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Research article

Note on the complete moment convergence of maximal partial sums for moving average process under sublinear expectations

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Abstract: In this paper, the complete moment convergence for the maximal partial sums of moving average processes generated by $\{Y_i, -\infty < i < \infty\}$ is proved under conditions that $C_{\mathbb{V}}\left(|Y_1|^p\left(1 \vee l(f^{-1}(Y_1))\right)\right) < \infty$, where f^{-1} is the inverse function of f, and $\{Y_i, -\infty < i < \infty\}$ is a double sequence of identically distributed, negatively dependent random variables under sublinear expectations. The results established in sublinear expectation spaces complement and extend the corresponding ones in probability space in some extent.

Keywords: complete moment convergence; moving average processes; negatively dependent; sub-linear expectations
Mathematics Subject Classification: 60F05, 60F15

1. Introduction

In order to study the uncertainty in probability, Peng [1, 2] initiated the concept of the sublinear expectations space. Inspired by the works of Peng [1, 2], many scholars have tried to prove the results under sublinear expectations space, extending the corresponding ones in probability space. Zhang [3–5] established Donsker's invariance principle, exponential inequalities, and Rosenthal's inequality under sub-linear expectations. Under sublinear expectations, Xu et al. [6] and Xu and Kong [7] studied complete convergence for weighted sums of negatively dependent (ND) random variables. Xu and Zhang [8, 9] found a three series theorem for independent random variables and the law of logarithm for arrays of random variables. Wu and Jiang [10] obtained a strong law of large numbers and Chover's law of the iterated logarithm, and Zhang and Lin [11] established Marcinkiewicz's strong law of large numbers. Zhong and Wu [12] studied complete convergence and complete moment convergence for weighted sums of extended negatively dependent (END) random variables, and Chen [13] obtained strong laws of large numbers for independent random variables. Chen and Wu [14] studied complete convergence and complete integral convergence of partial sums for a moving average process, and Zhang [15] got strong limit theorems for extended independent random variables and END random variables. Hu et al. [16] investigated how big the increments of G-Brownian motion are, and Gao and Xu [17] proved large deviations and moderate deviations for independent random variables. Kuczmaszewska [18] discussed complete convergence for widely acceptable random variables. Xu and Cheng [19] investigated convergence for sums of independent identically distributed (i.i.d.) random variables. For more limit theorems under sublinear expectations, interested readers could also refer to the references of the papers mentioned above.

In probability space, Hsu and Robbins [20] introduced the concept of complete convergence, and Chow [21] studied complete moment convergence for independent random variables. Li and Zhang [22] found complete moment convergence of moving average processes under dependence assumptions, and Baum and Katz [23] studied convergence rates in the law of large numbers. There is a vast body of literature on complete moment convergeces. Sung [24] found moment inequalities and complete moment convergence, Liu and Lin [25] studied precise asymptotics for a new kind of complete moment convergence, and Zhou [26] found complete moment convergence of moving average processes under a φ-mixing assumption. Qiu and Chen [27, 28] proved complete and complete moment convergence for weighted sums of widely orthant dependent random variables and complete moment convergence for i.i.d. random variables. Liang et al. [29] studied the complete moment and integral convergence for sums of negatively associated random variables. Shen et al. [30] obtained complete moment convergence for arrays of row-wise negatively superadditive dependent (NSD) random variables, and Guo and Zhu [31] established equivalent conditions of complete moment convergence of weighted sums for a ρ^*-mixing sequence of random variables. Kin and Ko [32] and Ko [33] investigated complete moment convergence of moving average processes. Meng et al. [34] discussed the convergence of asymptotically almost-negatively associated random variables with random coefficients, and Hosseini and Nezakati [35] investigated complete moment convergence for dependent linear processes with random coefficients. For references on complete moment convergence in linear expectation space, the interested reader could also refer to the refercences of the articles mentioned above. Recently, Zhang and Ding [36] studied the complete moment convergence of the partial sums of moving average processes under some proper assumptions. Chen and Wu [37] studied complete integral convergence for moving average processes of ND random variables under sublinear expectations. It is natural to study the complete moment convergence for maximal partial sums of moving average processes generated by identically distributed, ND random variables under sublinear expectations relevant to that of Zhang and Ding [36], which also extend that of Chen and Wu [37]. Encouraged by the works of Zhang and Ding [36], Chen and Wu [14, 37], and Xu et al. [6], we prove here the complete moment convergence for maximal partial sums of moving average processes generated by identically distributed, ND random variables under sublinear expectations, extending the corresponding results of Zhang and Ding [36] in classical probability space and Chen and Wu [37] under sublinear expectations. Our main contribution is that we extend the results of Zhang and Ding [36] and Li and Zhang [22] in classical probability space to that under sublinear expectations, and the conclusions of our results also extend Theorem 3.1 of Chen and Wu [14] and Theorem 3.1 of Xu and Kong [7] and of Chen and Wu [37], and our method of proof is heuristically inspired by that of Chen and Wu [14], Zhang [5], and Zhang and Ding [36], which is different from that of Zhang and Ding [36] and Chen and Wu [14, 37].

We construct the remainder of this paper as follows. We present necessary basic notions, concepts, and relevant properties and cite necessary lemmas under sublinear expectations in the next section. In Section 3, we present our main results, Theorems 3.1–3.4, and their proofs.

2. Preliminaries

We use similar notations as in the work by Peng [2], Chen [13], and Zhang [5]. Assume that $(\Omega, \mathcal{F})$ is a given measurable space. Suppose that $\mathcal{H}$ is a subset of all random variables on $(\Omega, \mathcal{F})$ such that $X_1, \cdots, X_n \in \mathcal{H}$ implies $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ stands for the linear space of a (local Lipschitz) function φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ.

Definition 2.1. *A sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ fulfilling the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) *If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;*

(b) $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$;

(c) $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$;

(d) $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ *whenever* $\mathbb{E}[X] + \mathbb{E}[Y]$ *is not of the form* $\infty - \infty$ *or* $-\infty + \infty$.

Remark 2.1. *Readers may wonder why sublinear expectation is useful for general practitioners. Readers could refer to Examples 1.1.5 and 1.1.6 of Peng [2] for simple, concrete examples, which said that the robust expectation of a loss of a gambler in a game has sublinearity but not linearity. As pointed out in Peng [1], [2], g-expectation introduced by Peng (see the introduction of Peng [1]) is also a typical example of sublinear expectation. Peng [1] used the concept of sublinear expectation to give solutions of nonlinear heat equations. The interested reader could also refer to Peng [38, 39] for the meaning of sublinear expectation. The facts above mean that the concept of sublinear expectation is very useful in practice.*

A set function $V : \mathcal{F} \mapsto [0, 1]$ is said to be a *capacity* if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \leq V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity V is called subadditive if $V(A + B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

In this sequel, given $(\Omega, \mathcal{H}, \mathbb{E})$, write $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in Zhang [4]). $\mathbb{V}$ is a subadditive capacity. Set

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables on $(\Omega, \mathcal{H}, \mathbb{E})$, and $a_n > 0$, $b_n > 0$. If for all $\epsilon > 0$, for some $q > 0$, $\sum_{n=1}^{\infty} a_n C_{\mathbb{V}}\left\{((Z_n - \epsilon b_n)^+)^q\right\} < \infty$ holds, then we name that $\{Z_n, n \geq 1\}$ satisfies the complete qth moment convergence under sublinear expectations. Here, the complete first moment convergence is also said to be the complete moment convergence.

As in 4.3 of Zhang [4], hereafter, define an extension of $\mathbb{E}$ on the space of all random variables by

$$\mathbb{E}^*(X) = \inf\{\mathbb{E}[Y] : X \le Y, Y \in \mathcal{H}\}.$$

Then, $\mathbb{E}^*$ is a sublinear expectation on the space of all random variables, $\mathbb{E}[X] = \mathbb{E}^*[X]$, $\forall X \in \mathcal{H}$, and $\mathbb{V}(A) = \mathbb{E}^*(I_A)$, $\forall A \in \mathcal{F}$. By the definition of $\mathbb{E}^*$, for any $x > 0$, for random variables X, $\mathbb{V}(|X| > x) \le \mathbb{E}^*|X|/x$, which is called Markov's inequality under sublinear expectations.

Assume that $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is said to be negatively dependent (ND) to $\mathbf{X}$ if for each $\psi_1 \in C_{l,Lip}(\mathbb{R}^m)$, $\psi_2 \in C_{l,Lip}(\mathbb{R}^n)$, we have $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \le \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \ge 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \ge 0$, $\mathbb{E}[|\psi_1(\mathbf{X})\psi_2(\mathbf{Y})|] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$; and either ψ_1 and ψ_2 are coordinatewise nondecreasing, or ψ_1 and ψ_2 are coordinatewise nonincreasing (see Definition 2.3 of Zhang [4], Definition 1.5 of Zhang [5]). $\{X_n\}_{n=1}^{\infty}$ is said to be a sequence of ND random variables if X_{n+1} is ND to $(X_1, \cdots, X_n)$ for each $n \ge 1$. The concept of ND comes from Definition 1.5 of Zhang [5], and the concept of negative associated (NA) random variables in Definition 2.1 of Joag-Dev and Proschan [40] in probability space. The difference between ND here and NA in probability space could be heuristically seen in Example 1.6 of Zhang [5] and Remark 3.1 of Xu [41], and the technical, key challenges of proofs of Theorems 3.1–3.4 are that subadditivity of capacities and the sublinear expectations $\mathbb{E}(Y_{xj})$ or $\mathbb{E}(-Y_{xj})$ (see the proof of Theorem 3.1) of the truncations of random variables Y_j occurs and needs to be controlled carefully by capacities, integrals, and inequalities. Rosenthal's inequalities for ND random variables under sublinear expectations are also key tools to obtain our results. Under sublinear expectations, the i.i.d. random variables are also ND. As discussed in Subsection 3.3 of Peng [38] and Subsection 5.2 of Peng [39], in practice, the concept of i.i.d. random variables under sublinear expectation could be used to constuct a φ-max-mean algorithm, which is a new nonlinear Monte Carlo approach.

Suppose that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors defined, respectively, in $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed if for every $\psi \in C_{l,Lip}(\mathbb{R}^n)$ such that $\psi(\mathbf{X}_1) \in \mathcal{H}_1, \psi(\mathbf{X}_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)]$$

whenever the sublinear expectations are finite. $\{X_n\}_{n=1}^{\infty}$ is said to be identically distributed if for each $i \ge 1$, X_i and X_1 are identically distributed.

In this paper, we suppose that $\mathbb{E}$ is countably subadditive; that is $\mathbb{E}(X) \le \sum_{n=1}^{\infty} \mathbb{E}(X_n)$ whenever $X \le \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \ge 0$, $X_n \ge 0$, $n = 1, 2, \ldots$. Hence, $\mathbb{E}^*$ is also countably subadditive. Let C stand for a positive constant which may change from place to place. $I(A)$ or I_A represents the indicator function of A.

Definition 2.2. *A real valued function $l(x)$, positive and measurable on $[0, \infty)$, is said to be slowly varying at infinity if for each $\lambda > 0$, $\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1$. A typical example is that $l(x) = (\log(x))^{\beta}(\log\log(x))^{\gamma}$, for some $\beta \ne 0$, $\gamma \in \mathbb{R}$.*

For the sake of completeness, we cite the necessary lemmas below.

Lemma 2.1. *(See Lemma 4.5 of Zhang [4]). For $X \in \mathcal{H}$, if $\mathbb{E}$ is countably subadditive, then*

$$\mathbb{E}|X| \le C_{\mathbb{V}}(|X|).$$

Lemma 2.2. *(See Proposition 1.3.7 of Peng [2]). For $Y \in \mathcal{H}$, if $\mathbb{E}(Y) = \mathbb{E}(-Y) = 0$, then for any $\xi \in \mathcal{H}$, for all $\alpha \in \mathbb{R}$,*

$$\mathbb{E}(\xi) = \mathbb{E}(\xi + \alpha Y).$$

We cite the following Rosenthal's inequality under sublinear expectations.

Lemma 2.3. *(See Theorem 2.1 of Zhang [5] and its proofs). Let $\{X_n; n \geq 1\}$ be a sequence of ND random variables under $(\Omega, \mathcal{H}, \mathbb{E})$. Then, there exists a positive constant $C = C_p$ depending on p such that for $n \geq 1$, and $p \geq 2$,*

$$\mathbb{E}\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right] \leq C_p\left\{\sum_{k=1}^{n}\mathbb{E}[|X_k|^p] + \left(\sum_{k=1}^{n}\mathbb{E}[|X_k|^2]\right)^{p/2} + \left(\sum_{i=1}^{n}(-\mathbb{E}(-X_i))^- + (\mathbb{E}(X_i))^+\right)^p\right\}, \tag{2.1}$$

$$\mathbb{E}\left[\left|\max_{1\le k\le n}\sum_{i=k}^{n} X_i\right|^p\right] \leq C_p\left\{\sum_{k=1}^{n}\mathbb{E}[|X_k|^p] + \left(\sum_{k=1}^{n}\mathbb{E}[|X_k|^2]\right)^{p/2} + \left(\sum_{i=1}^{n}(\mathbb{E}(X_i))^+\right)^p\right\}, \tag{2.2}$$

$$\mathbb{E}\left[\left|\max_{1\le k\le n}\sum_{i=k}^{n}(-X_i)\right|^p\right] \leq C_p\left\{\sum_{k=1}^{n}\mathbb{E}[|X_k|^p] + \left(\sum_{k=1}^{n}\mathbb{E}[|X_k|^2]\right)^{p/2} + \left(\sum_{i=1}^{n}(\mathbb{E}(-X_i))^+\right)^p\right\}, \tag{2.3}$$

where $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, $a \in \mathbb{R}$.

Remark 2.2. *The difference between Rosenthal's inequalities above and that in probability space is that $\mathbb{E}(X_i)$ and $(\mathbb{E}(-X_i))$ occur here but do not appear in classical probability space. Further, the inequalities above play an indispensable role in managing the capacities of the events that maximal partial sums of moving average processes are bigger than some number, as illustrated in the proofs $I_2 < \infty$ and $J_2 < \infty$ in Theorems 3.1, 3.3.*

In the rest of this paper, assume that $\{Y_i, -\infty < i < \infty\}$ is a sequence of ND random variables, identically distributed as Y under $(\Omega, \mathcal{H}, \mathbb{E})$ unless otherwise stated. We also suppose that $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty}|a_i| < \infty$, and $\{X_n, n \geq 1\}$ is defined by $X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}$. A typical example of $\{a_i, -\infty < i < \infty\}$ is $a_i = 1/|i|^{\zeta}$ for some $\zeta > 1$, $i \neq 0$, and $a_0 \in \mathbb{R}$. Another example of $\{a_i, -\infty < i < \infty\}$ is $a_i = 0$ for $i \neq 0$, and $a_0 = 1$, in this case as discussed in Remark 1.1 of Li and Zhang [22], $X_n = Y_n$.

3. Main results and their proofs

In this section, we give the following notations. Suppose that $\{g(n); n \geq 1\}$ and $\{f(n); n \geq 1\}$ are two sequences of positive constants such that, for some $r \geq \max\{2, p\}$, $p \geq 1$,

(C1) $f(n) \uparrow \infty$, $\frac{n}{f^p(n)} \to 0$;

(C2) $\sum_{m=1}^{k}\log\left(\frac{f(m+1)}{f(m)}\right)\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)} = O(f^{p-1}(k)l(k))$;

(C3) $\sum_{m=k}^{\infty}\left[f^{1-r}(m) - f^{1-r}(m+1)\right]\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)} = O(f^{p-r}(k)l(k))$;

(C4) $\sum_{m=1}^{k}\left[f(m+1)-f(m)\right]\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}=O(f^{p}(k)l(k))$;

(C5) $\sum_{m=1}^{\infty}\left[f^{1-r}(m)-f^{1-r}(m+1)\right]f^{t}(m+1)\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}<\infty$, where $t=\max\{0,2-p\}r/2$;

(C6) $\sum_{m=1}^{\infty}\left[f(m+1)-f(m)\right]f^{t'}(m+1)\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}<\infty$, where $t'=-\min\{2,p\}r/2$.

Our main results are the following.

Theorem 3.1. *Suppose that* (C1)-(C6) *hold. Assume that l is a function slowly varying at infinity, $\mathbb{E}(Y)=\mathbb{E}(-Y)=0$, and $C_{\mathbb{V}}\left(|Y|^{p}\left(1\vee l(f^{-1}(Y))\right)\right)<\infty$ under the sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$, where f^{-1} is the inverse function of f. Then, for all $\varepsilon>0$,*

$$\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}C_{\mathbb{V}}\left\{\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|-\varepsilon f(n)\right)^{+}\right\}<\infty. \tag{3.1}$$

Proof of Theorem 3.1. Write $Y_{xj}=-xI\{Y_j<-x\}+Y_jI\{|Y_j|\le x\}+xI\{Y_j>x\}$, $Y'_{xj}=Y_j-Y_{xj}$, $Y_x=-xI\{Y<-x\}+YI\{|Y|\le x\}+xI\{Y>x\}$, $Y'_x=Y-Y_x$. Note that $\sum_{k=1}^{n}X_k=\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j$. Notice that $\sum_{i=-\infty}^{\infty}|a_i|<\infty$, $\mathbb{E}(Y_i)=0$, $C_{\mathbb{V}}\left(|Y|^{p}(1\vee l(f^{-1}(|Y|)))\right)<\infty$; then, by the assumption of (C1), $|\mathbb{E}(X)-\mathbb{E}(Y)|\le\mathbb{E}|X-Y|$, and Lemma 2.1, for any $x>f(n)$, we see that

$$\begin{aligned}
&x^{-1}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\mathbb{E}(Y_{xj})\right|=x^{-1}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}[\mathbb{E}(Y_{xj})-\mathbb{E}(Y_j)]\right|\\
\le&x^{-1}\max_{1\le k\le n}C\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+k}\mathbb{E}|Y'_{xj}|\le Cx^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}|Y'_{xj}|\\
=&Cx^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}|Y'_{x}|\le Cx^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\{|Y'_{x}|\}\\
\le&Cx^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left(|Y|I\{|Y|>x\}\right)\\
\le&Cx^{-1}nC_{\mathbb{V}}\left(|Y|I\{|Y|>x\}\right)=Cx^{-1}n\int_{0}^{\infty}\mathbb{V}\left(|Y|I\{|Y|>x\}>y\right)\mathrm{d}y\\
=&Cx^{-1}n\left[\int_{0}^{x}\mathbb{V}\left(|Y|>x\right)\mathrm{d}y+\int_{x}^{\infty}\mathbb{V}\left(|Y|>y\right)\mathrm{d}y\right]\le Cx^{-1}n\Bigg[\mathbb{V}\left(|Y|>x\right)x\\
&+\int_{x}^{\infty}1\cdot\mathbb{V}\left(|Y|>y\right)\mathrm{d}y\Bigg]\le Cx^{-1}n\left[\int_{0}^{x}\frac{py^{p-1}}{x^{p-1}}\mathbb{V}\left(|Y|>x\right)\mathrm{d}y+\int_{x}^{\infty}\frac{py^{p-1}}{x^{p-1}}\mathbb{V}\left(|Y|>y\right)\mathrm{d}y\right]\\
\le&cnx^{-p}C_{\mathbb{V}}\left(|Y|^{p}I\{|Y|>x\}\right)\le C\frac{n}{f^{p}(n)}C_{\mathbb{V}}\left(|Y|^{p}I\{|Y|>x\}\right)\to0,\ \text{as } x\to\infty.
\end{aligned}$$

Hence, we obtain

$$x^{-1}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\mathbb{E}(Y_{xj})\right|<\varepsilon/4,$$

for any $\varepsilon > 0$ and $x > f(n)$ sufficiently large. Therefore, we conclude that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}C_{\mathbb{V}}\left\{\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|-\varepsilon f(n)\right)^{+}\right\}\\
\le&\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{\varepsilon f(n)}^{\infty}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|>x\right\}\mathrm{d}x\\
\le&\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|>\varepsilon x\right\}\mathrm{d}x\\
\le&\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}(Y_j-Y_{xj})\right|>\varepsilon x/2\right\}\mathrm{d}x\\
&+\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}(Y_{xj}-\mathbb{E}Y_{xj})\right|>\varepsilon x/4\right\}\mathrm{d}x\\
=:&I_1+I_2.
\end{aligned}\tag{3.2}$$

Now, we establish $I_1 < \infty$. Obviously, $|Y_j - Y_{xj}| \le |Y_j|I\{|Y_j| > x\}$. Write $H(m) = \sum_{k=1}^{m}\log\frac{f(k+1)}{f(k)}\sum_{n=1}^{k}\frac{ng(n)l(n)}{f(n)}$, $H(0) = 0$ below; by Markov's inequality under sublinear expectations, Assumptions (C1) and (C2), and the proof of Lemma 2.2 in Zhong and Wu [12], we see that

$$\begin{aligned}
I_1\le&C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-1}\mathbb{E}^*\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}(Y_j-Y_{xj})\right|\mathrm{d}x\\
\le&C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}^*\left|Y_j-Y_{xj}\right|\mathrm{d}x\\
\le&C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_j-Y_{xj}\right|\mathrm{d}x\\
=&C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-1}\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}[|Y_x'|]\mathrm{d}x\\
\le&C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-1}C_{\mathbb{V}}\left(|Y|I\{|Y|>x\}\right)\mathrm{d}x\\
=&C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\int_{f(m)}^{f(m+1)}x^{-1}C_{\mathbb{V}}\left(|Y|I\{|Y|>x\}\right)\mathrm{d}x\\
\le&C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\log\frac{f(m+1)}{f(m)}C_{\mathbb{V}}\left(|Y|I\{|Y|>f(m)\}\right)\\
=&C\sum_{m=1}^{\infty}\log\frac{f(m+1)}{f(m)}C_{\mathbb{V}}\left(|Y|I\{|Y|>f(m)\}\right)\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\\
=&C\sum_{m=1}^{\infty}\log\frac{f(m+1)}{f(m)}\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\left[\int_0^{f(m)}\mathbb{V}\left(|Y|>f(m)\right)\mathrm{d}y+\int_{f(m)}^{\infty}\mathbb{V}(|Y|>y)\mathrm{d}y\right]
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{m=1}^{\infty}\log\frac{f(m+1)}{f(m)}\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}f(m)\mathbb{V}\left(|Y|>f(m)\right)\\
&\quad + C\sum_{m=1}^{\infty}\log\frac{f(m+1)}{f(m)}\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\int_{f(m)}^{\infty}\mathbb{V}(|Y|>y)\mathrm{d}y\\
&= C\lim_{k\to\infty}\sum_{m=1}^{k}\left(H(m)-H(m-1)\right)f(m)\mathbb{V}\left(|Y|>f(m)\right)\\
&\quad + C\sum_{m=1}^{\infty}\log\frac{f(m+1)}{f(m)}\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\sum_{k=m}^{\infty}\int_{f(k)}^{f(k+1)}\mathbb{V}(|Y|>y)\mathrm{d}y\\
&\le C\lim_{k\to\infty}\left(\sum_{m=1}^{k-1}H(m)\left[f(m)\mathbb{V}\left(|Y|>f(m)\right)-f(m+1)\mathbb{V}\left(|Y|>f(m+1)\right)\right]\right.\\
&\quad \left.+ H(k)f(k)\mathbb{V}\left(|Y|>f(k)\right)\right)+C\sum_{k=1}^{\infty}\sum_{m=1}^{k}\log\frac{f(m+1)}{f(m)}\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\int_{f(k)}^{f(k+1)}\mathbb{V}(|Y|>y)\mathrm{d}y\\
&\le C\lim_{k\to\infty}\left(C\sum_{m=1}^{k-1}f^{p-1}(m)l(m)\left[f(m)\mathbb{V}\left(|Y|>f(m)\right)-f(m+1)\mathbb{V}\left(|Y|>f(m+1)\right)\right]\right.\\
&\quad \left.+ Cf^{p}(k)l(k)\mathbb{V}\left(|Y|>f(k)\right)\right)+C\sum_{k=1}^{\infty}f^{p-1}(k)l(k)\int_{f(k)}^{f(k+1)}\mathbb{V}(|Y|>y)\mathrm{d}y\\
&\le C\lim_{k\to\infty}\sum_{m=1}^{k}\left(f^{p-1}(m)l(m)-f^{p-1}(m-1)l(m-1)\right)f(m)\mathbb{V}\left(|Y|^{p}l(f^{-1}(|Y|))>f^{p}(m)l(m)\right)\\
&\quad + C\int_{f(1)}^{\infty}\mathbb{V}\left(|Y|>y\right)y^{p-1}l(f^{-1}(y))\mathrm{d}y\\
&\le CC_{\mathbb{V}}\left(|Y|^{p}\left(1\vee l(f^{-1}(|Y|))\right)\right)<\infty.
\end{aligned}
$$

Therefore, it remains to prove that $I_2 < \infty$. By Markov's inequality under sublinear expectations, Hölder's inequality, and (2.1), for $r > \max\{2, p\}$, we see that

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\mathbb{E}^{*}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}(Y_{xj}-\mathbb{E}Y_{xj})\right|^{r}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\mathbb{E}^{*}\left[\sum_{i=-\infty}^{\infty}\left(|a_i|^{\frac{r-1}{r}}\right)\left(|a_i|^{1/r}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}(Y_{xj}-\mathbb{E}Y_{xj})\right|\right)\right]^{r}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{r-1}\left(\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}^{*}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}(Y_{xj}-\mathbb{E}Y_{xj})\right|^{r}\right)\mathrm{d}x\\
&= C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{r-1}\left(\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}(Y_{xj}-\mathbb{E}Y_{xj})\right|^{r}\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_{xj}-\mathbb{E}Y_{xj}\right|^{r}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_{xj}-\mathbb{E}Y_{xj}\right|^2\right)^{r/2}\mathrm{d}x\\
&+ C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\left(-\mathbb{E}(-Y_{xj})-\mathbb{E}(Y_{xj})\right)^{-}\right)^{r}\mathrm{d}x\\
&=: I_{21}+I_{22}+I_{23}.
\end{aligned}
\tag{3.3}
$$

For I_{21}, by the C_r inequality, Assumptions (C1), (C3), and (C4), and Lemma 2.1, write

$$
\hat{H}(m)=\sum_{k=1}^{m}(f(k+1)-f(k))\sum_{n=1}^{k}\frac{ng(n)l(n)}{f(n)}, m\geq 1,
$$

$$
\hat{H}(0)=0.
$$

We conclude that

$$
\begin{aligned}
I_{21} &\leq C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}|Y_{xj}|^r\mathrm{d}x = C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\mathbb{E}|Y_x|^r\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}C_{\mathbb{V}}\left(|Y_x|^r\right)\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty}\frac{ng(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\int_{f(m)}^{f(m+1)}\left[x^{-r}C_{\mathbb{V}}\left(|Y|^rI\{|Y|\leq x\}\right)+\mathbb{V}\left(|Y|>x\right)\right]\mathrm{d}x\\
&\leq C\sum_{m=1}^{\infty}\left(f^{1-r}(m)-f^{1-r}(m+1)\right)\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\sum_{k=1}^{m}\int_{f(k)}^{f(k+1)}\mathbb{V}\left(|Y|>x\right)x^{r-1}r\mathrm{d}x\\
&\quad + C\sum_{m=1}^{\infty}\left[f(m+1)-f(m)\right]\mathbb{V}\left(|Y|>f(m)\right)\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\\
&\leq C\sum_{k=1}^{\infty}\int_{f(k)}^{f(k+1)}\mathbb{V}\left(|Y|>x\right)x^{r-1}r\mathrm{d}x\sum_{m=k}^{\infty}\left(f^{1-r}(m)-f^{1-r}(m+1)\right)\sum_{n=1}^{m}\frac{ng(n)l(n)}{f(n)}\\
&\quad + C\sum_{m=1}^{\infty}\left(\hat{H}(m)-\hat{H}(m-1)\right)\mathbb{V}\left(|Y|>f(m)\right)\\
&\leq C\sum_{k=1}^{\infty}f^{p-r}(k)l(k)\int_{f(k)}^{f(k+1)}\mathbb{V}\left(|Y|>x\right)x^{r-1}r\mathrm{d}x\\
&\quad + \lim_{k\to\infty}C\sum_{m=1}^{k-1}\hat{H}(m)\left(\mathbb{V}\left(|Y|>f(m)\right)-\mathbb{V}\left(|Y|>f(m+1)\right)\right)+\hat{H}(k)\mathbb{V}\left(|Y|>f(k)\right)\\
&\leq C\sum_{k=1}^{\infty}\int_{f(k)}^{f(k+1)}\mathbb{V}\left(|Y|>x\right)x^{p-1}l(f^{-1}(x))\mathrm{d}x\\
&\quad + \lim_{k\to\infty}C\sum_{m=1}^{k-1}f^{p}(m)l(m)\left(\mathbb{V}\left(|Y|>f(m)\right)-\mathbb{V}\left(|Y|>f(m+1)\right)\right)+f^{p}(k)l(k)\mathbb{V}\left(|Y|>f(k)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq CC_{\mathbb{V}}\left(|Y|^p\left(1\vee l(f^{-1}(|Y|))\right)\right) \\
&\quad + \lim_{k\to\infty} C\sum_{m=1}^{k}\left(f^p(m)l(m)-f^p(m-1)l(m-1)\right)\mathbb{V}\left(|Y|>f(m)\right) \\
&\leq CC_{\mathbb{V}}\left(|Y|^p\left(1\vee l(f^{-1}(|Y|))\right)\right)<\infty. \qquad (3.4)
\end{aligned}
$$

Next, we establish $I_{22}<\infty$. By the C_r ienquality, Lemma 2.1, and Assumptions (C1), (C5), and (C6), we see that

$$
\begin{aligned}
I_{22} &\leq C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left(\mathbb{E}|Y_{x1}|^2\right)^{r/2}\mathrm{d}x = C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left(\mathbb{E}|Y_{x}|^2\right)^{r/2}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left(C_{\mathbb{V}}\left\{|Y_x|^2\right\}\right)^{r/2}\mathrm{d}x \\
&= C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left[\int_0^{\infty}\mathbb{V}\left(|Y|^2I\{|Y|\leq x\}+x^2I\{|Y|>x\}>y\right)\mathrm{d}y\right]^{r/2}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left[\int_0^{\infty}\left[\mathbb{V}\left(|Y|^2I\{|Y|\leq x\}>y/2\right)+\mathbb{V}\left(x^2I\{|Y|>x\}>y/2\right)\right]\mathrm{d}y\right]^{r/2}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left[C_{\mathbb{V}}\{|Y|^2I\{|Y|\leq x\}\}+x^2\mathbb{V}\left(|Y|>x\right)\right]^{r/2}\mathrm{d}x \qquad (3.5) \\
&\leq C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\int_{f(m)}^{f(m+1)}\left[x^{-r}\left(C_{\mathbb{V}}\{|Y|^2I\{|Y|\leq x\}\}\right)^{r/2}+\mathbb{V}^{r/2}\left(|Y|>x\right)\right]\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\left[f^{1-r}(m)-f^{1-r}(m+1)\right]\left(C_{\mathbb{V}}\{|Y|^2I\{|Y|\leq f(m+1)\}\}\right)^{r/2} \\
&\quad + C\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\left[f(m+1)-f(m)\right]\mathbb{V}^{r/2}\left(|Y|>f(m)\right) \\
&= C\sum_{m=1}^{\infty}\left[f^{1-r}(m)-f^{1-r}(m+1)\right]\left(C_{\mathbb{V}}\{|Y|^2I\{|Y|\leq f(m+1)\}\}\right)^{r/2}\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)} \\
&\quad + \sum_{m=1}^{\infty}\left[f(m+1)-f(m)\right]\mathbb{V}^{r/2}\left(|Y|>f(m)\right)\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)} \\
&\leq C\sum_{m=1}^{\infty}\left[f^{1-r}(m)-f^{1-r}(m+1)\right]\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}f^{\max\{0,2-p\}r/2}(m+1)\left(C_{\mathbb{V}}\left\{|Y|^{\min\{p,2\}}\right\}\right)^{r/2} \\
&\quad + C\sum_{m=1}^{\infty}\left[f(m+1)-f(m)\right]\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}f^{-\min\{2,p\}r/2}(m)\left(\mathbb{E}|Y|^{\min\{2,p\}}\right)^{r/2}<\infty. \qquad (3.6)
\end{aligned}
$$

Finally, we aim to prove $I_{23}<\infty$. By $\mathbb{E}(Y)=\mathbb{E}(-Y)=0$, $|\mathbb{E}(X)-\mathbb{E}(Y)|\leq\mathbb{E}|X-Y|$, (C1), (C5), and Lemma 2.1, we see that

$$
I_{23}\leq C\sum_{n=1}^{\infty}\frac{n^r g(n)l(n)}{f(n)}\int_{f(n)}^{\infty}x^{-r}\left(|\mathbb{E}(-Y_{x1}|)+|\mathbb{E}(Y_{x1}|)\right)^r\mathrm{d}x
$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty}\frac{n^r g(n)l(n)}{f(n)}\int_{f(n)}^{\infty} x^{-r}\left(|\mathbb{E}(-Y_x)|+|\mathbb{E}(Y_x)|\right)^r \mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}\frac{n^r g(n)l(n)}{f(n)}\int_{f(n)}^{\infty} x^{-r}\left(\mathbb{E}|Y_x'|\right)^r \mathrm{d}x \le \sum_{n=1}^{\infty}\frac{n^r g(n)l(n)}{f(n)}\int_{f(n)}^{\infty} x^{-r}\left(C_{\mathbb{V}}\{|Y_x'|\}\right)^r \mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}\frac{n^r g(n)l(n)}{f(n)}\sum_{m=n}^{\infty}\int_{f(m)}^{f(m+1)} x^{-r}\left(C_{\mathbb{V}}[|Y|I\{|Y|>f(m)\}]\right)^r \mathrm{d}x\\
&\le C\sum_{m=1}^{\infty}\left(f^{1-r}(m)-f^{1-r}(m+1)\right)\sum_{n=1}^{m}\frac{n^r g(n)l(n)}{f(n)}\left(\frac{C_{\mathbb{V}}\{|Y|^p I\{|Y|>f(m)\}\}}{f^{p-1}(m)}\right)^r\\
&\le C\sum_{m=1}^{\infty}\left(f^{1-r}(m)-f^{1-r}(m+1)\right)f^{(2-p)r/2}(m)\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}\frac{n^{r/2}}{f^{pr/2}(m)}\left(C_{\mathbb{V}}\left(|Y|^p\right)\right)^r\\
&\le C\sum_{m=1}^{\infty}\left(f^{1-r}(m)-f^{1-r}(m+1)\right)f^{(2-p)r/2}(m)\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}\\
&\le C\sum_{m=1}^{\infty}\left(f^{1-r}(m)-f^{1-r}(m+1)\right)f^{\max\{0,(2-p)r/2\}}(m+1)\sum_{n=1}^{m}\frac{n^{r/2}g(n)l(n)}{f(n)}<\infty. \qquad (3.7)
\end{aligned}
$$

Therefore, combining (3.2)–(3.7) results in (3.1). The proof is complete. □

By the adapted proof of Theorem 3.1, as in Corollary 2.2 of Zhang and Ding [36], we see that the following corollary holds.

Corollary 3.1. *If the assumptions* (C2)–(C6) *in Theorem 3.1 are replaced by*

(C7) $\sum_{n=1}^{k}\frac{ng(n)l(n)}{f(n)}=O(f^{p-1}(k)l(k))$, $\sum_{n=1}^{\infty}\frac{n^{r/2}g(n)l(n)}{f^{\min\{2,p\}r}(n)}<\infty$, $\sum_{n=k}^{\infty}\frac{ng(n)l(n)}{f^r(n)}=O(f^{p-r}(k)l(k))$,

and the other conditions of Theorem 3.1 hold, then for all $\epsilon>0$, *we have*

$$\sum_{n=1}^{\infty} g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|>\epsilon f(n)\right\}<\infty. \qquad (3.8)$$

Theorem 3.2. *Assume that* $\{a_i,-\infty<i<\infty\}$ *is an absolutely summable sequence of non-negative real numbers. Suppose that the other conditions of Theorem 3.1 hold with* $\mathbb{E}(Y)=0$ *in place of* $\mathbb{E}(Y)=\mathbb{E}(-Y)=0$. *Then, for all* $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}C_{\mathbb{V}}\left\{\left(\max_{1\le k\le n}\sum_{j=k}^{n}X_j-\varepsilon f(n)\right)^+\right\}<\infty. \qquad (3.9)$$

Similarly, if the other conditions of Theorem 3.1 hold with $\mathbb{E}(-Y)=0$ *in place of* $\mathbb{E}(Y)=\mathbb{E}(-Y)=0$, *then for all* $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}C_{\mathbb{V}}\left\{\left(\max_{1\le k\le n}\sum_{j=k}^{n}(-X_j)-\varepsilon f(n)\right)^+\right\}<\infty. \qquad (3.10)$$

Proof of Theorem 3.2. By a similar proof of Theorem 3.1 with (2.2) or (2.3) in place of (2.1), we could deduce that Theorem 3.2 holds. □

Remark 3.1. *The novelty of Theorems 3.1 and 3.2 could be illustrated in the special cases below. As in Example 1.6 of Zhang [5], assume $\mathcal{P}$ is a family of probability measures defined on $(\Omega, \mathcal{F})$. For any random variables ξ, define the sublinear expectation $\mathbb{E}(\xi) = \sup_{P\in\mathcal{P}} E_P(\xi)$, where $E_P(\xi)$ means the linear expectation of ξ under P. Suppose that $\{Y_i, -\infty < i < \infty\}$ is a sequence of i.i.d. random variables under each $P \in \mathcal{P}$. Then, we see that $\{Y_i, -\infty < i < \infty\}$ is a sequence of ND random variables which are identically distributed under $\mathbb{E}$. In this case, suppose the assumptions of Theorem 3.1 or Theorem 3.2 hold. Then, (3.1) or (3.9) or (3.10) holds. When $\mathcal{P}$ contains only one probability measure P, then we obtain Theorem 2.1 of Zhang and Ding [36] in classical probability space. The same discussions also could apply to the cases of Remarks 3.2, 3.3, and 3.7 and Theorems 3.3 and 3.4. Here, we focus on ND random variables under sublinear expectations. Considering the results of widely acceptable (WA) random variables of Kuczmaszewska [18] and the END random variables of Zhang [15], we conjecture that the similar results here possibly hold for WA and END random variables under sublinear expectations as well.*

Remark 3.2. *In Theorem 3.1, let* $1 \le s < 2$, $p > s$, $f(n) = n^{1/s}$, $g(n) = n^{p/s-2}$, $r > \max\left\{2, p, \frac{p/s-1}{1/s-1/2}, \frac{p/s-1}{p/2-1/2}, \frac{p/s-1}{1/2}\right\}$, *and assume that* $\mathbb{E}(Y) = \mathbb{E}(-Y) = 0$, $C_{\mathbb{V}}\{|Y|^p h(|Y|^s)\} < \infty$. *Then, the conditions* (C1)–(C6) mean that $n/n^{p/s} \to 0$, $\sum_{m=1}^k m^{-1} \sum_{n=1}^m n^{p/s-1-1/s} l(n) \approx Ck^{p/s-1/s} l(k) = O(k^{(p-1)/s} l(k))$, $\sum_{m=k}^\infty m^{(1-r)/s-1} \sum_{n=1}^m n^{p/s-1-1/s} l(n) \approx Ck^{p/s-r/s} l(k) = O(k^{(p-r)/s} l(k))$, $\sum_{m=1}^k m^{1/s-1} \sum_{n=1}^m n^{p/s-1-1/s} l(n) \approx Ck^{p/s} l(k) = O(k^{p/s} l(k))$, $\sum_{m=1}^\infty m^{(1-r)/s-1} (m+1)^{t/s} \sum_{n=1}^m n^{r/2+p/s-2-1/s} l(n) = \sum_{m=1}^\infty m^{p/s-2+t/s+r/2-r/s} l(m) < \infty$, $\sum_{m=1}^\infty m^{1/s-1}(m+1)^{t'} \sum_{n=1}^m n^{r/2+p/s-2-1/s} l(n) \approx \sum_{m=1}^\infty m^{-\min\{2,p\}r/2+p/s-2+r/2} l(m) < \infty$, and therefore, we get

$$\sum_{n=1}^\infty n^{p/s-2-1/s} l(n) C_{\mathbb{V}} \left\{ \left(\max_{1\le k\le n} |S_k| - \varepsilon n^{1/s} \right)^+ \right\} < \infty, \text{ for every } \varepsilon > 0,$$

which extends Theorem 1.1 of Li and Zhang [22] in probability space to that under sublinear expectations space. $\mathbb{E}$ reduces to linear expectation E, with p and r in place of s and p here, respectively, and we recover Theorem 1.1 of Li and Zhang [22].

Remark 3.3. *In Corollary 3.1, let* $1 \le q < 2$, $\alpha > 1$, $f(n) = n^{1/q}$, $g(n) = n^{\alpha-2}$, $l(n) = 1$, *and* $p = q\alpha$, *and suppose that* $r \ge \max\{2, q\alpha\}$ *is large enough. Then, the assumptions* (C1) and (C7) respectively imply $n/n^\alpha \to 0$, $\sum_{n=1}^k n^{\alpha-1-1/q} \approx Ck^{\alpha-1/q} = O(k^{(q\alpha-1)/q})$, $\sum_{n=1}^\infty n^{r/2+\alpha-2-r\min\{2,q\alpha\}} < \infty$, $\sum_{n=k}^\infty n^{\alpha-1-rq\alpha} \approx Ck^{\alpha-rq\alpha} = O(k^{(q\alpha-r)/q})$, and therefore, we get

$$\sum_{n=1}^\infty n^{\alpha-2} \mathbb{V} \left\{ \max_{1\le k\le n} \left| \sum_{j=1}^k X_j \right| > \epsilon n^{1/q} \right\} < \infty, \text{ for any } \epsilon > 0, \tag{3.11}$$

which complements that of Theorem 3.1 of Chen and Wu [14]. With p in place of q here, we recover and extend Theorem 3.1 of Chen and Wu [14]. With r and p in place of α and q here, we recover the first part of Theorem 3.1 (iii) of Xu [42], but the assumptions here are weak.

Example 3.1. *Now, we give a concrete example illustrating numerical simulations of (3.11). As in Remark 3.1, we suppose that $\{Y_i, -\infty < i < \infty\}$ is a sequence of i.i.d. random variables under each* $P \in \mathcal{P}_1 = \{P_1, P_2\}$, *and* $P_1(Y_1 = -0.01) = P_1(Y_1 = 0.01) = 1/2$, $P_2(Y_1 \le x) = \frac{1}{\sqrt{0.02\pi}} \exp(-50x^2)\mathrm{d}x$, $\forall x \in \mathbb{R}$. *Then, $\{Y_i, -\infty < i < \infty\}$ is a sequence of ND random variables, which are identically distributed under*

$\mathbb{E} = \sup_{P\in\mathcal{P}_1} E_P$. In Remark 3.3, choose $q = 3/2$, $\epsilon = 0.1$, $a_0 = 1$, $a_i = 0$ for $i \neq 0$. By taking the sample size n as $n = 300, 600, 900, 1200, 1500$, respectively, we compute $n^{-1/q} \max_{1\le k\le n} \left|\sum_{j=1}^{k} X_j\right|$ for 10,000 times and calculate the frequencies of the event $\{n^{-1/q} \max_{1\le k\le n} \left|\sum_{j=1}^{k} X_j\right| > \epsilon\}$, which are 0.0191, 0.0061, 0.0037, 0.0016, and 0.0014, respectively. We see that $n^{-1/q} \max_{1\le k\le n} \left|\sum_{j=1}^{k} X_j\right|$ are round zero, whereas the volatilities become smaller and smaller as sample size increases. The simulations studies above support that (3.11) and hence, our main results, hold.

Remark 3.4. *In Theorems 3.1–3.4, we suppose that the coefficients $\{a_i, -\infty < i < \infty\}$ are absolutely summable. How do the particular structures of these coefficients influence convergence rates of our results? By the proofs of Theorems 3.1–3.4 and Jensen's inequality under sublinear expectations (see Proposition 2.1 of Chen et al. [43]), we see that the convergence rates of moving averages processes resemble that of ND random variables by the structures of these coefficients.*

Remark 3.5. *The subtle difference between the classic probability space and the sublinear expectation space here could be implied heuristically by the fact that although $\mathbb{E}[\alpha Y_1 + \beta Y_2] = \alpha\mathbb{E}[Y_1] + \beta\mathbb{E}[Y_2]$ holds for $\mathbb{E}(-Y) = \mathbb{E}(Y) = 0$, $\mathbb{E}[\alpha Y_1^r + \beta Y_2^s] \neq \alpha\mathbb{E}[Y_1^r] + \beta\mathbb{E}[Y_2^s]$ for any $r \neq 1$, $s \neq 1$, $\alpha, \beta \in \mathbb{R}$.*

As in Zhang and Ding [36], Assumptions (C1)–(C6) could be fulfilled by many sequences; the interested reader could refer to Zhang and Ding [36] for more examples of similar sequences.

Theorem 3.3. *Suppose l is a function slowly varying at infinity. Assume that $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \ge 1\}$ is a moving average process generated by a sequence of ND random variables $\{Y_i, -\infty < i < \infty\}$ with $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$, where $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. Suppose that $\{g(n), n \ge 1\}$ and $\{f(n), n \ge 1\}$ are two sequences of positive constants with $f(n) \uparrow \infty$, and $\{\Psi_n(t), n \ge 1\}$ is a sequence of even and non-negative functions such that for each $n \ge 1$, $t > 0$, $\Psi_n(t) > 0$. Suppose that*

$$\frac{\Psi_n(t)}{|t|^p} \uparrow, \frac{\Psi_n(t)}{|t|^q} \downarrow, \text{ as } |t| \uparrow, \tag{3.12}$$

for some $1 \le p < q < 2$, and

$$\sum_{i=-\infty}^{\infty} |a_i| \sum_{n=1}^{\infty} g(n)l(n) \sum_{j=i+1}^{j=i+n} \frac{C_{\mathbb{V}}(\Psi_j(Y_j))}{\Psi_j(f(n))} < \infty, \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{j=i+n} \frac{C_{\mathbb{V}}(\Psi_j(Y_j))}{\Psi_j(f(n))} \to 0, \text{ as } n \to \infty, \tag{3.13}$$

for any $j \ge 0$, and there exists a constant c fulfilling

$$\sum_{i=j+1}^{j+n} \frac{C_{\mathbb{V}}(\Psi_i(Y_i))}{\Psi_i(f(n))} \le c, \text{ for } n \ge 1. \tag{3.14}$$

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| > \varepsilon f(n)\right\} < \infty. \tag{3.15}$$

Proof of Theorem 3.3. Write $Y_{nj} = -f(n)I\{Y_j < f(n)\} + Y_j I\{|Y_j| \le f(n)\} + f(n)I\{Y_j > f(n)\}$, $Y'_{nj} = Y_{nj} - Y_j$. Obviously, $\sum_{j=1}^{k} X_j = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j$. Observing that $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, and $\mathbb{E}(Y_j) = 0$,

$|\mathbb{E}(X)-\mathbb{E}(Y)|\le \mathbb{E}|X-Y|$; then, by $C_{\mathbb{V}}\{\lambda|X|\}=\int_0^\infty \lambda\mathbb{V}\{|X|>x/\lambda\}\mathrm{d}x/\lambda=\lambda C_{\mathbb{V}}\{|X|\}$, $\forall\lambda>0$, (3.12), and (3.13), we see that

$$\begin{aligned}
&\frac{1}{f(n)}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\mathbb{E}(Y_{nj})\right|\le\frac{1}{f(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}|\mathbb{E}(Y_{nj})-\mathbb{E}(Y_j)|\\
\le&\frac{1}{f(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\{|Y'_{nj}|\}\le\frac{1}{f(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\{|Y_j|I\{|Y_j|>f(n)\}\}\\
=&\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left\{\frac{|Y_j|I\{|Y_j|>f(n)\}}{f(n)}\right\}\le\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left\{\frac{|Y_j|^pI\{|Y_j|>f(n)\}}{f^p(n)}\right\}\\
\le&C\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left[\frac{\Psi_j(Y_j)}{\Psi_j(f(n))}\left[I\left\{|Y_j|\le f(n)\right\}+I\left\{|Y_j|>f(n)\right\}\right]\right]\to 0\text{ as }n\to\infty.
\end{aligned}$$

Therefore, for n sufficiently large and any $\varepsilon>0$, we have

$$\frac{1}{f(n)}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\mathbb{E}(Y_{nj})\right|<\frac{\varepsilon}{4}.$$

Then, we conclude that

$$\begin{aligned}
&\sum_{n=1}^{\infty}g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k}X_j\right|>\varepsilon f(n)\right\}\\
\le&C\sum_{n=1}^{\infty}g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\left(Y_j-Y_{nj}\right)\right|>\varepsilon f(n)/2\right\}\\
&+C\sum_{n=1}^{\infty}g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\left(Y_{nj}-\mathbb{E}Y_{nj}\right)\right|>\varepsilon f(n)/4\right\}\\
=:&J_1+J_2.
\end{aligned}$$

By Markov's inequality under sublinear expectations, Lemma 2.1, (3.12), and (3.13), we see that

$$\begin{aligned}
J_1\le&C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\mathbb{E}^*\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\left(Y_j-Y_{nj}\right)\right|\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}|Y'_{nj}|\\
\le&C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\{|Y'_{nj}|\}\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left\{|Y_j|I\{|Y_j|>f(n)\}\right\}\\
\le&C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left\{\frac{|Y_j|I\{|Y_j|>f(n)\}}{f(n)}\right\}\le C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\\
&\times\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left\{\frac{|Y_j|^pI\{|Y_j|>f(n)\}}{f^p(n)}\right\}\le C\sum_{i=-\infty}^{\infty}|a_i|\sum_{n=1}^{\infty}g(n)l(n)\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left(\frac{\Psi_j(Y_j)}{\Psi_j(f(n))}\right)<\infty.
\end{aligned}$$

By Markov's inequality under sublinear expectations, Hölder's inequality, (2.1), $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0, i = 1, 2, \ldots$, Lemma 2.2, (3.12), (3.13), (3.14), and the proof of J_1, we obtain

$$\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f^2(n)}\mathbb{E}^*\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\left(Y_{nj}-\mathbb{E}Y_{nj}\right)\right|^2\\
&\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f^2(n)}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\mathbb{E}|Y_{nj}-\mathbb{E}Y_{nj}|^2+\left(\sum_{j=i+1}^{i+n}\left[\mathbb{E}(-Y_{nj})+\mathbb{E}(Y_{nj})\right]\right)^2\right)\\
&\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f^2(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}|Y_{nj}|^2\\
&\quad +C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f^2(n)}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\left[|\mathbb{E}(-Y_{nj}+Y_j)|+|\mathbb{E}(Y_{nj}-Y_j)|\right]\right)^2\\
&\le C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f^q(n)}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}|Y_{nj}|^q+C\sum_{n=1}^{\infty}\frac{g(n)l(n)}{f^2(n)}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\mathbb{E}|Y'_{nj}|\right)^2\\
&\le C\sum_{i=-\infty}^{\infty}|a_i|\sum_{n=1}^{\infty}g(n)l(n)\sum_{j=i+1}^{i+n}\mathbb{E}\frac{\Psi_j(Y_j)}{\Psi_j(f(n))}+C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\mathbb{E}\left[\frac{|Y'_{nj}|}{f(n)}\right]\right)^2\\
&\le C+C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left[\frac{|Y'_{nj}|}{f(n)}\right]\right)^2\\
&\le C+C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left[\frac{|Y_j|I\{|Y_j|>f(n)\}}{f(n)}\right]\right)^2\\
&\le C+C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left[\frac{|Y_j|^pI\{|Y_j|>f(n)\}}{f^p(n)}\right]\right)^2\\
&\le C+C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left[\frac{\Psi_j(|Y_j|)}{\Psi_j(f(n))}\right]\right)^2\\
&\le C+C\sum_{n=1}^{\infty}g(n)l(n)\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\frac{C_{\mathbb{V}}\{\Psi_j(|Y_j|)\}}{\Psi_j(f(n))}\right)^2\\
&\le C+C\sum_{i=-\infty}^{\infty}|a_i|\sum_{n=1}^{\infty}g(n)l(n)\sum_{j=i+1}^{i+n}\frac{C_{\mathbb{V}}\{\Psi_j(|Y_j|)\}}{\Psi_j(f(n))}<\infty.
\end{aligned}$$

Hence, the proof of Theorem 3.3 is complete. □

Theorem 3.4. *Assume that* $\{a_i, -\infty < i < \infty\}$ *is an absolutely summable sequence of non-negative real numbers. Suppose that the other conditions of Theorem 3.3 hold with* $\mathbb{E}(Y_i) = 0$ *in place of*

$\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$. *Then, for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\sum_{j=k}^{n} X_j > \varepsilon f(n)\right\} < \infty. \tag{3.16}$$

Similarly, if the other conditions of Theorem 3.3 hold with $\mathbb{E}(-Y_i) = 0$ *in place of* $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$, *then, for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} g(n)l(n)\mathbb{V}\left\{\max_{1\le k\le n}\sum_{j=k}^{n}(-X_j) > \varepsilon f(n)\right\} < \infty. \tag{3.17}$$

Proof of Theorem 3.4. By a similar proof of Theorem 3.3 with (2.2) or (2.3) in place of (2.1), we could conclude that Theorem 3.4 holds. □

Remark 3.6. *Theorems 3.3 and 3.4 here extend Theorem 3.1 of Chen and Wu [37], which obtained the relevant results of partial sums of moving average processes generated by ND random variables under sublinear expectations.*

Remark 3.7. *In Theorem 3.3, let* $1 \le s < 2$, $1 < \alpha < 2/s$, $f(n) = n^{1/s}$, $g(n) = n^{\alpha-2}$, $l(n) = 1$, $p = s\alpha$, $1 \le p < q < 2$, *and* $C_{\mathbb{V}}\{|Y|^q\} < \infty$, $\Psi_n(x) = |x|^q$. *Then, assumptions of Theorem 3.3 hold, and hence, we obtain*

$$\sum_{n=1}^{\infty} n^{\alpha-2}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| > \varepsilon n^{1/s}\right\} < \infty \text{ for any } \varepsilon > 0,$$

which complements Theorem 3.1 of Chen and Wu [14]. With p *in place of* s *here, we recover Theorem 3.1 of Chen and Wu [14] partially in some sense.*

4. Conclusions

We have obtained new results about the complete moment convergence for the maximal partial sum of moving average processes produced by ND random variables under sublinear expectations. Results obtained here generalize those for negatively dependent random variables in probability space, and Theorems 3.1–3.4 and Corollary 3.1 complement the results of Xu [42] and Chen and Wu [37]. Remarks 3.3 and 3.7 extend Theorem 3.1 of Chen and Wu [14]. The results here imply that the complete moment convergences for the maximal partial sums of ND random variables under sublinear expectations, including Theorem 3.1 of Xu and Kong [7], hold. A limitation here is that we do not obtain the complete qth moment convergence for the maximal partial sum of moving average processes produced by ND random variables under sublinear expectations. Future research directions include the study of the complete qth moment convergences for the maximal partial sum of moving average processes and complete moment convergences for the maximal partial sums of moving average processes with random coefficients under sublinear expectations.

Author contributions

Mingzhou Xu: Writing - original draft, writing - review & editing, conceptualization, formal analysis, funding acquisition, methodology, project administration, software. Wei Wang: Writing -

review & editing, conceptualization, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

All authors state no conflicts of interest in this article.

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