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Research article

Bipolar fuzzy rough aggregation operator hybrid with TOPSIS and their application in group decision-making

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Abstract: In this article, we applied the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) to multi-criteria group decision making (MCGDM) with bipolar fuzzy rough numbers (BFRN). We present the dominant concept to develop the model of bipolar fuzzy rough sets (BFRS) with a new score, accuracy functions, and essential operations. Based on the concept of BFRS, we proposed the notion of BFR averaging aggregation operators, such as bipolar fuzzy rough weighted averaging (BFRWA), bipolar fuzzy rough ordered weighted averaging (BFROWA), and bipolar fuzzy rough hybrid averaging (BFRHA) aggregation operators. Thereafter, the rudimentary properties of the mentioned aggregation operators (AOs) were given in detail. Further, based on the concept of BFRS, we developed the concept of bipolar fuzzy rough geometric (BFRG) aggregation operators, such as bipolar fuzzy rough weighted geometric (BFRWG), bipolar fuzzy rough ordered weighted geometric (BFROWG), and bipolar fuzzy rough hybrid geometric (BFRHG) AO. Thereafter, prominent properties of the geometric AO were given in detail. Moreover, based on the developed model, we present a stepwise algorithm for applying the TOPSIS approach. The proposed AOs were combined using the concept accumulated geometric operator (AGO) to transform the experts' assessments from the BFR decision matrix into a decision matrix in the form of BFNs to approximate the concept of lower and upper approximations to get a single aggregated BFN. Then, we illustrated a numeric example of the presented concept and discussed the applicability of the proposed approach with the literature to show the significance and consequences of the suggested model. Based on the overall comparative study, we concluded that the proposed approach is superior and more effective than existing methods.

Keywords: BFS; RS; TOPSIS; arithmetic and geometric aggregation operators; MCGDM
Mathematics Subject Classification: 03E72, 03E75, 94D05, 90B50

1. Introduction

Multi-criteria group decision making has played a significant role in scientific literature. Its purpose is to engage a group of individuals who collaboratively evaluate assessments or factors based on the expertise of professionals. In this process, decision makers use an organized and flexible framework to assess alternatives against a set of criteria and ultimately reach a rational decision. To make desirable and appropriate decisions that are free from uncertainty and vagueness, Zadeh [1] introduced the prominent paradigm of fuzzy sets (FSs). The fundamental component FS is the membership grade. In the FS framework, a membership degree (MD) is assigned to each alternative according to its characteristics. Since its inception, the FS theory has become a cornerstone for researchers and has been applied across domains. However, a limitation of this theory is that it lacks a non-membership degree (NMD). To address this shortcoming, Atanassov [2] developed the concept of intuitionistic fuzzy sets (IFSs). An IFS is a significant extension of FS that incorporates MD and NMD. In IFS, the membership and non-membership functions are defined on the closed unit interval. The constraint of IFS is that the summation of these two functions must lie between 0 and 1. Since its inception, this notion of IFS has had a wide range of applications and become a dominant model that has been applied by scholars in diverse directions. For instance, Xu [3] originated the averaging aggregation operators (AOs) with their desirable properties by using the notion of IFS. The concept of geometric AO was developed by Xu and Yager [4] to aggregate the multi-assessments of decision specialists into a single decision value to get the optimal solution. Ali et al. [5] proposed graphical results to rank the score and accuracy functions while using IFS.

Bipolar fuzzy sets: Zhang [6] introduced the novel concept of bipolar fuzzy sets (BFSs), characterized by two fundamental components: The membership degree (MD), mapped to the interval $[0,1]$, and the non-membership degree (NMD), mapped to the interval $[-1,0]$. Zhang [7] further examined logic and fuzzy logic within the bipolar fuzzy (BF) environment. Since its inception, BFS has remained an active research area, and scholars have applied this concept in several directions. For example, Mahmood et al. [8−10] developed hybrid models combining BFS and complex fuzzy sets to propose the generalized framework of bipolar complex fuzzy sets. They explored their applications using decision-making (DM) approaches. Akram and Arshad [11] introduced the concepts of BF numbers and BF linguistic variables as generalizations of BFS. Akram et al. [12] proposed TOPSIS and ELECTRE I techniques within the BF framework and demonstrated their applications in medical diagnosis. Alghamdi et al. [13] introduced a BF multi-criteria decision-making approach by applying BF values to evaluate objects with respect to matching criteria. Han et al. [14] proposed the concept of bipolar-valued rough fuzzy sets and demonstrated its relevance to decision making. Le [15] conducted a comparative study of interval-valued IFS, IFS, and BFS, noting that although these models are closely related, they differ in their respective value ranges. Gul [16] developed several BF aggregation operators, including the BF weighted averaging and BF weighted geometric operators, and illustrated their applications in decision making. Wei [17] introduced BF aggregation operators based on Hamacher operators, such as the BF Hamacher weighted averaging and weighted geometric operators and studied their fundamental properties. Sarwer et al. [18] developed a TOPSIS method based on competition graphs under a BF environment and applied the model to decision-making scenarios. Mahmood [19] further advanced the study of BF soft sets and investigated the decision-making applications of the proposed framework.

Rough set theory: The prominent paradigm of rough set (RS) theory was initially developed by Pawlak [20] as a significant mathematical tool for processing and modeling incomplete, vague, and imprecise data. RS theory generalizes the concept of a crisp set, where a subset of a universal set is represented by a pair of sets called the lower and upper approximations. Equivalence relations form the foundation of Pawlak's rough set approach, with equivalence classes playing a central role in constructing the lower and upper approximations. Dubois and Prade [21] established the relation between FSs and RSs, leading to the development of rough FS and fuzzy rough set approaches. Ali et al. [22,23] proposed the concept of generalized roughness in FS by applying algebraic structure of fuzzy filters and fuzzy ideals in an ordered semigroup. Cornelis et al. [24] extended this idea to intuitionistic fuzzy sets (IFSs), introducing the concept of intuitionistic fuzzy rough sets (IFRSs). Zhou and Wu [25] proposed an axiomatic and constructive model of IFRS using rough approximation operators. Zhang et al. [26] redefined IFRS based on general binary relations and presented its fundamental properties over two universes. Yun and Lee [27] further refined IFRS by replacing crisp relations with intuitionistic fuzzy relations. Zhang et al. [28] explored the integration of soft sets, RS, and IFS to develop hybrid models such as soft-rough IFS and IF soft RS. Zhang et al. [29] proposed a generalized framework to obtain generalized IF soft RS. Hussain et al. [30] introduced rough Pythagorean fuzzy (PF) sets through algebraic connections. Hussain et al. [31] applied PF soft RS in decision-making contexts. Hussain et al. [32] presented the hybrid structure of rough sets and q-rung orthopair fuzzy sets (qROFS) based on covering sets, to get a resulting concept of covering-based rough qROFS models integrated with the TOPSIS approach.

Bipolar fuzzy rough set: Han et al. [33] developed the study of Bipolar valued FZ with the RS, to get the concept of the bipolar-valued rough FS. Yang et al. [34] presented the concept of the bipolar fuzzy rough set model for two universes by applying bipolar fuzzy compatible relation. Yang et al. [35] proposed the composition of bipolar fuzzy approximation spaces and discussed some properties. They then presented the applications of BF rough set models. Malik et al. [36] investigated the idea of rough BF ideals in a semigroup, which is a generalization of rough BFSs using the algebraic structure of a semigroup. Mahmood et al. [37] presented the combination study of BFS and a complex FZ hybrid with rough set to achieve a novel concept of the bipolar complex fuzzy rough set (BCFRS). Based on the BCFRS, they defined arithmetic averaging and geometric averaging AO. Gul [38] proposed the concept of bipolar fuzzy preference delta-covering based BF rough set model by combining the VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje) scheme. Wang and Li [39] developed interactive aggregation operators based on the Bonferroni mean under the PF environment. Hussain et al. [40] linked soft sets and qROFS to propose generalized qROF soft averaging aggregation operators for decision-making applications. Chinram et al. [41] extended this work to qROF soft geometric aggregation operators, demonstrating their practical application. Wang et al. [42] investigated hybrid connections among rough, soft, and qROFS to develop novel aggregation operators such as qROFSRWA and qROFSRWG, along with their fundamental properties. Chinram et al. [43] established the link between RS and IFR aggregation operators, introducing IF rough AO integrated with the EDAS model. Yahya et al. [44] proposed Frank norms using qROF rough AO, including qROFFWA and qROFFWG, and examined their desirable properties. Shahzaib et al. [45] studied Einstein operators with the combined structure of qROFRS, applying them in robotic agrifarming.

The motivation and organization of our research:

We acknowledge different models in the literature for multi-criteria group decision making (MCGDM) methods like MARCOS (Measurement of Alternatives and Ranking according to COmpromise solution), VIKOR, MABAC (Multi-Attributive Border Approximation area Comparison), and AROMAN (Alternative Ranking Order Method Accounting for Two-Step Normalization) offering the best ranking perspective. These methods may be selected for evolution purposes, but they are less aligned with the uncertainties framework with the objectives and have different underlying mechanisms. For instance, VIKOR presents a strategy parameter that may impact ranking stability, while MARCOS and MABAC provide border approximation ideas that are less directly interpretable under rough and fuzzy uncertainty structures. In contrast, TOPSIS offers a parameter-free, geometrically intuitive, and computationally efficient structure with its primary objective being to identify the optimal alternative that is closest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS). The PIS represents the collection of best attribute values, while the NIS represents the collection of worst attribute values obtained during the evaluation process. This mechanism integrates naturally with lower and upper approximation modeling and preserves uncertainty dispersion without imposing additional behavioral assumptions. Furthermore, TOPSIS has a well-established theoretical foundation and extensive validation in uncertainty-based decision environments. For these methodological and practical reasons, TOPSIS is considered the most appropriate and theoretically consistent choice for the proposed model. The Dombi operation, with its natural flexibility via an operational parameter, provides additional adaptability for experts. Building on this, Hussain et al. [46] connected Dombi norms with IFRSs and qROF soft sets. They provided a resulting series of aggregation operators, including IFRDWA, IFRDWG, qROFSWA, and qROFSWG, integrated with the TOPSIS approach. The BFR aggregation models build the fundamental structure for the classical weighted and geometric AO. The MCGDM approach systematically aggregates the outcomes of developed models, including BFRWA and BFRWG aggregation operators. Our aim of the presented approach is to rank and identify the optimal alternative from the collection feasible options. The introduction of BFR averaging and geometric aggregation operators brings a significant contribution in decision making. This extension integrates the fundamental structure for averaging and geometric AO. The theoretical framework of BFS and RS is a well-established mathematical paradigm for modeling and analyzing imprecision, uncertainty, and vagueness inherent in real-world data. The integration of BFRS within MCDGM methodologies such as BFRWA and BFRWG aggregation operators provides a theoretically grounded enhancement to the decision-making framework, especially in research contexts where data is incomplete, uncertain, or imprecisely specified. This incorporation enables a more systematic and rigorous treatment of ambiguity inherent in complex evaluation environments. The studies mentioned above motivated our research presented in this paper, as no researcher has addressed the gap of rough aggregation operators within a bipolar fuzzy environment. Accordingly, we aim to establish a link between BFSs and rough sets by introducing pioneering rough averaging and geometric aggregation operators and presenting their essential properties.

The remainder of the paper is organized as follows: In Section 2, we introduce fundamental concepts and logics that establish the foundation for the developed models. In Section 3, the dominant concept of the proposed model is presented, along with new score and accuracy functions, and several essential operations. In Section 4, we develop the notion of BFRA aggregation operators based on the BFRS concept, including BFRWA, BFROWA, and BFRHA operators, with their fundamental properties discussed in detail. Similarly, in Section 5, we introduce BFRG aggregation operators, including BFRWG, BFROWG, and BFRHG aggregation operators, along with their essential properties. In Section 6, we present the framework of MCGDM and provide a stepwise algorithm for the TOPSIS approach applied to the developed model. Finally, in Section 7, we illustrate a numerical example of the proposed concepts and evaluate the applicability of the approach compared with other methods, demonstrating its superiority and significance.

2. Preliminaries

In this section, we present some fundamental concepts that will be helpful in establishing the proposed framework and guiding the subsequent sections.

Definition 1 [6]. Let U be a universal of discourse. A BFS $\boldsymbol{\mathcal{B}}$ on a ground set U is defined as:

$$\boldsymbol{\mathcal{B}} = \left\{\langle \wp, {\mu^+}_{\boldsymbol{\mathcal{B}}}(\wp), {\psi^-}_{\boldsymbol{\mathcal{B}}}(\wp)\rangle | \wp \in U\right\},$$

where ${\mu^+}_{\boldsymbol{\mathcal{B}}} : U \to [0,1]$ denotes a positive membership function and ${\psi^-}_{\boldsymbol{\mathcal{B}}}: U \to [-1, 0]$ represents negative membership function of an object $\wp \in U$, to the BFS set $\boldsymbol{\mathcal{B}}$. For simplicity, $\boldsymbol{\mathcal{B}} = \langle \wp, {\mu^+}_{\boldsymbol{\mathcal{B}}}(\wp), {\psi^-}_{\boldsymbol{\mathcal{B}}}(\wp)\rangle$ is represented by $\boldsymbol{\mathcal{B}} = (\mu, \psi)$ and is said to be a BF number (BFN) for alternative $\wp \in U$. The family of BF subsets are represented by $BFS(U)$.

Consider that $\boldsymbol{\mathcal{B}} = (\mu, \psi)$ and $\boldsymbol{\mathcal{B}}_1 = (\mu_1, \psi_1)$ are any two BFNs, then some elementary desirable operations for BFNs are given as [26]:

(i) $\boldsymbol{\mathcal{B}} \cup \boldsymbol{\mathcal{B}}_1 = \left(\max\left(\mu(\wp), \mu_1(\wp)\right), min\left((\psi(\wp), \psi_1(\wp))\right)\right)$;

(ii) $\boldsymbol{\mathcal{B}} \cap \boldsymbol{\mathcal{B}}_1 = \left(\min\left(\mu(\wp), \mu_1(\wp)\right), max\left(\psi(\wp), \psi_1(\wp)\right)\right)$;

(iii) $\boldsymbol{\mathcal{B}} \oplus \boldsymbol{\mathcal{B}}_1 = (\mu + \mu_1 - \mu\mu_1 , -|\psi||\psi_1|)$;

(iv) $\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{B}}_1 = (\mu\mu_1, \psi + \psi_1 - \psi\psi_1)$;

(v) $\boldsymbol{\mathcal{B}} \leq \boldsymbol{\mathcal{B}}_1$ if $\mu(\wp) \leq \mu_1(\wp)$, $\psi(\wp) \geq \psi_1(\wp)$ for all $\wp \in U$;

(vi) $\boldsymbol{\mathcal{B}}^c = (1 - \mu, |\psi| - 1)$ where $\boldsymbol{\mathcal{B}}^c$ characterizes the complement of $\boldsymbol{\mathcal{B}}$;

(vii) $\alpha\boldsymbol{\mathcal{B}} = (1 - (1 - \mu)^\alpha, -|\psi|^\alpha)$ for $\alpha \geq 1$;

(viii) $\boldsymbol{\mathcal{B}}^\alpha = ((\mu)^\alpha, -1 + |1 + \psi|^\alpha)$ for $\alpha \geq 1$.

Definition 2 [7]. Considering a BF relation $\mathcal{R}$ on a set U is a subset of $U \times U$, which is defined as:

$$\mathcal{R} = \{\langle(\wp, t), \mu(\wp, t), \psi(\wp, t)\rangle | (\wp, t) \in U \times U\}.$$

Definition 3 [7]. Consider a BF relation $\mathcal{R}$ on a set U, then

(a) The relation $\mathcal{R}$ is reflexive if $\mu(\wp, \wp) = 1$ and $\psi(\wp, \wp) = 0$ for each $\wp \in U$,

(b) The relation $\mathcal{R}$ is symmetric if $\mu(\wp, t) = \mu(t, \wp)$ and $\psi(\wp, t) = \psi(t, \wp)$ for each $\wp, t \in U$,

(c) The relation $\mathcal{R}$ is transitive if $\mu(\wp, t) = \mu(\wp, \mathrm{u})$ and $\mu(\wp, \mathrm{u}) = \mu(t, \mathrm{u})$ then $\mu(\wp, t) = \mu(t, \mathrm{u})$ and $\psi(\wp, t) = \psi(\wp, \mathrm{u})$ and $\psi(\wp, \mathrm{u}) = \psi(t, \mathrm{u})$ then $\psi(\wp, t) = \psi(t, \mathrm{u})$ for each $\wp, t, \mathrm{u} \in U$.

Definition 4 [17]. Consider $\boldsymbol{\mathcal{B}} = (\mu, \psi)$ be a BFN, then the score function is described as:

$$S(\boldsymbol{\mathcal{B}}) = \frac{1}{2}(1 + \mu + \psi) \quad for\ S(\boldsymbol{\mathcal{B}}) \in [0,1].$$

The lager score value implies the greater BFN.

Definition 5 [17]. Consider $\mathcal{B} = (\mu, \psi)$ be a BFN, then the accuracy function for $\boldsymbol{\mathcal{B}} = (\mu, \psi)$ is evaluated as:

$$A(\boldsymbol{\mathcal{B}}) = \frac{1}{2}(\mu - \psi) \quad for \ A(\boldsymbol{\mathcal{B}}) \in [0,1].$$

The larger accuracy implies the better BFN.

3. Bipolar fuzzy rough sets

Rough set theory is a nice tool to discuss uncertainty in which equivalence relation plays a significant role; that is, for any undefinable subset of a universal set, it may be approximated in the form of definable subsets, called lower approximation and upper approximation, and the pair of lower and upper approximation is a called rough set. However, due to the limited knowledge about the element of a set, it becomes difficult to check the equivalence relation. Thus, researchers have studied generalized models with less restrictions for defining lower and upper approximation by applying the concept of fuzzy relation, IF relation, and BF relation. The lower and upper approximation operators are key and primitive notions in the development of rough set theory, and one of the major directions for the study of rough set theory is the extensive definitions of rough approximation operators in various situations. In this section, we will put forward the basic notion of BFRS and present its important basic operations through which we will continue the on-word study.

Definition 6. Let U be a universe of set and $\mathcal{R} \in BFS(U \times U)$ be a BF fuzzy relation. The pair $(U, \mathcal{R})$ is known as BF approximation space. Now, for any $\xi \subseteq BFS(U)$, the lower and upper approximation of ξ based on BF approximation space $(U, \mathcal{R})$ are BFSs, evaluated as

$$\underline{\mathcal{R}}(\xi) = \{\langle \wp, \mu_{\underline{\mathcal{R}}(\xi)}(\wp), \psi_{\underline{\mathcal{R}}(\xi)}(\wp)\rangle | \wp \in U\}, \quad \text{(i)}$$

$$\overline{\mathcal{R}}(\xi) = \left\{\langle \wp, \mu_{\overline{\mathcal{R}}(\xi)}(\wp), \psi_{\overline{\mathcal{R}}(\xi)}(\wp)\rangle \,|\wp \in U\right\}. \quad \text{(ii)}$$

where:

$$\mu_{\underline{\mathcal{R}}(\xi)}(\wp) = \wedge_{t \in U}\{\mu(\wp, t) \wedge \mu_\xi(t)\},\ \psi_{\underline{\mathcal{R}}(\xi)}(\wp) = \vee_{t \in U}\{\psi(\wp, t) \vee \psi_\xi(t)\},$$

$$\mu_{\overline{\mathcal{R}}(\xi)}(\wp) = \vee_{t \in U}\{\mu(\wp, t) \vee \mu_\xi(t)\},\ \psi_{\overline{\mathcal{R}}(\xi)}(\wp) = \wedge_{t \in U}\{\psi(\wp, t) \wedge \psi_\xi(t)\}.$$

Since $\underline{\mathcal{R}}(\xi)$ and $\overline{\mathcal{R}}(\xi)$ are BFSs, $\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi): BFS(U) \to BFS(U)$ are lower and upper approximation operators. Hence, the pair $\mathcal{R}(\xi) = \left(\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi)\right) = \left\{\left(\wp, \langle \mu_{\underline{\mathcal{R}}(\xi)}(\wp), \psi_{\underline{\mathcal{R}}(\xi)}(\wp)\rangle, \langle \mu_{\overline{\mathcal{R}}(\xi)}(\wp),\ \psi_{\overline{\mathcal{R}}(\xi)}(\wp)\rangle\right) | \wp \in U\right\}$ is said to be BF rough set (BFRS). For straightforwardness, $\mathcal{R}(\xi) = \left(\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi)\right) = \left(\langle \underline{\mu}, \underline{\psi}\rangle, \langle\ \overline{\mu}, \overline{\psi}\ \rangle\right)$ represents the BF rough number (BFRN) if there is no confusion.

Definition 7. Let $\mathcal{R}(\xi) = \left(\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi)\right) = \left(\langle \underline{\mu}, \underline{\psi} \rangle, \langle \overline{\mu}, \overline{\psi} \rangle\right)$ be a BFRN. Then the score function is given as:

$$S(\mathcal{R}(\xi)) = \frac{1}{4}\left(2 + \underline{\mu} + \overline{\mu} + \underline{\psi} + \overline{\psi}\right) \quad for\ S(\mathcal{R}(\xi)) \in [0,1].$$

The BFRN with the maximum score value is regarded as the best alternative.

Definition 8. Let $\mathcal{R}(\xi) = \left(\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi)\right) = \left(\langle \underline{\mu}, \underline{\psi} \rangle, \langle \overline{\mu}, \overline{\psi} \rangle\right)$ be a BFRN. Then, if the score is level, the function for accuracy is given as:

$$A(\mathcal{R}(\xi)) = \frac{1}{4}\left(\underline{\mu} + \overline{\mu} - \underline{\psi} - \overline{\psi}\right) \quad for\ A(\mathcal{R}(\xi)) \in [0,1].$$

An alternative with maximum accuracy function value is considered the most superior.

Definition 9. Consider any two BFRNs $\mathcal{R}(\xi_1) = \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right)$ and $\mathcal{R}(\xi_2) = \left(\underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_2)\right)$. Then some desirable operations are given as.

(i) $\mathcal{R}(\xi_1) \cup \mathcal{R}(\xi_2) = \left\{\left(\underline{\mathcal{R}}(\xi_1) \cup \underline{\mathcal{R}}(\xi_2)\right), \left(\overline{\mathcal{R}}(\xi_2) \cup \overline{\mathcal{R}}(\xi_2)\right)\right\}$;

(ii) $\mathcal{R}(\xi_1) \cap \mathcal{R}(\xi_2) = \left\{\left(\underline{\mathcal{R}}(\xi_1) \cap \underline{\mathcal{R}}(\xi_2)\right), \left(\overline{\mathcal{R}}(\xi_1) \cap \overline{\mathcal{R}}(\xi_2)\right)\right\}$;

(iii) $\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi_2) = \left\{\left(\underline{\mathcal{R}}(\xi_1) \oplus \underline{\mathcal{R}}(\xi_2)\right), \left(\overline{\mathcal{R}}(\xi_1) \oplus \overline{\mathcal{R}}(\xi_2)\right)\right\}$;

(iv) $\mathcal{R}(\xi_1) \otimes \mathcal{R}(\xi_2) = \left\{\left(\underline{\mathcal{R}}(\xi_1) \otimes \underline{\mathcal{R}}(\xi_2)\right), \left(\overline{\mathcal{R}}(\xi_1) \otimes \overline{\mathcal{R}}(\xi_2)\right)\right\}$;

(v) $\mathcal{R}(\xi_1) \subseteq \mathcal{R}(\xi_2) = \left(\underline{\mathcal{R}}(\xi_1) \subseteq \underline{\mathcal{R}}(\xi_2)\right)$ *and* $\left(\overline{\mathcal{R}}(\xi_1) \subseteq \overline{\mathcal{R}}(\xi_2)\right)$;

(vi) $\alpha\mathcal{R}(\xi_1) = \left(\alpha\underline{\mathcal{R}}(\xi_1), \alpha\overline{\mathcal{R}}(\xi_1)\right)$ for $\alpha \geq 1$;

(vii) $\left(\mathcal{R}(\xi_1)\right)^\alpha = \left(\left(\underline{\mathcal{R}}(\xi_1)\right)^\alpha, \left(\overline{\mathcal{R}}(\xi_1)\right)^\alpha\right)$ for $\alpha \geq 1$.

(viii) $\mathcal{R}(\xi_1)^c = \left(\underline{\mathcal{R}}(\xi_1)^c, \overline{\mathcal{R}}(\xi_1)^c\right)$ where $\underline{\mathcal{R}}(\xi_1)^c$ *and* $\overline{\mathcal{R}}(\xi_1)^c$ denotes the complement of BFR approximation operators $\underline{\mathcal{R}}(\xi_1)$ *and* $\overline{\mathcal{R}}(\xi_1)$, i.e., $\underline{\mathcal{R}}(\xi_1)^c = \left(\underline{\psi_1}, \underline{\mu_1}\right)$.

(ix) $\mathcal{R}(\xi_1) = \mathcal{R}(\xi_2)$ *iff* $\underline{\mathcal{R}}(\xi_1) = \underline{\mathcal{R}}(\xi_2)$ *and* $\overline{\mathcal{R}}(\xi_1) = \overline{\mathcal{R}}(\xi_2)$.

Proposition 1. Let $(U, \mathcal{R})$ be a BF approximation space. Let $\mathcal{R}(\xi_1) = \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right)$ and $\mathcal{R}(\xi_2) = \left(\underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_2)\right)$ any two BFRNs on set U. Then some of the elementary properties are defined as BFRS.

(i) $\sim\left(\sim\mathcal{R}(\xi_1)\right) = \mathcal{R}(\xi_1)$ where $\sim\mathcal{R}(\xi_1)$ denotes the complement of $\mathcal{R}(\xi_1)$;

(ii) $\mathcal{R}(\xi_1) \cup \mathcal{R}(\xi_2) = \mathcal{R}(\xi_2) \cup \mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_1) \cap \mathcal{R}(\xi_2) = \mathcal{R}(\xi_2) \cap \mathcal{R}(\xi_1)$;

(iii) $\sim(\mathcal{R}(\xi_1) \cup \mathcal{R}(\xi_2)) = (\sim\mathcal{R}(\xi_1)) \cap (\sim\mathcal{R}(\xi_1))$;

(iv) $\sim(\mathcal{R}(\xi_1) \cap \mathcal{R}(\xi_2)) = (\sim\mathcal{R}(\xi_1)) \cup (\sim\mathcal{R}(\xi_1))$;

(v) If $\xi_1 \subseteq \xi_2$, then $\mathcal{R}(\xi_1) \subseteq \mathcal{R}(\xi_2)$;
(vi) $\mathcal{R}(\xi_1 \cup \xi_2) \supseteq \mathcal{R}(\xi_1) \cup \mathcal{R}(\xi_2)$;
(vii) $\mathcal{R}(\xi_1 \cap \xi_2) \subseteq \mathcal{R}(\xi_1) \cap \mathcal{R}(\xi_2)$.

Theorem 1. Let $\mathcal{R}(\xi_1) = \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right)$ and $\mathcal{R}(\xi_2) = \left(\underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_2)\right)$ be any two BFRNs with $\alpha_1, \alpha_2 > 0$. Then the results below are obvious for BFRNs:
(i) $\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi_2) = \mathcal{R}(\xi_2) \oplus \mathcal{R}(\xi_1)$;
(ii) $\mathcal{R}(\xi_1) \otimes \mathcal{R}(\xi_2) = \mathcal{R}(\xi_2) \otimes \mathcal{R}(\xi_1)$;

(iii) $\alpha_1(\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi_2)) = \alpha_1\mathcal{R}(\xi_1) \oplus \alpha_1\mathcal{R}(\xi_2)$;

(iv) $(\alpha_1 + \alpha_2)\mathcal{R}(\xi_1) = \alpha_1\mathcal{R}(\xi_1) \oplus \alpha_2\mathcal{R}(\xi_1)$;

(v) $(\mathcal{R}(\xi_1) \otimes \mathcal{R}(\xi_2))^{\alpha_1} = (\mathcal{R}(\xi_1))^{\alpha_1} \otimes (\mathcal{R}(\xi_2))^{\alpha_1}$;

(vi) $(\mathcal{R}(\xi_1))^{\alpha_1} \otimes (\mathcal{R}(\xi_1))^{\alpha_2} = (\mathcal{R}(\xi_1))^{(\alpha_1+\alpha_2)}$.

Proof. (i) As $\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi_2) = \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right) \oplus \left(\underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_2)\right)$

$$= \left(\underline{\mathcal{R}}(\xi_1) \oplus \underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_1) \oplus \overline{\mathcal{R}}(\xi_2)\right)$$

$$= \left[\left(\underline{\mu_1} + \underline{\mu_2} - \underline{\mu_1}\ \underline{\mu_2}, -\left|\underline{\psi_1}\right|\left|\underline{\psi_2}\right|\right), \left(\overline{\mu_1} + \overline{\mu_2} - \overline{\mu_1}\ \overline{\mu_2}, -\left|\overline{\psi_1}\right|\left|\overline{\psi_2}\right|\right)\right]$$

$$= \left[\left(\underline{\mu_2} + \underline{\mu_1} - \underline{\mu_2}\ \underline{\mu_1}, -\left|\underline{\psi_2}\right|\left|\underline{\psi_1}\right|\right), \left(\overline{\mu_2} + \overline{\mu_1} - \overline{\mu_2}\ \overline{\mu_1}, -\left|\overline{\psi_2}\right|\left|\overline{\psi_1}\right|\right)\right]$$

$$= \left(\underline{\mathcal{R}}(\xi_2) \oplus \underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_2) \oplus \overline{\mathcal{R}}(\xi_1)\right)$$

$$= \left(\underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_2)\right) \oplus \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right).$$

Proof. The remaining of the proofs follow from the above result.

4. Bipolar fuzzy rough averaging aggregation operators

The concept of AOs plays a crucial role in DM by combining evaluations from experts to obtain a single optimal value. In this section, we present several averaging AOs, including BFRWA, BFROWA, and BFRHA operators, and examine their fundamental properties.

4.1. BF rough weighted averaging operators

Definition 10. Suppose that $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ is the collection of BFRNs with weight vector (WV) $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then the aggregated form of the BFRWA operator is a mapping given by $\left(\mathcal{R}(\xi)\right)^n \to \mathcal{R}(\xi)$, and is defined as:

$$BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_n)\right) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}(\xi_i), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}(\xi_i)\right). \tag{iii}$$

Based on the definition of BFRWA operator, the aggregated result is illustrated in Theorem 2.

Theorem 2. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then the aggregated operator for BFRWA is given as:

$$BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_n)\right) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}(\xi_i), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}(\xi_i)\right)$$

$$= \left\{\left(1 - \prod_{i=1}^{n} \left(1 - \underline{\mu_i}\right)^{\varpi_i}, -\prod_{i=1}^{n} \left|\underline{\psi_i}\right|^{\varpi_i}\right), \left(1 - \prod_{i=1}^{n} (1 - \overline{\mu_i})^{\varpi_i}, -\prod_{i=1}^{n} \left|\overline{\psi_i}\right|^{\varpi_i}\right)\right\}. \tag{iv}$$

Proof. By applying the induction principle, we can find the proof of the required result.

As with the above BFR operational law given in Eq (iii):

$$\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi_2) = \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right) \oplus \left(\underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_2)\right) = \left(\underline{\mathcal{R}}(\xi_1) \oplus \underline{\mathcal{R}}(\xi_2), \overline{\mathcal{R}}(\xi_1) \oplus \overline{\mathcal{R}}(\xi_2)\right)$$

$$= \left[\left(\underline{\mu_1} + \underline{\mu_2} - \underline{\mu_1}\ \underline{\mu_2}, -\left|\underline{\psi_1}\right| \left|\underline{\psi_1}\right|\right), \left(\overline{\mu_1} + \overline{\mu_2} - \overline{\mu_1}\ \overline{\mu_2}, -\left|\overline{\psi_1}\right| \left|\overline{\psi_2}\right|\right)\right]$$

and

$$\alpha\mathcal{R}(\xi) = \left(\alpha\underline{\mathcal{R}}(\xi)\ , \alpha\overline{\mathcal{R}}(\xi)\right) = \left[\left(1 - \left(1 - \underline{\mu}\right)^{\alpha}, -\left|\underline{\psi}\right|^{\alpha}\right), \left(1 - (1 - \overline{\mu}\ )^{\alpha}, -\left|\overline{\psi}\right|^{\alpha}\right)\right]$$

Let the result is true for $n = 1$. Then the L.H.S of Eq (iv) is:

$$BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_n)\right) = \mathcal{R}(\xi_1) = \left(\underline{\mathcal{R}}(\xi_1), \overline{\mathcal{R}}(\xi_1)\right) = \left(\left(\underline{\mu_1}\ , \underline{\psi_1}\right), \left(\overline{\mu_1}, \overline{\psi_1}\right)\right)$$

and for R.H.S of Eq (iv) is:

$$= \left\{\left(1 - \left(1 - \underline{\mu_1}\right), -\left|\underline{\psi_1}\right|\right), \left(1 - (1 - \overline{\mu_1}), -\left|\overline{\psi_1}\right|\right)\right\} = \left(\left(\underline{\mu_1}\ , \underline{\psi_1}\right), \left(\overline{\mu_1}, \overline{\psi_1}\right)\right)$$

Thus, the result is true for $n = 1$.

Suppose Eq (iv) holds for $n = k$, so we have:

$$BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_k)\right) = \left(\oplus_{i=1}^{k} \varpi_i \underline{\mathcal{R}}(\xi_i), \oplus_{i=1}^{k} \varpi_i \overline{\mathcal{R}}(\xi_i)\right)$$

$$= \left\{ \left(1 - \prod_{i=1}^{k} \left(1 - \underline{\mu_i}\right)^{\varpi_i}, - \prod_{i=1}^{k} \left|\underline{\psi_i}\right|^{\varpi_i} \right), \left(1 - \prod_{i=1}^{k} (1 - \overline{\mu_i})^{\varpi_i}, - \prod_{i=1}^{k} \left|\overline{\psi_i}\right|^{\varpi_i} \right) \right\}.$$

Moreover, to verify that Eq (iv) is valid for $n = k + 1$,

$$BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_k), \mathcal{R}(\xi_{k+1})\right)$$

$$= \left[\left\{ \left(\oplus_{i=1}^{k} \varpi_i \underline{\mathcal{R}}(\xi_i) \right) \oplus \left(\varpi_{k+1} \underline{\mathcal{R}}(\xi_{k+1}) \right) \right\}, \left\{ \left(\oplus_{i=1}^{k} \varpi_i \overline{\mathcal{R}}(\xi_i) \right) \oplus \left(\varpi_{k+1} \overline{\mathcal{R}}(\xi_{k+1}) \right) \right\} \right]$$

$$= \left[\left\{ \left(1 - \prod_{i=1}^{k} \left(1 - \underline{\mu_i}\right)^{\varpi_i}, - \prod_{i=1}^{k} \left|\underline{\psi_i}\right|^{\varpi_i} \right) \oplus \left(1 - \left(1 - \underline{\mu_{k+1}}\right)^{\varpi_{k+1}}, - \left|\underline{\psi_{k+1}}\right|^{\varpi_{k+1}} \right) \right\}, \left\{ \left(1 - \prod_{i=1}^{k} (1 - \overline{\mu_i})^{\varpi_i}, - \prod_{i=1}^{k} \left|\overline{\psi_i}\right|^{\varpi_i} \right) \oplus \left(1 - (1 - \overline{\mu_{k+1}})^{\varpi_{k+1}}, - \left|\overline{\psi_{k+1}}\right|^{\varpi_{k+1}} \right) \right\} \right]$$

$$= \left\{ \left(1 - \prod_{i=1}^{k+1} \left(1 - \underline{\mu_i}\right)^{\varpi_i}, - \prod_{i=1}^{k+1} \left|\underline{\psi_i}\right|^{\varpi_i} \right), \left(1 - \prod_{i=1}^{k+1} (1 - \overline{\mu_i})^{\varpi_i}, - \prod_{i=1}^{k+1} \left|\overline{\psi_i}\right|^{\varpi_i} \right) \right\}.$$

Hence, the result holds for $n = k + 1$. Therefore, by applying the induction principle, Eq (iv) holds for any value of $n \geq 1$.

It is observed that $\underline{\mathcal{R}}(\xi)$ and $\overline{\mathcal{R}}(\xi)$ are BFRNs, implying that $\oplus_{i=1}^{k} \varpi_i \underline{\mathcal{R}}(\xi_i)$ and $\oplus_{i=1}^{k} \varpi_i \overline{\mathcal{R}}(\xi_i)$ are also BFRNs. Hence, we observe that $BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right)$ represents a BFRN based on the BF approximation space $(U, \mathcal{R})$.

The properties of the BFRWA operator are developed in Theorem 3.

Theorem 3. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then the BFRWA aggregated result satisfies the following desirable properties:

(i) **Idempotency**. Suppose $\mathcal{R}(\xi_i) = \mathcal{T}(\mathcal{G})$ for all $i = 1,2, \dots, n$ with $\mathcal{T}(\mathcal{G}) = \left(\underline{\mathcal{T}}(\mathcal{G}), \overline{\mathcal{T}}(\mathcal{G})\right) = \left(\langle \underline{\mathcal{u}}, \underline{\mathcal{v}} \rangle, \langle \overline{\mathcal{u}}, \overline{\mathcal{v}} \rangle\right)$. Then

$$BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) = \mathcal{T}(\xi).$$

(ii) **Boundedness**. Let $\left(\mathcal{R}(\xi_i)\right)^- = \left(\min_i \underline{\mathcal{R}}(\xi_i), \min_i \overline{\mathcal{R}}(\xi_i)\right)$ and $\left(\mathcal{R}(\xi_i)\right)^+ = \left(\max_i \underline{\mathcal{R}}(\xi_i), \max_i \overline{\mathcal{R}}(\xi_i)\right)$. Then

$$\left(\mathcal{R}(\xi_i)\right)^- \leq BFRWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) \leq \left(\mathcal{R}(\xi_i)\right)^+.$$

(iii) **Monotonicity**. Let the collection $\mathcal{R}(\xi_i') = \left(\underline{\mathcal{R}}(\xi_i'), \overline{\mathcal{R}}(\xi_i')\right)$ of BFRNs with $\underline{\mathcal{R}}(\xi_i') \leq \underline{\mathcal{R}}(\xi_i)$ and $\overline{\mathcal{R}}(\xi_i') \leq \overline{\mathcal{R}}(\xi_i)$. Then

$$BFRWA\big(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\big) \leq BFRWA\big(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\big).$$

(iv) **Shift invariance**. Let $\mathcal{R}(\xi') = \left(\underline{\mathcal{R}}(\xi'), \overline{\mathcal{R}}(\xi')\right)$ be another BFRNs. Then

$$BFRWA\big(\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi'), \mathcal{R}(\xi_2) \oplus \mathcal{R}(\xi'), \mathcal{R}(\xi_3) \oplus \mathcal{R}(\xi'), \dots, \mathcal{R}(\xi_n) \oplus \mathcal{R}(\xi')\big)$$

$$= BFRWA\big(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\big) \oplus \mathcal{R}(\xi').$$

(v) **Homogeneity**. Consider for any real number $\alpha > 0$,

$$BFRWA\big(\alpha\mathcal{R}(\xi_1), \alpha\mathcal{R}(\xi_2), \alpha\mathcal{R}(\xi_3), \dots, \alpha\mathcal{R}(\xi_n)\big) = \alpha BFRWA\big(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\big).$$

Proof. **(i) Idempotency**. As $\mathcal{R}(\xi_i) = \mathcal{T}(\mathcal{G})$ for all $i = 1,2,\dots,n$ with $\mathcal{T}(\mathcal{G}) = \left(\underline{\mathcal{T}}(\mathcal{G}),\ \overline{\mathcal{T}}(\mathcal{G})\right) = \left(\langle \underline{u}, \underline{v} \rangle, \langle \overline{u}, \overline{v} \rangle\right)$.

Now

$$BFRWA\big(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\big) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}(\xi_i), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}(\xi_i)\right)$$

$$= \left\{\left(1 - \prod_{i=1}^{n}\left(1 - \underline{\mu_i}\right)^{\varpi_i}, -\prod_{i=1}^{n}\left|\underline{\psi_i}\right|^{\varpi_i}\right), \left(1 - \prod_{i=1}^{n}(1 - \overline{\mu_i})^{\varpi_i}, -\prod_{i=1}^{n}\left|\overline{\psi_i}\right|^{\varpi_i}\right)\right\}.$$

Since for all i, $\mathcal{R}(\xi_i) = \mathcal{T}(\mathcal{G}) = \left(\underline{\mathcal{T}}(\mathcal{G}),\ \overline{\mathcal{T}}(\mathcal{G})\right) = \left(\langle \underline{u}, \underline{v} \rangle, \langle \overline{u}, \overline{v} \rangle\right)$.

Therefore,

$$= \left\{\left(1 - \prod_{i=1}^{n}(1 - \underline{u})^{\varpi_i}, -\prod_{i=1}^{n}\left|\underline{v}\right|^{\varpi_i}\right), \left(1 - \prod_{i=1}^{n}(1 - \overline{u})^{\varpi_i}, -\prod_{i=1}^{n}|\overline{v}|^{\varpi_i}\right)\right\}$$

$$= \left\{\left(1 - \left(1 - \underline{u}\right), -\left|\underline{v}\right|\right), (1 - (1 - \overline{u}), -|\overline{v}|)\right\}$$

$$= \left(\underline{\mathcal{T}}(\mathcal{G}), \overline{\mathcal{T}}(\mathcal{G})\right)$$

$$= \mathcal{T}(\mathcal{G}).$$

Hence,

$$BFRWA\big(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\big) = \mathcal{T}(\mathcal{G})$$

$$for\ \ all\ \ i,\ \mathcal{R}(\xi_i) = \mathcal{T}(\mathcal{G}) = \left(\underline{\mathcal{T}}(\mathcal{G}),\ \overline{\mathcal{T}}(\mathcal{G})\right) = \left(\langle \underline{u}, \underline{v} \rangle, \langle \overline{u}, \overline{v} \rangle\right).$$

Proof. The remaining proofs follow from Theorem 1 and Eq (iv).

4.2. BF rough ordered weighted averaging operators

From the analysis of the BFRWA operator, it is observed that the BFRWA operator only weighs the values of BFRN, while the BFROWA operator weighs the ordered positions of BFRN through scoring instead of weighting the BFR values. Here we will explore the logic properties of the proposed operator.

Definition 11. Suppose that $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ is the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Here the aggregated form of the BFROWA operator is a mapping given by $\left(\mathcal{R}(\xi)\right)^n \rightarrow \mathcal{R}(\xi)$, and is defined as:

$$BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}(\xi_{\sigma i}), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}(\xi_{\sigma i})\right). \qquad \text{(v)}$$

From the above definition of the BFROWA operator, the aggregated result is presented in Theorem 4.

Theorem 4. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Now the aggregated outcome for the BFROWA operator is as:

$$BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}(\xi_{\sigma i}), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}(\xi_{\sigma i})\right)$$

$$= \left\{\left(1 - \prod_{i=1}^{n}\left(1 - \underline{\mu_{\sigma i}}\right)^{\varpi_i}, -\prod_{i=1}^{n}\left|\underline{\psi_{\sigma i}}\right|^{\varpi_i}\right), \left(1 - \prod_{i=1}^{n}(1 - \overline{\mu_{\sigma i}})^{\varpi_i}, -\prod_{i=1}^{n}\left|\overline{\psi_{\sigma i}}\right|^{\varpi_i}\right)\right\}, \qquad \text{(vi)}$$

where $\mathcal{R}(\xi_\sigma) = \left(\underline{\mathcal{R}}(\xi_{\sigma i}), \overline{\mathcal{R}}(\xi_{\sigma i})\right)$ denotes the permutation with larger value from the collection of BSRNs.

The properties of the BFROWA operator are described in Theorem 5.

Theorem 5. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then the BFROWA aggregated result satisfies the following desirable properties:

(i) **Idempotency**. Suppose $\mathcal{R}(\xi_i) = \mathcal{T}(\mathcal{G})$ for all $i = 1,2, \dots, n$ with $\mathcal{T}(\mathcal{G}) = \left(\underline{\mathcal{T}}(\mathcal{G}), \overline{\mathcal{T}}(\mathcal{G})\right) = \left(\langle \underline{\mathcal{u}}, \underline{\mathcal{v}} \rangle, \langle \overline{\mathcal{u}}, \overline{\mathcal{v}} \rangle\right)$. Then

$$BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) = \mathcal{T}(\xi).$$

(ii) **Boundedness**. Let $\left(\mathcal{R}(\xi_i)\right)^{-} = \left(\min_i \underline{\mathcal{R}}(\xi_i), \min_i \overline{\mathcal{R}}(\xi_i)\right)$ and $\left(\mathcal{R}(\xi_i)\right)^{+} = \left(\max_i \underline{\mathcal{R}}(\xi_i),\right.$

$\max_i \overline{\mathcal{R}}(\xi_i)\Big)$. Then

$$\left(\mathcal{R}(\xi_i)\right)^{-} \leq BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) \leq \left(\mathcal{R}(\xi_i)\right)^{+}.$$

(iii) **Monotonicity**. Suppose that the collection $\mathcal{R}(\xi_i') = \left(\underline{\mathcal{R}}(\xi_i'), \overline{\mathcal{R}}(\xi_i')\right)$ of BFRNs such that $\underline{\mathcal{R}}(\xi_i') \leq \underline{\mathcal{R}}(\xi_i)$ and $\overline{\mathcal{R}}(\xi_i') \leq \overline{\mathcal{R}}(\xi_i)$. Then

$$BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) \leq BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right).$$

(iv)**Shift invariance**. Let $\mathcal{R}(\xi') = \left(\underline{\mathcal{R}}(\xi'), \overline{\mathcal{R}}(\xi')\right)$ be another BFRNs. Then

$$BFROWA\left(\mathcal{R}(\xi_1) \oplus \mathcal{R}(\xi'), \mathcal{R}(\xi_2) \oplus \mathcal{R}(\xi'), \mathcal{R}(\xi_3) \oplus \mathcal{R}(\xi'), \dots, \mathcal{R}(\xi_n) \oplus \mathcal{R}(\xi')\right)$$

$$= BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) \oplus \mathcal{R}(\xi').$$

(v) **Homogeneity**. Consider any constant α with the condition that $\alpha > 0$,

$$BFROWA\left(\alpha\mathcal{R}(\xi_1), \alpha\mathcal{R}(\xi_2), \alpha\mathcal{R}(\xi_3), \dots, \alpha\mathcal{R}(\xi_n)\right) = \alpha BFROWA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right).$$

Proof. Proof followed from Theorem 2.

4.3. BF rough hybrid averaging operators

In this section, we will develop the idea of the BFRHA operator. The BFRHA operator is a hybrid of the characteristics of the BFRWA and BFROWA operators. The notion of the BFRHA operator weighs the value and ordered position of a BF argument at the same time. The important properties of the initiated operator are presented in detail.

Definition 12. Suppose that $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ is the collection of BFRNs such that $\delta = (\delta_1, \delta_2, \delta_3, \dots, \delta_n)^T$ is the associated WV with $\sum_{i=1}^{n} \delta_i = 1$ and $\delta_i \in [0,1]$, and $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ is the WV such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Now the mathematical form of the BFRHA operator is a mapping given by $\left(\mathcal{R}(\xi)\right)^n \to \mathcal{R}(\xi)$, and is defined as:

$$BFRHA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}\left(\ddot{\xi}_{\sigma i}\right), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}\left(\ddot{\xi}_{\sigma i}\right)\right). \quad \text{(vii)}$$

By the definition of the BFRHA operator, their aggregated result originates in Theorem 6.

Theorem 6. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs such that $\delta = (\delta_1, \delta_2, \delta_3, \dots, \delta_n)^T$ is the associated WV with $\sum_{i=1}^{n} \delta_i = 1$ and $\delta_i \in [0,1]$, and $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ is the WV such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. The aggregated mathematical form of BFRHA is defined as:

$$BFRHA\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \dots, \mathcal{R}(\xi_n)\right) = \left(\oplus_{i=1}^{n} \varpi_i \underline{\mathcal{R}}\left(\ddot{\xi}_{\sigma i}\right), \oplus_{i=1}^{n} \varpi_i \overline{\mathcal{R}}\left(\ddot{\xi}_{\sigma i}\right)\right). \quad \text{(viii)}$$

$$=\left\{\left(1-\prod_{i=1}^{n}\left(1-\underline{\ddot{\mu}\ddot{\psi}_{\sigma i}}\right)^{\varpi_i},-\prod_{i=1}^{n}\left|\underline{\ddot{\psi}_{\sigma i}}\right|^{\varpi_i}\right),\left(1-\prod_{i=1}^{n}\left(1-\overline{\ddot{\mu}_{\sigma i}}\right)^{\varpi_i},-\prod_{i=1}^{n}\left|\overline{\ddot{\psi}_{\sigma i}}\right|^{\varpi_i}\right)\right\},$$

where $\mathcal{R}(\ddot{\xi}_{\sigma i})=\left(\underline{\mathcal{R}}(\ddot{\xi}_{\sigma i}),\overline{\mathcal{R}}(\ddot{\xi}_{\sigma i})\right)$ is the i^{th} largest element of BF argument from the collection of BSRNs and n represents a balancing coefficient.

Some basic and elementary properties of BFRHA operator are proposed in Theorem 7.

Theorem 7. Let $\mathcal{R}(\xi_i)=\left(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs such that $\delta=(\delta_1,\delta_2,\delta_3,\dots,\delta_n)^T$ is the associated WV with $\sum_{i=1}^{n}\delta_i=1$ and $\delta_i\in[0,1]$, and $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ is the WV such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. The operational form of the BFRHA operators satisfies the following desirable properties:

(i) **Idempotency**. Suppose $\mathcal{R}(\xi_i)=\mathcal{T}(\mathcal{G})$ for all $i=1,2,\dots,n$ with $\mathcal{T}(\mathcal{G})=\left(\underline{\mathcal{T}}(\mathcal{G}),\overline{\mathcal{T}}(\mathcal{G})\right)=\left(\langle\underline{u},\underline{v}\rangle,\langle\overline{u},\overline{v}\rangle\right)$. Then

$$BFRHA\left(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\right)=\mathcal{T}(\xi).$$

(ii) **Boundedness**. Let $\left(\mathcal{R}(\xi_i)\right)^-=\left(\min_i\underline{\mathcal{R}}(\xi_i),\min_i\overline{\mathcal{R}}(\xi_i)\right)$ and $\left(\mathcal{R}(\xi_i)\right)^+=\left(\max_i\underline{\mathcal{R}}(\xi_i),\max_i\overline{\mathcal{R}}(\xi_i)\right)$. Then

$$\left(\mathcal{R}(\xi_i)\right)^-\leq BFRHA\left(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\right)\leq\left(\mathcal{R}(\xi_i)\right)^+.$$

(iii) **Monotonicity**. Let the collection $\mathcal{R}(\xi_i')=\left(\underline{\mathcal{R}}(\xi_i'),\overline{\mathcal{R}}(\xi_i')\right)$ of BFRNs with $\underline{\mathcal{R}}(\xi_i')\leq\underline{\mathcal{R}}(\xi_i)$ and $\overline{\mathcal{R}}(\xi_i')\leq\overline{\mathcal{R}}(\xi_i)$. Then

$$BFRHA\left(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\right)\leq BFRHA\left(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\right).$$

(iv) **Shift invariance**. Let $\mathcal{R}(\xi')=\left(\underline{\mathcal{R}}(\xi'),\overline{\mathcal{R}}(\xi')\right)$ be another BFRNs. Then

$$BFRHA\left(\mathcal{R}(\xi_1)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_2)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_3)\oplus\mathcal{R}(\xi'),\dots,\mathcal{R}(\xi_n)\oplus\mathcal{R}(\xi')\right)$$
$$=BFRHA\left(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\right)\oplus\mathcal{R}(\xi').$$

(v) **Homogeneity**. Consider for any real number $\alpha>0$,

$$BFRHA\left(\alpha\mathcal{R}(\xi_1),\alpha\mathcal{R}(\xi_2),\alpha\mathcal{R}(\xi_3),\dots,\alpha\mathcal{R}(\xi_n)\right)$$
$$=\alpha BFRHA\left(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\right).$$

Proof. Proof followed from Theorem 2.

5. Bipolar fuzzy rough geometric aggregation operators

In this section, we originate the logic of geometric AO, which has a significant role in decision making to aggregate an expert's assessment to obtain an output of a single value. This study consists of the BFRWG, BFROWG, and BFRHG operators, and we will initiate the important properties.

5.1. BF rough weighted geometric operators

Definition 13. Suppose that $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ is the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. The mathematical output of the BFRWG operator is a mapping given by $\left(\mathcal{R}(\xi)\right)^n \to \mathcal{R}(\xi)$ and is defined as:

$$BFRWG\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_n)\right) = \left(\otimes_{i=1}^{n} \left(\underline{\mathcal{R}}(\xi_i)\right)^{\varpi_i}, \otimes_{i=1}^{n} \left(\overline{\mathcal{R}}(\xi_i)\right)^{\varpi_i}\right). \tag{ix}$$

Based on the definition of the BFRWG operator, the aggregated result is illustrated in Theorem 8.

Theorem 8. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. The result for the BFRWG operator is given as:

$$BFRWG\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_n)\right) = \left(\otimes_{i=1}^{n} \left(\underline{\mathcal{R}}(\xi_i)\right)^{\varpi_i}, \otimes_{i=1}^{n} \left(\overline{\mathcal{R}}(\xi_i)\right)^{\varpi_i}\right).$$

$$= \left\{ \left(\prod_{i=1}^{n} \left(\underline{\mu_i}\right)^{\varpi_i}, -\left(1 - \prod_{i=1}^{n} \left(1 - \underline{\psi_i}\right)^{\varpi_i}\right) \right), \left(\prod_{i=1}^{n} (\overline{\mu_i})^{\varpi_i}, -\left(1 - \prod_{i=1}^{n} \left(1 - \overline{\psi_i}\right)^{\varpi_i}\right) \right) \right\}. \tag{x}$$

Proof. Proof followed from Theorem 2.

It is observed that $\underline{\mathcal{R}}(\xi)$ and $\overline{\mathcal{R}}(\xi)$ are BFRNs, implying that $\otimes_{i=1}^{n} \left(\underline{\mathcal{R}}(\xi_i)\right)^{\varpi_i}$ and $\otimes_{i=1}^{n} \left(\overline{\mathcal{R}}(\xi_i)\right)^{\varpi_i}$ are also BFRNs. Hence, we observe that $BFRWG\left(\mathcal{R}(\xi_1), \mathcal{R}(\xi_2), \mathcal{R}(\xi_3), \ldots, \mathcal{R}(\xi_n)\right)$ represents a BFRN based on the BF approximation space $(U, \mathcal{R})$.

The elementary properties of the BFRWG operator are described in Theorem 9.

Theorem 9. Let $\mathcal{R}(\xi_i) = \left(\underline{\mathcal{R}}(\xi_i), \overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n)^T$ such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then the BFRWG aggregated result satisfies the following desirable properties:

(i) **Idempotency**. Suppose $\mathcal{R}(\xi_i) = \mathcal{T}(\mathcal{G})$ for all $i = 1,2,\ldots,n$ with $\mathcal{T}(\mathcal{G}) = \left(\underline{\mathcal{T}}(\mathcal{G}), \overline{\mathcal{T}}(\mathcal{G})\right) = \left(\langle \underline{u}, \underline{v} \rangle, \langle \overline{u}, \overline{v} \rangle\right)$. Then

$$BFRWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)=\mathcal{T}(\xi).$$

(ii) **Boundedness**. Let $\big(\mathcal{R}(\xi_i)\big)^{-}=\Big(\min_i \underline{\mathcal{R}}(\xi_i),\min_i \overline{\mathcal{R}}(\xi_i)\Big)$ and $\big(\mathcal{R}(\xi_i)\big)^{+}=\Big(\max_i \underline{\mathcal{R}}(\xi_i),$ $\max_i \overline{\mathcal{R}}(\xi_i)\Big)$. Then

$$\big(\mathcal{R}(\xi_i)\big)^{-}\leq BFRWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)\leq\big(\mathcal{R}(\xi_i)\big)^{+}.$$

(iii) **Monotonicity**. Consider the collection $\mathcal{R}(\xi_i')=\Big(\underline{\mathcal{R}}(\xi_i'),\overline{\mathcal{R}}(\xi_i')\Big)$ of BFRNs such that $\underline{\mathcal{R}}(\xi_i')\leq\underline{\mathcal{R}}(\xi_i)$ and $\overline{\mathcal{R}}(\xi_i')\leq\overline{\mathcal{R}}(\xi_i)$. Then

$$BFRWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)\leq BFRWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big).$$

(iv) **Shift Invariance**. Let $\mathcal{R}(\xi')=\Big(\underline{\mathcal{R}}(\xi'),\overline{\mathcal{R}}(\xi')\Big)$ be another BFRNs. Then

$$BFRWG\big(\mathcal{R}(\xi_1)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_2)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_3)\oplus\mathcal{R}(\xi'),\dots,\mathcal{R}(\xi_n)\oplus\mathcal{R}(\xi')\big)$$

$$=BFRWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)\oplus\mathcal{R}(\xi').$$

(v) **Homogeneity**. Consider any real number $\alpha>0$,

$$BFRWG\big(\alpha\mathcal{R}(\xi_1),\alpha\mathcal{R}(\xi_2),\alpha\mathcal{R}(\xi_3),\dots,\alpha\mathcal{R}(\xi_n)\big)$$

$$=\alpha BFRWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big).$$

Proof. Proofs are easy and followed from Theorem 2.

5.2. BF rough ordered weighted geometric operators

From the analysis of the BFRWG operator, it is observed that the BFRWG operator weighs only the values of BFRN, while the BFROWG operator weighs the ordered positions of BFRN through scoring instead of weighing the BFR value. Here we will explore the logic properties of the proposed operator.

Definition 14. Suppose that $\mathcal{R}(\xi_i)=\Big(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\Big)$ is the collection of BFRNs with WV $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. The result for BFROWG AO is a mapping given by $\big(\mathcal{R}(\xi)\big)^n\rightarrow\mathcal{R}(\xi)$ and is described below.

$$BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)=\Big(\otimes_{i=1}^{n}\Big(\underline{\mathcal{R}}(\xi_{\sigma i})\Big)^{\varpi_i},\otimes_{i=1}^{n}\Big(\overline{\mathcal{R}}(\xi_{\sigma i})\Big)^{\varpi_i}\Big).\qquad\text{(xi)}$$

From the above definition of the BFROWG operator, the aggregated result is presented in

Theorem 10.

Theorem 10. Let $\mathcal{R}(\xi_i)=\left(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi=(\varpi_1,\varpi_2,\varpi_3,\ldots,\varpi_n)^T$ such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. The result described for BFROWG AO is given below.

$$BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\ldots,\mathcal{R}(\xi_n)\big)=\left(\otimes_{i=1}^{n}\left(\underline{\mathcal{R}}(\xi_{\sigma i})\right)^{\varpi_i},\otimes_{i=1}^{n}\left(\overline{\mathcal{R}}(\xi_{\sigma i})\right)^{\varpi_i}\right)$$

$$=\left\{\left(\prod_{i=1}^{n}\left(\underline{\mu_{\sigma i}}\right)^{\varpi_i},-\left(1-\prod_{i=1}^{n}\left(1-\underline{\psi_{\sigma i}}\right)^{\varpi_i}\right)\right),\left(\prod_{i=1}^{n}(\overline{\mu_{\sigma i}})^{\varpi_i},-\left(1-\prod_{i=1}^{n}\left(1-\overline{\psi_{\sigma i}}\right)^{\varpi_i}\right)\right)\right\}, \quad \text{(xii)}$$

where $\mathcal{R}(\xi_\sigma)=\left(\underline{\mathcal{R}}(\xi_{\sigma i}),\overline{\mathcal{R}}(\xi_{\sigma i})\right)$ denotes the largest value of the permutation from the collection of BSRNs.

The fundamental results of the BFROWG operator are explained in Theorem 11.

Theorem 11. Let $\mathcal{R}(\xi_i)=\left(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs with WV $\varpi=(\varpi_1,\varpi_2,\varpi_3,\ldots,\varpi_n)^T$ such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. Then the BFROWG aggregated result satisfies the following desirable properties:

(i) **Idempotency**. Suppose $\mathcal{R}(\xi_i)=\mathcal{T}(\mathcal{G})$ for all $i=1,2,\ldots,n$ with $\mathcal{T}(\mathcal{G})=\left(\underline{\mathcal{T}}(\mathcal{G}),\overline{\mathcal{T}}(\mathcal{G})\right)=\left(\langle\underline{u},\underline{v}\rangle,\langle\overline{u},\overline{v}\rangle\right)$. Then

$$BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\ldots,\mathcal{R}(\xi_n)\big)=\mathcal{T}(\xi).$$

(ii) **Boundedness**. Let $\left(\mathcal{R}(\xi_i)\right)^-=\left(\min_i\underline{\mathcal{R}}(\xi_i),\min_i\overline{\mathcal{R}}(\xi_i)\right)$ and $\left(\mathcal{R}(\xi_i)\right)^+=\left(\max_i\underline{\mathcal{R}}(\xi_i),\max_i\overline{\mathcal{R}}(\xi_i)\right)$. Then

$$\left(\mathcal{R}(\xi_i)\right)^-\leq BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\ldots,\mathcal{R}(\xi_n)\big)\leq\left(\mathcal{R}(\xi_i)\right)^+.$$

(iii) **Monotonicity**. Consider the collection $\mathcal{R}(\xi_i')=\left(\underline{\mathcal{R}}(\xi_i'),\overline{\mathcal{R}}(\xi_i')\right)$ of BFRNs such that $\underline{\mathcal{R}}(\xi_i')\leq\underline{\mathcal{R}}(\xi_i)$ and $\overline{\mathcal{R}}(\xi_i')\leq\overline{\mathcal{R}}(\xi_i)$. Then

$$BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\ldots,\mathcal{R}(\xi_n)\big)\leq BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\ldots,\mathcal{R}(\xi_n)\big).$$

(iv) **Shift invariance**. Let $\mathcal{R}(\xi')=\left(\underline{\mathcal{R}}(\xi'),\overline{\mathcal{R}}(\xi')\right)$ be another BFRNs. Then

$$BFROWG\big(\mathcal{R}(\xi_1)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_2)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_3)\oplus\mathcal{R}(\xi'),\ldots,\mathcal{R}(\xi_n)\oplus\mathcal{R}(\xi')\big)$$
$$=BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\ldots,\mathcal{R}(\xi_n)\big)\oplus\mathcal{R}(\xi').$$

(v) **Homogeneity**. Consider any real number $\alpha>0$,

$$BFROWG\big(\alpha\mathcal{R}(\xi_1),\alpha\mathcal{R}(\xi_2),\alpha\mathcal{R}(\xi_3),\dots,\alpha\mathcal{R}(\xi_n)\big)=\alpha\ BFROWG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big).$$

Proof. Proof followed from Theorem 2.

5.3. BF rough hybrid geometric operators

In this section, we will develop the idea of the BFRHG operator. The BFRHG operator is a hybrid of the BFRWG and BFROWG operators. The notion of BFRHG operators is that they weigh the value and their ordered position of BF arguments at the same time. The important properties of the initiated operator are presented in detail.

Definition 15. Suppose that $\mathcal{R}(\xi_i)=\left(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\right)$ is the collection of BFRNs such that $\delta=(\delta_1,\delta_2,\delta_3,\dots,\delta_n)^T$ is the associated WV with $\sum_{i=1}^{n}\delta_i=1$ and $\delta_i\in[0,1]$, and $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ is the WV such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. The mathematical form of BFRHG is a mapping given by $\left(\mathcal{R}(\xi)\right)^n\to\mathcal{R}(\xi)$ and is presented as:

$$BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)=\left(\otimes_{i=1}^{n}\left(\underline{\mathcal{R}}(\ddot{\xi}_{\sigma i})\right)^{\varpi_i},\otimes_{i=1}^{n}\left(\overline{\mathcal{R}}(\ddot{\xi}_{\sigma i})\right)^{\varpi_i}\right).\qquad\text{(xiii)}$$

By the definition of the BFRHG operator, their aggregated result originates in Theorem 12.

Theorem 12. Let $\mathcal{R}(\xi_i)=\left(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs such that $\delta=(\delta_1,\delta_2,\delta_3,\dots,\delta_n)^T$ is the associated WV with $\sum_{i=1}^{n}\delta_i=1$ and $\delta_i\in[0,1]$, and $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ is the WV such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. The result for the BFRHG operator is described below.

$$BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)=\left(\otimes_{i=1}^{n}\left(\underline{\mathcal{R}}(\ddot{\xi}_{\sigma i})\right)^{\varpi_i},\otimes_{i=1}^{n}\left(\overline{\mathcal{R}}(\ddot{\xi}_{\sigma i})\right)^{\varpi_i}\right)$$

$$=\left\{\left(\prod_{i=1}^{n}\left(\underline{\ddot{\mu}_{\sigma i}}\right)^{\varpi_i},-\left(1-\prod_{i=1}^{n}\left(1-\underline{\ddot{\psi}_{\sigma i}}\right)^{\varpi_i}\right)\right),\left(\prod_{i=1}^{n}\left(\overline{\ddot{\mu}_{\sigma i}}\right)^{\varpi_i},-\left(1-\prod_{i=1}^{n}\left(1-\overline{\ddot{\psi}_{\sigma i}}\right)^{\varpi_i}\right)\right)\right\},\qquad\text{(xiv)}$$

where $\mathcal{R}(\ddot{\xi}_{\sigma i})=\left(\underline{\mathcal{R}}(\ddot{\xi}_{\sigma i}),\overline{\mathcal{R}}(\ddot{\xi}_{\sigma i})\right)$ is the i^{th} largest element of BF argument from the collection of BSRNs and n represents a balancing coefficient.

Some important properties of the BFRHG operator are proposed in Theorem 13.

Theorem 13. Let $\mathcal{R}(\xi_i)=\left(\underline{\mathcal{R}}(\xi_i),\overline{\mathcal{R}}(\xi_i)\right)$ be the collection of BFRNs such that $\delta=(\delta_1,\delta_2,\delta_3,\dots,\delta_n)^T$ is the associated WV with $\sum_{i=1}^{n}\delta_i=1$ and $\delta_i\in[0,1]$, and $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ is the WV such that $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i\in[0,1]$. The mathematical form of the BFRHG operator satisfies the following desirable properties:

(i) **Idempotency**. Suppose $\mathcal{R}(\xi_i)=\mathcal{T}(\mathcal{G})$ for all $i=1,2,\dots,n$ with $\mathcal{T}(\mathcal{G})=\left(\underline{\mathcal{T}}(\mathcal{G}),\overline{\mathcal{T}}(\mathcal{G})\right)=\left(\langle\underline{u},\underline{v}\rangle,\langle\overline{u},\overline{v}\rangle\right)$. Then

$$BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)=\mathcal{T}(\xi).$$

(ii) **Boundedness**. Let $\big(\mathcal{R}(\xi_i)\big)^{-}=\Big(\min_i \underline{\mathcal{R}}(\xi_i),\min_i \overline{\mathcal{R}}(\xi_i)\Big)$ and $\big(\mathcal{R}(\xi_i)\big)^{+}=\Big(\max_i \underline{\mathcal{R}}(\xi_i),$ $\max_i \overline{\mathcal{R}}(\xi_i)\Big)$. Then

$$\big(\mathcal{R}(\xi_i)\big)^{-}\le BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)\le\big(\mathcal{R}(\xi_i)\big)^{+}.$$

(iii) **Monotonicity**. Consider the collection $\mathcal{R}(\xi_i')=\Big(\underline{\mathcal{R}}(\xi_i'),\overline{\mathcal{R}}(\xi_i')\Big)$ of BFRNs such that $\underline{\mathcal{R}}(\xi_i')\le\underline{\mathcal{R}}(\xi_i)$ and $\overline{\mathcal{R}}(\xi_i')\le\overline{\mathcal{R}}(\xi_i)$. Then

$$BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)\le BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big).$$

(iv) **Shift invariance**. Let $\mathcal{R}(\xi')=\Big(\underline{\mathcal{R}}(\xi'),\overline{\mathcal{R}}(\xi')\Big)$ be another BFRNs. Then

$$BFRHG\big(\mathcal{R}(\xi_1)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_2)\oplus\mathcal{R}(\xi'),\mathcal{R}(\xi_3)\oplus\mathcal{R}(\xi'),\dots,\mathcal{R}(\xi_n)\oplus\mathcal{R}(\xi')\big)$$

$$=BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big)\oplus\mathcal{R}(\xi').$$

(v) **Homogeneity**. Consider for any real number $\alpha>0$,

$$BFRHG\big(\alpha\mathcal{R}(\xi_1),\alpha\mathcal{R}(\xi_2),\alpha\mathcal{R}(\xi_3),\dots,\alpha\mathcal{R}(\xi_n)\big)$$

$$=\alpha\ BFRHG\big(\mathcal{R}(\xi_1),\mathcal{R}(\xi_2),\mathcal{R}(\xi_3),\dots,\mathcal{R}(\xi_n)\big).$$

Proof. Proof follows from Theorem 2.

Table 1. BFR evaluation information D_1.

	$\widetilde{\mathfrak{C}}_1$	$\widetilde{\mathfrak{C}}_2$	$\widetilde{\mathfrak{C}}_3$	$\widetilde{\mathfrak{C}}_4$
$\wp_1$	$((0.8,-0.2),(0.9,-0.1))$	$((0.6,-0.3),(0.7,-0.3))$	$((0.6,-0.2),(0.9,-0.1))$	$((0.8,-0.2),(0.8,-0.1))$
$\wp_2$	$((0.7,-0.1),(0.8,-0.2))$	$((0.9,-0.1),(0.6,-0.4))$	$((0.5,-0.3),(0.4,-0.4))$	$((0.6,-0.3),(0.5,-0.4))$
$\wp_3$	$((0.6,-0.2),(0.6,-0.1))$	$((0.8,-0.2),(0.4,-0.2))$	$((0.2,-0.1),(0.8,-0.2))$	$((0.5,-0.2),(0.6,-0.3))$
$\wp_4$	$((0.5,-0.3),(0.7,-0.2))$	$((0.5,-0.4),(0.8,-0.1))$	$((0.6,-0.4),(0.9,-0.1))$	$((0.7,-0.3),(0.9,-0.1))$

6. MCGDM hybrid with the TOPSIS technique

In this section, we present the general model and formulate the mathematical framework for group DM integrated with the TOPSIS method. DM plays a critical role in real life, where a group of professional experts from multiple fields provide their evaluations of each alternative against relevant criteria to determine the optimal outcome. Consider a discrete set $U=\{\wp_1,\wp_2,\dots,\wp_n\}$ of n alternatives and assume that $\widetilde{\mathfrak{C}}=\{\widetilde{\mathfrak{C}}_1,\widetilde{\mathfrak{C}}_2,\dots,\widetilde{\mathfrak{C}}_m\}$ is the corresponding set of desirable criteria with WV $\varpi=(\varpi_1,\varpi_2,\dots,\varpi_m)^T$ such that $\sum_{j=1}^{m}\varpi_j$ with $\varpi_j\in[0,1]$. The team of professional decision experts $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_t\}$ who propose their assessments for every object considering the given criteria with WV $v = (v_1, v_2, \dots, v_t)^T$ such that $\sum_{l=1}^{t} v_l$ with $v_l \in [0,1]$. The experts (decision makers) express their opinions with regard to BFRN and present the cumulative statement in form of decision matrix $\mathcal{M} = [\mathcal{R}(\xi_{ij})]_{m\times n}$. Then, the proposed AOs are combined using an accumulated geometric operator (AGO) to transform the experts' assessments from the BFR decision matrix into a decision matrix in the form of bipolar fuzzy numbers (BFNs). The definition and formula for the AGO are provided below.

Definition 16. Consider that $\mathcal{R}(\xi) = \left(\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi)\right) = \left(\langle \underline{\mu}, \underline{\psi}\rangle, \langle \overline{\mu}, \overline{\psi}\rangle\right)$ is the BFRN. Now, by means of AGO, convert the BFRN into the basic form of BFN, which is stated below.

$$\boldsymbol{\mathcal{B}} = (\mu_{\mathcal{B}}, \psi_{\mathcal{B}}) = \left(\underline{\mathcal{R}}(\xi), \overline{\mathcal{R}}(\xi)\right)^{0.5} = \left(\left(\underline{\mu}\overline{\mu}\right)^{0.5}, -\left(\underline{\psi}\overline{\psi}\right)^{0.5}\right), \tag{xv}$$

Based on the analysis of the above definition of AGO, change the defined BFR decision matrix $\mathcal{M} = [\mathcal{R}(\xi_{ij})]_{m\times n}$ to BF decision matrix $\mathcal{M} = \left[\boldsymbol{\mathcal{B}}\left(\mu_{\mathcal{B}_{ij}}, \psi_{\mathcal{B}_{ij}}\right)\right]_{m\times n}$ to get the aggregated outcomes through the proposed model.

Next, we compute the BF-PIS P^+ and BF-NIS P^- through the TOPSIS approach of the transform BF decision matrix by applying a score function through Definition 7, which is given below.

$$P^+ = \left\{\langle \tilde{\mathfrak{C}}_j, max\left\{S\left(\tilde{\mathfrak{C}}_j(\wp_i)\right)\right\}\rangle \,|\, i = 1, \dots, n, \quad j = 1, \dots, m\right\}$$

$$= \left\{\langle \tilde{\mathfrak{C}}_1, (\mu_1^+, \psi_1^+)\rangle, \langle \tilde{\mathfrak{C}}_2, (\mu_2^+, \psi_2^+)\rangle, \dots, \langle \tilde{\mathfrak{C}}_m, (\mu_m^+, \psi_m^+)\rangle\right\},$$

$$P^- = \left\{\langle \tilde{\mathfrak{C}}_j, min\left\{S\left(\tilde{\mathfrak{C}}_j(\wp_i)\right)\right\}\rangle \,|\, i = 1, \dots, n, \quad j = 1, \dots, m\right\}$$

$$= \left\{\langle \tilde{\mathfrak{C}}_1, (\mu_1^-, \psi_1^-)\rangle, \langle \tilde{\mathfrak{C}}_2, (\mu_2^-, \psi_2^-)\rangle, \dots, \langle \tilde{\mathfrak{C}}_m, (\mu_m^-, \psi_m^-)\rangle\right\}.$$

Table 2. BFR evaluation information D_2.

	$\tilde{\mathfrak{C}}_1$	$\tilde{\mathfrak{C}}_2$	$\tilde{\mathfrak{C}}_3$	$\tilde{\mathfrak{C}}_4$
$\wp_1$	$((0.5, -0.3), (0.4, -0.5))$	$((0.9, -0.1), (0.6, -0.3))$	$((0.8, -0.2), (0.8, -0.1))$	$((0.7, -0.3), (0.5, -0.4))$
$\wp_2$	$((0.7, -0.2), (0.5, -0.3))$	$((0.6, -0.3), (0.5, -0.4))$	$((0.6, -0.4), (0.5, -0.3))$	$((0.9, -0.1), (0.4, -0.3))$
$\wp_3$	$((0.8, -0.1), (0.7, -0.1))$	$((0.7, -0.1), (0.4, -0.3))$	$((0.9, -0.1), (0.7, -0.2))$	$((0.7, -0.2), (0.9, -0.1))$
$\wp_4$	$((0.6, -0.4), (0.6, -0.4))$	$((0.4, -0.5), (0.5, -0.1))$	$((0.7, -0.2), (0.5, -0.4))$	$((0.6, -0.3), (0.5, -0.4))$

Table 3. BFR evaluation information D_3.

	$\tilde{\mathfrak{C}}_1$	$\tilde{\mathfrak{C}}_2$	$\tilde{\mathfrak{C}}_3$	$\tilde{\mathfrak{C}}_4$
$\wp_1$	$((0.7, -0.2), (0.6, -0.4))$	$((0.8, -0.1), (0.7, -0.2))$	$((0.9, -0.1), (0.7, -0.2))$	$((0.7, -0.2), (0.6, -0.2))$
$\wp_2$	$((0.9, -0.1), (0.7, -0.2))$	$((0.7, -0.2), (0.5, -0.1))$	$((0.7, -0.3), (0.6, -0.4))$	$((0.4, -0.3), (0.4, -0.1))$
$\wp_3$	$((0.7, -0.3), (0.6, -0.3))$	$((0.9, -0.1), (0.6, -0.2))$	$((0.8, -0.2), (0.5, -0.1))$	$((0.6, -0.1), (0.8, -0.2))$
$\wp_4$	$((0.5, -0.4), (0.8, -0.1))$	$((0.5, -0.3), (0.4, -0.3))$	$((0.6, -0.1), (0.4, -0.5))$	$((0.5, -0.4), (0.5, -0.3))$

Find the shortest and farthest distance that is D^+ and D^- for each alternative $\wp_i$, and the BF-

PIS P^+ and BF-NIS P^- and are given below.

$$D^+(\wp_i, P^+) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}\left(\left(\mu_{ij}-\mu_j^+\right)^2+\left(\psi_{ij}-\psi_j^+\right)^2\right)},$$

$$D^-(\wp_i, P^-) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}\left(\left(\mu_{ij}-\mu_j^-\right)^2+\left(\psi_{ij}-\psi_j^-\right)^2\right)}.$$

The shorter the distance, $D^+(\wp_i, P^+)$ will measure the better option for an object, and the further the distance $D^-(\wp_i, P^-)$, will measure the better of the alternative.

The conclusion for the proposed model can be calculated through the relative closeness degree, which measures the rank of each alternative against their relevant criteria and properly orders them to obtain the best optimal alternative and the ranking formula $\mathfrak{F}(\wp_i)$, which is stated below.

$$\mathfrak{F}(\wp_i) = \frac{D^-(\wp_i, P^-)}{D^+(\wp_i, P^+) + D^-(\wp_i, P^-)}, \quad \text{(xvi)}$$

The alternative will be considered the best, having the highest relative closeness degree.

Algorithm

The detailed algorithmic procedure of the proposed model using the TOPSIS technique is presented below, providing a systematic framework to identify the optimal alternative in a DM problem.

Step 1. The experts (decision makers) express their opinions with regard to BFRN and present the cumulative statement in the form of decision matrix $\mathcal{M} = [\mathcal{R}(\xi_{ij})]_{m\times n}$.

Step 2. Aggregate the decision information $\mathcal{M} = [\mathcal{R}(\xi_{ij})]_{m\times n}$ of the experts through the developed AO to achieve a BFR decision matrix.

Step 3. Based on the analysis of step 2, change the aggregated assessments BFR decision matrix $\mathcal{M} = [\mathcal{R}(\xi_{ij})]_{m\times n}$ to BF decision matrix $\mathcal{M} = \left[\boldsymbol{\mathcal{B}}\left(\mu_{\boldsymbol{\mathcal{B}}_{ij}}, \psi_{\boldsymbol{\mathcal{B}}_{ij}}\right)\right]_{m\times n}$ by applying Definition 16 of AGO.

Table 4. Aggregated result by applying the proposed BFRWA model.

	$\widetilde{\mathfrak{C}}_1$	$\widetilde{\mathfrak{C}}_2$
$\wp_1$	$((0.6963, -0.2268), (0.7246, -0.2602))$	$((0.6930, -0.1485), (0.6720, -0.2624))$
$\wp_2$	$((0.7912, -0.124), (0.6963, -0.2268))$	$((0.7792, -0.1767), (0.5386, -0.2532))$
$\wp_3$	$((0.7066, -0.1844), (0.6341, -0.1437))$	$((0.8196, -0.1283), (0.4751, -0.2268))$
$\wp_4$	$((0.5334, -0.3607), (0.7131, -0.1973))$	$((0.4709, -0.3898), (0.6182, -0.1437))$
	$\widetilde{\mathfrak{C}}_3$	$\widetilde{\mathfrak{C}}_4$
$\wp_1$	$((0.7958, -0.1591), (0.8219, -0.1257))$	$((0.7407, -0.2268), (0.6660, -0.1932))$
$\wp_2$	$((0.6058, -0.3280), (0.5040, -0.3659))$	$((0.7025, -0.2134), (0.4381, -0.2316))$
$\wp_3$	$((0.7343, -0.1257), (0.6931, -0.1591))$	$((0.6035, -0.1591), (0.7930, -0.1867))$
$\wp_4$	$((0.6341, -0.2042), (0.7025, -0.2614))$	$((0.6118, -0.3299), (0.7199, -0.2209))$

Step 4. Find the BF-PIS P^+ and BF-NIS P^- from the outcome obtained from BF decision matrix $\mathcal{M} = \left[\boldsymbol{\mathcal{B}}\left(\mu_{\boldsymbol{\mathcal{B}}_{ij}}, \psi_{\boldsymbol{\mathcal{B}}_{ij}}\right)\right]_{m\times n}$ through a score function.

Step 5. Find the shortest and farthest distance, that is D^+ and D^-, respectively, for each alternative $\wp_i$ from the BF decision matrix $\mathcal{M} = \left[\boldsymbol{\mathcal{B}}\left(\mu_{\boldsymbol{\mathcal{B}}_{ij}}, \psi_{\boldsymbol{\mathcal{B}}_{ij}}\right)\right]_{m\times n}$ and the BF-PIS P^+ and BF-NIS P^-, respectively.

Step 6. The conclusion for the proposed model is derived through the relative closeness degree, which measures the rank of each alternative against their relevant criteria and properly orders them to obtain the best optimal alternative by utilizing the ranking formula $\mathfrak{F}(\wp_i)$.

7. Illustrative example

The process of multi-criteria group decision making (MCGDM) involves multiple factors or criteria and is conducted by a group of professional stakeholders. This approach is designed to assist decision makers in evaluating complex criteria that require inputs from diverse perspectives while considering competing alternatives. In the following, we present an illustrative example of MCGDM using the proposed approach.

Consider a multinational company in Pakistan designing a plan for its monetary system to achieve their goal of a strategy target for the upcoming year. A group of professional business analysts $\mathcal{H}_i$ $(i = 1,2,3)$ with WV $v = (0.36,0.31,0.33)^T$ are invited to present their preliminary evaluations to identify and solve problems and improve their overall performance. They apply a range of techniques and tools to analyze four alternatives, $\wp_1, \wp_2, \wp_3, \wp_4$, for business processes, with the goal of identifying inefficiencies, highlighting areas for improvement, and implementing effective solutions. The alternatives are defined as follows: $\wp_1$ investing in the central Asian markets, $\wp_2$ investing in the East Asian markets, $\wp_3$ investing in the south Asian markets, and $\wp_4$ investing in the west Asian markets. The group of analysts continues their evaluation process from the four criteria, given as $\widetilde{\mathfrak{C}}_1$: "the risk analysis", $\widetilde{\mathfrak{C}}_2$: "the growth analysis", $\widetilde{\mathfrak{C}}_3$: "the environmental analysis", and $\widetilde{\mathfrak{C}}_4$: "the social political impact analysis". The analysts collaborate with stakeholders across all aspects of the company, including management, employees, and customers, to gain a comprehensive understanding of their fundamental needs and requirements. The team of experts (decision makers) demonstrate their opinions with regard to BFRN and present the cumulative statement in the form of decision matrix $\mathcal{M} = [\mathcal{R}(\xi_{ij})]_{m\times n}$. Next, the developed AOs are combined using an AGO to transform the experts' assessments from the BFR decision matrix into a decision matrix in the form of BFNs. The AGO is defined with the aim of identifying the most suitable alternative among $\wp_i$ $(i = 1, \ldots 4)$, as illustrated through the algorithm of the proposed approach.

Table 5. BF decision metric by applying AGO.

	$\widetilde{\mathfrak{C}}_1$	$\widetilde{\mathfrak{C}}_2$	$\widetilde{\mathfrak{C}}_3$	$\widetilde{\mathfrak{C}}_4$
$\wp_1$	$(0.7103, -0.2429)$	$(0.7300, -0.1974)$	$(0.8087, -0.1414)$	$(0.7034, -0.2093)$
$\wp_2$	$(0.7423, -0.1677)$	$(0.6478, -0.2115)$	$(0.5526, -0.3464)$	$(0.5548, -0.2223)$
$\wp_3$	$(0.6694, -0.1628)$	$(0.2640, -0.1706)$	$(0.7134, -0.1414)$	$(0.6918, -0.1723)$
$\wp_4$	$(0.6168, -0.2667)$	$(0.5356, -0.2367)$	$(0.6674, -0.3210)$	$(0.6636, -0.2699)$

Table 6. Result obtained for the BFRWA operator by using the BF TOPSIS approach.

	$D^{+}(\wp_i, P^{+})$	$D^{-}(\wp_i, P^{-})$	$\mathfrak{F}(\wp_i)$	Ranking
$\wp_1$	0.0639	0.2973	0.8231	1
$\wp_2$	0.2606	0.1377	0.3456	4
$\wp_3$	0.1148	0.2390	0.6755	2
$\wp_4$	0.2255	0.1425	0.3873	3

Table 7. Result obtained for the BFRWG operator by using the BF TOPSIS approach.

	$D^{+}(\wp_i, P^{+})$	$D^{-}(\wp_i, P^{-})$	$\mathfrak{F}(\wp_i)$	Ranking
$\wp_1$	0.0982	0.2907	0.7475	1
$\wp_2$	0.2582	0.1396	0.3509	3
$\wp_3$	0.1646	0.2263	0.5789	2
$\wp_4$	0.2583	0.0965	0.2719	4

Step 1. The team of authorized expert decision makers demonstrate their opinions with regard to BFRN and present the cumulative statement in form of decision matrix $\mathcal{M} = \left[\mathcal{R}(\xi_{ij})\right]_{m\times n}$, as shown in Table 13.

Step 2. Aggregate the expert information $\mathcal{M} = \left[\mathcal{R}(\xi_{ij})\right]_{m\times n}$ of the experts through the developed AO to achieve a BFR decision matrix, which is presented in Table 4.

Step 3. Based on the analysis of step 2, change the aggregated assessment BFR decision matrix $\mathcal{M} = \left[\mathcal{R}(\xi_{ij})\right]_{m\times n}$ to the BF decision matrix $\mathcal{M} = \left[\boldsymbol{\mathcal{B}}\left(\mu_{\boldsymbol{\mathcal{B}}_{ij}}, \psi_{\boldsymbol{\mathcal{B}}_{ij}}\right)\right]_{m\times n}$ by applying Definition 16 of AGO. The final outcomes are given in Table 5.

Step 4. Calculate the BF-PIS P^{+} and BF-NIS P^{-} of the converted decision matrix via the score function, that is:

$$P^{+} = \{(0.7983,0.1358),(0.7393,0.1755)(0.7739,0.1503)(0.7711,0.1503)\}$$

$$P^{-} = \{(0.5590,0.3186),(0.5111,0.2998)(0.4908,0.3347)(0.5315,0.3513)\}.$$

Step 5. Find the shortest and farthest distance, that is D^{+} and D^{-}, respectively, for each alternative $\wp_i$ from the BF decision matrix $\mathcal{M} = \left[\boldsymbol{\mathcal{B}}\left(\mu_{\boldsymbol{\mathcal{B}}_{ij}}, \psi_{\boldsymbol{\mathcal{B}}_{ij}}\right)\right]_{m\times n}$ and the BF-PIS P^{+} and BF-NIS P^{-}, respectively. Their report is given in Table 6.

Step 6. The conclusion for the proposed model can be calculated through the relative closeness degree, which measures the rank of each alternative against their relevant criteria and properly orders them to obtain the best optimal alternative from $\wp_i$ $(i = 1, \dots 4)$ by applying the proposed model of the BFRWA aggregation operators, $\wp_1$, which is investing in the central Asian markets. The results are given in Table 6.

Similarly, by applying the relative closeness degree to ranking function $\mathfrak{F}(\wp_i)$ $(i = 1, \dots 4)$ and the proper order of the ranking outcome in a specific order, we get the most appropriate monetary strategy from $\wp_i$ $(i = 1, \dots 4)$, by applying the proposed model of the BFRWG aggregation operators, which is $\wp_1$: Investing in the central Asian markets. The results are given in Table 7. Moreover, the same example is solved by applying the developed model for BFROWA and BFROWG aggregation operators, and the optimal result is the same in all cases. The overall results are given in Table 8. Similarly, by applying the relative closeness degree to ranking function $\mathfrak{F}(\wp_i)$ $(i = 1, \dots 4)$ and the proper order of the ranking outcome in a specific order, we get the most appropriate monetary strategy from $\wp_i$ $(i = 1, \dots 4)$, by applying proposed model of BFRWG aggregation operators, which is $\wp_1$: Investing in the central Asian markets. The results are given in Table 7. Moreover, the same example is solved by applying the developed model for the BFROWA and BFROWG aggregation operators, and the optimal result is the same in all cases. The overall results are given in Table 8.

Comparative analysis

In this paper, we extend the TOPSIS approach for multi-criteria group decision making (MCGDM) based on bipolar fuzzy rough numbers (BPRN). The TOPSIS method is widely used due to its simplicity and effectiveness, with its primary objective being to identify the optimal alternative that is closest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS). The PIS represents the collection of best attribute values, while the NIS represents the collection of worst attribute values obtained during the evaluation process.

To demonstrate the effectiveness of the proposed approach, several methods are considered benchmark models for comparative analysis. Gul [16] introduced BF weighted averaging (BFWA) and BF weighted geometric (BFWG) aggregation operators; however, the method was unable to handle the proposed model. Akram et al. [12] conducted a comparative study of BF-TOPSIS and BF-ELECTRE-I to assess patients' health status and determine influencing factors using BF information, but these methods are not suitable for the developed model. Similarly, the approach proposed in [11] lacks the capability to process BPRNs, highlighting the limitations of existing models. In the domain of aggregation operators (AOs), Chinram et al. [43] introduced IF rough aggregation operators combined with the EDAS approach. However, comparative analysis shows that this method cannot address the concepts investigated in this study. Moreover, Yahya et al. [44] developed qROFFWA and qROFFWG operators based on Frank norms integrated with EDAS, but these operators are also insufficient for handling the proposed model. Mahmood and Rehman [9] presented bipolar complex fuzzy (BCF) AOs using Dombi norms in a BF environment, but this method cannot solve the application of the developed model due to the absence of BF rough information. Furthermore, Mahmood et al. [8] proposed BCF AOs based on Hamacher norms, including BCFHWA, BCFHOWA, BCFHHA, BCFHWG, BCFHOWG, and BCFHHG for MADM applications, but these operators also fail to address the proposed model. Similarly, Wei et al. [17] introduced BF AOs based on Hamacher t-norms and t-conorms, including BF Hamacher weighted averaging and geometric operators, but these are not applicable to the investigated concepts. Similarly, compared with other existing methods, details can be found in [18,45,46]. The corresponding ranking graph of proposed models presented in Table 8 are given in Figure 1. From Table 8 and Figure 1, the rankings are slightly different, but their best optimal alternative against BFRWA, BFROWA, BFRWG, and BFROWG aggregation operators remain the same: $\wp_1$. From the comparisons of the proposed methods, it is evident that earlier models fail to handle the application of the investigated approach due to the absence of BFR information. Therefore, based on the overall comparative study, we conclude that the proposed approach is superior and more capable than existing methods. The ranking results of the proposed model are presented in Table 8.

Table 8. The most appropriate monetary strategy by applying the Relative closeness degree.

Proposed models	*Relative closeness degree*				Ranking
	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$	
BCFHWA [8]	Not applicable				×
BCFDWA [9]	Not applicable				×
TBFN TOPSIS [11]	Not applicable				×
BF TOPSIS and BF ELECTRE-I [12]	Not applicable				×
BFWA [16]	Not applicable				×
BFHWA [17]	Not applicable				×
IFRWA EDAS [43]	Not applicable				×
q-ROFFWA EDAS [44]	Not applicable				×
BFRWA (proposed)	0.8231,	0.3456,	0.6755,	0.3873	$\wp_1 > \wp_4 > \wp_2 > \wp_3$
BFROWA (proposed)	1.0000,	0.7479,	0.3083,	0.0000	$\wp_1 > \wp_2 > \wp_3 > \wp_4$
BCFHWG [8]	Not applicable				×
BCFDWG [9]	Not applicable				×
BFWG [16]	Not applicable				×
BFHWG [17]	Not applicable				×
IFRWA EDAS [43]	Not applicable				×
q-ROFFWG EDAS [44]	Not applicable				×
BFRWG (proposed)	0.7475,	0.3509,	0.5789,	0.2719	$\wp_1 > \wp_3 > \wp_2 > \wp_4$
BFROWG (proposed)	1.0000,	0.6320,	0.2475,	0.0000	$\wp_1 > \wp_2 > \wp_3 > \wp_4$

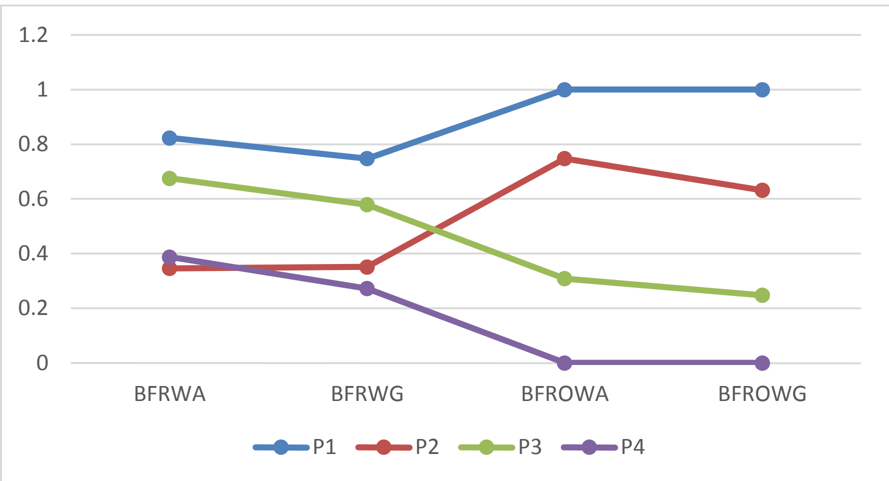

Figure1. Comparative study of the proposed averaging and geometric operators in Table 8.

8. Conclusions

In this paper, we introduce several key concepts to develop the model of BFRS, including new score and accuracy functions and essential operations. Based on the BFRS framework, we proposed BFRA aggregation operators, including BFRWA, BFROWA, and BFRHA, and examined their fundamental properties in detail. Similarly, we developed BFRG aggregation operators, including BFRWG, BFROWG, and BFRHG, and discussed their prominent properties. Furthermore, we presented a stepwise algorithm for the TOPSIS approach based on the developed model and illustrated the methodology with a numerical example. The applicability of the proposed approach was compared with several other methods. Based on the overall comparative study, we conclude that the proposed approach is superior and more effective than existing methods.

Regarding future directions, we will study the approximations of the BFS based on the generalized rough set (and its applications) and discuss attribute reductions in the BFRS model, covering basic BFRS models. We will also study real-world applications of the designed technique in solving a wider variety of selection problems using averaging and geometric aggregation operators, such as Bipolar complex fuzzy rough aggregation operators, Hamacher norms and t-conorms, Dombi norms and t-conorms, and Einstein norms and t-conorms and other related methods.

Author contributions

Azmat Hussain: Conceptualization, Methodology, Writing – original draft, Validation, Writing – review and editing; Vassilis C. Gerogiannis Methodology, Writing – original draft, Validation, Writing – review and editing, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

The data used in this article are artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

Conflict of interest

Prof. Vassilis C. Gerogiannis is an editorial board member for AIMS Mathematics, and was not involved in the editorial review and the decision to publish this article. The authors declare that they have no conflicts of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Inform Control*, **8** (1965), 338–356. https://doi.org/10.1016/S0019-9958(65)90241-X
2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
3. Z. S. Xu, Intuitionistic fuzzy aggregation operators, *IEEE Trans. Fuzzy Syst.*, **15** (2007), 1179–1187. https://doi.org/10.1109/TFUZZ.2006.890678
4. Z. S. Xu, R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, *Int. J. Gen Syst.*, **35** (2006), 417–433. https://doi.org/10.1080/03081070600574353
5. M. I. Ali, F. Feng, T. Mahmood, I. Mahmood, H. Faizan, A graphical method for ranking Atanassov's intuitionistic fuzzy values using the uncertainty index and entropy, *Int. J. Intell. Syst.*, **34** (2019), 2692–2712. https://doi.org/10.1002/int.22174
6. W. R. Zhang, Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis, *NAFIPS/IFIS/NASA'94, Proceedings of the First International Joint Conference of the North American Fuzzy Information Processing Society Biannual Conference*, 1994, 305–309. https://doi.org/10.1109/IJCF.1994.375115
7. W. R. Zhang, L. Zhang, YinYang bipolar logic and bipolar fuzzy logic, *Inf. Sci.*, **165** (2004), 265–287. https://doi.org/10.1016/j.ins.2003.05.010
8. T. Mahmood, U. Ur Rehman, J. Ahmmad, G. Santos-García, Bipolar complex fuzzy Hamacher aggregation operators and their applications in multi attribute decision making, *Mathematics*, **10** (2021), 23. https://doi.org/10.3390/math10010023
9. T. Mahmood, U. Ur Rehman, A method to multi attribute decision making technique based on Dombi aggregation operators under bipolar complex fuzzy information, *Comput. Appl. Math.*, **41** (2022), 47. https://doi.org/10.1007/s40314-021-01735-9
10. T. Mahmood, U. Ur Rehman, A. Jaleel, J. Ahmmad, R. Chinram, Bipolar complex fuzzy soft sets and their applications in decision-making, *Maths*, **10** (2022), 1048. https://doi.org/10.3390/math10071048
11. M. Akram, M. Arshad, A novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making, *Group Decis. Negot*, **28** (2019), 565–584. https://doi.org/10.1007/s10726-018-9606-6
12. M. Akram, Shumaiza, M. Arshad, Bipolar fuzzy TOPSIS and bipolar fuzzy ELECTRE-I methods to diagnosis, *Comput. Appl. Math.*, **39** (2020), 7. https://doi.org/10.1007/s40314-019-0980-8
13. M. A. Alghamdi, N. O. Alshehri, M. Akram, Multi-criteria decision-making methods in bipolar fuzzy environment, *Int. J. Fuzzy Syst.*, **20** (2018), 2057–2064. https://doi.org/10.1007/s40815-018-0499-y
14. Y. Han, P. Shi, S. Chen, Bipolar-valued rough fuzzy set and its applications to decision information system, *IEEE Trans Fuzzy Syst.*, **23** (2015), 2358–2370. https://doi.org/10.1109/TFUZZ.2015.2423707
15. K. M. Lee, Comparison of Interval-valued fuzzy sets, Intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, *J. Korean Inst. Intell. Syst.*, **14** (2004), 125–129. https://doi.org/10.5391/JKIIS.2004.14.2.125
16. Z. Gul, Some bipolar fuzzy aggregations operators and their applications in multi criteria group decision making, *M. Phil Thesis*, 2015.

17. G. W. Wei, F. E. Alsaadi, H. Tasawar, A. Alsaedi, Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision Making, *Int J Fuzzy Syst.*, **20** (2018), 1–12. https://doi.org/10.1007/s40815-017-0338-6
18. M. Sarwar, M. Akram, F. Zafar, Decision Making Approach Based on Competition Graphs and Extended TOPSIS Method under Bipolar Fuzzy Environment, *Math. Comput. Appl.*, **23** (2018), 68. https://doi.org/10.3390/mca23040068
19. T. Mahmood, A Novel Approach towards Bipolar Soft Sets and Their Applications, *J. Math.*, (2020), 4690808. https://doi.org/10.1155/2020/4690808
20. Z. A. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
21. D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *Int J Gen Syst.*, **17** (1990), 191–209. https://doi.org/10.1080/03081079008935107
22. M. I. Ali, T. Mahmood, A. Hussain, A study of generalized roughness in-fuzzy filters of ordered semigroups, *J. Taibah Uni. Sci.*, **12** (2018), 163–172. https://doi.org/10.1080/16583655.2018.1451067
23. Mahmood, T., Ali, M. I., A. Hussain, Generalized roughness in fuzzy filters and fuzzy ideals with thresholds in ordered semigroups, *Comput. Appl. Math.*, **37** (2018), 5013–5033. https://doi.org/10.1007/s40314-018-0615-5
24. C. Cornelis, M. De Cock, E. E. Kerre, Intuitionistic fuzzy rough sets: At the crossroads of imperfect knowledge, *Expert Syst.*, **20** (2003), 260–270. https://doi.org/10.1111/1468-0394.00250
25. L. Zhou, W. Wu, On generalized intuitionistic fuzzy rough approximation operators, *Inf. Sci.*, **178** (2018), 2448–2465.
26. X. Zhang, B. Zhou, P. Li, A general frame for intuitionistic fuzzy rough sets, *Inf. Sci.*, **216** (2012), 34–49. https://doi.org/10.1016/j.ins.2012.04.018
27. S. M. Yun, S. J. Lee, Intuitionistic fuzzy rough approximation operators, *Int. J. Fuzzy Log. Intell. Syst.*, **15** (2015), 208–215. https://doi.org/10.5391/IJFIS.2015.15.3.208
28. H. Zhang, L. Shu, S. Liao, Intuitionistic fuzzy soft rough set and its application in decision making, *Abstract Appl. Anal.*, **2014** (2014), 0287314. https://doi.org/10.1155/2014/287314
29. H. Zhang, L. Xiong, W. Ma, Generalized intuitionistic fuzzy soft rough set and its application in decision making, *J. Comput. Anal. Appl.*, **20** (2016), 750–766.
30. A. Hussain, T. Mahmood, M. I. Ali, Rough Pythagorean fuzzy ideals in semigroups, *Comput. Appl. Math.*, **38** (2019), 67. https://doi.org/10.1007/s40314-019-0824-6
31. A. Hussain, M. I. Ali, T. Mahmood, Pythagorean fuzzy soft rough sets and their applications in decision making, *J. Taibah Univ. Sci.*, **14** (2020), 101–113. https://doi.org/10.1080/16583655.2019.1708541
32. A. Hussain, M. I. Ali, T. Mahmood, Covering based q-rung orthopair fuzzy rough set model hybrid with TOPSIS for multi-attribute decision making, *J. Intell. Fuzzy Syst.*, **37** (2019), 981–993. https://doi.org/10.3233/JIFS-181832
33. Y. Han, P. Shi, S. Chen, Bipolar-valued rough fuzzy set and its applications to the decision information system, *IEEE Trans Fuzzy Syst.*, **23** (2015), 2358–2370. https://doi.org/10.1109/TFUZZ.2015.2423707
34. H. L. Yang, S. G. Li, S. Wang, J. Wang, Bipolar fuzzy rough set model on two different universes and its application, *Know. Based Syst.*, **35** (2012), 94–101. https://doi.org/10.1016/j.knosys.2012.01.001

35. H. L. Yang, S. G. Li, Z. L. Guo, C. H. Ma, Transformation of bipolar fuzzy rough set models, *Know. Based Syst.*, **27** (2012), 60–68. https://doi.org/10.1016/j.knosys.2011.07.012
36. N. Malik, M. Shabir, T. M. Al-shami, R. Gul, M. Arar, M. Hosny, Rough bipolar fuzzy ideals in semigroups, *Complex Intell. Syst.*, **9** (2023), 7197–7212. https://doi.org/10.1007/s40747-023-01132-1
37. T. Mahmood, U. U. Rehman, J. Ahmmad, Bipolar complex fuzzy rough sets and their applications in multicriteria decision making, *Punjab Univ. J. Math.*, **56** (2024), 175–207. https://doi.org/10.52280/pujm.2024.56(5)04
38. R. Gul, An extension of VIKOR approach for MCDM using bipolar fuzzy preference δ-covering based bipolar fuzzy rough set model, *Spectrum Oper. Res.*, **2** (2025), 72–91. https://doi.org/10.31181/sor21202511
39. L. Wang, N. Li, Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making, *Int. J. Intell. Syst.*, **35** (2020), 150–183. https://doi.org/10.1002/int.22204
40. A. Hussain, M. I. Ali, T. Mahmood M. Munir, q-Rung orthopair fuzzy soft average aggregation operators and their application in multi-criteria decision making, *Int. J. Intell. Syst.*, **35** (2020), 571–599. https://doi.org/10.1002/int.22217
41. R. Chinram, A. Hussian, M. I. Ali, T. Mahmood, Some geometric aggregation operators under q-Rung orthopair fuzzy soft information with their applications in multi-criteria decision making, *IEEE Access.*, **9** (2021), 31975–31993. https://doi.org/10.1109/ACCESS.2021.3059683
42. Y. Wang, A. Hussain, T. Mahmood, M. I. Ali, H. Wu, Y. Jin, Decision making based on q-rung orthopair fuzzy soft rough sets, *Math Probl Eng.*, (2020) 1–21. https://doi.org/10.1155/2020/6671001
43. R. Chinram, A. Hussian, T. Mahmood, M. I. Ali, EDAS method for multi-criteria group decision making based on intuitionistic fuzzy rough aggregation operators, *IEEE Access*, **9** (2021), 10199–10216. https://doi.org/10.1109/ACCESS.2021.3049605
44. M. Yahya, M. Naeem, S. Abdullah, M. Qiyas, M. Aamir, A Novel approach on the intuitionistic fuzzy rough frank aggregation operator-based EDAS method for multi criteria group decision-making, *Complexity*, **2021** (2021), 176. https://doi.org/10.1155/2021/5534381
45. A. Shahzaib, N. Rehman, A. Hussain, H. AlSalman, A. H. Gumaei, q-rung orthopair fuzzy rough Einstein aggregation information-based EDAS method: Applications in robotic Agrifarming, *Comput. Intell. Neurosci.*, (2021), 5520264. https://doi.org/10.1155/2021/5520264
46. A. Hussain, T. Mahmood, F. Smarandache, S. Ashraf, TOPSIS approach for MCGDM based on intuitionistic fuzzy rough Dombi aggregation operations, *Comput. Appl. Math.*, **42** (2023), 176. https://doi.org/10.1007/s40314-023-02266-1