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Research article

Geometric generalizations of the E. Study maps and dual curve theory in the dual space $\mathbb{D}^3$

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Abstract: The correspondence between points on the unit dual sphere and lines in the Euclidean space $\mathbb{R}^3$ was first expressed via E. Study maps, which has served as a foundation for numerous studies in the theory of ruled surfaces and kinematics. The significance of this theorem lies in the correspondence it establishes between the curves on the unit dual sphere and the ruled surfaces in the Euclidean space $\mathbb{R}^3$. However, it has led to a strong focus on the unit dual sphere, leading to the neglect of the broader $\mathbb{D}^3$ and leaving the theory of curves, surface theory, and kinematics in the dual space $\mathbb{D}^3$ largely unexplored. To fill this gap, the present study introduced "generalized E. Study maps", which proved that for every dual curve in the dual space $\mathbb{D}^3$, there existed a corresponding ruled surface in the Euclidean space $\mathbb{R}^3$. Furthermore, the study constructed the theory of curves in the dual space $\mathbb{D}^3$ via the theory of real curves. The results were expected to guide future research on the dual curve theory, dual surface theory, and kinematics in the dual space $\mathbb{D}^3$, and pave the way for exploring the striking correspondence between the dual space $\mathbb{D}^3$ and the Euclidean space $\mathbb{R}^3$ from an expanded viewpoint.

Keywords: dual space; E. Study maps; special curves; dual curve; dual spherical curve; ruled surfaces
Mathematics Subject Classification: 14J26, 53A04, 53A05

1. Introduction

The correspondence between points on the unit dual sphere and lines in the Euclidean space was first established through E. Study maps, which have formed the basis for extensive research in the theory of ruled surfaces and kinematics for over a century. This theorem continues to hold fundamental importance due to its establishment of a one-to-one correspondence between the curves on the unit dual sphere and the ruled surfaces of the Euclidean space. In the literature, extensive studies have been conducted on E. Study Maps [1, 2]. Its generalizations have also attracted considerable scholarly attention [3, 4]. These studies primarily focus on applications in the geometry of ruled surfaces. The determination of ruled and developable surfaces based on the E. Study map is introduced in [5]. The geodesic curvature and the second fundamental form of the ruled surfaces are thoroughly examined in [6], providing important differential geometric characterizations. A developable surface normal to a surface along a curve on the surface is constructed in [7]. The spatial quaternionic expression of ruled surfaces is analyzed in [8], highlighting their structural behavior. Several geometric properties of closed ruled surfaces are further investigated in [9]. Each natural lift curve of the main curve corresponding to a ruled surface obtained by exploiting the E. Study map is studied in [10]. Dual numbers constitute another essential concept [11–14]. In addition, Jacobi theorem and integral invariants of closed ruled surfaces are examined in [15, 16], respectively. Furthermore, some studies adopt a Lorentzian approach [17, 18]. Surface pairs also hold significant importance within this framework [19–21]. Bonnet and orthogonal ruled surfaces are investigated in [22, 23].

Moreover, the generalization of Holditch theorem for ruled surfaces has attracted further attention, particularly in the field of kinematics [24]. Among these studies, one of the central focuses is the theory of curves [25–27]. Direction involves the theory of curves in dual space, which facilitates the analysis and enables such investigations [28–30]. In addition, dual spherical curves also play a crucial role in this framework [31–33]. An explicit characterization of dual spherical curve and evolutes of dual spherical curves for ruled surfaces are discussed in [34, 35], respectively. However, a substantial portion of the existing literature is predominantly confined to the unit dual sphere [32, 35, 36]. From a mathematical perspective, this framework investigates the relationship between dual curve:

$$E(s) = e(s) + \varepsilon e^*(s), \qquad \|E(s)\| = 1,\ \|e(s)\| = 1, \qquad \langle e(s), e^*(s)\rangle = 0$$

on the unit dual sphere and ruled surfaces such that $e(s)$ is the ruling vector and $e(s) \times e^*(s)$ is the base curve, where the first norm is the Euclidean norm in the dual space $\mathbb{D}^3$, the second norm is the Euclidean norm in the real space $\mathbb{R}^3$, the dot product is the Euclidean inner product, and the vector product is the Euclidean vector product.

At the same time, this theorem prompted fixation on the unit dual sphere, leading to the neglect of the broader dual space $\mathbb{D}^3$ and leaving the theory of curves, surface theory, and kinematics in $\mathbb{D}^3$ largely unexplored. To fill this gap, the present study introduces “generalized E. Study maps” as a theorem that proves that for every dual curve in the dual space, there exists a corresponding ruled surface in the Euclidean space. To establish this proof, to eliminate the said fixation, and to introduce to the literature the concepts related to the neglected areas, the following steps are undertaken: First, the concepts of the dual point, the dual line, and the dual plane are defined in $\mathbb{D}^3$ and it is proved that certain geometric structures in the Euclidean space correspond to these. It is well known that points, lines, planes, and curves are the fundamental elements of geometry, providing the basic framework for constructing and understanding spatial structures. Their interrelationships form the foundation for exploring more complex geometric concepts and theorems. Second, the theory of curves in $\mathbb{D}^3$ is constructed and the Frenet elements and formulas of the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ are obtained by using the Frenet elements and formulas of the real curves α and α^*. Third, helix curve, involute-evolute offsets, Bertrand offsets, and Darboux frame, which are significant concepts of the theory of curves, are introduced to the theory of dual curves. Moreover, the fundamental properties of the ruled surfaces that correspond to the dual curve $\boldsymbol{\alpha}$ are provided in a simplified form to serve as a reference for future work. Thus, the approach developed in the present study is expected to pave the way for future studies on the dual curve theory, dual surface theory, and kinematics in $\mathbb{D}^3$. Most particularly, the correspondence between the ruled surfaces in the Euclidean space and dual curves can also be analyzed and studies on dual sphere motions can be extended to the investigations into the kinematics in $\mathbb{D}^3$. Accordingly, generalized E. Study maps allows for a broader examination of the relationship between the dual space and the Euclidean space, moving beyond the constraints of unit dual sphere.

2. Preliminaries

The set of dual numbers* is denoted by $\mathbb{D}$ and given by

$$\mathbb{D} = \{A = a + \varepsilon a^* : a, a^* \in \mathbb{R}, \varepsilon^2 = 0, \ldots, \varepsilon^n = 0, \varepsilon \neq 0\},$$

where $\varepsilon \cdot 0 = 0 \cdot \varepsilon = 0$, $\varepsilon \cdot 1 = 1 \cdot \varepsilon = \varepsilon$. Here, ε is called dual unit. For any $A = a + \varepsilon a^*$, $B = b + \varepsilon b^* \in \mathbb{D}$, equality, addition, scalar multiplication, and multiplication over dual numbers can be given as follows:

$$\begin{aligned} a + \varepsilon a^* = b + \varepsilon b^* &\Leftrightarrow a = b,\ a^* = b^*, \\ (a + \varepsilon a^*) + (b + \varepsilon b^*) &= (a + b) + \varepsilon(a^* + b^*), \\ c\,(a + \varepsilon a^*) &= ca + \varepsilon(ca^*),\ c \in \mathbb{R}, \\ (a + \varepsilon a^*)(b + \varepsilon b^*) &= ab + \varepsilon(ab^* + a^* b). \end{aligned}$$

The set $\mathbb{D}$ forms a commutative ring with unity. Moreover, for a differentiable function f, we have

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x), \qquad \frac{df}{dx} = f'(x), \tag{2.1}$$

where $x + \varepsilon x^* \in \mathbb{D}$. Therefore,

$$\begin{aligned} \sin(x + \varepsilon x^*) &= \sin(x) + \varepsilon x^* \cos(x), \\ \cos(x + \varepsilon x^*) &= \cos(x) - \varepsilon x^* \sin(x), \\ \sqrt{x + \varepsilon x^*} &= \sqrt{x} + \varepsilon \frac{x^*}{2\sqrt{x}},\ x > 0, \\ |x + \varepsilon x^*| &= |x| + \varepsilon\, \mathrm{sgn}(x)\, x^*. \end{aligned}$$

Moreover, the dual space $\mathbb{D}^3$ is defined by

$$\mathbb{D}^3 = \{\boldsymbol{A} = (A_1, A_2, A_3) \mid A_1, A_2, A_3 \in \mathbb{D}\} = \{\boldsymbol{a} + \varepsilon \boldsymbol{a}^* \mid \boldsymbol{a}, \boldsymbol{a}^* \in \mathbb{R}^3,\ \varepsilon^2 = 0\}.$$

Every element of $\mathbb{D}^3$ is called dual vector and A_1, A_2, A_3 are called coordinates of $\boldsymbol{A}$. Besides, $\mathbb{D}^3$ is a module over $\mathbb{D}$. For all $\boldsymbol{A} = (A_1, A_2, A_3) = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*, \boldsymbol{B} = (B_1, B_2, B_3) = \boldsymbol{b} + \varepsilon \boldsymbol{b}^* \in \mathbb{D}^3, A_i,\ B_i \in \mathbb{D}$, $i = 1, 2, 3$, $\boldsymbol{a}, \boldsymbol{a}^* \in \mathbb{R}^3$. The following fundamental concepts hold ([2, 12–14]):

- **Scalar or inner product:**

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \sum_{i=1}^{3} A_i B_i = \langle \boldsymbol{a}, \boldsymbol{b} \rangle + \varepsilon(\langle \boldsymbol{a}, \boldsymbol{b}^* \rangle + \langle \boldsymbol{a}^*, \boldsymbol{b} \rangle).$$

*Dual numbers and their applications have been extensively covered in the literature, [1, 2, 11–14]. These numbers play a significant role in fields such as kinematics, automatic differentiation, etc.

- **Cross-product:**

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix} e_1 & e_2 & e_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}=(\boldsymbol{a}\times\boldsymbol{b})+\varepsilon(\boldsymbol{a}\times\boldsymbol{b}^*+\boldsymbol{a}^*\times\boldsymbol{b}),$$

where $\{e_1, e_2, e_3\}$ is standard basis of $\mathbb{D}^3$.

- **Norm:** By Eq (2.1),

$$\|\boldsymbol{A}\|=\|\boldsymbol{a}+\varepsilon\boldsymbol{a}^*\|=\|\boldsymbol{a}\|+\varepsilon\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{\|\boldsymbol{a}\|}\cdot \tag{2.2}$$

- **Unit dual vector:** By Eq (2.2),

$$\|\boldsymbol{A}\|=1 \quad\Leftrightarrow\quad \langle\boldsymbol{a},\boldsymbol{a}\rangle=1,\ \langle\boldsymbol{a},\boldsymbol{a}^*\rangle=0.$$

- **Unit dual sphere:**

$$\mathbb{S}^2=\{\boldsymbol{A}\in\mathbb{D}^3 \mid \|\boldsymbol{A}\|=1\}.$$

(The real sphere is a subset of the unit dual sphere.)

Theorem 1. *(Classical E. Study maps) The points of the unit dual sphere in $\mathbb{D}^3$ are mapped one-to-one to the directional lines of $\mathbb{R}^3$. Also, there exists a one-to-one correspondence between the set of all dual curves on the unit dual sphere and the set of ruled surfaces in $\mathbb{R}^3$ [1, 2].*

All dual curves $\boldsymbol{A}(s)=\boldsymbol{a}(s)+\varepsilon\boldsymbol{a}^*(s)\in\mathbb{S}^2$, $s\in I\subset\mathbb{R}$ $\Leftrightarrow$ ruled surfaces $\phi(s,v)=\big(\boldsymbol{a}(s)\times\boldsymbol{a}^*(s)\big)+v\boldsymbol{a}(s)$.

Proof. For all points (or vectors) $\boldsymbol{A}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^*\in\mathbb{S}^2$, we can write $\langle\boldsymbol{a},\boldsymbol{a}\rangle=1$, $\langle\boldsymbol{a},\boldsymbol{a}^*\rangle=0$. Then, we obtain homogeneous Plücker coordinates $(\boldsymbol{a},\boldsymbol{a}^*)$ in $\mathbb{R}^3$, where $\boldsymbol{a}^*$ is a vector moment about the origin of the coordinate system and $\boldsymbol{a}$ is the direction vector of the line (see Figure 1). The converse also holds.

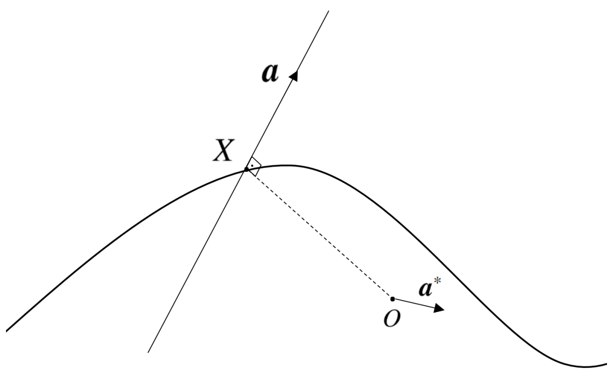

Figure 1. Plücker coordinates.

Also, for the dual spherical curve $\boldsymbol{A}(s)=\boldsymbol{a}(s)+\varepsilon\boldsymbol{a}^*(s)\in\mathbb{S}^2$, $s\in I\subset\mathbb{R}$, we obtain the line family (Plücker coordinates) $(\boldsymbol{a}(s),\boldsymbol{a}^*(s))$ and it is defined as a ruled surface ϕ in $\mathbb{R}^3$. By equations $\|\boldsymbol{OX}\|=\|\boldsymbol{a}^*\|$, $\boldsymbol{X}=\boldsymbol{a}\times\boldsymbol{a}^*$, we get the ruled surface $\phi(s,v)=\big(\boldsymbol{a}(s)\times\boldsymbol{a}^*(s)\big)+v\boldsymbol{a}(s)$. □

Remark 1. *For a dual spherical curve $\boldsymbol{A}(s) = \boldsymbol{a}(s) + \varepsilon \boldsymbol{a}^*(s) \in \mathbb{S}^2$ parameterized by arc length $\hat{s}$,*

$$\frac{d\hat{s}}{ds} = \langle \boldsymbol{a}'(s), \boldsymbol{a}^{*\prime}(s) \rangle = 0 \Leftrightarrow \text{ the ruled surface } \phi \text{ is developable.}$$

- **Dual angle:** For the points (or lines in $\mathbb{R}^3$) $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^2$, we can write

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \cos(\Phi) = \cos(\varphi + \varepsilon \varphi^*),$$

where φ is the angle between two lines $\boldsymbol{A}$ and $\boldsymbol{B}$, and φ^* is the directed distance between $\boldsymbol{A}$ and $\boldsymbol{B}$. The dual number $\Phi = \varphi + \varepsilon \varphi^*$ is called the dual angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Comment 1. *Thanks to the E. Study maps, every point on the unit dual sphere corresponds to a directed line in $\mathbb{R}^3$, and therefore the relative positions of lines in $\mathbb{R}^3$ can be constructed using the dual angle. Also, both dual spherical curve theory and ruled surface theory in $\mathbb{R}^3$ can be reconstructed.*

3. Generalized E. Study maps for dual point in $\mathbb{D}^3$

While the classical E. Study maps establishes a bijection between dual spherical curves and ruled surfaces in $\mathbb{R}^3$, it remains restricted to the unit dual sphere.

In this section, we introduce a generalized study mapping defined for arbitrary regular dual curves and prove that every unit speed curve in $\mathbb{D}^3$ uniquely generates a ruled surface in the Euclidean space.

The results provide a unified geometric framework linking dual curve theory, ruled surface geometry, and kinematics beyond the classical spherical restriction, opening new directions for dual geometry and computational line geometry.

Theorem 2. *(Generalized E. Study maps) There exists a one-to-one correspondence between the set of dual directions, defined as equivalence classes of nonzero dual vector under positive scaling in $\mathbb{D}^3 - \{0\}$ and the set of the directed lines in $\mathbb{R}^3$. Moreover, there exists a one-to-one correspondence between equivalence classes of regular dual curves in $\mathbb{D}^3$, considered with respect to reparameterization and common positive scaling and the set of ruled surfaces in $\mathbb{R}^3$.*

Proof. Let the dual direction $\boldsymbol{X} = \boldsymbol{x} + \varepsilon \boldsymbol{x}^* \in \mathbb{D}^3 - \{0\}$ be given. The normalization of this direction is $\boldsymbol{E} = \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}$ and it follows directly from definition of unit dual sphere that $\boldsymbol{E} \in \mathbb{S}^2$. Then, there exists one-to-one correspondence between the set of dual directions, defined as equivalence classes of nonzero dual vectors under positive scaling, and the points on the dual unit sphere.

According to the classical E. Study maps, the point $\boldsymbol{E} = \boldsymbol{e} + \varepsilon \boldsymbol{e}^*$ with $\langle \boldsymbol{e}, \boldsymbol{e} \rangle = 1$, $\langle \boldsymbol{e}, \boldsymbol{e}^* \rangle = 0$, corresponds to a line whose direction is $\boldsymbol{e}$ and which passes through the point $\boldsymbol{e} \times \boldsymbol{e}^*$ in $\mathbb{R}^3$.

Therefore, the line in $\mathbb{R}^3$ corresponding each dual direction $\boldsymbol{X} = \boldsymbol{x} + \varepsilon \boldsymbol{x}^*$ has direction

$$\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \tag{3.1}$$

and the vector moment of this vector with respect to the origin O is

$$\frac{\boldsymbol{x}^*}{\|\boldsymbol{x}\|} - \frac{\langle \boldsymbol{x}, \boldsymbol{x}^* \rangle}{\|\boldsymbol{x}\|^3} \boldsymbol{x} \tag{3.2}$$

i.e., it has $\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \frac{\boldsymbol{x}^*}{\|\boldsymbol{x}\|} - \frac{\langle \boldsymbol{x}, \boldsymbol{x}^* \rangle}{\|\boldsymbol{x}\|^3}\boldsymbol{x}\right)$-Plücker line coordinates. Conversely, all (δ, δ^*)-Plücker lines in $\mathbb{R}^3$ ($\langle \delta, \delta \rangle = 1$, $\langle \delta, \delta^* \rangle = 0$, δ^* is the vector moment of δ with respect to the origin O) correspond the dual unit vector $\boldsymbol{E} = \delta + \varepsilon\delta^*$ in $\mathbb{S}^2$. Thus, a corresponding dual direction can be found for the line (δ, δ^*).

Similarly, let us conduct the proof for a dual curve considered with respect to reparameterization and common positive scaling.

In $\mathbb{D}^3 - \{0\}$, let regular dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$, ($\|\boldsymbol{\alpha}'(s)\| = 1$) parameterized by arc length s be given. Here,

$$\boldsymbol{\alpha}'(s) = \frac{d\boldsymbol{\alpha}}{ds} = \alpha'(s) + \varepsilon\alpha^{*\prime}(s)$$

and by using Eq (2.2), we get

$$\|\boldsymbol{\alpha}'(s)\| = \|\alpha'(s)\| + \varepsilon\frac{\langle \alpha'(s), \alpha^{*\prime}(s) \rangle}{\|\alpha'(s)\|} = 1 + \varepsilon\, 0,$$

and it follows that (see [30])

$$\|\alpha'(s)\| = 1, \qquad \langle \alpha'(s), \alpha^{*\prime}(s) \rangle = 0. \tag{3.3}$$

Hence, the real curve α is parameterized by arc-length s and the pair (α, α^*) forms involute-evolute offsets[†] in $\mathbb{R}^3$. In particular, we can write (see [26]):

$$\alpha^*(s) = \alpha(s) + (c - s)\alpha'(s), \qquad c \in \mathbb{R}.$$

For each point on the dual curve, by Eqs (3.1) and (3.2), the corresponding line in $\mathbb{R}^3$ has direction

$$\delta(s) = \frac{\alpha(s)}{\|\alpha(s)\|}, \tag{3.4}$$

and the vector moment

$$\delta^*(s) = \frac{\alpha^*(s)}{\|\alpha(s)\|} - \frac{\langle \alpha(s), \alpha^*(s) \rangle}{\|\alpha(s)\|^3}\,\alpha(s).$$

Then, the ruled surfaces $\phi(s, v) = \gamma(s) + v\,\delta(s)$ correspond to the dual curves $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ parameterized by arc-length s. Here,

$$\gamma(s) = \frac{\alpha(s) \times \alpha^*(s)}{\|\alpha(s)\|^2} \tag{3.5}$$

is the directrix curve, and

$$\delta(s) = \frac{\alpha(s)}{\|\alpha(s)\|}$$

is the ruling direction. Also, we obtain for the vector moment $\delta^*(s)$ of the ruling direction $\delta(s)$ with respect to the origin

$$\|\delta^*(s)\| = \frac{\|\alpha^*(s)\|}{\|\alpha(s)\|}\sin(\theta)$$

where θ is the angle between α and α^*.

[†]For the real involute-evolute offsets, see [26].

Conversely, let the ruled surface be $\phi(s, v) = \gamma(s) + v\delta(s)$, $\|\delta(s)\| = 1$. Then, the Plücker line (δ, δ^*) corresponds to the ruled surface ϕ, where the vector moment δ^* of δ with respect to the origin O is $\delta^* = \gamma \times \delta$. Thus, the dual spherical curve $\delta(s) + \varepsilon\delta^*(s)$ corresponds to (δ, δ^*)-line, or regular dual curves considered with respect to reparameterization and common positive scaling corresponds to the ruled surface ϕ. □

Note 1. *The proof for a regular unit speed curve in $\mathbb{D}^3$ is also valid for a regular non-unit speed curve, since every regular dual curve can be transformed into a unit speed curve.*

3.1. Ruled surface theory

The geodesic Frenet trihedron[‡]

$$\left\{\frac{\alpha(s)}{\|\alpha(s)\|}, -\frac{\langle\alpha(s), \alpha'(s)\rangle}{\|\alpha(s)\|\,\|\alpha(s) \times \alpha'(s)\|}\alpha(s) + \frac{\|\alpha(s)\|}{\|\alpha(s) \times \alpha'(s)\|}\alpha'(s), \frac{\alpha(s) \times \alpha'(s)}{\|\alpha(s) \times \alpha'(s)\|}\right\}$$

of the ruled surface

$$\phi(s, v) = \frac{\alpha(s) \times \alpha^*(s)}{\|\alpha(s)\|^2} + v\frac{\alpha(s)}{\|\alpha(s)\|},$$

corresponding to the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ is simply obtained. Thus, analogously to the study of [20], the other geometric proportion of the ruled surface $\phi(s, v)$ can be investigated.

In this section, however, we restrict our attention to the striction curve, the distribution parameter, and the developability of ϕ.

Theorem 3. *Let the ruled surface $\phi(s, v) = \gamma(s) + v\delta(s)$ in $\mathbb{R}^3$ corresponding to the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ parameterized by arc-length s in $\mathbb{D}^3$ be given. The striction curve $C(s)$ and the pitch of the ruled surface ϕ, respectively, are:*

$$C(s) = \frac{\alpha(s) \times \alpha^*(s)}{\|\alpha(s)\|^2} - \left(\frac{\|\alpha(s)\|^2\ \det(\alpha(s), \alpha^{*\prime}(s), \alpha'(s)) - \langle\alpha(s), \alpha^*(s)\rangle \det(\alpha(s), \alpha^*(s), \alpha'(s))}{\|\alpha(s)\|^2\,\|\alpha(s) \times \alpha'(s)\|^2}\right)\alpha(s)$$

and

$$P = -\frac{\langle\alpha(s), \alpha'(s)\rangle}{\|\alpha(s) \times \alpha'(s)\|^2}\left(\langle\alpha(s), \alpha^*(s)\rangle\right)' + \left(\frac{2\langle\alpha(s), \alpha'(s)\rangle^2 - \|\alpha(s)\|^2}{\|\alpha(s)\|^2\,\|\alpha(s) \times \alpha'(s)\|^2}\right)\langle\alpha(s), \alpha^*(s)\rangle. \tag{3.6}$$

Proof. For the striction curve $C(s)$, we can write (see [20]):

$$C(s) = \gamma(s) - \frac{\langle\gamma'(s), \delta'(s)\rangle}{\langle\delta'(s), \delta'(s)\rangle}\delta(s). \tag{3.7}$$

From Eqs (3.4) and (3.5), we obtain

$$\gamma'(s) = \frac{\alpha'(s) \times \alpha^*(s) + \alpha(s) \times \alpha^{*\prime}(s)}{\|\alpha(s)\|^2} - \frac{2\langle\alpha(s), \alpha'(s)\rangle}{\|\alpha(s)\|^2} \cdot \frac{\alpha(s) \times \alpha^*(s)}{\|\alpha(s)\|^2},$$

$$\delta'(s) = \frac{\alpha'(s)}{\|\alpha(s)\|} - \frac{\langle\alpha(s), \alpha'(s)\rangle}{\|\alpha(s)\|^3}\alpha(s),$$

[‡]For the definition of the geodesic Frenet trihedron in $\mathbb{R}^3$ and its geometric properties, see [20].

and

$$\|\delta'(s)\|^2 = \frac{\|\alpha(s) \times \alpha'(s)\|^2}{\|\alpha(s)\|^4}.$$

Then, we get Eq (3.7), or by Eq (3.3), we write:

$$C(s) = \frac{c - s}{\|\alpha(s)\|^2}(\alpha(s) \times \alpha'(s)) + \left(\frac{(c - s)\det(\alpha(s), \alpha'(s), \alpha''(s))}{\|\alpha(s) \times \alpha'(s)\|^2}\right)\alpha(s).$$

Furthermore, the pitch of the ruled surface ϕ is:

$$P = \frac{\langle \gamma'(s), \delta(s) \times \delta'(s)\rangle}{\|\delta'(s)\|^2} = \frac{\langle C'(s), \delta(s) \times \delta'(s)\rangle}{\|\delta'(s)\|^2}.$$

Finally, we have Eq (3.6). □

By using Eq (3.6), the following corollary can be stated immediately.

Corollary 1. *The ruled surface $\phi(s, v) = \gamma(s) + v\delta(s)$ corresponding to the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ parameterized by arc-length s, (i.e., $\|\boldsymbol{\alpha}'(s)\| = 1$ or $\|\alpha'(s)\| = 1$, $\langle \alpha'(s), \alpha^{*\prime}(s)\rangle = 0$) is developable ($P = 0$) if $\langle \alpha(s), \alpha^*(s)\rangle = 0$. Also $P = 0$, if and only if,*

$$\langle \alpha(s), \alpha^*(s)\rangle' = \frac{2\langle \alpha(s), \alpha'(s)\rangle^2 - \|\alpha(s)\|^2}{\|\alpha(s)\|^2 \langle \alpha(s), \alpha'(s)\rangle}\langle \alpha(s), \alpha^*(s)\rangle, \ \langle \alpha(s), \alpha'(s)\rangle \neq 0. \tag{3.8}$$

Proof. If we substitute $\langle \alpha(s), \alpha^*(s)\rangle = 0$ into Eq (3.6), it is clear that $P = 0$. Conversely for $P = 0$ in Eq (3.6), the numerator of Eq (3.6) must be 0. Solving this equation reveals that (3.8) is required for this necessary and sufficient condition. □

Comment 2. *By generalized E. Study maps, it is possible to move away from the traditional approach focused on the unit dual sphere and investigate the geometry of the dual space $\mathbb{D}^3$ and its connection with $\mathbb{R}^3$.*

4. Generalized E. Study maps for the dual line and dual plane in $\mathbb{D}^3$

In this section, the concepts of the dual line and the dual plane are defined in $\mathbb{D}^3$ and that certain geometric structures in the Euclidean space correspond to them is proved.

Let us continue with the *dual line* in $\mathbb{D}^3$. The vector equation of the line d, with direction vector $\boldsymbol{U} = \boldsymbol{u} + \varepsilon\boldsymbol{u}^*$, and passing through the point $\boldsymbol{A} = \boldsymbol{a} + \varepsilon\boldsymbol{a}^*$ in $\mathbb{D}^3$, is given by:

$$\boldsymbol{OX} = \boldsymbol{OA} + \Lambda\boldsymbol{U},$$

where $\boldsymbol{X} = \boldsymbol{x} + \varepsilon\boldsymbol{x}^*$ is a point on the line and $\Lambda = \lambda + \varepsilon\lambda^* \in \mathbb{D}$. From this vector equation, we obtain

$$\boldsymbol{x} = \boldsymbol{a} + \lambda\boldsymbol{u}, \tag{4.1}$$

$$\boldsymbol{x}^* = \boldsymbol{a}^* + (\lambda\boldsymbol{u}^* + \lambda^*\boldsymbol{u}). \tag{4.2}$$

Theorem 4. *(Generalized E. Study maps for the dual line) Each dual line d in $\mathbb{D}^3$ is mapped one-to-one to a directional real line d_x and a plane P_x parallel to d_x in $\mathbb{R}^3$.*

Proof. Let d be dual line with vector equation $\boldsymbol{OX} = \boldsymbol{OA} + \Lambda \boldsymbol{U}$, where $\boldsymbol{X} = \boldsymbol{x} + \varepsilon \boldsymbol{x}^*$, $\boldsymbol{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$, $\boldsymbol{U} = \boldsymbol{u} + \varepsilon \boldsymbol{u}^* \in \mathbb{D}^3$. Then, for a dual line d in $\mathbb{D}^3$, there corresponds, in $\mathbb{R}^3$, a line d_x with vector equation $\boldsymbol{x} = \boldsymbol{a} + \lambda \boldsymbol{u}$ and plane P_x parallel to d_x given by the vector equation $\boldsymbol{x}^* = \boldsymbol{a}^* + (\lambda \boldsymbol{u}^* + \lambda^* \boldsymbol{u})$. □

Theorem 5. *Let us consider dual lines*

$$\boldsymbol{OX} = \boldsymbol{OA} + \Lambda \boldsymbol{U}, \qquad \boldsymbol{OY} = \boldsymbol{OB} + \Omega \boldsymbol{V}, \quad \Lambda, \Omega \in \mathbb{D}.$$

Then, the corresponding lines and planes in $\mathbb{R}^3$ are denoted by d_x, P_x and d_y, P_y, respectively, and we have

(1) If $\boldsymbol{U} \parallel \boldsymbol{V}$, then the associated lines satisfy $d_x \parallel d_y$ and $P_x \parallel P_y$.
(2) If the lines $\boldsymbol{U}$ and $\boldsymbol{V}$ intersect in $\mathbb{D}^3$, then the corresponding lines d_x and d_y also intersect in $\mathbb{R}^3$.
(3) If $\boldsymbol{U} \perp \boldsymbol{V}$, then the associated lines satisfy $d_x \perp d_y$.

Proof. From Theorem 4, for the dual line $\boldsymbol{OX} = \boldsymbol{OA} + \Lambda \boldsymbol{U}$ that corresponds the real line d_x with $\boldsymbol{x} = \boldsymbol{a} + \lambda \boldsymbol{u}$ and real plane P_x with $\boldsymbol{x}^* = \boldsymbol{a}^* + (\lambda \boldsymbol{u}^* + \lambda^* \boldsymbol{u})$; for the line $\boldsymbol{OY} = \boldsymbol{OB} + \Omega \boldsymbol{V}$ that corresponds the real line d_y with $\boldsymbol{y} = \boldsymbol{b} + \omega \boldsymbol{v}$ and real plane P_y with $\boldsymbol{y}^* = \boldsymbol{b}^* + (\omega \boldsymbol{v}^* + \omega^* \boldsymbol{v})$, where $\Lambda = \lambda + \varepsilon \lambda^*$, $\Omega = \omega + \varepsilon \omega^* \in \mathbb{D}$.

(1) If $\boldsymbol{U} \parallel \boldsymbol{V}$, then we can write $\boldsymbol{u} = \boldsymbol{v}$ and $\boldsymbol{u}^* = \boldsymbol{v}^*$. Then, $d_x \parallel d_y$, and also by $\mathrm{sp}\{\boldsymbol{u}, \boldsymbol{u}^*\} = \mathrm{sp}\{\boldsymbol{v}, \boldsymbol{v}^*\}$ and $\boldsymbol{a}^* \neq \boldsymbol{b}^*$, we obtain $P_x \parallel P_y$.
(2) The proof follows similarly.
(3) If $\langle \boldsymbol{U}, \boldsymbol{V} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \varepsilon (\langle \boldsymbol{u}, \boldsymbol{v}^* \rangle + \langle \boldsymbol{u}^*, \boldsymbol{v} \rangle) = 0$, then for $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$ we obtain $d_x \perp d_y$.

This completes the proof. □

Theorem 6. *For every dual line $\boldsymbol{X} = \boldsymbol{A} + \Lambda \boldsymbol{U}$ in $\mathbb{D}^3$, one can associate the pair $(\boldsymbol{U}, \boldsymbol{U}^*)$, which provides a representation of the line in $\mathbb{D}^6$, where the vector moment $\boldsymbol{U}^* = \boldsymbol{X} \times \boldsymbol{U} = \boldsymbol{A} \times \boldsymbol{U}$ of $\boldsymbol{U}$ with respect to its initial point. Also, the pair $(\boldsymbol{U}, \boldsymbol{U}^*)$ provides the following properties by analogy, the Plücker coordinates of a line in $\mathbb{R}^3$:*

(1) $\boldsymbol{U}^*$ is independent of the choice of the representative point on the dual line.
(2) $\boldsymbol{U}^*$ is orthogonal to $\boldsymbol{U}$, i.e., $\boldsymbol{U}^* \perp \boldsymbol{U}$.
(3) $\|\boldsymbol{U}^*\| = \|\boldsymbol{OZ}\|$, where Z denotes the foot of the perpendicular from the origin onto the dual line.

Proof. The proof can be done simply, similar to the properties provided by Plücker coordinates in $\mathbb{R}^3$ [2]. □

Theorem 7. *(Generalized E. Study maps for the dual plane) For every dual plane $\boldsymbol{OX} = \boldsymbol{OA} + \Lambda \boldsymbol{U} + \Omega \boldsymbol{V}$, ($\Lambda = \lambda + \varepsilon \lambda^*$, $\Omega = \omega + \varepsilon \omega^* \in \mathbb{D}$) which is passing through a point A and determined by two nonparallel dual vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ corresponding to the line congruence (two parameter family of directed lines) in the Euclidean space $\mathbb{R}^3$ such that*

$$\boldsymbol{x} = \boldsymbol{a} + \lambda \boldsymbol{u} + \omega \boldsymbol{v} \tag{4.3}$$

and

$$\boldsymbol{x}^* = \boldsymbol{a}^* + \lambda \boldsymbol{u}^* + \lambda^* \boldsymbol{u} + \omega \boldsymbol{v}^* + \omega^* \boldsymbol{v}. \tag{4.4}$$

Proof. Let $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{D}^3$ and $\Lambda = \lambda + \varepsilon\lambda^*$, $\Omega = \omega + \varepsilon\omega^* \in \mathbb{D}$. For equation of dual plane determined by two nonparallel dual vectors $\boldsymbol{U}$ and $\boldsymbol{V}$, we can write $\boldsymbol{OX} = \boldsymbol{OA} + \Lambda\boldsymbol{U} + \Omega\boldsymbol{V}$ which is passing through a point A. By equations $\boldsymbol{OX} = \boldsymbol{x} + \varepsilon\boldsymbol{x}^*, \boldsymbol{OA} = \boldsymbol{a} + \varepsilon\boldsymbol{a}^*, \boldsymbol{U} = \boldsymbol{u} + \varepsilon\boldsymbol{u}^*, \boldsymbol{V} = \boldsymbol{v} + \varepsilon\boldsymbol{v}^*$, we obtain Eqs (4.3) and (4.4). Equation (4.3) represents the equation of a real plane in $\mathbb{R}^3$, and Eq (4.4) represents the equation of a line congruence in $\mathbb{R}^3$. □

Comment 3. *In Section 3, it was proven that every point in the dual space $\mathbb{D}^3$ corresponds to a line in $\mathbb{R}^3$. In this section, it is shown that every line in $\mathbb{D}^3$ corresponds to a (line-planes) pair in $\mathbb{R}^3$ and also that every dual plane in $\mathbb{D}^3$ corresponds to a line congruence in $\mathbb{R}^3$. Furthermore, thanks to dual Plücker coordinates, it has been shown that the set of all lines in dual space $\mathbb{D}^3$ is a 4-dimensional subspace of a 6-dimensional space on $\mathbb{D}$.*

5. Theory of curves in $\mathbb{D}^3$

In this section, the theory of curves in $\mathbb{D}^3$ is constructed and the Frenet elements and formulas of the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ are obtained by using the Frenet elements and formulas of the real curves α and α^*. Moreover, helix curve, involute-evolute offsets, Bertrand offsets, and Darboux frame, which are significant concepts of the theory of curves, are introduced to the theory of dual curves.

Let $\boldsymbol{\alpha}\colon I \subset \mathbb{R} \longrightarrow \mathbb{D}^3$ be a differentiable mapping where $\boldsymbol{\alpha}(I) \subset \mathbb{D}^3$ is a dual curve. This curve can be represented in terms of the regular real curves α and α^* as

$$\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s).$$

Assume that s denotes the arc-length parameter of the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$, i.e., $\|\boldsymbol{\alpha}'(s)\| = 1$ or $\|\alpha'(s)\| = 1$, $\langle\alpha'(s), \alpha^{*\prime}(s)\rangle = 0$. Then, s is arc-length of α and α is the evolute of the curve α^* (i.e., $\alpha^*(s) = \alpha(s) + (c - s)\alpha'(s)$, $c \in \mathbb{R}$). Let $\{T_\alpha, N_\alpha, B_\alpha\}$ denote the Frenet frame of the real curve α. Then, we have

$$\alpha^{*\prime}(s) = (c - s)T'_\alpha \tag{5.1}$$

or

$$T = \boldsymbol{\alpha}'(s) = \alpha'(s) + \varepsilon\alpha^{*\prime}(s) = T_\alpha + \varepsilon(c - s)T'_\alpha. \tag{5.2}$$

Let the Frenet elements of the dual curve $\boldsymbol{\alpha}$ be $\{T, N, B, \kappa, \tau\}$, and $\kappa_\alpha, \tau_\alpha$ denote the curvature and torsion of the real curve α. Then, we obtain

$$\boldsymbol{\alpha}''(s) = \kappa_\alpha N_\alpha + \varepsilon\Big(-(c - s)\kappa_\alpha^2 T_\alpha + \big(-\kappa_\alpha + (c - s)\kappa'_\alpha\big)N_\alpha + (c - s)\kappa_\alpha\tau_\alpha B_\alpha\Big).$$

Thus,

$$\kappa = \|\boldsymbol{\alpha}''(s)\| = \kappa_\alpha + \varepsilon\big(-\kappa_\alpha + (c - s)\kappa'_\alpha\big). \tag{5.3}$$

Furthermore, defining

$$N = \frac{\boldsymbol{\alpha}''(s)}{\|\boldsymbol{\alpha}''(s)\|}, \qquad B = T \times N, \tag{5.4}$$

we obtain

$$\begin{cases} T = T_\alpha + \varepsilon(c - s)T'_\alpha, \\ N = N_\alpha + \varepsilon(c - s)N'_\alpha, \\ B = B_\alpha + \varepsilon(c - s)B'_\alpha. \end{cases} \tag{5.5}$$

The torsion τ is given by

$$\tau = \frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{\alpha}'', \boldsymbol{\alpha}''' \rangle}{\|\boldsymbol{\alpha}''\|^2}.$$

Then, we have

$$\tau = \tau_\alpha + \varepsilon(-\tau_\alpha + (c-s)\tau'_\alpha). \tag{5.6}$$

Then, using Eq (5.5) and the Frenet formulas of the real curve α as

$$\begin{cases} T'_\alpha = \kappa_\alpha N_\alpha, \\ N'_\alpha = -\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha, \\ B'_\alpha = -\tau_\alpha N_\alpha, \end{cases} \tag{5.7}$$

we get

$$\begin{cases} T' = \kappa_\alpha N_\alpha + \varepsilon\Big(-(c-s)\kappa_\alpha^2 T_\alpha + (-\kappa_\alpha + (c-s)\kappa'_\alpha)N_\alpha + (c-s)\kappa_\alpha\tau_\alpha B_\alpha\Big), \\ N' = (-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha) + \varepsilon\Big((\kappa_\alpha - (c-s)\kappa'_\alpha)T_\alpha - (c-s)(\kappa_\alpha^2 + \tau_\alpha^2)N_\alpha - (\tau'_\alpha - (c-s)\tau'_\alpha)B_\alpha\Big), \\ B' = -\tau_\alpha N_\alpha + \varepsilon\Big((c-s)\kappa_\alpha\tau_\alpha T_\alpha - (-\tau_\alpha + (c-s)\tau'_\alpha)N_\alpha - (c-s)\tau_\alpha^2 B_\alpha\Big). \end{cases} \tag{5.8}$$

By analogy with the Frenet formulas of real curves, we get the following theorem:

Theorem 8. *For the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ parameterized by arc-length s, we obtain dual Frenet formulas as:*

$$\begin{cases} T' = \kappa N, \\ N' = -\kappa T + \tau B, \\ B' = -\tau N. \end{cases} \tag{5.9}$$

Proof. By using Eqs (5.3) and (5.5)–(5.8), we can easily provide the proof. □

Example 1. *Let $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$, where*

$$\alpha(s) = \left(\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\sin(s), \frac{1}{\sqrt{2}}s\right)$$

and $\alpha^*(s) = (-\cos(s), \sin(s), -\sin(s)\cos(s))$ (see α and α^* in Figure 2, plotted in Python).

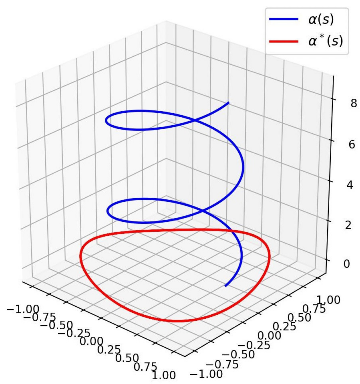

Figure 2. The curve α (red) and the curve α^* (blue) for the dual curve given in Example 1.

Via the classical curve theory, the Frenet elements of the real curve α are obtained as:

$$\begin{cases} \|\alpha'(s)\| = 1, \quad \kappa_\alpha = \dfrac{1}{\sqrt{2}}, \qquad \tau_\alpha = \dfrac{1}{\sqrt{2}}, \\ T_\alpha(s) = \alpha'(s) = \left(-\dfrac{1}{\sqrt{2}}\sin(s), \dfrac{1}{\sqrt{2}}\cos(s), \dfrac{1}{\sqrt{2}}\right), \\ N_\alpha(s) = (-\cos(s), -\sin(s), 0), \\ B_\alpha(s) = \left(\dfrac{1}{\sqrt{2}}\sin(s), -\dfrac{1}{\sqrt{2}}\cos(s), \dfrac{1}{\sqrt{2}}\right). \end{cases}$$

If we consider Eqs (2.2)*,* (5.3)*,* (5.5)*, and* (5.6) *for the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$, we have the Frenet elements as:*

$$\begin{cases} \|\boldsymbol{\alpha}'(s)\| = 1, \quad \kappa = \dfrac{1}{\sqrt{2}} - \varepsilon\dfrac{1}{\sqrt{2}}, \quad \tau = \dfrac{1}{\sqrt{2}} - \varepsilon\dfrac{1}{\sqrt{2}}, \\ T(s) = \dfrac{1}{\sqrt{2}}\left(-\sin(s) - \varepsilon(c-s)\cos(s), \cos(s) - \varepsilon(c-s)\sin(s), 1\right), \\ N(s) = \left(-\cos(s) + \varepsilon(c-s)\sin(s), -\sin(s) - \varepsilon(c-s)\cos(s), 0\right), \\ B(s) = \dfrac{1}{\sqrt{2}}\left(\sin(s) + \varepsilon(c-s)\cos(s), -\cos(s) + \varepsilon(c-s)\sin(s), 1\right). \end{cases}$$

One can see that Eqs (5.3)*,* (5.5)*, and* (5.6) *are satisfied. Hence, the theoretical findings derived for the theory of dual curves are confirmed by this example.*

Also, as in Theorem 2, the equation of the ruled surface $\phi(s, v) = \gamma(s) + v\delta(s)$ corresponding to the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ is obtained as:

$$\phi(s, v) = \frac{\sqrt{2}}{1+s^2}\left(-\sin^2(s)\cos(s) - s\sin(s), s\cos(s) + \sin(s)\cos^2(s), \sin(2s)\right) + v\frac{1}{\sqrt{1+s^2}}(\cos(s), \sin(s), s).$$

Figure 3 shows the graph of the ruled surface plotted using Matlab.

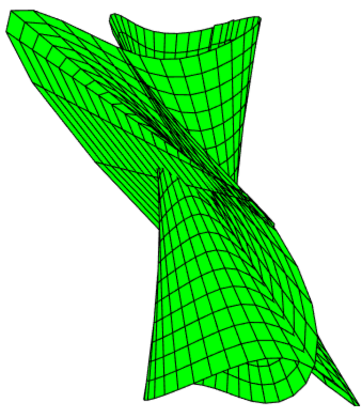

Figure 3. The graph of the ruled surface corresponding to the dual curve given in Example 1.

Remark 2. *For the constructed ruled surface associated with the given dual curve, the distribution parameter does not vanish, implying that the surface is non-developable. Furthermore, the Gaussian curvature is nonpositive, and the striction curve differs from the base curve, confirming the general nature of the ruled surface generated by the generalized E. Study maps.*

5.1. Dual helix curve

Let a dual curve in $\mathbb{D}^3$ be given by the expression:

$$\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s),$$

where s denotes the arc-length parameter of $\boldsymbol{\alpha}$. Let the Frenet elements of the dual curve $\boldsymbol{\alpha}$ be $\{T, N, B, \kappa, \tau\}$. If the unit tangent vector T of $\boldsymbol{\alpha}$ makes a constant dual angle with a fixed dual vector $\boldsymbol{U} = \boldsymbol{u} + \varepsilon\boldsymbol{u}^*$, then $\boldsymbol{\alpha}$ is called a dual helix in $\mathbb{D}^3$.

Equivalently, the condition can be expressed as

$$\langle T, \boldsymbol{U}\rangle = \cos(\Phi), \qquad \Phi = \varphi + \varepsilon\varphi^* = \text{constant},$$

where Φ denotes the dual angle between T and $\boldsymbol{U}$.

Theorem 9. *A necessary and sufficient condition for the dual curve $\boldsymbol{\alpha}$ to be a helix is that $\frac{\tau}{\kappa}$ is constant.*

Proof. Let $\boldsymbol{\alpha}$ be a helix. Then, $\langle T, \boldsymbol{U}\rangle = \cos(\Phi) = \text{constant}$. Then,

$$\langle T', \boldsymbol{U}\rangle = 0 \qquad \text{or} \qquad \langle N, \boldsymbol{U}\rangle = 0$$

are obtained. Hence, from Eq (5.5), we have

$$\langle N_\alpha, \boldsymbol{u}\rangle = 0, \quad \langle N_\alpha, \boldsymbol{u}^*\rangle = 0, \quad \langle T_\alpha, \boldsymbol{u}\rangle = \cos(\varphi), \quad \langle T_\alpha, \boldsymbol{u}^*\rangle = -\varphi^* \sin(\varphi).$$

Moreover, let $\boldsymbol{U} = \cos(\Phi)\, T + \sin(\Phi)\, B$. Then, we obtain

$$\boldsymbol{U} = \boldsymbol{u} + \varepsilon\boldsymbol{u}^* = \big(\cos(\varphi)\, T_\alpha + \sin(\varphi)\, B_\alpha\big) + \varepsilon\varphi^*\big(-\sin(\varphi)\, T_\alpha + \cos(\varphi)\, B_\alpha\big).$$

Using equation

$$\boldsymbol{U}' = \cos(\Phi)\, T' + \sin(\Phi)\, B'$$

and Eq (5.9), we have

$$\kappa\cos(\Phi) - \tau\sin(\Phi) = 0 \qquad \text{or} \qquad \frac{\tau}{\kappa} = \cot(\Phi) = \text{constant}. \tag{5.10}$$

This completes the proof. □

Note 2. *By the help of Eq* (5.10)*, we obtain*

$$\kappa_\alpha\cos(\varphi) - \tau_\alpha\sin(\varphi) = 0, \qquad \textit{and} \qquad (\tau_\alpha - \kappa_\alpha)\sin(\varphi) - (\tau_\alpha + \kappa_\alpha)\cos(\varphi) = 0.$$

Thus, if $\boldsymbol{\alpha}$ is a helix, then the real curve α is also a helix.

Note 3. *The vector pair $\{\boldsymbol{U}, \boldsymbol{U}^*\}$ can be obtained by rotating the vector pair $\{T_\alpha, N_\alpha\}$ through the angle φ.*

Let $\alpha^*(s) = \alpha(s) + \varepsilon(c - s)T_\alpha$. Since $\alpha^{*\prime}(s) = (c - s)T'_\alpha = (c - s)\kappa_\alpha N_\alpha$, it follows that

$$\langle \alpha^{*\prime}, \boldsymbol{U}^*\rangle = (c - s)\kappa_\alpha\langle N_\alpha, u^*\rangle = 0 = \cos(\tfrac{\pi}{2}).$$

Consequently, T_{α^*} and $\boldsymbol{U}^*$ are fixed vectors forming a constant angle of $\pi/2$. Hence, α^* is also a helix.

5.2. Dual involute-evolute offsets

In the dual space $\mathbb{D}^3$, let

$$\begin{aligned} \boldsymbol{\alpha}(s) &= \alpha(s) + \varepsilon\alpha^*(s), \qquad \|\boldsymbol{\alpha}'(s)\| = 1, \\ \boldsymbol{\beta}(t) &= \beta(t) + \varepsilon\beta^*(t), \qquad \|\boldsymbol{\beta}'(t)\| = 1, \end{aligned} \tag{5.11}$$

be two dual curves. Then, s is the arc-length parameter of both the dual curve $\boldsymbol{\alpha}$ and the real curve α, and the pair (α, α^*) forms real involute–evolute offsets. Similarly, t is the arc-length parameter of both the dual curve $\boldsymbol{\beta}$ and the real curve β, and the pair (β, β^*) forms real involute–evolute offsets.

Let the Frenet elements of the dual curve $\boldsymbol{\alpha}$ be $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ and those of the dual curve $\boldsymbol{\beta}$ be $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$. If $\langle T_\alpha, T_\beta\rangle = 0$, then the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is called dual involute–evolute offsets. In this case, we have

$$\langle T_\alpha, T_\beta\rangle = 0, \qquad \boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \Lambda(s)T_\alpha(s), \qquad \Lambda = \lambda + \varepsilon\lambda^* \in \mathbb{D}.$$

Together with Eq (5.5), this yields

$$\langle T_\alpha, T_\beta\rangle = 0, \qquad (d-t)\kappa_\beta\langle T_\alpha, N_\beta\rangle + (c-s)\kappa_\alpha\langle N_\alpha, T_\beta\rangle = 0.$$

Moreover, since $\Lambda'(s) = -1$, it follows that $c, d \in \mathbb{R}$. With $\lambda(s) = c_1 - s$ and $\lambda^*(s) = c_2$, we obtain

$$\Lambda(s) = C - s \in \mathbb{D}.$$

Theorem 10. *Let the pair* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *be the dual involute-evolute offsets with Eq* (5.11)*. Then, we have*

$$d(\boldsymbol{\alpha}(s), \boldsymbol{\beta}(s)) = |C - s|, \; C \in \mathbb{D}.$$

Remark 3. *By Theorem 10, the real curves* α *and* β *form involute-evolute offsets in* $\mathbb{R}^3$*, and*

$$d(\alpha(s), \beta(s)) = |c_1 - s|.$$

Hence, all properties valid for involute–evolute offsets in the Euclidean space remain valid in dual case.

Theorem 11. *Let* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *be dual involute-evolute offsets with the dual Frenet elements* $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ *and* $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$*, respectively. Then, the following relations hold:*

$$\begin{aligned} (C - s)\kappa_\alpha &= \frac{dt}{ds}, \\ T_\beta &= N_\alpha, \\ \kappa_\beta^2 &= \frac{\kappa_\alpha^2 + \tau_\alpha^2}{\kappa_\alpha^2 (C-s)^2}, \\ N_\beta &= -\frac{\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha}{\kappa_\alpha\kappa_\beta (C-s)}, \\ B_\beta &= T_\beta \times N_\beta = \frac{\tau_\alpha T_\alpha + \kappa_\alpha B_\alpha}{\kappa_\alpha\kappa_\beta (C-s)}. \end{aligned}$$

5.3. Dual bertrand offset

In the dual space $\mathbb{D}^3$, let

$$\begin{aligned} \boldsymbol{\alpha}(s) &= \alpha(s) + \varepsilon\alpha^*(s), \qquad \|\boldsymbol{\alpha}'(s)\| = 1, \\ \boldsymbol{\beta}(t) &= \beta(t) + \varepsilon\beta^*(t), \qquad \|\boldsymbol{\beta}'(t)\| = 1, \end{aligned} \tag{5.12}$$

be given curves. From this, s is the arc-length parameter of both the curve $\boldsymbol{\alpha}$ and the real curve α, and α together with α^* form real involute-evolute offsets. Similarly, t is the arc-length parameter of both the curve $\boldsymbol{\beta}$ and the real curve β, and β together with β^* form real involute-evolute offsets.

If $\{N_{\boldsymbol{\alpha}}, N_{\boldsymbol{\beta}}\}$ are linearly dependent, then the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is called a dual Bertrand offset. Let

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \Lambda(s)N_{\boldsymbol{\alpha}}(s), \qquad \Lambda = \lambda + \varepsilon\lambda^* \in \mathbb{D},$$

so that $\langle T_{\boldsymbol{\beta}}, N_{\boldsymbol{\beta}}\rangle = 0$, which implies $\langle T_{\boldsymbol{\beta}}, N_{\boldsymbol{\alpha}}\rangle = 0$. Moreover,

$$\frac{d\boldsymbol{\beta}}{ds} = (1 - \Lambda\kappa_{\boldsymbol{\alpha}})T_{\boldsymbol{\alpha}} + \Lambda' N_{\boldsymbol{\alpha}} + \Lambda\tau_{\boldsymbol{\alpha}}B_{\boldsymbol{\alpha}},$$

and, hence,

$$\Lambda'(s) = 0$$

is obtained. For $\lambda'(s) + \varepsilon\lambda^{*\prime}(s) = 0$, we have

$$\lambda(s) = \lambda^*(s) = \text{constant},$$

thus, Λ is found to be constant. Hence, $|\Lambda| = \text{constant}$ is obtained.

Theorem 12. *Let the pair* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *be the dual Bertrand offsets with Eq* (5.12). *Then, we have*

$$d(\boldsymbol{\alpha}(s), \boldsymbol{\beta}(s)) = \textit{constant}.$$

Remark 4. *From the equality* $\langle T_{\boldsymbol{\beta}}, N_{\boldsymbol{\alpha}}\rangle = 0$, *it follows that* $\langle T_\beta, N_\alpha\rangle = 0$. *Since* $N_{\boldsymbol{\alpha}}$ *and* $N_{\boldsymbol{\beta}}$ *are linearly dependent,* $\{N_\alpha, N_\beta\}$ *are also linearly dependent, and thus* α *and* β *form a real Bertrand offset, for which the concepts of differential geometry are valid:*

$$d(\alpha(s), \beta(s)) = \lambda = \textit{constant}.$$

Since α *and* α^* *form real involute-evolute offsets and* α *and* β *form a real Bertrand offset, it follows that* α^* *and* β *also satisfy the properties of involute-evolute offsets.*

5.4. Dual Darboux frame

In the dual space $\mathbb{D}^3$, the projection of the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ onto the unit dual sphere $\mathbb{S}^2$ is the dual curve $E(s) = e(s) + \varepsilon e^*(s)$, where s is arc-length parameter (see Eq (3.3)). In this case, we write:

$$E(s) = \frac{\boldsymbol{\alpha}(s)}{\|\boldsymbol{\alpha}(s)\|},$$

so that

$$E(s) = \frac{\alpha(s)}{\|\alpha(s)\|} + \varepsilon\left(\frac{\alpha^*(s)}{\|\alpha(s)\|} - \frac{\langle \alpha(s), \alpha^*(s)\rangle}{\|\alpha(s)\|^3}\,\alpha(s)\right).$$

The unit tangent vector of the dual curve E is given by $T_E = E'$. Since $\langle E, E\rangle = 1$, it follows that $\langle E, T_E\rangle = 0$. If we define $G = E \times T_E$, then $\{T_E, G, E\}$ forms the so-called dual Darboux frame of the dual curve E, with E serving as the dual spherical normal. The curve E admits the following invariants:

i. Dual geodesic curvature: $\kappa_g = \langle E'', E \times E'\rangle = \langle T_E', G\rangle$;
ii. Dual normal curvature: $\kappa_n = \langle E'', E\rangle = \langle T_E', E\rangle$;
iii. Dual geodesic torsion: $\tau_g = \langle E', G\rangle = -\langle E, G'\rangle$.

Accordingly, the dual Darboux frame satisfies the relations

$$\begin{cases} T_E' = -\kappa_g G + \kappa_n E, \\ G' = -\kappa_g T_E - \tau_g E, \\ E' = -\kappa_n T_E + \tau_g G. \end{cases}$$

Moreover, the dual Darboux frame $\{T_E, G, E\}$ and the dual Frenet frame $\{T_E, N_E, B_E\}$ are related by

$$\begin{pmatrix} T_E \\ G \\ E \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Phi) & \sin(\Phi) \\ 0 & -\sin(\Phi) & \cos(\Phi) \end{pmatrix} \begin{pmatrix} T_E \\ N_E \\ B_E \end{pmatrix}.$$

Hence, we obtain the following:

$$\kappa_g = \kappa\cos(\Phi), \qquad \kappa_n = -\kappa\sin(\Phi), \qquad \tau_g = -\tau - \Phi'.$$

6. Discussion and conclusions

In this study, the concepts of the dual point, the dual line, and the dual plane are defined in $\mathbb{D}^3$ and the fact that certain geometric structures in the Euclidean space correspond to them is proved. Besides, the theory of curves in $\mathbb{D}^3$ is constructed, and the Frenet elements and formulas of the dual curve $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$ are obtained by using the Frenet elements and formulas of the real curves α and α^*. Additionally, helix curve, involute-evolute offsets, Bertrand offsets, and Darboux frame, the significant concepts of the theory of curves, are introduced to the theory of dual curves. Moreover, generalized E. Study maps is introduced. This map states that for every dual curve in the dual space $\mathbb{D}^3$ of the form $\boldsymbol{\alpha}(s) = \alpha(s) + \varepsilon\alpha^*(s)$, there exists a corresponding ruled surface.

The findings of this study are expected to contribute to future studies on the dual curve theory, dual surface theory, and kinematics in $\mathbb{D}^3$. Notably, the correspondence between the ruled surfaces in the Euclidean space and dual curves can also be analyzed and studies on dual sphere motions can be extended to the investigations into the kinematics in $\mathbb{D}^3$. Thus, thanks to the generalized E. Study Maps, the striking correspondence between the dual space and the Euclidean space can be explored from an expanded viewpoint, without confining our attention to the unit dual sphere.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author states no conflict of interest.

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