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Research article

Bifurcation and persistence in a stochastic seasonal dynamical system

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Abstract: We studied stochastic bifurcation and persistence in seasonally forced dynamical systems governed by stochastic differential equations with periodically switching drift in this paper. By using Lyapunov exponent techniques, we characterized extinction and persistence via a noise dependent threshold determined by the top Lyapunov exponent. Futhermore, we proved that variation of the season length parameter induces a stochastic bifurcation, where stability of the extinction state is lost and nontrivial invariant probability measures emerge. Analytical results were illustrated through a stochastic Lotka-Volterra model, showing that environmental noise shifts extinction persistence thresholds and fundamentally alters long-term dynamics.

Keywords: stochastic bifurcation; seasonal dynamical systems; stochastic persistence; invariant probability measures; Lyapunov exponents
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Abbreviations

t: Time variable; τ: Season length (period parameter); ε: Transition/smoothing parameter for seasonal switching; $\chi_g^\varepsilon(t,\tau)$: Indicator function for the favorable (growth) season; $\chi_d^\varepsilon(t,\tau)$: Indicator function for the unfavorable (decay) season; $U(t)$: State vector of the stochastic system; $X(t)$: Solution of the linearized system; $f_g(u)$: Drift during the favorable season; $f_d(u)$: Drift during the unfavorable season; $Df_g(0)$: Jacobian matrix of f_g at the origin; $Df_d(0)$: Jacobian matrix of f_d at the origin; $A(t,\tau)$: Time-dependent linearized drift matrix; $\bar{A}(\tau)$: Time-averaged drift matrix; $r_\varepsilon(\tau)$: Fraction of time spent in the favorable season; $\Sigma(U)$: Diffusion matrix; B: Noise coefficient matrix; σ: Noise intensity (when $B = \sigma I$); $W(t)$: Standard Brownian motion; $\lambda(\tau)$: Top Lyapunov exponent; $\lambda_i(\tau)$: Lyapunov exponent of the i-th mode; τ_σ^*: Critical threshold for stochastic bifurcation; μ_τ: Invariant probability measure; μ_T: Time-averaged empirical measure; Δt: Time step for numerical simulations; T: Total simulation time; T_{burn}: Burn-in (transient) period.

1. Introduction

Seasonal variation is a fundamental feature of many natural and engineered systems, influencing population dynamics, epidemiological processes, and ecological interactions [1]. In mathematical modeling, seasonality is commonly represented through periodically varying coefficients or by the alternation of distinct dynamical regimes, such as phases of growth and decline. Deterministic models incorporating such mechanisms have been widely used to explain population persistence, competitive exclusion, and coexistence driven by temporal heterogeneity [2].

In realistic settings, however, seasonal systems are inevitably affected by random fluctuations arising from environmental variability, demographic stochasticity, or unresolved external factors. These stochastic perturbations may substantially influence long-term dynamics, modifying stability properties, shifting bifurcation thresholds, and even inducing qualitative transitions that are absent in purely deterministic models [3]. Consequently, understanding the interplay between seasonal forcing and stochastic effects is essential for developing realistic and predictive mathematical descriptions of such systems [4].

In this work, we study stochastic seasonal dynamical systems in which the evolution alternates between distinct growth and decline dynamics while being continuously perturbed by noise. The duration of the favorable season is governed by a parameter that serves as a natural control parameter for the system. Our primary objective is to understand how variation of this season length parameter affects extinction, persistence, and the emergence of nontrivial, long-term stochastic behavior.

From a deterministic viewpoint, bifurcation theory provides a powerful framework for describing qualitative changes in dynamics as parameters vary. Classical results, such as the Crandall-Rabinowitz theorem, rigorously characterize the emergence of new equilibria when a simple eigenvalue crosses zero [5]. In stochastic systems, however, deterministic equilibria are typically replaced by invariant probability measures, random equilibria, or random periodic solutions, and stability must be interpreted in probabilistic or almost sure terms. As a result, classical bifurcation concepts require careful adaptation to stochastic settings.

Several notions of stochastic bifurcation have been proposed in the literature, including phenomenological, dynamical, and distributional bifurcations [6]. In the presence of seasonal forcing, stochastic perturbations may either suppress or enhance deterministic bifurcations and may give rise to noise induced persistence or extinction phenomena [7]. Despite increasing interest in stochastic bifurcation theory, comparatively few studies address bifurcation and persistence in systems with explicit seasonal switching or alternating dynamical regimes [8].

The aim of the present paper is to contribute to this direction by developing a systematic framework for the analysis of extinction, persistence, and bifurcation in stochastic seasonal dynamical systems. We consider models in which the drift alternates periodically between growth and decline regimes, while stochastic perturbations act continuously in time. Both sharp seasonal switching and smoothly regularized transitions are allowed, enabling the treatment of a broad class of biologically and physically relevant models [9].

Our analysis begins with general results on existence, uniqueness, and boundedness of solutions. We then focus on the extinction state, which persists as a solution in the stochastic setting, and investigate its stability in probability and, where appropriate, almost surely. By analyzing the associated linearized stochastic system, we identify conditions under which the extinction state loses stability as the season-length parameter varies. This loss of stability marks a stochastic bifurcation associated with the emergence of persistent stochastic dynamics, described in terms of nontrivial invariant probability measures and, in certain cases, random periodic behavior.

We further demonstrate that stochastic perturbations can significantly modify deterministic predictions. In particular, noise may shift critical season lengths required for persistence, alter extinction persistence thresholds, and modify coexistence regions. These effects are illustrated through a stochastic competitive Lotka-Volterra model with seasonal forcing, where numerical simulations complement the theoretical analysis and highlight noise-induced changes in bifurcation structure and coexistence behavior [10, 11].

Additional applications and motivation. Beyond ecological and population dynamics, the class of stochastic seasonal systems studied here is closely related to several important applied frameworks. In particular, functional differential equations with non-canonical operators exhibit oscillatory behavior driven by memory and delay effects, which can be naturally combined with seasonal forcing. Moreover, adjoint nonlinear impulsive neutral mixed integral equations arise in control theory and engineering systems with discontinuities and aftereffects, where stability analysis parallels the stochastic persistence framework developed in this paper.

These connections highlight that the interplay between stochastic perturbations, seasonal switching, and memory effects is fundamental across a wide range of applied mathematical models, motivating the generality of the present analysis.

Connection with biomathematical modeling and threshold dynamics. Threshold phenomena and bifurcation mechanisms play a central role in biological dynamical systems, particularly in epidemiology and population biology. Recently, the authors of [12] demonstrated how immune pathogen interactions can exhibit oscillatory patterns and qualitative transitions driven by threshold conditions and Hopf bifurcation. Similarly, the authors of [13] investigated delayed epidemiological systems with environmental transmission, emphasizing the role of threshold criteria, Lyapunov methods, and control strategies in determining long-term dynamics.

Although the present work focuses on stochastic seasonally forced systems, it shares key structural features with these models, including threshold-induced transitions, bifurcation-driven changes in asymptotic behavior, and the integration of biological interpretation with mathematical analysis. In particular, our results show how environmental variability and seasonal switching can shift critical thresholds and generate qualitatively different long-term regimes, providing a stochastic counterpart to deterministic bifurcation phenomena observed in biomathematical models.

The paper is organized as follows. In Section 2, we introduce the stochastic seasonal model and state the main assumptions. Section 3 is devoted to the analysis of extinction and stochastic persistence, including Lyapunov exponent criteria for stability. In Section 4, we investigate stochastic bifurcation phenomena and establish the existence of nontrivial invariant probability measures. Section 5 applies the theory to a stochastic Lotka-Volterra competition model and presents numerical experiments. Finally, Section 6 summarizes the main results and discusses directions for future research.

2. Stochastic seasonal model and preliminaries

In this section, we introduce the class of stochastic seasonal dynamical systems studied in this work and collect the basic assumptions and definitions that will be used throughout the paper.

2.1. *Probability setting*

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. Let

$$W(t) = (W_1(t), \dots, W_m(t))$$

be an m-dimensional standard Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. All stochastic processes considered below are assumed to be $\{\mathcal{F}_t\}$-adapted.

2.2. *Seasonal structure*

We normalize the seasonal period to one. For a given season-length parameter $\tau \in (0, 1)$, the growth and decline seasons are defined by

$$G_n = (n, n+\tau], \qquad D_n = (n+\tau, n+1], \qquad n \in \mathbb{N}. \tag{2.1}$$

To allow both sharp and smooth transitions between seasons, we introduce regularized seasonal indicator functions. Let $\rho \in C_c^\infty(\mathbb{R})$ be a nonnegative, symmetric mollifier satisfying

$$\rho(s) = \rho(-s), \qquad \int_{\mathbb{R}} \rho(s)\, ds = 1. \tag{2.2}$$

For $\varepsilon \geq 0$, define the rescaled kernel

$$\rho_\varepsilon(s) = \varepsilon^{-1}\rho(s/\varepsilon),$$

with the convention that $\varepsilon = 0$ corresponds to the Dirac delta distribution. We define

$$\chi_g^\varepsilon(t,\tau) = \int_0^\tau \rho_\varepsilon(t-s)\, ds, \qquad \chi_d^\varepsilon(t,\tau) = \int_\tau^1 \rho_\varepsilon(t-s)\, ds. \tag{2.3}$$

These functions satisfy

$$\chi_g^\varepsilon(t,\tau) + \chi_d^\varepsilon(t,\tau) \equiv 1, \tag{2.4}$$

and

$$\chi_g^\varepsilon(t,\tau) \to \mathbf{1}_{(0,\tau)}(t) \quad \text{as } \varepsilon \to 0,$$

pointwise for almost every t.

2.3. *Stochastic seasonal dynamical system*

Let $f_g, f_d : \mathbb{R}^N \to \mathbb{R}^N$ be continuously differentiable vector fields describing the growth and decline dynamics, respectively, and let $\Sigma : \mathbb{R}^N \to \mathbb{R}^{N\times m}$ be a diffusion coefficient.

We consider the stochastic differential equation

$$dU(t) = \Big(f_g(U(t))\,\chi_g^{\varepsilon}(t,\tau) + f_d(U(t))\,\chi_d^{\varepsilon}(t,\tau)\Big)\,dt \\ + \Sigma(U(t))\,dW(t), \qquad t > 0, \tag{2.5}$$

with the initial condition

$$U(0) = U_0 \in \mathbb{R}^N. \tag{2.6}$$

Equation (2.5) describes a system whose drift alternates seasonally between growth and decline regimes, while stochastic perturbations act continuously in time.

2.4. Structural assumptions

Throughout the paper, we impose the following assumptions.

A1. **Regularity.** The functions $f_g, f_d \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ and $\Sigma \in C^1(\mathbb{R}^N;\mathbb{R}^{N\times m})$.

A2. **Equilibrium at extinction.**

$$f_g(0) = f_d(0) = 0, \qquad \Sigma(0) = 0. \tag{2.7}$$

Thus, the extinction state $U(t) \equiv 0$ is a solution of (2.5).

A3. **Linear growth condition.** There exists $C > 0$ such that

$$|f_g(u)| + |f_d(u)| + \|\Sigma(u)\| \le C(1 + |u|), \qquad \forall u \in \mathbb{R}^N. \tag{2.8}$$

A4. **Forward invariance and boundedness.** There exists a compact set $K \subset \mathbb{R}_+^N$ such that, for any initial condition $U_0 \in K$, the corresponding solution of (2.5) remains in K almost surely for all $t \ge 0$.

Remark 2.1. ***(Role and verification of Assumption (A4)).*** *Assumption (A4) plays a key role in the analysis by ensuring dissipativity of the system for large values of the state variable. More precisely, it guarantees that there exist constants* $c_1, c_2 > 0$ *such that*

$$\langle u, f_g(u)\rangle \le -c_1|u|^2 + c_2, \quad \langle u, f_d(u)\rangle \le -c_1|u|^2 + c_2,$$

for all sufficiently large $|u|$.

This condition implies that trajectories are eventually driven toward a compact set, which is essential for establishing uniform moment bounds and tightness of time-averaged probability measures. In particular, it ensures that

$$\sup_{t\ge 0} \mathbb{E}|U(t)|^p < \infty$$

for some $p > 0$*, a key ingredient in the Krylov-Bogolyubov argument for the existence of invariant measures.*

Verification in applications. In many biological and ecological models, Assumption (A4) follows naturally from the presence of density-dependent regulation. For example, in the stochastic Lotka-Volterra model considered in Section 5, the drift has the form

$$f(u) = u(a - Bu),$$

where B is a matrix with positive entries. Then

$$\langle u, f(u)\rangle = \langle u, a \circ u\rangle - \langle u, Bu \circ u\rangle,$$

and the quadratic term dominates for large $|u|$, yielding

$$\langle u, f(u)\rangle \leq -c|u|^2$$

for some $c > 0$.

Thus, Assumption (A4) is satisfied in a broad class of population models with self limiting growth.

2.5. Well-posedness

Under Assumptions (A1)–(A3), standard results for stochastic differential equations imply the following.

Proposition 2.2 (Existence and uniqueness)**.** *For any initial condition $U_0 \in \mathbb{R}^N$, Eqs* (2.5) *and* (2.6) *admit a unique global strong solution.*

If, in addition, $U_0 \in K$, then the solution remains bounded and nonnegative almost surely.

2.6. Random dynamical system formulation

Equation (2.5) generates a random dynamical system (RDS) $\varphi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\varphi(t, \omega)U_0 = U(t, \omega; U_0), \tag{2.9}$$

where $U(t, \omega; U_0)$ denotes the solution starting from U_0.

Due to the periodic dependence of the coefficients on time, the associated Markov transition semigroup is time-periodic with period one. This structure allows the introduction of Poincaré-type operators and facilitates the analysis of long-term statistical behavior.

2.7. Stochastic equilibria and persistence

We recall several notions relevant to the stochastic setting.

Definition 2.3 (Stochastic extinction state)**.** *The identically zero process $U(t) \equiv 0$ is called the stochastic extinction state.*

Definition 2.4 (Persistence in probability)**.** *Equation* (2.5) *is said to be* persistent in probability *if, for every $\varepsilon > 0$,*

$$\liminf_{t\to\infty} \mathbb{P}(|U(t)| > \varepsilon) > 0. \tag{2.10}$$

Definition 2.5 (Invariant probability measure)**.** *A probability measure μ on $\mathbb{R}^N$ is called* invariant *for* (2.5) *if*

$$\mu P_t = \mu \qquad \text{for all } t \geq 0,$$

where $\{P_t\}_{t\geq 0}$ denotes the Markov transition semigroup associated with Eq (2.5)*.*

In the following sections, we analyze how the stability of the extinction state and the existence of nontrivial invariant probability measures depend on the season length parameter τ [14, 15].

3. Stability of the extinction state and stochastic persistence

In this section, we analyze the stability of the extinction state for the stochastic seasonal equation (2.5) and derive conditions under which stochastic persistence occurs. The season length parameter τ acts as a bifurcation parameter governing the transition between extinction and persistence.

3.1. Linearization around the extinction state

Under Assumption (A2), the extinction state $U(t) \equiv 0$ is a solution of (2.5). Linearizing the system around the origin yields the stochastic linear equation

$$dX(t) = A(t, \tau)X(t)\,dt + B\,X(t)\,dW(t), \tag{3.1}$$

where

$$A(t, \tau) = Df_g(0)\,\chi_g^{\varepsilon}(t, \tau) + Df_d(0)\,\chi_d^{\varepsilon}(t, \tau), \tag{3.2}$$

and

$$B = D\Sigma(0). \tag{3.3}$$

Equation (3.1) is a linear stochastic differential equation with bounded, time-periodic coefficients. Its long-term growth properties determine the local stability of the extinction state.

3.2. Lyapunov exponent and stochastic stability

The stability of the zero solution of (3.1) is characterized by its top Lyapunov exponent.

Definition 3.1 (Top Lyapunov exponent)**.** *The top Lyapunov exponent associated with Eq* (3.1) *is defined by*

$$\lambda(\tau) = \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}[\,\log|X(t)|], \tag{3.4}$$

for any nonzero initial condition $X(0)$*.*

Proposition 3.2 (Stochastic stability of extinction)**.** *If* $\lambda(\tau) < 0$*, then the extinction state is exponentially stable in probability. If* $\lambda(\tau) > 0$*, then the extinction state is unstable in probability.*

This result follows from standard stochastic stability theory for linear stochastic differential equations with bounded coefficients.

3.3. Seasonal averaging and the effective growth rate

Due to the periodic structure of the coefficients, it is natural to introduce the seasonally averaged drift matrix

$$\bar{A}(\tau) = \int_0^1 A(t, \tau)\,dt = r_\varepsilon(\tau)\,Df_g(0) + (1 - r_\varepsilon(\tau))\,Df_d(0), \tag{3.5}$$

where

$$r_\varepsilon(\tau) = \int_0^\tau \int_0^1 \rho_\varepsilon(t - s)\,dt\,ds. \tag{3.6}$$

In general, the Lyapunov exponent $\lambda(\tau)$ cannot be computed explicitly. However, for scalar or commutative systems, or for isotropic multiplicative noise of the form $B = \sigma I$, one obtains the approximation

$$\lambda(\tau) \approx \max \Re(\mathrm{spec}(\bar{A}(\tau))) - \frac{\sigma^2}{2}. \tag{3.7}$$

Remark 3.3. *(Approximate Lyapunov exponent). The expression in Eq (3.7) should be interpreted as an approximation of the top Lyapunov exponent. In general, for non-commuting matrices $A(t,\tau)$, no closed form expression for $\lambda(\tau)$ is available.*

However, if the matrices $Df_g(0)$ and $Df_d(0)$ commute, then $A(t,\tau)$ is simultaneously diagonalizable, and the stochastic linear system reduces to a collection of scalar equations. In this case, for isotropic multiplicative noise $B = \sigma I$, Ito's formula applied componentwise yields

$$\lambda(\tau) = \max \mathrm{Re}(\mathrm{spec}(\bar{A}(\tau))) - \frac{\sigma^2}{2}.$$

Thus, under these assumptions, Eq (3.7) becomes an exact expression rather than an approximation.

This approximation captures the leading-order effect of seasonality and noise on the growth rate near extinction.

Throughout the paper, whenever Eq (3.7) is used in a rigorous context, the commutativity and isotropic noise assumptions are explicitly imposed.

3.4. Noise-shifted extinction threshold

Let τ^* denote the deterministic critical season length satisfying

$$\max \Re(\mathrm{spec}(\bar{A}(\tau^*))) = 0.$$

Lemma 3.4. *(Monotonicity of the averaged growth rate). Assume that $Df_g(0)$ and $Df_d(0)$ commute. Then*

$$\bar{A}(\tau) = r_\varepsilon(\tau)Df_g(0) + (1 - r_\varepsilon(\tau))Df_d(0), \tag{3.8}$$

where $r_\varepsilon(\tau)$ is increasing in τ.

If, in addition, the matrix

$$Df_g(0) - Df_d(0)$$

is positive definite (or has strictly larger real parts of eigenvalues), then the function

$$\tau \mapsto \max \mathrm{Re}(\mathrm{spec}(\bar{A}(\tau)))$$

is strictly increasing.

Proof. Since $r_\varepsilon(\tau)$ is increasing in τ, the matrix $\bar{A}(\tau)$ is an affine combination of $Df_g(0)$ and $Df_d(0)$ with increasing weight on $Df_g(0)$.

By commutativity, the eigenvalues of $\bar{A}(\tau)$ are convex combinations of the eigenvalues of $Df_g(0)$ and $Df_d(0)$. Hence,

$$\lambda_i(\tau) = r_\varepsilon(\tau)\lambda_i^g + (1 - r_\varepsilon(\tau))\lambda_i^d. \tag{3.9}$$

If $\lambda_i^g > \lambda_i^d$ for all i, then each $\lambda_i(\tau)$ is increasing, and therefore

$$\max_i \mathrm{Re}(\lambda_i(\tau))$$

is increasing. □

Theorem 3.5 (Noise-shifted extinction threshold)**.** *Assume that:*

- *$B = \sigma I$ (isotropic multiplicative noise);*
- *the matrices $Df_g(0)$ and $Df_d(0)$ commute.*

Then the top Lyapunov exponent satisfies

$$\lambda(\tau) = \max \mathrm{Re}(\mathrm{spec}(\bar{A}(\tau))) - \frac{\sigma^2}{2}. \tag{3.10}$$

Moreover, there exists a critical value τ_σ^* such that:

- *$\lambda(\tau) < 0$ for $\tau < \tau_\sigma^*$;*
- *$\lambda(\tau) > 0$ for $\tau > \tau_\sigma^*$.*

Proof. Since the matrices $Df_g(0)$ and $Df_d(0)$ commute, they are simultaneously triangularizable over $\mathbb{C}$; there exists an invertible matrix P such that $P^{-1}Df_g(0)P = \Lambda_g$ and $P^{-1}Df_d(0)P = \Lambda_d$, where Λ_g and Λ_d are upper triangular matrices with eigenvalues $\{\lambda_i^g\}$ and $\{\lambda_i^d\}$ on the diagonal. Consequently, $P^{-1}A(t,\tau)P = \Lambda(t,\tau) = \Lambda_g\chi_g^\varepsilon(t,\tau) + \Lambda_d\chi_d^\varepsilon(t,\tau)$, which remains upper triangular. Hence the system decouples coordinatewise up to a triangular structure, and the top Lyapunov exponent equals the maximal exponent among the diagonal entries.

Set $Y(t) = P^{-1}X(t)$. Then $Y(t)$ satisfies $dY(t) = \Lambda(t,\tau)Y(t)\,dt + \sigma Y(t)\,dW(t)$. Writing componentwise, the i-th coordinate obeys $dY_i(t) = \alpha_i(t,\tau)Y_i(t)\,dt + \sigma Y_i(t)\,dW(t)$, with $\alpha_i(t,\tau) = \lambda_i^g\chi_g^\varepsilon(t,\tau) + \lambda_i^d\chi_d^\varepsilon(t,\tau)$.

Applying Ito's formula to $\log|Y_i(t)|$ yields

$$d\log|Y_i(t)| = (\alpha_i(t,\tau) - \frac{\sigma^2}{2})dt + \sigma\,dW(t). \tag{3.11}$$

Integrating from 0 to t gives

$$\log|Y_i(t)| = \log|Y_i(0)| + \int_0^t (\alpha_i(s,\tau) - \frac{\sigma^2}{2})ds + \sigma W(t). \tag{3.12}$$

Dividing by t and taking limits,

$$\frac{1}{t}\log|Y_i(t)| = \frac{1}{t}\int_0^t \alpha_i(s,\tau)\,ds - \frac{\sigma^2}{2} + \frac{\sigma W(t)}{t} + o(1). \tag{3.13}$$

Since $W(t)/t \to 0$ almost surely, we obtain the Lyapunov exponent

$$\lambda_i(\tau) = \lim_{t\to\infty}\frac{1}{t}\log|Y_i(t)| = \bar{\alpha}_i(\tau) - \frac{\sigma^2}{2}, \qquad \bar{\alpha}_i(\tau) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \alpha_i(s,\tau)\,ds. \tag{3.14}$$

By the periodicity of χ_g^ε and χ_d^ε, the time average simplifies to $\bar{\alpha}_i(\tau) = r_\varepsilon(\tau)\lambda_i^g + (1 - r_\varepsilon(\tau))\lambda_i^d$, where $r_\varepsilon(\tau)$ is the fraction of time spent in the favorable season. Hence

$$\lambda_i(\tau) = r_\varepsilon(\tau)\lambda_i^g + (1 - r_\varepsilon(\tau))\lambda_i^d - \frac{\sigma^2}{2}. \tag{3.15}$$

The top Lyapunov exponent is therefore

$$\lambda(\tau) = \max_i \lambda_i(\tau) = \max \operatorname{Re}(\operatorname{spec}(\bar{A}(\tau))) - \frac{\sigma^2}{2}, \tag{3.16}$$

with $\bar{A}(\tau) = r_\varepsilon(\tau)Df_g(0) + (1 - r_\varepsilon(\tau))Df_d(0)$.

By Lemma 3.4, the map $\tau \mapsto \max \operatorname{Re}(\operatorname{spec}(\bar{A}(\tau)))$ is continuous and monotone under the stated assumptions; thus $\lambda(\tau)$ is continuous and strictly increasing [16]. Consequently, there exists a unique threshold τ_σ^* such that $\lambda(\tau_\sigma^*) = 0$, and

$$\lambda(\tau) < 0 \text{ for } \tau < \tau_\sigma^*, \qquad \lambda(\tau) > 0 \text{ for } \tau > \tau_\sigma^*.$$

□

3.5. Persistence in probability

Theorem 3.6 (Stochastic persistence)**.** *Assume that the top Lyapunov exponent satisfies $\lambda(\tau) > 0$ and that the drift is dissipative outside a neighborhood of the origin. Then there exists $\varepsilon_0 > 0$ such that*

$$\liminf_{t\to\infty} \mathbb{P}(|U(t)| > \varepsilon_0) > 0.$$

Proof. Let $U(t)$ denote the solution of (2.5). By C^1 regularity, we write

$$f_g(u)\chi_g^\varepsilon(t,\tau) + f_d(u)\chi_d^\varepsilon(t,\tau) = A(t,\tau)u + R(t,u), \tag{3.17}$$

where there exists $\delta_1 > 0$ and $K > 0$ such that for all $|u| < \delta_1$,

$$|R(t,u)| \le K|u|^2, \quad \text{uniformly in } t.$$

Let $X(t)$ solve the linearized system. Since $\lambda(\tau) > 0$, there exists $\gamma > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}[\log|X(t)|] \ge \gamma.$$

Define the stopping time

$$\tau_\delta = \inf\{t > 0 : |U(t)| \ge \delta_1\}. \tag{3.18}$$

For $t \le \tau_\delta$, we estimate using Ito's formula applied to $\log|U(t)|$:

$$d\log|U(t)| = \frac{\langle U, A(t,\tau)U\rangle}{|U|^2}dt + \frac{\langle U, R(t,U)\rangle}{|U|^2}dt - \frac{1}{2}\frac{\|\Sigma(U)^\top U\|^2}{|U|^4}dt + dM(t), \tag{3.19}$$

where $M(t)$ is a local martingale.

Using the bound on $R(t,u)$, we obtain

$$\left|\frac{\langle U, R(t, U)\rangle}{|U|^2}\right| \le K|U(t)|.$$

Thus, for $|U(t)|$ sufficiently small,

$$\frac{d}{dt}\mathbb{E}[\log|U(t)|] \ge \gamma - C\mathbb{E}|U(t)|.$$

Choosing initial data sufficiently small ensures

$$\frac{d}{dt}\mathbb{E}[\log|U(t)|] \ge \frac{\gamma}{2}. \tag{3.20}$$

Integrating (3.20) yields

$$\mathbb{E}[\log|U(t \wedge \tau_\delta)|] \ge \log|U(0)| + \frac{\gamma}{2}t. \tag{3.21}$$

By Markov's inequality, there exists $\varepsilon > 0$ and $p > 0$ such that

$$\mathbb{P}(\tau_\delta < T) \ge p, \tag{3.22}$$

for some finite $T > 0$.

Thus, trajectories exit any neighborhood of the origin with positive probability [17].

Finally, by dissipativity, solutions remain in a compact set K, and recurrence arguments imply

$$\liminf_{t\to\infty} \mathbb{P}(|U(t)| > \varepsilon_0) > 0.$$

□

3.6. *Interpretation as stochastic bifurcation*

The transition at $\tau = \tau^*_\sigma$ constitutes a stochastic bifurcation: the extinction state loses stochastic stability, and the system admits nontrivial invariant probability measures or random periodic behavior. Unlike deterministic bifurcations, this transition is driven by changes in the sign of the Lyapunov exponent rather than the appearance of classical equilibria.

This bifurcation will be analyzed in greater detail in the next section.

4. Stochastic bifurcation and emergence of invariant measures

In this section, we characterize the stochastic bifurcation associated with the loss of stability of the extinction state at the critical season length τ^*_σ identified in Section 3. We show that this loss of stability is accompanied by the existence of nontrivial invariant probability measures, which describe persistent stochastic dynamics.

Definition 4.1 (Stochastic bifurcation)**.** *A stochastic bifurcation is said to occur at $\tau = \tau^*_\sigma$ if:*

1) *the extinction state $U(t) \equiv 0$ is exponentially stable in probability for $\tau < \tau^*_\sigma$;*
2) *the extinction state loses stability in probability at $\tau = \tau^*_\sigma$;*
3) *for $\tau > \tau^*_\sigma$, the system admits at least one invariant probability measure not concentrated at the extinction state [18].*

4.1. Tightness and existence of invariant measures

We first establish uniform moment bounds, which ensure tightness of time-averaged distributions.

Lemma 4.2 (Moment boundedness)**.** *Assume* (A1)–(A4). *For any* $\tau \in (0,1)$*, there exist* $p > 0$ *and* $C_p > 0$ *such that*

$$\sup_{t\geq 0} \mathbb{E}|U(t)|^p \leq C_p. \tag{4.1}$$

Proof. By Assumption (A4), solutions starting in the compact invariant set $K \subset \mathbb{R}_+^N$ remain in K almost surely. Hence both the drift and diffusion coefficients are uniformly bounded on K. Applying Itô's formula to $|U(t)|^p$ for $p \geq 2$, taking expectations, and using Grönwall's inequality yield the bound (4.1). □

4.2. Existence of invariant probability measures

Theorem 4.3 (Existence of invariant measures)**.** *For every* $\tau > \tau_\sigma^*$*, Eq* (2.5) *admits at least one invariant probability measure* μ_τ*.*

Proof. Define

$$\mu_T(A) = \frac{1}{T}\int_0^T \mathbb{P}(U(t) \in A)\, dt. \tag{4.2}$$

By Lemma 4.2, for some $p > 0$,

$$\sup_{t\geq 0} \mathbb{E}|U(t)|^p < \infty.$$

Thus, for any $R > 0$,

$$\sup_{T>0} \mu_T(\{|u| > R\}) \leq \frac{1}{R^p} \sup_{t\geq 0} \mathbb{E}|U(t)|^p. \tag{4.3}$$

Hence $\{\mu_T\}$ is tight. By Prokhorov's theorem, there exists μ_τ such that

$$\mu_{T_n} \Rightarrow \mu_\tau.$$

To prove invariance, we verify that for any bounded continuous function φ,

$$\int \varphi(u)\, d\mu_\tau(u) = \int P_t\varphi(u)\, d\mu_\tau(u). \tag{4.4}$$

This follows by standard semigroup arguments. Finally, from Theorem 3.6,

$$\inf_{T>0} \mu_T(\{|u| > \varepsilon_0\}) > 0, \tag{4.5}$$

hence $\mu_\tau(\{0\}) = 0$. □

4.3. Persistence and nontriviality of invariant measures

Theorem 4.4 (Nontrivial invariant measures)**.** *Assume $\tau > \tau_\sigma^*$. Then any invariant probability measure μ_τ associated with* (2.5) *is not concentrated at the extinction state, that is,*

$$\mu_\tau(\{0\}) = 0.$$

Proof. Assume by contradiction that $\mu_\tau = \delta_0$. Then for any bounded measurable φ,

$$\int \varphi(u)\, d\mu_\tau(u) = \varphi(0). \tag{4.6}$$

By invariance,

$$\varphi(0) = \int P_t\varphi(u)\, d\mu_\tau(u) = \mathbb{E}[\varphi(U(t;0))]. \tag{4.7}$$

Thus, $U(t) \equiv 0$ almost surely [19]. However, since $\lambda(\tau) > 0$, for any $\delta > 0$,

$$\mathbb{P}(|U(t)| > \delta) > 0$$

for some $t > 0$, which contradicts the above. Therefore $\mu_\tau \neq \delta_0$. □

4.4. Interpretation as stochastic bifurcation

Theorem 4.5 (Stochastic bifurcation)**.** *Equation* (2.5) *undergoes a stochastic bifurcation at $\tau = \tau_\sigma^*$ in the sense of Definition 4.1.*

Proof. Define $\lambda(\tau)$ as in (14).

- If $\lambda(\tau) < 0$, then by standard stochastic stability theory, $U(t) \to 0$ in probability.
- If $\lambda(\tau) > 0$, then by Theorem 3.6,

$$\liminf_{t\to\infty} \mathbb{P}(|U(t)| > \varepsilon_0) > 0.$$

  Thus extinction is unstable.

By Theorem 4.3, invariant measures exist, and by Theorem 4.4, they are nontrivial. Hence the qualitative behavior changes at $\lambda(\tau) = 0$, which defines a stochastic bifurcation. □

4.5. Remark on random periodic behavior

Remark 4.6. *Under additional assumptions, such as nondegenerate diffusion and smooth seasonal regularization ($\varepsilon > 0$), the periodic structure of the coefficients allows for the existence of random periodic solutions associated with invariant measures. A detailed analysis of random periodic solutions is beyond the scope of this paper.*

Remark 4.7. *Fractional-order extension. An important generalization of system (5) consists in replacing the classical time derivative by a Caputo fractional derivative of order $\alpha \in (0, 1)$:*

$$D_t^\alpha U(t) = \Big(f_g(U(t))\chi_g^\varepsilon(t,\tau) + f_d(U(t))\chi_d^\varepsilon(t,\tau)\Big) + \Sigma(U(t))\dot{W}(t).$$

In this setting, the system exhibits memory effects due to the nonlocal nature of the fractional derivative:

$$D_t^\alpha U(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{U}(s)}{(t-s)^\alpha} ds.$$

Consequences: (1) The Markov property is generally lost, and the system becomes non Markovian, which prevents direct application of Krylov-Bogolyubov theory.

(2) The notion of an invariant probability measure must be replaced by generalized invariant measures on path space or stationary distributions of the associated non-Markovian process.

(3) Lyapunov exponent theory must be modified, as exponential growth rates are replaced by Mittag–Leffler–type growth:

$$|U(t)| \sim E_\alpha(\lambda t^\alpha),$$

where E_α denotes the Mittag–Leffler function.

On invariant measures: Nontrivial invariant measures may still exist under suitable compactness and dissipativity assumptions, but their existence requires alternative approaches such as fractional semigroup theory or infinite dimensional embedding techniques. In particular, persistence can still be defined in probability, but its characterization becomes more delicate due to memory effects.

Importantly, while the qualitative notion of stochastic persistence remains meaningful, the existence and characterization of invariant measures in the fractional setting require fundamentally different analytical tools and are not directly covered by the present framework.

This extension represents a significant direction for future research.

5. Application to a stochastic Lotka–Volterra model and numerical experiments

In this section, we apply the general theory developed in Sections 3 and 4 to a stochastic Lotka–Volterra competition model with seasonal forcing. The aim is to illustrate how the season length parameter and environmental noise jointly influence extinction, persistence, and stochastic bifurcation phenomena.

5.1. Model formulation

We consider the two-species stochastic Lotka–Volterra competition model

$$\begin{aligned} dx(t) &= x(t)\big(a(t) - b_{11}x(t) - b_{12}y(t)\big)\,dt + \sigma_x x(t)\,dW_1(t), \\ dy(t) &= y(t)\big(d(t) - b_{22}y(t) - b_{21}x(t)\big)\,dt + \sigma_y y(t)\,dW_2(t), \end{aligned} \tag{5.1}$$

where W_1 and W_2 are independent standard Brownian motions, and $b_{ij} > 0$ represent inter- and intraspecific competition coefficients.

The intrinsic growth rates vary seasonally according to

$$a(t) = a_g\chi_g^\varepsilon(t,\tau) + a_d\chi_d^\varepsilon(t,\tau), \qquad d(t) = d_g\chi_g^\varepsilon(t,\tau) + d_d\chi_d^\varepsilon(t,\tau), \tag{5.2}$$

where $a_g, d_g > 0$ and $a_d, d_d < 0$.

System (5.1) preserves positivity and admits a unique global strong solution for any initial condition in $\mathbb{R}_+^2$.

5.2. Extinction equilibrium and linearization

The extinction equilibrium $(x, y) = (0, 0)$ is a solution of (5.1). Linearizing around the origin yields the diagonal linear system

$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a(t,\tau) & 0 \\ 0 & d(t,\tau) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} dt + \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} dW(t), \tag{5.3}$$

where $a(t, \tau)$ and $d(t, \tau)$ are defined in (5.2). This form allows explicit characterization of extinction thresholds via Lyapunov exponents.

5.3. Noise-shifted extinction persistence thresholds

Define the averaged growth rates

$$\bar{a}(\tau) = r_\varepsilon(\tau)a_g + (1 - r_\varepsilon(\tau))a_d, \qquad \bar{d}(\tau) = r_\varepsilon(\tau)d_g + (1 - r_\varepsilon(\tau))d_d. \tag{5.4}$$

Theorem 5.1 (Noise-shifted extinction thresholds)**.** *1) If $\bar{a}(\tau) < \sigma_x^2/2$, then the extinction state of species x is exponentially stable in probability.*

2) If $\bar{a}(\tau) > \sigma_x^2/2$, then the extinction of species x is unstable in probability.

An analogous statement holds for species y with $\bar{d}(\tau)$ and σ_y.

Proof. Near the boundary $x = 0$, the x equation satisfies

$$dx(t) = x(t)a(t)\,dt + \sigma_x x(t)\,dW_1(t) + O(x^2).$$

Applying Itô's formula to $\log x(t)$ yields

$$d\log x(t) = \big(a(t) - \tfrac{\sigma_x^2}{2}\big)\,dt + \sigma_x\,dW_1(t) + O(x). \tag{5.5}$$

Averaging over time and using the periodicity of the coefficients gives

$$\lambda_x(\tau) = \bar{a}(\tau) - \frac{\sigma_x^2}{2}.$$

The conclusions follow from the sign of $\lambda_x(\tau)$ and Proposition 3.2. □

5.4. Stochastic persistence and invariant measures

Theorem 5.2 (Stochastic persistence)**.** *Assume*

$$\bar{a}(\tau) > \frac{\sigma_x^2}{2}, \qquad \bar{d}(\tau) > \frac{\sigma_y^2}{2}. \tag{5.6}$$

Then system (5.1) *is persistent in probability and admits at least one invariant probability measure supported in* $\mathbb{R}_+^2 \setminus \{(0, 0)\}$.

Proof. Under condition (5.6), both boundary subsystems ($x = 0$ and $y = 0$) are unstable in probability. Consider the Lyapunov function

$$V(x, y) = \log x + \log y, \qquad (x, y) \in \mathbb{R}_+^2.$$

Applying Itô's formula yields

$$\mathcal{L}V = a(t) + d(t) - \frac{\sigma_x^2 + \sigma_y^2}{2} - (b_{11} + b_{21})x - (b_{22} + b_{12})y. \tag{5.7}$$

Averaging over one period, the drift of V is positive near the boundary of $\mathbb{R}_+^2$. Dissipativity of the nonlinear terms ensures tightness of trajectories. The existence of an invariant probability measure then follows from the Krylov–Bogolyubov procedure, and persistence in probability follows as in Theorem 3.6. □

5.5. Stochastic coexistence

Theorem 5.3 (Stochastic coexistence)**.** *Assume* (5.6) *and*

$$\frac{b_{12}}{b_{22}} < \frac{a_g}{d_g}, \qquad \frac{b_{21}}{b_{11}} < \frac{d_g}{a_g}. \tag{5.8}$$

Then system (5.1) *admits at least one invariant probability measure supported in the interior of* $\mathbb{R}_+^2$.

Proof. Conditions (5.8) ensure coexistence during the favorable season in the deterministic system. Combined with stochastic persistence and nondegenerate diffusion, this implies the existence of an invariant probability measure supported in the interior. Uniqueness is not claimed. □

5.6. Numerical experiments

We present numerical simulations to illustrate the analytical results above. All simulations are performed using the Euler-Maruyama scheme with a sufficiently small time step $\Delta t = 10^{-3}$. The simulations are intended to be illustrative and do not constitute proofs [20, 21].

5.6.1. Noise-delayed persistence

For parameters

$$a_g = 2, \quad a_d = -1, \quad \sigma_x = 0.5, \quad b_{11} = b_{12} = 0.5,$$

the deterministic system predicts persistence when $\bar{a}(\tau) > 0$, corresponding to $\tau \approx 0.33$. With noise, Theorem 5.1 predicts instability of extinction only when $\bar{a}(\tau) > 0.125$, yielding $\tau \approx 0.41$. Simulations are consistent with a noise-induced delay of persistence.

5.6.2. Noise-induced extinction

Increasing the noise intensity to $\sigma_x = 1$, the threshold $\bar{a}(\tau) > \sigma_x^2/2 = 0.5$ is not satisfied for any $\tau \in (0, 1)$. Numerical sample paths indicate convergence toward extinction for all τ, illustrating noise-induced extinction.

5.6.3. Stochastic coexistence

For symmetric parameters

$$a_g = d_g = 2, \quad a_d = d_d = -1, \quad \sigma_x = \sigma_y = 0.4,$$

satisfying (5.8), numerical simulations show both species persisting with fluctuating densities. Empirical distributions appear to stabilize, consistent with the existence of an interior invariant probability measure.

5.6.4. Summary

The numerical experiments provide quantitative validation of the theoretical results:

(1) *Noise-shifted thresholds:* The critical condition for persistence is no longer given by $\bar{a}(\tau) > 0$, but instead by $\bar{a}(\tau) > \frac{\sigma^2}{2}$, showing that stochastic perturbations effectively reduce the growth rate through variance-induced drift corrections.

(2) *Delay and suppression of persistence:* Even when deterministic models predict survival, stochastic fluctuations can delay the onset of persistence or completely suppress it by increasing the effective extinction region in parameter space.

(3) *Invariant measure-driven dynamics:* Instead of convergence to equilibria, long-term behavior is characterized by invariant probability distributions, reflecting sustained stochastic fluctuations and ergodic dynamics.

6. Numerical visualization and interpretation

In this section, we present numerical visualizations illustrating the theoretical results developed in Sections 3–5. The figures demonstrate extinction, stochastic persistence, noise-shifted bifurcation thresholds, and the emergence of invariant probability measures in the stochastic seasonal Lotka-Volterra system. All simulations are performed using the Euler-Maruyama scheme with a sufficiently small time step and are intended to be illustrative of the analytical results.

6.1. Extinction below the stochastic threshold

Figure 1 corresponds to a parameter regime in which the averaged seasonal growth rate is insufficient to overcome environmental noise. Despite intermittent favorable seasons, stochastic fluctuations dominate the dynamics, driving the population toward extinction. This behavior is consistent with the stochastic stability of the extinction state predicted by Theorem 5.1.

6.2. Persistence above the stochastic threshold

In Figure 2, when the averaged growth rate exceeds the noise-corrected threshold, extinction becomes unstable in probability. Rather than converging to a deterministic equilibrium, the population exhibits persistent stochastic fluctuations. This qualitative change reflects a stochastic bifurcation and is consistent with the persistence results established in Theorems 3.6 and 5.2.

6.3. Noise-induced extinction

Figure 3 highlights a purely stochastic effect. Increasing the noise intensity shifts the extinction persistence threshold beyond biologically feasible season lengths. As predicted by Theorem 5.1, persistence is lost even though deterministic averaging suggests survival.

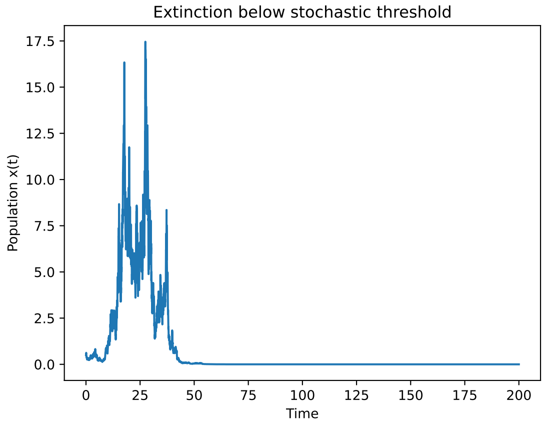

Figure 1. Extinction below the noise-shifted threshold. Time series of the population density $x(t)$ for $\bar{a}(\tau) < \sigma_x^2/2$. The population decays toward extinction, illustrating stochastic stability of the extinction state.

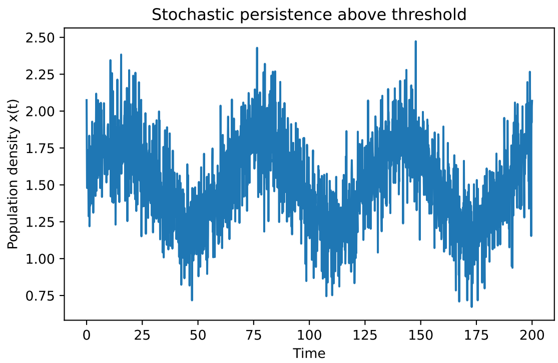

Figure 2. Stochastic persistence above the threshold. Time series of $x(t)$ for $\bar{a}(\tau) > \sigma_x^2/2$. The population exhibits sustained stochastic fluctuations and remains away from extinction over long time intervals.

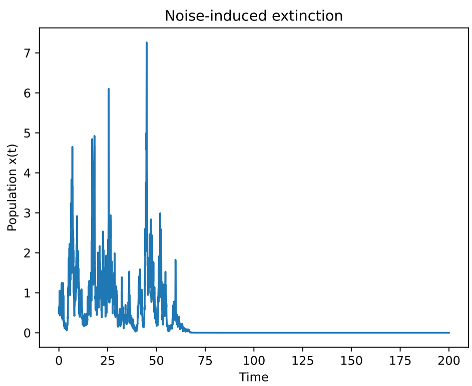

Figure 3. Noise-induced extinction. Increasing noise intensity leads to extinction even in parameter regimes where the deterministic model predicts persistence.

6.4. Noise-shifted bifurcation threshold

Figure 4 provides a direct visualization of the stochastic bifurcation mechanism. Environmental noise shifts the extinction–persistence threshold to larger values of the favorable season length, delaying persistence relative to the deterministic case. This visualization complements the analytical threshold conditions derived in Sections 3 and 5.

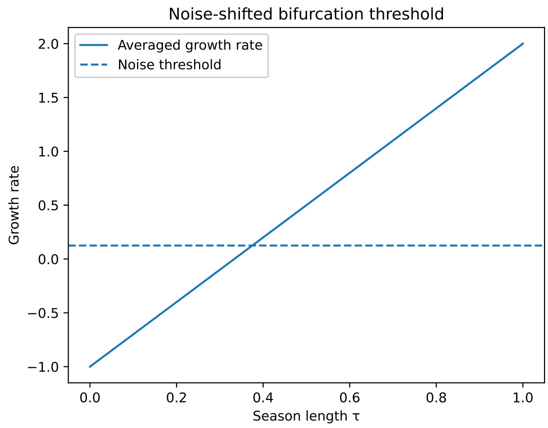

Figure 4. Noise-shifted bifurcation threshold. The averaged seasonal growth rate $\bar{a}(\tau)$ is plotted as a function of the season length τ. The dashed line represents the stochastic threshold $\sigma_x^2/2$, whose intersection with $\bar{a}(\tau)$ determines the critical value τ_σ^*.

6.5. Invariant probability measure

Figure 5 illustrates that, unlike deterministic systems, stochastic persistence does not correspond to convergence toward a fixed equilibrium. Instead, trajectories exhibit long-term fluctuations whose empirical distribution stabilizes. This behavior is consistent with the existence of nontrivial invariant probability measures established in Section 4.

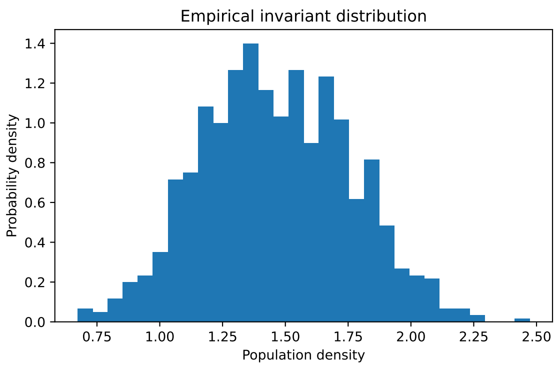

Figure 5. Empirical invariant distribution. Histogram of long-term population values for $\bar{a}(\tau) > \sigma_x^2/2$, illustrating convergence in distribution toward a nontrivial invariant probability measure.

6.6. *Stochastic bifurcation diagram*

Figure 6 summarizes the global effect of stochasticity on the system dynamics. Instead of sharp deterministic bifurcation branches, stochastic fluctuations produce a smooth transition between extinction and persistence. The qualitative change near τ_σ^* represents a stochastic bifurcation in the sense of Definition 4.1.

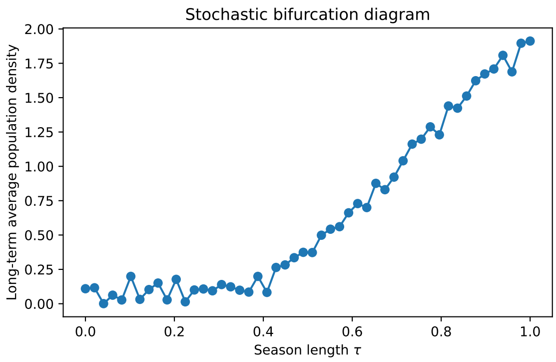

Figure 6. Stochastic bifurcation diagram showing the long-term average population density as a function of the season length τ. The transition near τ_σ^* is smoothed by stochastic fluctuations.

6.7. *Summary of visual evidence*

Taken together, Figures 1–6 provide numerical illustrations of the theoretical framework developed in this paper:

- Extinction is stochastically stable below the noise-shifted threshold;
- persistence emerges through invariant probability measures rather than fixed equilibria;
- stochastic bifurcation replaces classical deterministic bifurcation;
- seasonal forcing amplifies the impact of noise near critical parameter values.

These observations highlight that stochastic seasonal effects can fundamentally alter long-term dynamics and should be incorporated into realistic ecological and biological models.

7. Conclusion and perspectives

We established a rigorous framework for analyzing extinction, persistence, and stochastic bifurcation in seasonally forced stochastic systems. The transition from extinction to persistence is governed by the sign of the top Lyapunov exponent, yielding a noise-shifted critical season length τ_σ^*. Beyond this threshold, dynamics are described by nontrivial invariant probability measures rather than deterministic equilibria.

Future work includes extending the analysis to fractional-order stochastic systems, incorporating Levy noise, and studying uniqueness and ergodic properties of invariant measures. Applications to high-dimensional ecological and epidemiological systems with seasonal forcing remain an important direction.

We believe that the framework developed here provides a flexible foundation for analyzing stochastic seasonal systems across a broad range of applications, including ecology, epidemiology, and climate-driven population dynamics.

Connections with biological applications and future directions. The theoretical framework developed in this paper can be applied to a wide class of biological systems, including epidemiological and ecological models with seasonal forcing. In particular, the threshold-based characterization of persistence and extinction is closely related to the basic reproduction number and stability thresholds commonly studied in infectious disease models.

Author contributions

Shah Hussain contributed to writing the original draft, software, methodology, and formal analysis; Thoraya N. Alharthi contributed to writing, review and editing, visualization, validation, supervision, and conceptualization. All authors have agreed to and given their consent for the publication of this research paper.

Use of Generative-AI tools declaration

All authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. P. Ashwin, S. Wieczorek, R. Vitolo, P. Cox, Tipping points in open systems: Bifurcation, noise-induced and rate-dependent examples in the climate system, *Philos. Trans. A Math. Phys. Eng. Sci.*, **370** (2012), 1166–1184. https://doi.org/10.1098/rsta.2011.0306
2. R. Bastiaansen, H. A. Dijkstra, A. S. von der Heydt, Fragmented tipping in a spatially heterogeneous world, *Environ. Res. Lett.*, **17** (2022), 045006. https://doi.org/10.1088/1748-9326/ac59a8
3. A. Kumar, U. S. Dubey, B. Dubey, Bifurcation and chaotic dynamics in a spatiotemporal epidemic model with delayed optimal control, stochastic process, and sensitivity analysis, *Chaos*, **35** (2025), 033160. https://doi.org/10.1063/5.0251992
4. X. E. Zhao, J. Shi, On determination of the bifurcation type for a free boundary problem modeling tumor growth, *J. Differ. Equ.*, **436** (2025), 113352. https://doi.org/10.1016/j.jde.2025.113352

5. M. G. Crandall, P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.*, **8** (1971), 321–340. https://doi.org/10.1016/0022-1236(71)90015-2
6. J. C. Huang, Z. S. Qu, R. Varennes, Y. W. Cho, X. Garbet, C. G. WanG, et al., Extracting a stochastic model for predator-prey dynamic of turbulence and zonal flows with limited data, *Nucl. Fusion*, **66** (2026), 036010. https://doi.org/10.1088/1741-4326/ae36c1
7. F. Brauer, Backward bifurcations in simple vaccination/treatment models, *J. Biol. Dynam.*, **5** (2011), 410–418. https://doi.org/10.1080/17513758.2010.510584
8. G. Rozhnova, A. Nunes, Stochastic effects in a seasonally forced epidemic model, *Phys. Rev. E*, **82** (2010), 041906. https://doi.org/10.1103/PhysRevE.82.041906
9. M. Liu, M. Fan, Permanence of stochastic Lotka-Volterra systems, *J. Nonlinear Sci.*, **27** (2017), 425–452. https://doi.org/10.1007/s00332-016-9337-2
10. T. Reichenbach, M. Mobilia, E. Frey, Coexistence versus extinction in the stochastic cyclic Lotka-Volterra model, *Phys. Rev. E*, **74** (2006), 051907. https://doi.org/10.1103/PhysRevE.74.051907
11. G. Galiano, J. Velasco, Bifurcation in dynamic problems with seasonal succession, *Nonlinear Anal.-Real*, **92** (2026), 104648. https://doi.org/10.1016/j.nonrwa.2026.104648
12. Z. Li, J. Liu, L. Zhou, C. Shan, M. Fan, Model formulation and oscillatory patterns in immune-pathogen dynamics during Brucellosis infection, *Math. Biosci.*, **390** (2025), 109543. https://doi.org/10.1016/j.mbs.2025.109543
13. X. He, T. Lv, J. Liu, L. Zhou, M. Fan, Dynamical modeling and optimal control of brucellosis with mixed delays and environmental transmission, *Int. J. Biomath.*, **11** (2025), 2550136. https://doi.org/10.1142/S1793524525501360
14. C. Ji, D. Jiang, Q. Yang, N. Shi, Dynamics of a multigroup SIR epidemic model with stochastic perturbation, *Automatica*, **48** (2012), 121–131. https://doi.org/10.1016/j.automatica.2011.09.044
15. C. A. Braumann, *Introduction to stochastic differential equations with applications to modelling in biology and finance*, 1 Eds., John Wiley and Sons, 2019. 10.1002/9781119166092
16. A. Novikov, Lévy processes and stochastic calculus, *J. Am. Stat. Assoc.*, **100** (2005), 1461. https://doi.org/10.1198/jasa.2005.s53
17. A. Ashyralyev, U. Okur, Stability of stochastic partial differential equations, *Axioms.*, **12** (2023), 718. https://doi.org/10.3390/axioms12070718
18. X. T. Ong, R. Van Handel, Conditional ergodicity in infinite dimension, *Ann. Probab.*, **42** (2014), 2243–2313. https://doi.org/10.1214/13-AOP879
19. L. Arnold, N. Sri Namachchivaya, K. R. Schenk-Hoppé, Toward an understanding of stochastic Hopf bifurcation: A case study, *Int. J. Bifurcat. Chaos*, **6** (1996), 1947–1975. https://doi.org/10.1142/S0218127496001272
20. A. Hening, D. H. Nguyen, G. Yin, Stochastic population growth in spatially heterogeneous environments: The density-dependent case, *J. Math. Biol.*, **76** (2018), 697–754. https://doi.org/10.1007/s00285-017-1153-2
21. S. J. Schreiber, M. Benaïm, K. A. Atchadé, Persistence in fluctuating environments, *J. Math. Biol.*, **62** (2011), 655–683. https://doi.org/10.1007/s00285-010-0349-5

Appendix

Numerical implementation details

All numerical simulations in Sections 5 and 6 were performed using the Euler-Maruyama scheme for stochastic differential equations.

Time discretization. The time step is fixed at $\Delta t = 10^{-3}$, which is sufficiently small to ensure numerical stability and accuracy of the stochastic integrator.

Simulation horizon. Each trajectory is simulated over the time interval $t \in [0, T], \quad T = 200$.

Burn-in period. To eliminate transient effects, an initial burn-in period of length $T_{\text{burn}} = 50$ is discarded. All statistical quantities are computed using data from $t \geq T_{\text{burn}}$.

Initial conditions. Unless otherwise specified, simulations are initialized at $U(0) = (1, 1) \in \mathbb{R}^2_+$.

Number of realizations. Each experiment is repeated over $N = 500$ independent sample paths in order to estimate statistical quantities such as empirical distributions and long-term averages.

Random number generation. All stochastic simulations use pseudo-random numbers generated with a fixed seed for reproducibility. Specifically, we use the seed value `seed` = 12345.

Statistical estimation. Empirical invariant distributions are approximated using normalized histograms constructed from pooled trajectory data after the burn-in period. Long-term averages are computed as $\frac{1}{T-T_{\text{burn}}} \int_{T_{\text{burn}}}^{T} U(t)\, dt$.

Remarks on numerical accuracy. The chosen time step and number of realizations were verified to produce stable and consistent results under refinement, ensuring that the observed phenomena are not numerical artifacts [11].

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