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Research article

A stochastic enterprise cluster model with mean-reverting Ornstein-Uhlenbeck process

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Abstract: In this paper, we construct and analyze a stochastic enterprise cluster model with a mean-reverting Ornstein-Uhlenbeck process. We first investigate the existence of a global unique positive solution for the system. After that, the stochastically ultimate boundedness of the system is considered. Based on a series of Lyapunov functions, sufficient criteria for the system's dynamic behaviors, including exponential extinction and persistence in the mean of the system, are established. Two numerical examples validate the theoretical results of this paper. This article provides a theoretical basis for promoting the development of enterprise clusters.

Keywords: enterprise cluster model; Ornstein-Uhlenbeck process; stochastic; Lyapunov function; extinction and persistence
Mathematics Subject Classification: 34B13, 60H10

1. Introduction

In 2006, Tian and Nie [1] considered the following autonomous enterprise cluster model:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\Big(a_1 - b_{11}x(t) - b_{12}(y(t)-c_2)^2\Big) \\ \frac{dy(t)}{dt} &= y(t)\Big(a_2 - b_{21}y(t) + b_{22}(x(t)-c_1)^2\Big), \end{aligned} \tag{1.1}$$

where x and y denote the output of two enterprises, respectively, a_i $(i = 1, 2)$ is intrinsic growth rate, b_{ij} $(i, j = 1, 2)$ denotes the competitive coefficients, and c_i $(i = 1, 2)$ denotes the initial production of the enterprise. There is competition and cooperation between different enterprise clusters, so the establishment of a biological population model can be used to construct enterprise cluster models. Liao, Xu, and Tang [2] studied the competition and cooperation system of two enterprises with delay,

$$\frac{dx(t)}{dt} = x(t)\Big(a_1 - b_{11}x(t-\tau) - b_{12}(y(t-\tau) - c_2)^2\Big)$$
$$\frac{dy(t)}{dt} = y(t)\Big(a_2 - b_{21}y(t-\tau) + b_{22}(x(t-\tau) - c_1)^2\Big),$$

where $\tau > 0$ is the delay. If $b_{21}^2 d_1 > b_{12} d_2^2$, the above system exists a unique positive equilibrium, where $d_1 = a_1 - b_{11}c_1$, and $d_2 = a_2 - b_{21}c_2$. Furthermore, when $\tau = 0$, the positive equilibrium of the system is asymptotically stable. The authors also studied the complex Hopf bifurcation phenomenon at the positive equilibrium under different delays. In order to investigate the impact of different time delays on the dynamic behavior of enterprise cluster models, they further studied enterprise cluster models with two time delays in [3]. Using the continuation theorem of coincidence degree theory and the Lyapunov functional method, Xu and Shao [4] investigated an enterprise cluster model with impulse and obtained the existence and global attractivity of the positive periodic solution. The authors of [5] considered an enterprise cluster model on time scales. Based on the comparison theorem on time scales and the construction of a suitable Lyapunov functional, the permanence of an asymptotically stable and almost periodic solution of the system was obtained. Muhammadhaji and Maimaiti [6] dealt with an enterprise cluster model involving discrete time delays and feedback controls. They proposed new criteria for the permanence, periodic solution, and global attractiveness of the model.

Enterprise clusters are located in complex external environments and are subject to various uncontrollable factors, making the study of stochastic enterprise cluster models particularly important. For studying the impact of stochastic environmental noise on the intrinsic growth rate a_i, we can take two ways: The first way to interfere with a_i is by linear Gaussian white noise

$$a_i = \bar{a}_i + \nu_i \frac{dB_i(t)}{dt}, \ i = 1, 2,$$

where $\bar{a}_i$ denotes the mean level of the maximal intrinsic growth rate a_i, which is a positive constant; $B_i(t)$ is a standard Wiener process; and ν_i represents the intensity of environmental noise. The second way to interfere with a_i is by a classical mean-reverting Ornstein-Uhlenbeck process [7], where we have

$$da_i(t) = \alpha_1(\bar{a}_i - a_i(t))dt + \alpha_2 dB_i(t), \tag{1.2}$$

where α_1 and α_2 are positive constants which denote the speed of reversion and the intensity of volatility, respectively. From the discussions in [8] and [9], we have

$$a_i(t) = \bar{a}_i + (a_i(0) - \bar{a}_i)e^{-\alpha_1 t} + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1 t}}\frac{dB_i(t)}{dt}. \tag{1.3}$$

Combining systems (1.1) and (1.2), we can get the following system:

$$\begin{aligned} dx(t) &= x(t)\Big(a_1 - b_{11}x(t) - b_{12}(y(t) - c_2)^2\Big)dt \\ dy(t) &= y(t)\Big(a_2 - b_{21}y(t) + b_{22}(x(t) - c_1)^2\Big)dt, \\ da_i(t) &= \alpha_1(\bar{a}_i - a_i(t))dt + \alpha_2 dB_i(t). \end{aligned} \tag{1.4}$$

Furthermore, combining systems (1.3) and (1.4), we can get the following system:

$$\begin{aligned} dx(t) &= x(t)\Big(\bar{a}_1 + (a_1(0) - \bar{a}_1)e^{-\alpha_1 t} - b_{11}x(t) - b_{12}(y(t) - c_2)^2\Big)dt \\ &\quad + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1 t}}x(t)dB_1(t) \\ dy(t) &= y(t)\Big(\bar{a}_2 + (a_2(0) - \bar{a}_2)e^{-\alpha_1 t} - b_{21}y(t) + b_{22}(x(t) - c_1)^2\Big)dt \\ &\quad + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1 t}}y(t)dB_2(t). \end{aligned} \tag{1.5}$$

The mean-reverting Ornstein-Uhlenbeck process was first introduced by Dixit and Pindyck in [10] for studying investment. Allen [11] studied a stochastic Ornstein-Uhlenbeck process (also known as the mean-reverting process), which has many advantages over linear functions of white noise. Song and Zhang [12] established a new epidemic model incorporating the Ornstein-Uhlenbeck process and obtained stationary distribution and extinction of the system. In [13], assuming that the intrinsic growth rate and the death rate are governed by the Ornstein-Uhlenbeck process, the authors studied a stochastic predator-prey model with distributed delay and obtained rich dynamic properties, including stationary distribution, a probability density function, and extinction. For more recent advances concerning the Ornstein-Uhlenbeck process and stochastic models, see [14–18] and related references.

To be best our knowledge, few people have studied the enterprise cluster under the influence of color noise. Therefore, System (1.5) is a new breakthrough in the dynamical behavior of the stochastic enterprise cluster model. Here, we study the solution characteristics of this model and explore its rich dynamic behaviors.

In this paper, let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ which satisfies the usual conditions, and $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$.

We list the main innovations of this paper as follows:

(1) There are no papers for studying a stochastic enterprise cluster model with a mean-reverting Ornstein-Uhlenbeck process. This study fills the gap in the aforementioned research.

(2) We use Lyapunov functions and the stochastic analysis method for obtaining the existence, stochastically ultimate boundedness, exponential extinction, and persistence in the mean of the system.

(3) System (1.5) is a stochastic enterprise cluster model which can more accurately reflect the actual situation than previous models.

This paper is organized as follows. In Section 2 , we show the existence and uniqueness of global positive solutions and the stochastically ultimate boundedness of System (1.5). In Section 3, sufficient conditions for the exponential extinction of the enterprise x are given. In Section 4, we study persistence in the mean of System (1.5). In Section 5, two numerical simulations are given for verifying our theoretical results. In Section 6, we conclude the paper and discuss future work.

2. Existence and stochastically ultimate boundedness

In this section, we will give the global positive existence conditions and stochastically ultimate boundedness for System (1.5).

Theorem 2.1. For each initial value $(x(0), y(0)) \in \mathbb{R}^2_+$, System (1.5) has a unique positive solution

$(x(t), y(t))$ on $t \geq 0$, and the solution will stay in $\mathbb{R}^2_+$ with probability 1.

Proof. The initial part of the proof is similar to that of [9]. Hence, we omit it here and only provide the most critical step, that is, to construct the C^2-function $\mathcal{W}_0 : \mathbb{R}^2_+ \to \mathbb{R}_+$:

$$\mathcal{W}_0(x, y) = x^{p_0} - 1 - p_1 \ln x + p_2(y - 1 - \ln y),$$

where $p_i (i = 0, 1, 2)$ is a positive constant which will be determined later. It is easy to see that $\mathcal{W}_0(x, y)$ is continuous and goes to ∞ as (x, y) goes to the boundary of $\mathbb{R}^2_+$. Obviously, $\mathcal{W}_0(x, y)$ has a minimum value N_0. Then, we can define the C^2-function $\mathcal{W}$ by

$$\mathcal{W}(x, y) = x^{p_0} - 1 - p_1 \ln x + p_2(y - 1 - \ln y) - N_0.$$

It follows by Itô′s formula to the above function that

$$\begin{aligned} d\mathcal{W} &= \mathcal{L}\mathcal{W}dt + (p_0 x^{p_0-1} - p_1 x^{-1})\frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1 t}}x(t)dB_1(t) \\ &+ p_2(1 - \frac{1}{y})\frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1 t}}y(t)dB_2(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}\mathcal{W} &= (p_0 x^{p_0-1} - p_1 x^{-1})x(t)\Big(\bar{a}_1 + (a_1(0) - \bar{a}_1)e^{-\alpha_1 t} - b_{11}x(t) - b_{12}(y(t) - c_2)^2\Big) \\ &+ \Big(p_0(p_0 - 1)x^{p_0} + p_1\Big)\frac{\alpha_2^2}{4\alpha_1}(1 - e^{-2\alpha_1 t}) \\ &+ p_2(1 - \frac{1}{y})y(t)\Big(\bar{a}_2 + (a_2(0) - \bar{a}_2)e^{-\alpha_1 t} - b_{21}y(t) + b_{22}(x(t) - c_1)^2\Big) + \frac{p_2\alpha_2^2}{4\alpha_1}(1 - e^{-2\alpha_1 t}) \\ &\leq \bar{a}_1 p_0 x^{p_0} + (a_1(0) - \bar{a}_1)e^{-\alpha_1 t}p_0 x^{p_0} - b_{11}p_0 x^{p_0+1} + p_1 b_{11}x + p_1 b_{12}(y - c_2)^2 \\ &+ p_0(p_0 - 1)\frac{\alpha_2^2}{4\alpha_1}x^{p_0} + p_1\frac{\alpha_2^2}{4\alpha_1} \\ &+ p_2\bar{a}_2 y + p_2 a_2(0)y + p_2\bar{a}_2 + p_2 b_{21}y - p_2 b_{21}y^2 + p_2 y b_{22}(x - c_1)^2 + p_2\frac{\alpha_2^2}{4\alpha_1} \\ &\leq [\bar{a}_1 p_0 + a_1(0)p_0 + p_1 b_{11} + p_0^2\frac{\alpha_2^2}{4\alpha_1}]x^{p_0} - p_0 b_{11}x^{p_0+1} + 0.5p_2 b_{22}x^4 \\ &+ [\bar{a}_2 p_2 + a_2(0)p_2 + p_2 b_{22}c_1^2 + p_2 b_{21}]y + [p_1 b_{12} + 0.5p_2 b_{22} - p_2 b_{21}]y^2 \\ &+ p_1\bar{a}_1 + p_1\frac{\alpha_2^2}{4\alpha_1} + p_1 b_{12}c_2^2 + \bar{a}_2 p_2 + p_2\frac{\alpha_2^2}{4\alpha_1}. \end{aligned}$$

We can choose p_i $(i = 0, 1, 2)$ satisfying

$$p_0 > 3, \ p_1 b_{12} + 0.5p_2 b_{22} - p_2 b_{21} < 0.$$

Thus,

$$\mathcal{L}\mathcal{W} \leq C,$$

where C is a positive constant. The remaining proof is similar to the one of Theorem 2.1 in [9], so we omit it. This completes the proof.

The ultimate boundedness of solutions demonstrates the global existence of the system. Therefore, studying the properties of these solutions can provide a deeper understanding of the system. For System (1.5), we give the following definition:

Definition 2.1. System (1.5) is said to be stochastically ultimately bounded if for each $\varsigma \in (0,1)$ and initial data $(x(0), y(0)) \in \mathbb{R}^2_+$, there exists a positive constant $L(\varsigma)$ such that the solution $z(t) = (x(t), y(t))$ of System (1.5) satisfies the following property:

$$\lim_{t\to\infty} \sup P(|z(t)| \le L) \ge 1 - \varsigma.$$

In this section, because the outputs of x and y are limited, we assume that x and y have maximum value $K > 0$.

Theorem 2.2. For any given initial data $(x(0), y(0)) \in \mathbb{R}^2_+$. Then, System (1.5) is stochastically ultimately bounded.

Proof. Let

$$V(x,y) = \frac{1}{\theta}\Big(x^\theta + y^\theta\Big),\ x, y \in \mathbb{R}_+,\ \theta \in (0,1].$$

Using Itô′s formula, System (1.5) leads to

$$dV(x,y) = \mathcal{L}V(x,y)dt + x^\theta \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t) + y^\theta \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_2(t), \tag{2.1}$$

where

$$\begin{aligned}
\mathcal{L}V(x,y) = {} & x^\theta\Big(\bar{a}_1 + (a_1(0) - \bar{a}_1)e^{-\alpha_1 t} - b_{11}x(t) - b_{12}(y(t) - c_2)^2\Big) \\
& + 0.25(\theta - 1)\frac{\alpha_2^2}{4\alpha_1}(1 - e^{-2\alpha_1 t})x^\theta \\
& + y^\theta\Big(\bar{a}_2 + (a_2(0) - \bar{a}_2)e^{-\alpha_1 t} - b_{21}y(t) + b_{22}(x(t) - c_1)^2\Big) \\
& + 0.25(\theta - 1)\frac{\alpha_2^2}{4\alpha_1}(1 - e^{-2\alpha_1 t})y^\theta \\
& \le (\bar{a}_1 + a_1(0))x^\theta - b_{11}x^{1+\theta} + (\bar{a}_2 + a_2(0) + b_{22}c_1^2)y^\theta - b_{21}y^{1+\theta} + 0.5b_{22}y^{2\theta} + 0.5b_{22}K^{4-2\theta}x^{2\theta} \\
& \le L,
\end{aligned}$$

where L is a positive constant. Substituting this into (2.1) gives

$$dV(x,y) \le Ldt + x^\theta \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t) + y^\theta \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_2(t). \tag{2.2}$$

By (2.2) we have

$$d\Big[e^t V(x,y)\Big] \le Le^t dt + x^\theta \frac{e^t\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t) + y^\theta \frac{e^t\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_2(t).$$

Thus,

$$e^t\mathbb{E}V(x,y) \le e^t V(x(0), y(0)) + Le^t - L$$

which implies that

$$\lim_{t\to\infty}\sup \mathbb{E}V(x(t), y(t)) \leq L.$$

Let $z = (x, y)$. We also have

$$|z|^2 \leq 2\Big(\max\{x, y\}\Big)^2$$

and

$$|z|^\theta \leq 2^{\frac{\theta}{2}}\Big(\max\{x, y\}\Big)^\theta \leq 2^{\frac{\theta}{2}}\Big(\frac{1}{\theta}\Big)^\theta V(x, y).$$

Thus,

$$\lim_{t\to\infty}\sup \mathbb{E}|z|^\theta \leq 2^{\frac{\theta}{2}}\Big(\frac{1}{\theta}\Big)^\theta L := \widetilde{L}.$$

For any $\varsigma > 0$. let $k = \frac{\widetilde{L}}{\varsigma}$. By Chebyshev's inequality, we get

$$P(|z| > k) \leq \frac{\mathbb{E}|z|}{k} \leq \frac{\widetilde{L}}{k} = \varsigma,$$

which results in

$$P(|z| \leq k) \geq 1 - \varsigma.$$

The proof is completed.

3. Extinction

In this section, we investigate the extinction of x. Denote $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(s)ds$. Using Theorem 2.2, we can assume that

$$x \vee y \leq H \text{ almost surely,} \tag{3.1}$$

where $H > 0$ is a constant.

Definition 3.1. [19] The population $Z(t)$ is called
(1) Extinct if $\lim_{t\to\infty} Z(t) = 0$ almost surely;
(2) Weakly persistent in the mean if $\lim_{t\to\infty}\sup \frac{1}{t}\int_0^t Z(s)ds > 0$ almost surely;
(3) Persistent in the mean if $\lim_{t\to\infty}\inf \frac{1}{t}\int_0^t Z(s)ds > 0$ almost surely.

Lemma 3.1. [20] Let $Y = \{Y_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. It follows that:
(1) If $\lim_{t\to\infty}\langle Y, Y\rangle = \infty$ almost surely, then $\lim_{t\to\infty}\frac{Y_t}{\langle Y,Y\rangle} = 0$ almost surely, and
(2) If $\lim_{t\to\infty}\frac{\langle Y,Y\rangle}{t} < \infty$ almost surely, then $\lim_{t\to\infty}\frac{Y_t}{t} = 0$ almost surely.

Theorem 3.1. Let $(x(t), y(t))$ be the positive solution of System (1.5) with initial value $(x(0), y(0)) \in \mathbb{R}^2_+$. If $\bar{a}_1 + a_1(0) - b_{12}c_2^2 + 2b_{12}H < 0$, where H is defined by (3.1), then we have

$$\lim_{t\to\infty} x(t) = 0 \text{ almost surely.}$$

Proof. Let $\mathcal{W}_1(t) = \ln x(t)$; then, we have

$$\begin{aligned} d\mathcal{W}_1(t) = &\Big(\bar{a}_1 + (a_1(0) - \bar{a}_1)e^{-\alpha_1 t} - b_{11}x(t) - b_{12}(y(t) - c_2)^2 - \frac{\alpha_2^2}{4\alpha_1}(1 - e^{-2\alpha_1 t})\Big)dt \\ &+ \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1 - e^{-2\alpha_1 t}}dB_1(t). \end{aligned} \tag{3.2}$$

By (3.2), we have

$$d\ln x(t) \leq \left(\bar{a}_1 + a_1(0) + 2b_{12}y(t) - b_{12}c_2^2\right)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t). \tag{3.3}$$

Integrating both sides of (3.3) from 0 to t, we obtain that

$$\begin{aligned}&\frac{\ln x(t)}{t} - \frac{\ln x(0)}{t}\\ &\leq \bar{a}_1 + a_1(0) - b_{12}c_2^2 + \frac{1}{t}\int_0^t 2b_{12}y(s)ds + \frac{1}{t}\int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 s}}dB_1(s)\end{aligned}$$

and

$$\frac{\ln x(t)}{t} \leq \bar{a}_1 + a_1(0) - b_{12}c_2^2 + \frac{1}{t}\int_0^t 2b_{12}y(s)ds + \lambda_1(t), \tag{3.4}$$

where $\lambda_1(t) = \frac{\ln x(0)}{t} + \frac{1}{t}\int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 s}}dB_1(s)$. It follows by Lemma 3.1 that

$$\lim_{t\to\infty}\lambda_1(t) = 0. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \leq \bar{a}_1 + a_1(0) - b_{12}c_2^2 + 2b_{12}H < 0 \text{ almost surely.}$$

Using similar analysis to that in [21] and [22], we have $\lim_{t\to\infty} x(t) = 0$ almost surely.

4. Persistence in the mean

In this section, we consider the conditions for the survival of enterprise clusters x and y. As is well-known, there are many ways for enterprise clusters to persist, such as stochastic persistence, persistence in mean, strong stochastic persistence, weak persistence, and so on. For convenience of calculation, we will investigate persistence in the mean of x and y in System (1.5). The definition of persistence in the mean can be found in Definition 3.1.

Lemma 4.1. [23] Let $u \in C[[0,\infty)\times\Theta,[0,\infty)]$. For all $t \geq 0$ and $U \in C[[0,\infty)\times\Theta,\mathbb{R}]$ with $\lim_{t\to\infty}\frac{U(t)}{t} = 0$, if there exist constants $k_0, k_1 > 0$ such that

$$\ln u(t) \geq k_1 t - k_0\int_0^t u(s)ds + U(t) \text{ almost surely,}$$

then

$$\lim_{t\to\infty}\inf\langle u(t)\rangle \geq \frac{k_1}{k_0} \text{ almost surely.}$$

Theorem 4.1. For any initial value $(x(0), y(0)) \in \mathbb{R}_+^2$, if

$$\min\{a_1(0), \bar{a}_1\} - b_{12}(H + c_2)^2 - \frac{\alpha_2^2}{4\alpha_1} > 0,$$

then we have

$$\lim_{t\to\infty}\inf\langle x(t)\rangle \geq \frac{1}{b_{11}}\min\{a_1(0),\bar{a}_1\} - \frac{b_{12}}{b_{11}}(H+c_2)^2 - \frac{\alpha_2^2}{4b_{11}\alpha_1} \text{ almost surely.}$$

Furthermore, if $\min\{a_2(0),\bar{a}_2\} - \frac{\alpha_2^2}{4\alpha_1} > 0$, then we have

$$\lim_{t\to\infty}\inf\langle y(t)\rangle \geq \frac{1}{b_{21}}\min\{a_2(0),\bar{a}_2\} - \frac{\alpha_2^2}{4b_{21}\alpha_1} \text{ almost surely.}$$

Proof. Let $\mathcal{W}_2(t) = \ln x(t)$. Applying Itô's formula to System (1.5), we have

$$\begin{aligned} d\mathcal{W}_2(t) &= \left(\bar{a}_1 + (a_1(0)-\bar{a}_1)e^{-\alpha_1 t} - b_{11}x(t) - b_{12}(y(t)-c_2)^2 - \frac{\alpha_2^2}{4\alpha_1}(1-e^{-2\alpha_1 t})\right)dt \\ &\quad + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t) \\ &\geq \left(\bar{a}_1 + (a_1(0)-\bar{a}_1)e^{-\alpha_1 t} - b_{11}x(t) - b_{12}(H+c_2)^2 - \frac{\alpha_2^2}{4\alpha_1}\right)dt \\ &\quad + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t). \end{aligned} \tag{4.1}$$

Case 1: When $a_1(0) < \bar{a}_1$, by (4.1), we have

$$d\mathcal{W}_2(t) \geq \left(a_1(0) - b_{11}x(t) - b_{12}(H+c_2)^2 - \frac{\alpha_2^2}{4\alpha_1}\right)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t).$$

Case 2: When $a_1(0) > \bar{a}_1$, by (4.1), we have

$$d\mathcal{W}_2(t) \geq \left(\bar{a}_1 - b_{11}x(t) - b_{12}(H+c_2)^2 - \frac{\alpha_2^2}{4\alpha_1}\right)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t).$$

From the above two situations, we get

$$d\mathcal{W}_2(t) \geq \left(\min\{a_1(0),\bar{a}_1\} - b_{11}x(t) - b_{12}(H+c_2)^2 - \frac{\alpha_2^2}{4\alpha_1}\right)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 t}}dB_1(t). \tag{4.2}$$

Integrating both sides of (4.2) from 0 to t, we have

$$\begin{aligned} \frac{\mathcal{W}_2(t)}{t} - \frac{\mathcal{W}_2(0)}{t} &\geq \min\{a_1(0),\bar{a}_1\} - b_{11}\langle x(t)\rangle - b_{12}(H+c_2)^2 - \frac{\alpha_2^2}{4\alpha_1} \\ &\quad + \frac{1}{t}\int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 s}}dB_1(s). \end{aligned}$$

This shows that

$$\ln x(t) \geq \rho_1 t - b_{11}\int_0^t x(s)ds + \int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}}\sqrt{1-e^{-2\alpha_1 s}}dB_1(s) + \ln x(0), \tag{4.3}$$

where $\rho_1 = \min\{a_1(0), \bar{a}_1\} - b_{12}(H + c_2)^2 - \frac{\alpha_2^2}{4\alpha_1} > 0$. It follows by Lemma 3.1 that

$$\lim_{t\to\infty} \frac{1}{t}\Big(\int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 s}} dB_1(s) + \ln x(0)\Big) = 0. \tag{4.4}$$

Hence, from Lemma 4.1, (4.3), and (4.4), we have

$$\lim_{t\to\infty} \inf\langle x(t)\rangle \geq \frac{1}{b_{11}} \min\{a_1(0), \bar{a}_1\} - \frac{b_{12}}{b_{11}}(H + c_2)^2 - \frac{\alpha_2^2}{4b_{11}\alpha_1} \text{ almost surely.}$$

On the other hand, let $\mathcal{W}_3(t) = \ln y(t)$. Applying Itô's formula to System (1.5), we have

$$\begin{aligned} d\mathcal{W}_3(t) &= \Big(\bar{a}_2 + (a_2(0) - \bar{a}_2)e^{-\alpha_1 t} - b_{21}y(t) + b_{22}(x(t) - c_1)^2 - \frac{\alpha_2^2}{4\alpha_1}(1 - e^{-2\alpha_1 t})\Big)dt \\ &\quad + \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 t}} dB_2(t) \\ &\geq \Big(\bar{a}_2 + (a_2(0) - \bar{a}_2)e^{-\alpha_1 t} - b_{21}y(t) - \frac{\alpha_2^2}{4\alpha_1}\Big)dt \\ &\quad + \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 t}} dB_2(t). \end{aligned} \tag{4.5}$$

Case 1: When $a_2(0) < \bar{a}_2$, by (4.5), we have

$$d\mathcal{W}_3(t) \geq \Big(a_2(0) - b_{21}y(t) - \frac{\alpha_2^2}{4\alpha_1}\Big)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 t}} dB_2(t).$$

Case 2: When $a_2(0) > \bar{a}_2$, by (4.5), we have

$$d\mathcal{W}_3(t) \geq \Big(\bar{a}_2 - b_{21}y(t) - \frac{\alpha_2^2}{4\alpha_1}\Big)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 t}} dB_2(t).$$

From the above two situations, we get

$$d\mathcal{W}_3(t) \geq \Big(\min\{a_2(0), \bar{a}_2\} - b_{21}y(t) - \frac{\alpha_2^2}{4\alpha_1}\Big)dt + \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 t}} dB_2(t). \tag{4.6}$$

Integrating both sides of (4.6) from 0 to t, we have

$$\begin{aligned} \frac{\mathcal{W}_3(t)}{t} - \frac{\mathcal{W}_3(0)}{t} &\geq \min\{a_2(0), \bar{a}_2\} - b_{21}\langle y(t)\rangle - \frac{\alpha_2^2}{4\alpha_1} \\ &\quad + \frac{1}{t}\int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 s}} dB_2(s). \end{aligned}$$

This shows that

$$\ln y(t) \geq \rho_2 t - b_{21}\int_0^t y(s)ds + \int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 s}} dB_2(s) + \ln y(0), \tag{4.7}$$

where $\rho_2 = \min\{a_2(0), \bar{a}_2\} - \frac{\alpha_2^2}{4\alpha_1} > 0$. It follows by Lemma 3.1 that

$$\lim_{t\to\infty} \frac{1}{t}\Big(\int_0^t \frac{\alpha_2}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 s}} dB_2(s) + \ln y(0) \Big) = 0. \tag{4.8}$$

Hence, from Lemma 4.1, (4.7), and (4.8), we have

$$\liminf_{t\to\infty} \langle y(t)\rangle \geq \frac{1}{b_{21}} \min\{a_2(0), \bar{a}_2\} - \frac{\alpha_2^2}{4b_{21}\alpha_1} \text{ almost surely.}$$

Remark 4.1. From Theorem 4.1, it is easy to see that the enterprises x and y will be persistent under the conditions of Theorem 4.1, which implies that if intrinsic growth rates a_1 and a_2 are larger, the enterprises x and y will survive.

5. Two examples

In this section, we will provide two examples of the stochastic enterprise cluster model (1.5). Our goal is to verify the correctness of Theorems 3.1 and 4.1 and to show the rich dynamical behavior of System (1.5).

Example 5.1. Let us demonstrate the extinction of the enterprise cluster x of System (1.5) in Theorem 3.1. Consider the parameters in Table 1, and select a value of H.

State 1: $H_1 = 2$. By computation, it has

$$\bar{a}_1 + a_1(0) - b_{12}c_2^2 + 2b_{12}H_1 = -41.7 < 0.$$

State 2: $H_2 = 2.3$. By computation, it has

$$\bar{a}_1 + a_1(0) - b_{12}c_2^2 + 2b_{12}H_2 = -38.13 < 0.$$

State 3: $H_3 = 2.5$. By computation, it has

$$\bar{a}_1 + a_1(0) - b_{12}c_2^2 + 2b_{12}H_3 = -15.59 < 0.$$

Based on the presumptions established in State 1–State 3, all conditions of Theorem 3.1 hold, and we have $\lim_{t\to\infty} x(t) = 0$ almost surely, which means the enterprise cluster x will become extinct almost surely. The matching figure is displayed in Figure 1. This discovery emphasizes that the level of the initial production c_2 of the enterprise and the competitive coefficient b_{12} has a significant impact on System (1.5), with bigger c_2 and b_{12} intensities producing stronger impacts.

Example 5.2. Let us demonstrate the stochastic persistence in the mean of System (1.5) in Theorem 4.1. Consider the parameters in Table 2, and select a value of H.

State 1: $H_1 = 3.5$. By computation, we have

$$\min\{a_1(0), \bar{a}_1\} - b_{12}(H_1 + c_2)^2 - \frac{\alpha_2^2}{4\alpha_1} \approx 3.38 > 0 \text{ and } \min\{a_2(0), \bar{a}_2\} - \frac{\alpha_2^2}{4\alpha_1} = 5.75 > 0.$$

State 2: $H_2 = 4$. By computation, we have

$$\min\{a_1(0), \bar{a}_1\} - b_{12}(H_2 + c_2)^2 - \frac{\alpha_2^2}{4\alpha_1} \approx 3.89 > 0 \text{ and } \min\{a_2(0), \bar{a}_2\} - \frac{\alpha_2^2}{4\alpha_1} = 6.125 > 0.$$

Table 1. Parameter values of System (1.5).

	$\bar{a}_1$	$a_1(0)$	b_{11}	b_{12}	$\bar{a}_2$	$a_2(0)$	b_{21}	b_{22}	c_1	c_2	α_1	α_2
state1	0.2	0.1	2.0	2.0	0.3	0.2	3.0	3.0	1	5	2.5	1.5
state2	0.25	0.12	2.5	2.5	0.33	0.25	3.5	3.5	1.2	4	3	1.3
state3	0.26	0.15	3	2	0.35	0.24	3.3	2.2	0.97	2	2.7	1.2

Table 2. Parameter values of System (1.5).

	$\bar{a}_1$	$a_1(0)$	b_{11}	b_{12}	$\bar{a}_2$	$a_2(0)$	b_{21}	b_{22}	c_1	c_2	α_1	α_2
state1	5	7	0.1	0.1	6	8	3.0	3.0	0.3	0.2	1	1
state2	6.2	8.5	0.2	0.1	6.5	7.5	3.5	3.5	0.2	0.4	1.5	1.5
state3	5.5	6	0.3	0.2	5.35	6.5	3.5	3.2	0.3	0.2	1.2	1.2

State 3: $H_3 = 4.5$. By computation, we have

$$\min\{a_1(0), \bar{a}_1\} - b_{12}(H_3 + c_2)^2 - \frac{\alpha_2^2}{4\alpha_1} \approx 0.78 > 0 \text{ and } \min\{a_2(0), \bar{a}_2\} - \frac{\alpha_2^2}{4\alpha_1} = 5.2 > 0.$$

Based on State 1 and Theorem 4.1, we have

$$\lim_{t\to\infty} \inf\langle x(t)\rangle \geq 33.8, \ \lim_{t\to\infty} \inf\langle y(t)\rangle \geq 1.92 \ \text{ almost surely.}$$

Based on State 2 and Theorem 4.1, we have

$$\lim_{t\to\infty} \inf\langle x(t)\rangle \geq 19.45, \ \lim_{t\to\infty} \inf\langle y(t)\rangle \geq 1.75 \ \text{ almost surely.}$$

Based on State 3 and Theorem 4.1, we have

$$\lim_{t\to\infty} \inf\langle x(t)\rangle \geq 2.6, \ \lim_{t\to\infty} \inf\langle y(t)\rangle \geq 1.64 \ \text{ almost surely.}$$

Hence, the enterprise cluster (1.5) has stochastic persistence in the mean, and the corresponding results are presented in Figures 2 and 3.

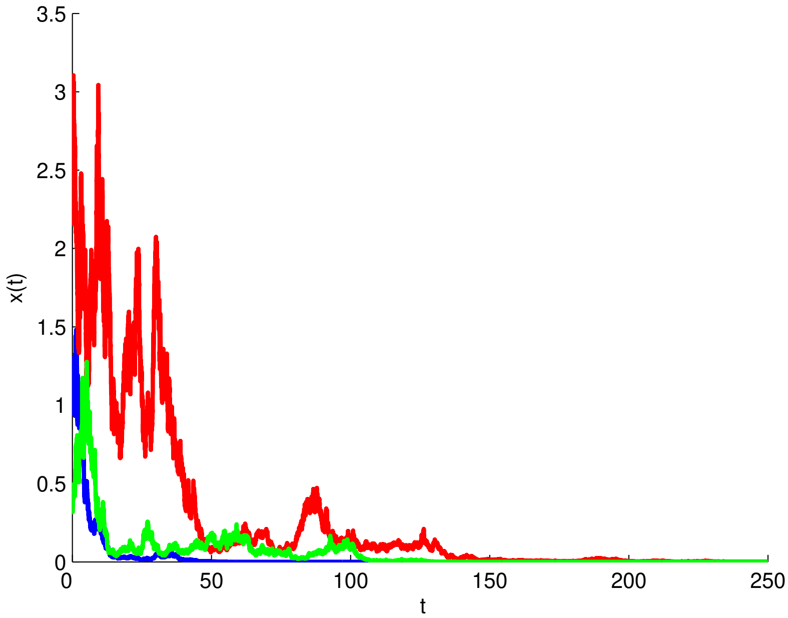

Figure 1. The extinction of the enterprise cluster x in System (1.5).

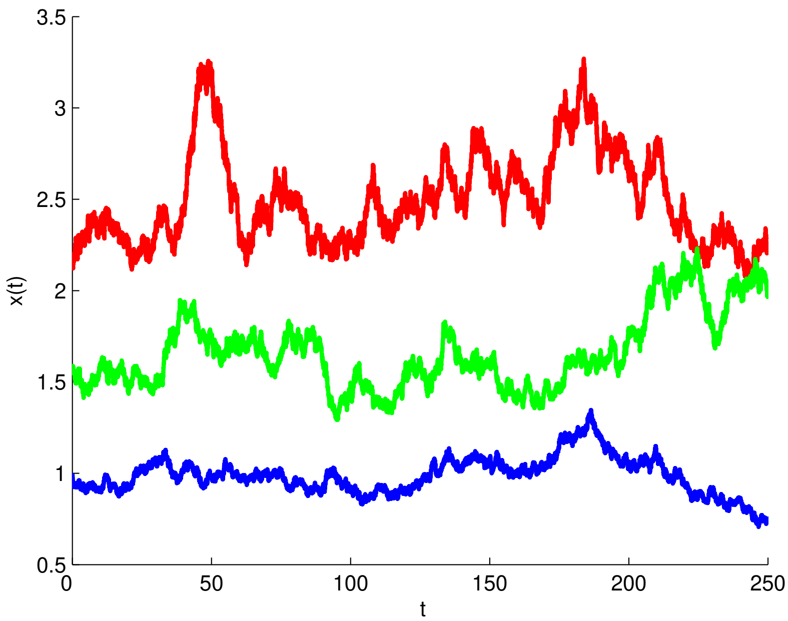

Figure 2. The stochastic persistence in the mean of the enterprise cluster x in System (1.5).

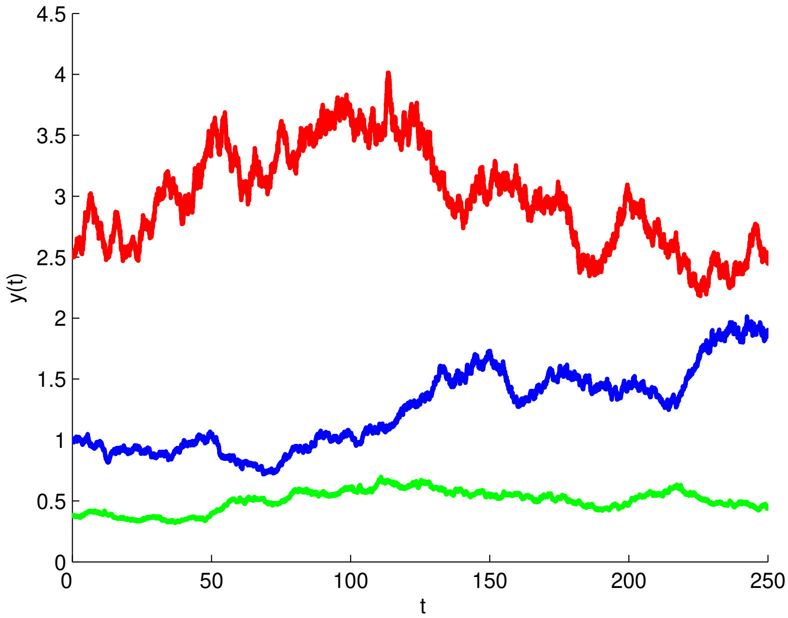

Figure 3. The stochastic persistence in the mean of the enterprise cluster y in System (1.5).

6. Conclusions

The survival and development of enterprise clusters have a direct contribution to the development of the economy. Therefore, studying the dynamic mechanism of the random enterprise cluster model can provide a theoretical basis for the decision-making of management departments and thus formulate scientific policies to promote economic and social development. Using a reasonable method to introduce white noise into an enterprise cluster model, we construct a new stochastic enterprise cluster system with a mean-reverting Ornstein-Uhlenbeck process. Because the mean-reverting Ornstein-Uhlenbeck process has good properties, including non-negativity, continuity, practicality, and asymptotic distribution, the considered model has important research value. From a series of Lyapunov functions, we investigate the existence, stochastically ultimate boundedness, exponential extinction, and persistence in the mean of the system. The main progress of this work compared to previous work may be listed as follows:

• The model presented in this article extends the previous work on deterministic enterprise cluster models (see [2–4]). System (1.5) provides a more detailed understanding of the interactions between enterprise clusters than prior models and improves the accuracy of determining control model parameters for enterprise cluster systems.

• In this paper, we first construct and analyze a stochastic enterprise cluster system with a mean-reverting Ornstein-Uhlenbeck process. We study the dynamic mechanism of enterprise clusters in a changing environment, especially the influence of system parameters on system dynamic behavior. In Theorem 2.2, we obtain the stochastically ultimate boundedness of System (1.5). In Theorem 3.1, sufficient conditions are derived for the extinction of the enterprise cluster x. In Theorem 4.1, we obtain persistence in the mean of System (1.5).

• Due to the fact that a time delay can determine certain dynamic mechanisms of enterprise cluster models, there have been numerous studies on deterministic enterprise cluster models with a time delay (see [3, 6]). However, no one has studied stochastic models that contain both time delays and the Ornstein-Uhlenbeck process so far. Next, we will conduct research on the aforementioned work.

Author contributions

Bo Du: Conceptualization, Writing-original draft; Shuxiang Shao: Investigation, Writing-review and editing; Xiaoliang Li: Model validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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