$ is a homeomorphism, the preimage $X_0 := \varphi^{-1}(Y_0)$ is dense in X. Take any $(\gamma, x) \in \mathcal{G} * X$. By density of X_0 and the étale property of $\mathcal{G}$, nets $\{x_i\} \subseteq X_0$ and $\{\gamma_i\} \subseteq \mathcal{G}$ can be chosen such that $(\gamma_i, x_i) \to (\gamma, x)$ and $\rho_X(x_i) \to \rho_X(x)$. For each i, since $x_i \in X_0$, it follows that $\varphi(x_i) \in Y_0$, so $\widetilde{\mathcal{H}}_{\varphi(x_i)} = \{\rho_Y(\varphi(x_i))\}$. Hence, $\eta(\gamma_i, x_i) = \rho_Y(\varphi(x_i))$ for all i. By continuity of η, one has $\eta(\gamma_i, x_i) \to \eta(\gamma, x)$. Additionally, since $\rho_Y(\varphi(x_i)) \to \rho_Y(\varphi(x))$, it follows that $\eta(\gamma, x) = \rho_Y(\varphi(x))$. Thus, $a'(\gamma, x)^{-1} a(\gamma, x) = \rho_Y(\varphi(x))$, and multiplying on the left by $a'(\gamma, x)$ yields $a(\gamma, x) = a'(\gamma, x)$. □

Lemma 4.2. *In Definition 4.3, if $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are topologically free, then,*

$$b(a(\gamma, x), \varphi(x)) = \gamma, \quad \textit{for} \;\; (\gamma, x) \in \mathcal{G} * X,$$

$$a(b(\eta, y), \varphi^{-1}(y)) = \eta, \quad \textit{for} \;\; (\eta, y) \in \mathcal{H} * Y.$$

Proof. For an arbitrary $(\gamma, x) \in \mathcal{G} * X$, one has $\varphi(\gamma \cdot x) = a(\gamma, x) \cdot \varphi(x)$. Applying φ^{-1} to both sides gives $\gamma \cdot x = b(a(\gamma, x), \varphi(x)) \cdot x$. This implies that $(\gamma^{-1} b(a(\gamma, x), \varphi(x))) \cdot x = x$. Let $\zeta = \gamma^{-1} b(a(\gamma, x), \varphi(x))$. Then $\zeta \in \widetilde{\mathcal{G}}_x$.

Since $\mathcal{G}$ is étale, there exists an open neighborhood U of γ in $\mathcal{G}$ such that $s|_U : U \to s(U)$ is a homeomorphism. Then for any $z \in \rho^{-1}(s(U))$, there is a unique element $\gamma_z \in U$ with $s(\gamma_z) = \rho(z)$. This defines a continuous map $z \mapsto \gamma_z$ on $\rho^{-1}(s(U))$. Now, for each $z \in \rho^{-1}(s(U))$, $(\gamma_z, z) \in \mathcal{G} * X$, and thus $(\gamma_z^{-1} b(a(\gamma_z, z), \varphi(z))) \cdot z = z$. Let $\zeta(z) = \gamma_z^{-1} b(a(\gamma_z, z), \varphi(z))$. Then $\zeta(z) \in \widetilde{\mathcal{G}}_z$ for all $z \in \rho^{-1}(s(U))$.

Since the action $\mathcal{G} \curvearrowright X$ is topologically free, the set $\{z \in X : \widetilde{\mathcal{G}}_z = \{\rho(z)\}\}$ is dense in X. In particular, there exists a net $\{z_i\}$ in $\rho^{-1}(s(U))$ converging to x such that $\widetilde{\mathcal{G}}_{z_i} = \{\rho(z_i)\}$ for all i. For each such z_i, one has $\zeta(z_i) \in \widetilde{\mathcal{G}}_{z_i} = \{\rho(z_i)\}$, so $\zeta(z_i) = \rho(z_i)$. By continuity of the maps involved,

$$\zeta(z_i) = \gamma_{z_i}^{-1} b(a(\gamma_{z_i}, z_i), \varphi(z_i)) \to \gamma^{-1} b(a(\gamma, x), \varphi(x)) = \zeta.$$

However, $\zeta(z_i) = \rho(z_i) \to \rho(x) = s(\gamma)$. Therefore, $\zeta = s(\gamma)$. This shows that $\gamma^{-1} b(a(\gamma, x), \varphi(x)) = s(\gamma)$, and hence $b(a(\gamma, x), \varphi(x)) = \gamma$. The second equality follows through a similar way. □

Lemma 4.3. *In Definition 4.3, if $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are topologically free, then,*

$$a(\gamma, \eta \cdot x) a(\eta, x) = a(\gamma\eta, x) \; \textit{for} \; (\gamma, \eta \cdot x), (\eta, x) \in \mathcal{G} * X,$$

$$b(\alpha, \beta \cdot y) b(\beta, y) = b(\alpha\beta, y) \; \textit{for} \; (\alpha, \beta \cdot y), (\beta, y) \in \mathcal{H} * Y.$$

Proof. Let $(\gamma, \eta \cdot x), (\eta, x) \in \mathcal{G} * X$ be arbitrary. Then $(\gamma\eta, x) = (\gamma, \eta \cdot x)(\eta, x) \in \mathcal{G} * X$ and $\varphi(\gamma\eta \cdot x) = a(\gamma\eta, x) \cdot \varphi(x)$. On the other hand, $\varphi(\gamma\eta \cdot x) = \varphi(\gamma \cdot (\eta \cdot x)) = a(\gamma, \eta \cdot x) \cdot \varphi(\eta \cdot x) = a(\gamma, \eta \cdot x) a(\eta, x) \cdot \varphi(x)$. Thus $a(\gamma\eta, x) \cdot \varphi(x) = a(\gamma, \eta \cdot x) a(\eta, x) \cdot \varphi(x)$. This implies that $[a(\gamma\eta, x)]^{-1} a(\gamma, \eta \cdot x) a(\eta, x) \cdot \varphi(x) = \varphi(x)$. Let $\zeta(x) = [a(\gamma\eta, x)]^{-1} a(\gamma, \eta \cdot x) a(\eta, x)$. Then $\zeta(x) \in \widetilde{\mathcal{H}}_{\varphi(x)}$.

Since $\mathcal{H}$ is étale, there exists an open neighborhood U of $\gamma\eta$ in $\mathcal{G}$ such that $s|_U : U \to s(U)$ is a homeomorphism. Consider the set $V = \rho^{-1}(s(U))$. Then V is an open neighborhood of x in X, and for each $z \in V$, there exists a unique element $(\gamma\eta)_z \in U$ such that $s((\gamma\eta)_z) = \rho(z)$. This defines a continuous map $\sigma : V \to U$ given by $\sigma(z) = (\gamma\eta)_z$, with $\sigma(x) = \gamma\eta$.

Now, for each $z \in V$, $((\gamma\eta)_z, z) \in \mathcal{G} * X$. One can similarly define continuous maps that give γ_z and η_z such that $(\gamma_z, \eta_z \cdot z)$ and (η_z, z) are in $\mathcal{G} * X$ with $\gamma_z\eta_z = (\gamma\eta)_z$. This is possible because $\mathcal{G}$ is étale, and one can use local sections of the source map. Define the function $F\colon V \to \mathcal{H}$ by $F(z) = [a((\gamma\eta)_z, z)]^{-1} a(\gamma_z, \eta_z \cdot z) a(\eta_z, z)$. By continuity of a and the groupoid operations, F is continuous. Moreover, for each $z \in V$, $F(z) \cdot \varphi(z) = \varphi(z)$, so $F(z) \in \widetilde{\mathcal{H}}_{\varphi(z)}$.

Since the action $\mathcal{H} \curvearrowright Y$ is topologically free, the set $\{y \in Y : \widetilde{\mathcal{H}}_y = \{\rho(y)\}\}$ is dense in Y. As φ is a homeomorphism, the set $\{z \in X : \widetilde{\mathcal{H}}_{\varphi(z)} = \{\rho(\varphi(z))\}\}$ is dense in X. In particular, there exists a net $\{z_i\}$ in V converging to x such that $\widetilde{\mathcal{H}}_{\varphi(z_i)} = \{\rho(\varphi(z_i))\}$ for all i. For each z_i, one has $F(z_i) \in \widetilde{\mathcal{H}}_{\varphi(z_i)} = \{\rho(\varphi(z_i))\}$, so $F(z_i) = \rho(\varphi(z_i))$. By continuity of F, it follows that $F(z_i) \to F(x) = \zeta(x)$. Additionally, $\rho(\varphi(z_i)) \to \rho(\varphi(x)) = s(a(\gamma\eta, x))$. Therefore, $\zeta(x) = s(a(\gamma\eta, x))$. This shows that $[a(\gamma\eta, x)]^{-1} a(\gamma, \eta \cdot x) a(\eta, x) = s(a(\gamma\eta, x))$. Multiplying on the left by $a(\gamma\eta, x)$ yields $a(\gamma, \eta \cdot x) a(\eta, x) = a(\gamma\eta, x)$. The proof of the second equality is similar. □

Corollary 4.1. *In the situation of Definition 4.3, assume that $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are topologically free. For every $x \in X$, the map $a_x : \gamma \in \mathcal{G}_{\rho_X(x)} \to a(\gamma, x) \in \mathcal{H}_{\rho_Y(\varphi(x))}$ is a bijection with inverse $b_{\varphi(x)} : \eta \in \mathcal{H}_{\rho_Y(\varphi(x))} \to b(\eta, \varphi(x)) \in \mathcal{G}_{\rho_X(x)}$. Moreover, $a_x(\rho_X(x)) = \rho_Y(\varphi(x))$.*

Proof. For $y \in Y$, define $b_y : \eta \in \mathcal{H}_{\rho_Y(y)} \to b(\eta, y) \in \mathcal{G}_{\rho_X(\varphi^{-1}(y))}$. It follows from Lemma 4.2 that $a_x(b_{\varphi(x)}(\eta)) = a(b(\eta, \varphi(x)), x) = \eta$ and $b_{\varphi(x)}(a_x(\gamma)) = b(a(\gamma, x), \varphi(x)) = \gamma$. Therefore, a_x and $b_{\varphi(x)}$ are inverse to each other, and hence a_x is a bijection.

To show that $a_x(\rho_X(x)) = \rho_Y(\varphi(x))$, note that $\varphi(x) = \varphi(\rho_X(x) \cdot x) = a(\rho_X(x), x) \cdot \varphi(x) = a_x(\rho_X(x)) \cdot \varphi(x)$. Thus, $a_x(\rho_X(x))$ fixes $\varphi(x)$. Now, define the map $F : X \to \mathcal{H}$ by $F(x) = a(\rho_X(x), x)$. Then F is continuous.

Because $\mathcal{H} \curvearrowright Y$ is topologically free and φ is a homeomorphism, the set $\{x \in X : \widetilde{\mathcal{H}}_{\varphi(x)} = \{\rho_Y(\varphi(x))\}\}$ is dense in X. In particular, there exists a net $\{x_i\}$ in X converging to x such that for each x_i, $\widetilde{\mathcal{H}}_{\varphi(x_i)} = \{\rho_Y(\varphi(x_i))\}$. Then $\varphi(x_i) = a(\rho_X(x_i), x_i) \cdot \varphi(x_i) = F(x_i) \cdot \varphi(x_i)$, so $F(x_i) \in \widetilde{\mathcal{H}}_{\varphi(x_i)} = \{\rho_Y(\varphi(x_i))\}$. By continuity of F and $\rho_Y \circ \varphi$, one has $F(x) = \lim F(x_i) = \lim \rho_Y(\varphi(x_i)) = \rho_Y(\varphi(x))$. Therefore, $a_x(\rho_X(x)) = \rho_Y(\varphi(x))$. □

Proposition 4.1. *If two groupoid actions $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are conjugate, then they are continuously orbit equivalent. Conversely, if the actions are topologically free and every fiber of the moment maps $\rho_X : X \to \mathcal{G}^{(0)}$ and $\rho_Y : Y \to \mathcal{H}^{(0)}$ is connected, then continuous orbit equivalence implies conjugacy.*

Proof. Assume that $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are conjugate by homeomorphism φ and groupoid isomorphism Λ. Define $a(\gamma, x) = \Lambda(\gamma)$, $b(\eta, y) = \Lambda^{-1}(\eta)$ for $(\gamma, x) \in \mathcal{G} * X$ and $(\eta, y) \in \mathcal{H} * Y$. Since Λ is a groupoid isomorphism, it is continuous, and hence a and b are continuous. Moreover,

$$\varphi(\gamma \cdot x) = \Lambda(\gamma) \cdot \varphi(x) = a(\gamma, x) \cdot \varphi(x),$$

and

$$\varphi^{-1}(\eta \cdot y) = \Lambda^{-1}(\eta) \cdot \varphi^{-1}(y) = b(\eta, y) \cdot \varphi^{-1}(y).$$

Thus, φ, a, and b satisfy Definition 4.3, and the actions are continuous orbit equivalent.

Conversely, suppose the actions are topologically free, every fiber of $\rho_X : X \to \mathcal{G}^{(0)}$ and $\rho_Y : Y \to \mathcal{H}^{(0)}$ is connected, and that they are continuously orbit equivalent via φ, a, and b. Define $\Lambda\colon \mathcal{G} \to \mathcal{H}$

as follows: For $\gamma \in \mathcal{G}$, choose any $x \in X$ with $s(\gamma) = \rho_X(x)$, and set $\Lambda(\gamma) = a(\gamma, x)$. Fix $\gamma \in \mathcal{G}$, consider the map $f\colon X_{s(\gamma)} \to \mathcal{H}$ defined by $f(x) = a(\gamma, x)$, where $X_{s(\gamma)} = \rho_X^{-1}(s(\gamma))$. Since $\mathcal{H}$ is étale, the source fibers of $\mathcal{H}$ are discrete. By the continuity of f and the connectedness of $X_{s(\gamma)}$, f must be constant. Thus, $\Lambda(\gamma)$ is well-defined.

By Lemma 4.3 and Corollary 4.1, Λ is a bijection, and $\Lambda(\gamma\eta) = a(\gamma\eta, x) = a(\gamma, \eta \cdot x)a(\eta, x) = \Lambda(\gamma)\Lambda(\eta)$, so Λ is a groupoid homomorphism. Similarly, Λ^{-1} defined by $\Lambda^{-1}(\eta) = b(\eta, y)$ for any y with $s(\eta) = \rho_Y(y)$ is the inverse of Λ. Thus, Λ is a groupoid isomorphism. Finally, the conjugacy conditions are verified: $\rho_Y(\varphi(x)) = s(a(\gamma, x)) = s(\Lambda(\gamma)) = \Lambda(s(\gamma)) = \Lambda(\rho_X(x))$ and $\varphi(\gamma \cdot x) = a(\gamma, x) \cdot \varphi(x) = \Lambda(\gamma) \cdot \varphi(x)$. Hence, the actions are conjugate. □

If an étale groupoid $\mathcal{G}$ is topologically principal, then $C_0(\mathcal{G}^{(0)})$ is a Cartan subalgebra of $C_r^*(\mathcal{G})$. Moreover, two topologically principal étale groupoids $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if and only if there is a C^*-isomomorphism $\Phi : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{H})$ such that $\Phi(C_0(\mathcal{G}^{(0)})) = C_0(\mathcal{H}^{(0)})$ [30]. Combining these results yields the following theorem:

Theorem 4.1. *Let $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ be two topologically free continuous groupoid actions. Then the following are equivalent:*

(1) *$\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are continuous orbit equivalent;*
(2) *$\mathcal{G} * X$ and $\mathcal{H} * Y$ are isomorphic as étale groupoid;*
(3) *$\mathcal{G} \ltimes X$ and $\mathcal{H} \ltimes Y$ are isomorphic as étale groupoid;*
(4) *There exists a C^*-isomorphism $\Phi : C_r^*(\mathcal{G} \ltimes X) \to C_r^*(\mathcal{H} \ltimes Y)$ such that $\Phi(C(X)) = C(Y)$.*

Proof. Only the equivalence of (1) and (2) is proved.

Assume (1) holds, i.e., $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are continuous orbit equivalent via a homeomorphism φ and continuous maps a and b. Define

$$\Phi : (\gamma, x) \in \mathcal{G} * X \to (a(\gamma, x), \varphi(x)) \in \mathcal{H} * Y,$$

$$\Psi : (\eta, y) \in \mathcal{H} * Y \to (b(\eta, y), \varphi^{-1}(y)) \in \mathcal{G} * X.$$

From Lemma 4.3, one can show that Φ and Ψ are continuous groupoid homomorphisms. Moreover, it follows from Lemma 4.2 that $\Phi(\Psi(\eta, y)) = (\eta, y)$ and $\Psi(\Phi(\gamma, x)) = (\gamma, x)$. Then Φ and Ψ are converse with each other. Hence, $\mathcal{G} * X$ and $\mathcal{H} * Y$ are étale isomomorphic.

Assume (2) holds. Let $\Lambda : \mathcal{G} * X \to \mathcal{H} * Y$ be an isomorphism of étale groupoids. Identify the unit spaces of $\mathcal{G} * X$ and $\mathcal{H} * Y$ with X and Y, respectively. Define $\varphi = \Lambda|_X$, which is a homeomorphism from X to Y. Define a and b as follows:

$$a : \mathcal{G} * X \xrightarrow{\Lambda} \mathcal{H} * Y \to \mathcal{H}, \quad (\gamma, x) \mapsto a(\gamma, x),$$

$$b : \mathcal{H} * Y \xrightarrow{\Lambda^{-1}} \mathcal{G} * X \to \mathcal{G}, \quad (\eta, y) \mapsto b(\eta, y),$$

where the maps to $\mathcal{H}$ and $\mathcal{G}$ are the projections onto the first component. Then a and b are continuous. For $(\gamma, x) \in \mathcal{G}*X$ and $(\eta, y) \in \mathcal{H}*Y$, one has $\Lambda(\gamma, x) = (a(\gamma, x), \varphi(x))$, $\Lambda^{-1}(\eta, y) = (b(\eta, y), \varphi^{-1}(y))$. Since Λ is a groupoid homomorphism, it preserves the source and range maps. Thus, $s(a(\gamma, x)) = \rho_Y(\varphi(x))$, $s(b(\eta, y)) = \rho_X(\varphi^{-1}(y))$, and $\varphi(\gamma{\cdot}x) = r(\Lambda(\gamma, x)) = a(\gamma, x){\cdot}\varphi(x)$, $\varphi^{-1}(\eta{\cdot}y) = r(\Lambda^{-1}(\eta, y)) = b(\eta, y){\cdot}\varphi^{-1}(y)$. Hence, $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ are continuous orbit equivalent. □

5. Conclusions

In this paper, the theory of continuous orbit equivalence for continuous actions of étale groupoids on compact topological spaces is studied. This framework provides a natural generalization of both partial group homeomorphism actions and actions of reversible semigroups. Under the assumption of topological freeness, a profound correspondence is established among three domains: The dynamical system, its associated transformation groupoid, and the corresponding groupoid C^*-algebra. Specifically, it is proved that continuous orbit equivalence of the actions, étale groupoid isomorphism of the transformation groupoids, and C^*-isomorphism of the reduced algebras that preserve the canonical Cartan subalgebras are all equivalent. Furthermore, for the case of actions by discrete groupoids, the isomorphism $C_r^*(\mathcal{G} \ltimes X) \cong C(X) \rtimes_r^\alpha \mathcal{G}$ between the reduced groupoid C^*-algebra and the reduced crossed product algebra is proved.

In future research, I will focus on two major directions. The first is to extend the fundamental C^*-algebraic isomorphism established in this work to the setting of general (non-discrete) locally compact Hausdorff groupoid actions. The second direction aims to further generalize the theory by considering continuous actions that are genuine homeomorphisms (rather than local homeomorphisms). Developing a complete theory of continuous orbit equivalence for such groupoid homeomorphism actions would directly generalize the classical theory for group actions and offer a more unified framework for understanding the interplay between dynamical systems and operator algebras.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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