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Research article

g-Averaged Halpern-type convergence theorems for enriched g-nonexpansive mappings in Hilbert spaces

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Abstract: We introduce the set of g-attractive points of a mapping f and study a g-averaged Halpern-type iterative scheme for b-enriched g-nonexpansive mappings in real Hilbert spaces. For $0 < \lambda \leq 1/(b+1)$, we prove that every sequence generated by the scheme converges strongly-after applying g-to the metric projection of the anchor onto the set $A^g(f_{g,\lambda})$, provided this set is nonempty. As a consequence, we obtain the corresponding strong convergence theorem for b-enriched nonexpansive mappings. A numerical example is given to illustrate the method and the role of the averaging parameter.

Keywords: fixed point; attractive point; enriched g-nonexpansive mapping; metric projection; Halpern-type iteration; averaged g-mapping
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1. Introduction and preliminaries

Fixed point theory for nonexpansive mappings in Hilbert spaces has attracted considerable attention due to its fundamental role in nonlinear analysis and its wide range of applications. These include finding zeros of accretive operators, split feasibility and multiple-set split feasibility problems, and variational inequality problems (see, for instance, [1–3]), as well as minimax inequality problems and optimization problems (see, for instance, [4, 5]). Moreover, related results have been extended to CAT(0) spaces (see, for example, [6]).

Among the foundational results in this area, the Browder-Göhde-Kirk Theorem [7] plays a central role by guaranteeing the existence of fixed points for nonexpansive mappings defined on closed, convex, and bounded subsets of Hilbert spaces. This theorem provides the theoretical basis for the development of iterative algorithms and convergence theory.

To approximate such fixed points, constructive methods have been extensively studied (see, for instance, [4, 8–10]). One of the most effective methods for approximating fixed points while ensuring the strong convergence of the generated sequence is the Halpern iterative algorithm. The method was originally introduced by Halpern [11] in 1967 for the special case where $\mathcal{M}$ is the closed unit ball of a real Hilbert space $\mathcal{H}$ and an anchor $w = 0$ (see also [12, 13]).

Definition 1. *Let $\mathcal{M}$ be a nonempty convex subset of a Hilbert space $\mathcal{H}$ and let $f : \mathcal{M} \to \mathcal{M}$. Given a fixed element $w \in \mathcal{M}$, called the anchor, the Halpern iteration is the sequence $\{u_n\}$ in $\mathcal{M}$ defined by an initial point $u_0 \in \mathcal{M}$ and*

$$u_{n+1} = \alpha_n w + (1 - \alpha_n) f u_n, \qquad \forall n \in \mathbb{W}, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$ is a sequence of real numbers, called the control sequence.

Halpern [11] established a strong convergence result for the Halpern iteration in Hilbert spaces. In particular, for a nonexpansive mapping f defined on a nonempty, bounded, closed, and convex subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, and for a control sequence $\{\alpha_n\}$ given by $\alpha_n = \frac{1}{n^\theta}$ with $\theta \in (0, 1)$, the Halpern iteration (1.1), starting from an initial point $u_0 \in \mathcal{M}$ and an anchor $w \in \mathcal{M}$, converges strongly to $a = P_{\mathrm{Fix}(f)} w$, where $P_{\mathrm{Fix}(f)}$ is the metric projection of w onto Fix(f). Halpern noticed that the conditions

$$\lim_{n\to\infty} \alpha_n = 0, \tag{C1}$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \tag{C2}$$

are necessary for the convergence of the sequence $\{u_n\}$ defined by (1.1) to an element of Fix(f).

In 1977, Lions [14] improved Halpern's result by establishing strong convergence under an additional condition on the control sequence. Specifically, for a nonexpansive mapping f on a nonempty, bounded, closed, and convex subset $\mathcal{M}$ of a real Hilbert space $\mathcal{H}$, the Halpern iteration (1.1), starting from an initial point $u_0 \in \mathcal{M}$ and an anchor $w \in \mathcal{M}$, converges strongly to $a = P_{\mathrm{Fix}(f)} w$, provided that the control sequence $\{\alpha_n\} \subset (0, 1)$ satisfies ($C1$), ($C2$), and

$$\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0. \tag{C3}$$

The conditions imposed on the control sequence $\{\alpha_n\}$ by Halpern's and Lions' results exclude the choice $\alpha_n = \frac{1}{n}$. To address this restriction, in 1992, Wittmann [15] established a strong convergence result by replacing Condition ($C3$) with the more general assumption

$$\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty. \tag{C4}$$

In particular, for a nonexpansive mapping $f : \mathcal{M} \to \mathcal{M}$ defined on a nonempty, closed, and convex subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ with Fix(f) $\neq \emptyset$, the Halpern iteration (1.1), starting from $u_0 = w \in \mathcal{M}$ and generated by a control sequence $\{\alpha_n\} \subset [0, 1]$ satisfying ($C1$), ($C2$), and ($C4$), converges strongly to $a = P_{\mathrm{Fix}(f)} w$.

In 2002, Xu [16] improved Lions' result by relaxing Condition ($C3$) to

$$\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0, \tag{C3$'$}$$

thereby allowing the choice $\alpha_n = \frac{1}{n}$.

In 2007, Suzuki [17] introduced an explicit form of a Halpern-type iteration and showed that conditions ($C1$) and ($C2$) are necessary and sufficient for strong convergence. For a nonexpansive mapping f on a nonempty, bounded, closed, and convex subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, given $w \in \mathcal{M}$, $\lambda \in (0, 1)$, and an initial point $u_0 \in \mathcal{M}$, the iteration

$$u_{n+1} = \alpha_n w + (1 - \alpha_n)((1 - \lambda)u_n + \lambda f u_n), \qquad n \in \mathbb{W}, \tag{1.2}$$

converges strongly to $a = P_{\mathrm{Fix}(f)}w$ if and only if the sequence $\{\alpha_n\}$ satisfies ($C1$) and ($C2$).

Takahashi et al. [18] proposed the following iterative algorithm and established the strong convergence of the generated sequence for nonexpansive mappings.

Theorem 1. *Let $\mathcal{M}$ be a nonempty convex subset of a Hilbert space $\mathcal{H}$. Let $f : \mathcal{M} \to \mathcal{M}$ be a nonexpansive mapping with a nonempty set of attractive points, i.e., $A(f) \neq \emptyset$. Fix $w \in \mathcal{M}$ and define a sequence $\{u_n\}$ in $\mathcal{M}$ by $u_0 \in \mathcal{M}$ and*

$$u_{n+1} = \alpha_n w + (1 - \alpha_n)(\beta_n u_n + (1 - \beta_n) f u_n), \ \forall n \in \mathbb{W}, \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying conditions ($C1$), ($C2$), and

$$\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0. \tag{C5}$$

Then $\{u_n\}$ converges strongly to the metric projection of w onto $A(f)$. Moreover, if $\mathcal{M}$ is closed, then $\{u_n\}$ converges strongly to the metric projection of w onto $\mathrm{Fix}(f)$.

Before proceeding, we recall some notation used throughout the paper. We denote $\mathbb{N} := \{1, 2, 3, \dots\}$ as the set of natural numbers, $\mathbb{W} := \{0, 1, 2, \dots\}$ as the set of non-negative integers, and $\mathbb{R}$ as the set of real numbers. Unless otherwise specified, $\mathcal{M}$ denotes a nonempty subset of a real Hilbert space $\mathcal{H}$ equipped with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. The identity mapping on $\mathcal{M}$ is denoted by I. Strong convergence of $\{u_n\}$ to a is written as $u_n \to a$, while weak convergence to a is denoted by $u_n \rightharpoonup a$.

Definition 2. *[2] Let $f, g : \mathcal{M} \to \mathcal{M}$. If $u = fu = gu$, then $u \in \mathcal{M}$ is a common fixed point of f and g. The set of common fixed points of f and g is denoted by* $\mathrm{Fix}(f, g)$.

In [19], we introduced the class of b-enriched g-nonexpansive mappings of the second kind. For simplicity, throughout this paper, we refer to them as b-enriched g-nonexpansive mappings.

Definition 3. *Let $f, g : \mathcal{M} \to \mathcal{M}$. A mapping f is called a b-enriched g-nonexpansive mapping if $b \in [0, \infty)$ exists such that*

$$\|b(gu - gv) + fu - fv\| \leq (b + 1)\|gu - gv\|, \qquad \forall u, v \in \mathcal{M}. \tag{1.4}$$

Remark 1. *(i) If $b = 0$, then f is called a g-nonexpansive mapping (see [2]).*

(ii) If $g = I$, f is called a b-enriched nonexpansive mapping (see [20, 21]).

(iii) If $g = I$ and $b = 0$, f is called a nonexpansive mapping (see [2]).

In [19], we also introduced the following averaged g-mapping.

Definition 4. *Let $\mathcal{M}$ be a convex subset of $\mathcal{H}$ and $f, g : \mathcal{M} \to \mathcal{M}$. Then for any $\lambda \in (0, 1]$, the averaged g-mapping (or averaged mapping with respect to g) is defined as*

$$f_{g,\lambda}u = (1 - \lambda)gu + \lambda fu, \qquad \forall u \in \mathcal{M}.$$

Note that $\mathrm{Fix}(f_{g,\lambda}, g) = \mathrm{Fix}(f, g)$.

Remark 2. *If $g = I$, then*

$$f_\lambda u = (1 - \lambda)u + \lambda fu, \qquad u \in \mathcal{M},$$

and $\mathrm{Fix}(f_\lambda) = \mathrm{Fix}(f)$ *whenever* $0 < \lambda \le 1$.

Remark 3. *Suppose that f is b-enriched g-nonexpansive. Then for every $b_1 \ge b$, f is also b_1-enriched g-nonexpansive. Indeed, we have*

$$\begin{aligned}\|b_1(gu - gv) + fu - fv\| &\le (b_1 - b)\|gu - gv\| + \|b(gu - gv) + fu - fv\| \\ &\le (b_1 - b)\|gu - gv\| + (b + 1)\|gu - gv\| \\ &= (b_1 + 1)\|gu - gv\|.\end{aligned}$$

Consequently, if $0 < \lambda \le 1/(b + 1)$, *then* $b_1 = 1/\lambda - 1 \ge b$ *and*

$$\|f_{g,\lambda}u - f_{g,\lambda}v\| \le \|gu - gv\|, \qquad u, v \in \mathcal{M}.$$

Thus $f_{g,\lambda}$ is g-nonexpansive (see [19]).

Takahashi and Takeuchi [23] introduced the concept of attractive points in Hilbert spaces.

Definition 5. *Let $f : \mathcal{M} \to \mathcal{H}$. The set*

$$A(f) = \{z \in \mathcal{H} : \|fu - z\| \le \|u - z\|, \forall u \in \mathcal{M}\}$$

is called the set of attractive points of f.

Definition 6. *[2] Let $f : \mathcal{M} \to \mathcal{M}$. Then*

(i) The set $\mathcal{M}$ is called convex if $ku + (1 - k)v \in \mathcal{M}$ for all $u, v \in \mathcal{M}$ and all $k \in [0, 1]$.

(ii) The mapping f is called affine if $\mathcal{M}$ is convex and

$$f(ku + (1 - k)v) = kfu + (1 - k)fv,$$

for all $u, v \in \mathcal{M}$ and all $k \in [0, 1]$.

Definition 7. *[22] Let $\mathcal{M}$ be a nonempty subset of a Hilbert space $\mathcal{H}$ and $u \in \mathcal{H}$. The metric projection of u onto $\mathcal{M}$ is defined by*

$$P_{\mathcal{M}}u = \{z \in \mathcal{M} : \|u - z\| = \inf_{v \in \mathcal{M}} \|u - v\|\}.$$

Theorem 2. *[24] Let $\mathcal{M}$ be a closed and convex subset of $\mathcal{H}$. Then for each $u \in \mathcal{H}$, a unique nearest point $P_{\mathcal{M}}u \in \mathcal{M}$ exists such that $\|u - P_{\mathcal{M}}u\| = \inf_{v \in \mathcal{M}} \|u - v\|$.*

Remark 4. *[24] Let $\mathcal{M}$ be a closed and convex subset of $\mathcal{H}$. Then the metric projection $P_{\mathcal{M}} : \mathcal{H} \to \mathcal{M}$ is nonexpansive.*

Lemma 1. *[24] Let $\mathcal{M}$ be a closed and convex subset of $\mathcal{H}$. Then for every $u \in \mathcal{H}$ and $w \in \mathcal{M}$, we have $w = P_{\mathcal{M}}u$ if and only if $0 \leq \langle u - w, w - v \rangle, \forall v \in \mathcal{M}$.*

Lemma 2. *[18] Let $f : \mathcal{M} \to \mathcal{H}$. Then the following hold:*

(i) $A(f)$ is closed and convex.

(ii) If f is a nonexpansive mapping, $u_n \rightharpoonup z$, and $u_n - fu_n \to 0$, then $z \in A(f)$.

Lemma 3. *[23] Let $\mathcal{M}$ be a closed and convex subset of $\mathcal{H}$ and $z \in A(f)$. Then $P_{\mathcal{M}}z \in \mathrm{Fix}(f)$.*

Lemma 4. *[12,25] For $u, v \in \mathcal{H}$ and $\mu \in \mathbb{R}$, the following hold:*

(i) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v \rangle$.

(ii) $\|\mu u + (1 - \mu)v\|^2 = \mu\|u\|^2 + (1 - \mu)\|v\|^2 - \mu(1 - \mu)\|u - v\|^2$.

(iii) $\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle$.

Lemma 5. *[25] Let $\{\rho_n\}$ be a sequence of nonnegative real numbers such that*

$$\rho_{n+1} \leq (1 - \alpha_n)\rho_n + \alpha_n\gamma_n + \beta_n, \qquad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\gamma_n\}$ is a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$, and $\{\beta_n\}$ is a sequence of non-negative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$. Then $\lim_{n\to\infty} \rho_n = 0$.

We next recall a useful lemma that will play a key role in the proof of our main result.

Lemma 6. *[26] Let $\{\Gamma_n\}$ be a sequence of real numbers that has a subsequence $\{\Gamma_{n_k}\}$ satisfying $\Gamma_{n_k} < \Gamma_{n_k+1}, \forall k \in \mathbb{N}$. Then an increasing sequence of integers $\{\tau(n)\}_{n \geq n_0}$ exists such that*

$$\begin{gathered} \lim_{n\to\infty} \tau(n) = \infty, \\ \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \\ \Gamma_n \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0. \end{gathered} \tag{1.5}$$

2. The main results

We define the following set of g-attractive points of f.

Definition 8. *Let $f, g : \mathcal{M} \to \mathcal{H}$. The set*

$$A^g(f) = \{z \in \mathcal{H} : \|fu - z\| \leq \|gu - z\|, \forall u \in \mathcal{M}\},$$

is called the set of g-attractive points of f.

Lemma 7. *Let $f, g : \mathcal{M} \to \mathcal{H}$, and denote by $A^g(f_{g,\lambda})$ the set of g-attractive points of $f_{g,\lambda}$, where $\lambda \in (0, 1]$. We then have*

(i) $A^g(f)$ is closed and convex.

(ii) $A^g(f_{g,\lambda})$ is closed and convex for every $\lambda \in (0, 1]$.

Proof. (i) **Convexity.** Let $z_1, z_2 \in A^g(f)$. Then for each $u \in \mathcal{M}$, we have $\|fu - z_1\| \leq \|gu - z_1\|$ and $\|fu - z_2\| \leq \|gu - z_2\|$. Let $\mu \in [0, 1]$ and define $z = (1 - \mu)z_1 + \mu z_2$. Using Lemma 4 Part (ii), we get

$$\begin{aligned} \|fu - z\|^2 &= \|fu - ((1 - \mu)z_1 + \mu z_2)\|^2 \\ &= \|(1 - \mu)(fu - z_1) + \mu(fu - z_2)\|^2 \\ &= (1 - \mu)\|fu - z_1\|^2 + \mu\|fu - z_2\|^2 - \mu(1 - \mu)\|(fu - z_1) - (fu - z_2)\|^2 \\ &= (1 - \mu)\|fu - z_1\|^2 + \mu\|fu - z_2\|^2 - \mu(1 - \mu)\|z_1 - z_2\|^2. \end{aligned} \tag{2.1}$$

Similarly

$$\|gu - z\|^2 = (1 - \mu)\|gu - z_1\|^2 + \mu\|gu - z_2\|^2 - \mu(1 - \mu)\|z_2 - z_1\|^2. \tag{2.2}$$

From (2.1), since $z_1, z_2 \in A^g(f)$, it follows that

$$\|fu - z\|^2 \leq (1 - \mu)\|gu - z_1\|^2 + \mu\|gu - z_2\|^2 - \mu(1 - \mu)\|z_2 - z_1\|^2. \tag{2.3}$$

From (2.2) and (2.3), we have $\|fu - z\|^2 \leq \|gu - z\|^2$ which implies $\|fu - z\| \leq \|gu - z\|$. Since this holds for all $u \in \mathcal{M}$, we have $z \in A^g(f)$. Therefore, $A^g(f)$ is convex.

Closedness. Let $\{z_n\} \subset A^g(f)$ and assume that $z_n \to z$ in $\mathcal{H}$. Then for each $u \in \mathcal{M}$, we have

$$\|fu - z_n\| \leq \|gu - z_n\|, \qquad \forall n.$$

By continuity of the norm, letting $n \to \infty$ gives

$$\|fu - z\| \leq \|gu - z\|.$$

Since this holds for all $u \in \mathcal{M}$, we have $z \in A^g(f)$. Therefore, $A^g(f)$ is closed.

(ii) Apply Part (i) to the mapping $f_{g,\lambda}$.

□

Lemma 8. *Let $f, g : \mathcal{M} \to \mathcal{H}$. Assume that f is a g-nonexpansive mapping and $\{u_n\} \subset \mathcal{M}$. If $gu_n \rightharpoonup z$ and $gu_n - fu_n \to 0$, then $z \in A^g(f)$.*

Proof. Let $v \in \mathcal{M}$ be arbitrary. Since f is a g-nonexpansive mapping, we have

$$\|fu_n - fv\|^2 \leq \|gu_n - gv\|^2. \tag{2.4}$$

It follows that

$$(\|fu_n - gu_n\|^2 + \|gu_n - fv\|^2 + 2\langle fu_n - gu_n,\ gu_n - fv\rangle) \leq \|gu_n - gv\|^2.$$

Since $gu_n \rightharpoonup z$, the sequence $\{gu_n\}$ is bounded. Moreover, since $gu_n - fu_n \to 0$ and the sequence $\{gu_n - fv\}$ is bounded, it follows that $\langle fu_n - gu_n,\ gu_n - fv\rangle \to 0$. Hence

$$\limsup_{n\to\infty} \|gu_n - fv\|^2 \leq \limsup_{n\to\infty} \|gu_n - gv\|^2.$$

Since $gu_n \rightharpoonup z$, we have

$$\langle gu_n - gv,\ gv - fv\rangle \to \langle z - gv,\ gv - fv\rangle.$$

Moreover,

$$\|gu_n - fv\|^2 = \|gu_n - gv\|^2 + \|gv - fv\|^2 + 2\langle gu_n - gv,\ gv - fv\rangle.$$

Combining this identity with the preceding lim sup inequality gives

$$\|gv - fv\|^2 + 2\langle z - gv,\ gv - fv\rangle \leq 0.$$

Since

$$\|z - fv\|^2 = \|(z - gv) + (gv - fv)\|^2 = \|z - gv\|^2 + 2\langle z - gv, gv - fv\rangle + \|gv - fv\|^2,$$

we obtain

$$\|z - fv\|^2 - \|z - gv\|^2 = 2\langle z - gv, gv - fv\rangle + \|gv - fv\|^2.$$

Thus

$$\|z - fv\|^2 - \|z - gv\|^2 \leq 0.$$

Hence

$$\|z - fv\| \leq \|z - gv\|.$$

Since this holds for all $v \in \mathcal{M}$, we conclude that $z \in A^g(f)$. □

Lemma 9. *Let $\mathcal{M}$ be a convex subset of $\mathcal{H}$, let $f, g : \mathcal{M} \to \mathcal{H}$, and let $A(g)$ denote the set of attractive points of g. Assume that f is a b-enriched g-nonexpansive mapping. Then for every $0 < \lambda \leq \frac{1}{b+1}$, we have*

(i) $\mathrm{Fix}(f, g) \subseteq \mathcal{M} \cap A^g(f_{g,\lambda})$. *Furthermore, if* $A^g(f_{g,\lambda}) \subseteq \mathrm{Fix}(f, g)$, *then* $\mathrm{Fix}(f, g) = \mathcal{M} \cap A^g(f_{g,\lambda})$.

(ii) $\mathrm{Fix}(f, g) \cap A(g) = A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M}$.

Proof. (i) The inclusion is trivial if $\mathrm{Fix}(f, g) = \emptyset$. Assume that $\mathrm{Fix}(f, g) \neq \emptyset$ and let $z \in \mathrm{Fix}(f, g)$. Since f is a b-enriched g-nonexpansive mapping, then $f_{g,\lambda}$ is g-nonexpansive. Thus, for any $u \in \mathcal{M}$, we have

$$\|f_{g,\lambda}u - f_{g,\lambda}z\| \leq \|gu - gz\|.$$

Since $z \in \mathrm{Fix}(f, g) = \mathrm{Fix}(f_{g,\lambda}, g)$, it follows that

$$\|f_{g,\lambda}u - z\| = \|f_{g,\lambda}u - f_{g,\lambda}z\| \leq \|gu - gz\| = \|gu - z\|.$$

Thus, $z \in \mathcal{M} \cap A^g(f_{g,\lambda})$. Hence, $\mathrm{Fix}(f, g) \subseteq \mathcal{M} \cap A^g(f_{g,\lambda})$.

(ii) If $\mathrm{Fix}(f, g) \cap A(g) = \emptyset$, then we are done. Assume that $\mathrm{Fix}(f, g) \cap A(g) \neq \emptyset$ and let $z \in \mathrm{Fix}(f, g) \cap A(g)$. Then $z \in \mathcal{M}$, $z = fz = gz$, and $\|gu - z\| \leq \|u - z\|$, $\forall u \in \mathcal{M}$. Since $f_{g,\lambda}$ is g-nonexpansive, we have

$$\|f_{g,\lambda}u - z\| = \|f_{g,\lambda}u - f_{g,\lambda}z\| \leq \|gu - gz\| = \|gu - z\| \leq \|u - z\|, \qquad \forall u \in \mathcal{M}.$$

This implies $z \in A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M}$. Hence, we have

$$\mathrm{Fix}(f, g) \cap A(g) \subseteq A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M}.$$

For reverse inclusion, if $A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M} = \emptyset$, then we are done. Assume that $A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M} \neq \emptyset$ and let $z \in A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M}$. Since $z \in A(g) \cap \mathcal{M}$, we get

$$\|gz - z\| \leq \|z - z\| = 0,$$

and so $z \in \mathrm{Fix}(g)$. Moreover, since $z \in A^g(f_{g,\lambda}) \cap \mathcal{M}$, then $\|f_{g,\lambda}z - z\| \leq \|gz - z\| = 0$ and so $z \in \mathrm{Fix}(f)$. Thus, $z \in \mathrm{Fix}(f, g) \cap A(g)$. Hence, we have

$$A^g(f_{g,\lambda}) \cap A(g) \cap \mathcal{M} \subseteq \mathrm{Fix}(f, g) \cap A(g).$$

□

Lemma 10. *Let $\mathcal{M}$ be a convex subset of $\mathcal{H}$ and $f, g : \mathcal{M} \to \mathcal{H}$. We then have*

$$A^g(f) \subseteq A^g(f_{g,\lambda}), \qquad 0 < \lambda \leq 1.$$

Proof. Let $z \in A^g(f)$. If $0 < \lambda < 1$, then, for every $u \in \mathcal{M}$, we have

$$\begin{aligned}
\|f_{g,\lambda}u - z\|^2 &= \|(1 - \lambda)(gu - z) + \lambda(fu - z)\|^2 \\
&\leq (1 - \lambda)\|gu - z\|^2 + \lambda\|fu - z\|^2 \\
&\leq (1 - \lambda)\|gu - z\|^2 + \lambda\|gu - z\|^2 \\
&= \|gu - z\|^2.
\end{aligned}$$

Thus $z \in A^g(f_{g,\lambda})$. If $\lambda = 1$, then $f_{g,1} = f$, so the assertion is immediate. □

Example 1. *Consider $\mathcal{H} = \mathbb{R}^2$ equipped with the Euclidean norm $\|(u_1, u_2)\| = \sqrt{|u_1|^2 + |u_2|^2}$, $(u_1, u_2) \in \mathcal{H}$ and consider $\mathcal{M} = \left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right]$. For each $(u_1, u_2) \in \mathcal{M}$, define $f, g : \mathcal{M} \to \mathcal{H}$ by*

$$f(u_1, u_2) = (\frac{1}{u_1}, 0), \quad g(u_1, u_2) = (u_1, 0).$$

Notice that f is an enriched g-nonexpansive mapping on $\mathcal{M}$. To verify this, consider Condition (1.4). This implies

$$\|b(g(u_1, u_2) - g(v_1, v_2)) + f(u_1, u_2) - f(v_1, v_2)\| = \left\|(b(u_1 - v_1) + \frac{1}{u_1} - \frac{1}{v_1}, 0)\right\| = |(u_1 - v_1)(b - \frac{1}{u_1v_1})|,$$

for all $(u_1, u_2), (v_1, v_2) \in \mathcal{M}$*. Since, for any* $b \geq \frac{3}{2}$, $\left|b - \frac{1}{u_1 v_1}\right| \leq b + 1$ *holds, we conclude that*

$$\begin{aligned}\|b(g(u_1, u_2) - g(v_1, v_2)) + f(u_1, u_2) - f(v_1, v_2)\| &\leq (b + 1)\,|u_1 - v_1| \\ &= (b + 1)\,\|g(u_1, u_2) - g(v_1, v_2)\|\,,\end{aligned}$$

for all $(u_1, u_2), (v_1, v_2) \in \mathcal{M}$*. Therefore,* f *is a* $\frac{3}{2}$*-enriched g-nonexpansive mapping on* $\mathcal{M}$*. Hence, we get* $\lambda = 0.4$.

Notice that

$$A^g(f_{g,0.4}) \cap \mathcal{M} = \{(1, z_2) : z_2 \in [\frac{1}{2}, 2]\},$$

whereas

$$\mathrm{Fix}(f, g) = \emptyset.$$

Consequently

$$\mathcal{M} \cap A^g(f_{g,0.4}) \nsubseteq \mathrm{Fix}(f, g).$$

Hence, the reverse inclusion in Lemma 9 Part (i) does not hold in general.

We now compute $A^g(f)$*. Let* $z = (a, c) \in \mathbb{R}^2$*. Then* $z \in A^g(f)$ *if and only if*

$$\|fu - z\| \leq \|gu - z\|, \qquad u \in \mathcal{M}.$$

Since f *and* g *depend only on* u_1*, this is equivalent to*

$$\left(\frac{1}{u_1} - a\right)^2 + c^2 \leq (u_1 - a)^2 + c^2, \qquad u_1 \in \left[\frac{1}{2}, 2\right],$$

which implies

$$\left(\frac{1}{u_1} - a\right)^2 \leq (u_1 - a)^2.$$

Equivalently

$$\left(\frac{1}{u_1} - u_1\right)\left(\frac{1}{u_1} + u_1 - 2a\right) \leq 0.$$

If $1/2 \leq u_1 < 1$*, then*

$$\frac{1}{u_1} - u_1 > 0,$$

and so

$$a \geq \frac{1}{2}\left(u_1 + \frac{1}{u_1}\right).$$

Taking $u_1 = 1/2$*, we obtain*

$$a \geq \frac{1}{2}\left(\frac{1}{2} + 2\right) = \frac{5}{4}.$$

On the other hand, if $1 < u_1 \leq 2$*, then*

$$\frac{1}{u_1} - u_1 < 0,$$

and so

$$a \leq \frac{1}{2}\left(u_1 + \frac{1}{u_1}\right).$$

Letting $u_1 \downarrow 1$, *we obtain*

$$a \leq 1.$$

Thus, $a \geq 5/4$ *and* $a \leq 1$, *which is impossible. Therefore*

$$A^g(f) = \emptyset.$$

Thus, $A^g(f_{g,0.4}) \not\subseteq A^g(f)$. *Hence, the reverse inclusion in Lemma 10 does not hold in general.*

Lemma 11. *Let* $\mathcal{M}$ *be a closed and convex subset of* $\mathcal{H}$, *and let* $f, g : \mathcal{M} \to \mathcal{M}$. *Assume that* f *is b-enriched g-nonexpansive, and g is affine and continuous. Then* $\mathrm{Fix}(f, g)$ *is closed and convex.*

Proof. Put $F = f_{g,\lambda}$, where $0 < \lambda \leq 1/(b+1)$. By Remark 3, F is g-nonexpansive. Moreover, we have,

$$\mathrm{Fix}(F, g) = \mathrm{Fix}(f, g).$$

We first prove convexity. If $\mathrm{Fix}(F, g) = \emptyset$, there is nothing to prove. Let $z_1, z_2 \in \mathrm{Fix}(F, g)$, and let $\mu \in [0, 1]$. Put

$$z = (1-\mu)z_1 + \mu z_2.$$

Since $\mathcal{M}$ is convex, $z \in \mathcal{M}$. Since g is affine and $gz_1 = z_1$, $gz_2 = z_2$, we have

$$gz = (1-\mu)gz_1 + \mu gz_2 = (1-\mu)z_1 + \mu z_2 = z.$$

Thus, $z \in \mathrm{Fix}(g)$.

It remains to prove $Fz = z$. Since F is g-nonexpansive, we have

$$\|Fz - z_1\| = \|Fz - Fz_1\| \leq \|gz - gz_1\| = \|z - z_1\|,$$

and similarly

$$\|Fz - z_2\| \leq \|z - z_2\|.$$

If $z_1 = z_2$, then $z = z_1$ and there is nothing to prove. Assume $z_1 \neq z_2$ and write $D = \|z_1 - z_2\|$. Then

$$\|z - z_1\| = \mu D, \qquad \|z - z_2\| = (1-\mu)D.$$

Therefore

$$D = \|z_1 - z_2\| \leq \|z_1 - Fz\| + \|Fz - z_2\| \leq \mu D + (1-\mu)D = D.$$

Hence, equality holds throughout. Equality in the triangle inequality in a Hilbert space implies that Fz lies on the line segment joining z_1 and z_2. Moreover, the two distance equalities force

$$\|Fz - z_1\| = \mu D, \qquad \|Fz - z_2\| = (1-\mu)D.$$

The unique point of $[z_1, z_2]$ with these two distances is

$$(1-\mu)z_1 + \mu z_2 = z.$$

Hence, $Fz = z$ and so $z \in \mathrm{Fix}(F, g)$. Thus, $\mathrm{Fix}(f, g)$ is convex.

We now prove closedness. Let $\{x_n\} \subset \mathrm{Fix}(F, g)$ and assume $x_n \to x$ in $\mathcal{H}$. Since $\mathcal{M}$ is closed, $x \in \mathcal{M}$. Since g is continuous, we have

$$gx = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} x_n = x.$$

Thus, $x \in \mathrm{Fix}(g)$. Moreover, by g-nonexpansiveness of F, we have

$$\|Fx_n - Fx\| \leq \|gx_n - gx\| \to 0.$$

Since $Fx_n = x_n \to x$, it follows that $Fx = x$. Therefore, $x \in \mathrm{Fix}(F, g) = \mathrm{Fix}(f, g)$. Hence, $\mathrm{Fix}(f, g)$ is closed. □

We define the following g-averaged Halpern-type iterative algorithm.

Definition 9. *Let $\mathcal{M}$ be a convex subset of $\mathcal{H}$ and let $f, g : \mathcal{M} \to \mathcal{M}$ satisfy $f(\mathcal{M}) \subseteq g(\mathcal{M})$. Assume that $g(\mathcal{M})$ is convex (or, in particular, g is affine). Let $w \in g(\mathcal{M})$, $u_0 \in \mathcal{M}$, $0 < \lambda \leq 1$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.*

A sequence $\{u_n\} \subset \mathcal{M}$ is called a g-averaged Halpern-type iteration generated by $f, g, w, \lambda, \{\alpha_n\}, \{\beta_n\}$ if, for each $n \geq 0$, after u_n has been chosen, one chooses $u_{n+1} \in \mathcal{M}$ such that

$$gu_{n+1} = \alpha_n w + (1 - \alpha_n)(\beta_n gu_n + (1 - \beta_n)[(1 - \lambda)gu_n + \lambda fu_n]), \qquad \forall n \in \mathbb{W}. \tag{2.5}$$

Remark 5. *The choice of u_{n+1} is possible because the right-hand side belongs to $g(\mathcal{M})$. If g is not one-to-one, this u_{n+1} need not be unique. This does not affect the convergence theorem, because the theorem is stated for any sequence generated by (2.5), and its conclusion concerns the image sequence $\{gu_n\}$.*

Remark 6. *If $g = I$ and $\lambda = 1$, then the iteration (2.5) reduces to the Takahashi, Wong, Yao iteration (1.3).*

In the following theorem, we extend Theorem 1 by establishing the strong convergence of the g-averaged Halpern-type iteration to a g-attractive point of the mapping $f_{g,\lambda}$ in real Hilbert spaces.

Theorem 3. *Let $\mathcal{M}$ be a convex subset of $\mathcal{H}$ and $f, g : \mathcal{M} \to \mathcal{M}$ such that $f(\mathcal{M}) \subseteq g(\mathcal{M})$. Assume that $g(\mathcal{M})$ is convex (or, in particular, g is affine), $0 < \lambda \leq \frac{1}{b+1}$, and f is a b-enriched g-nonexpansive mapping with $A^g(f_{g,\lambda}) \neq \emptyset$. Fix $w \in g(\mathcal{M})$ and let $\{u_n\}$ in $\mathcal{M}$ be any sequence generated by Definition 9, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that (C1), (C2), and (C5) hold. Then $\{gu_n\}$ converges strongly to $a = P_{A^g(f_{g,\lambda})}w$. In particular, the conclusion holds whenever $A^g(f) \neq \emptyset$.*

Proof. Since $0 < \lambda \leq 1/(b + 1)$, Remark 3 implies that $f_{g,\lambda}$ is g-nonexpansive.

We first note that the recursive construction is possible. Suppose that $u_n \in \mathcal{M}$ has been chosen. Since $w \in g(\mathcal{M})$, $gu_n \in g(\mathcal{M})$, $fu_n \in f(\mathcal{M}) \subseteq g(\mathcal{M})$, and $g(\mathcal{M})$ is convex, the point

$$\alpha_n w + (1 - \alpha_n)(\beta_n gu_n + (1 - \beta_n)[(1 - \lambda)gu_n + \lambda fu_n]),$$

belongs to $g(\mathcal{M})$. Hence, there is a $u_{n+1} \in \mathcal{M}$ satisfying (2.5).

Let $u \in A^g(f_{g,\lambda})$. Such a point exists by assumption. Define

$$z_n = \beta_n gu_n + (1 - \beta_n)f_{g,\lambda}u_n.$$

Then

$$\begin{aligned}\|z_n - u\|^2 &= \|\beta_n(gu_n - u) + (1-\beta_n)(f_{g,\lambda}u_n - u)\|^2\\ &\le \beta_n\|gu_n - u\|^2 + (1-\beta_n)\|gu_n - u\|^2 - \beta_n(1-\beta_n)\|gu_n - f_{g,\lambda}u_n\|^2\\ &= \|gu_n - u\|^2 - \beta_n(1-\beta_n)\|gu_n - f_{g,\lambda}u_n\|^2\\ &\le \|gu_n - u\|^2. \end{aligned} \tag{2.6}$$

Put $k = \|gu_0 - u\| + \|w - u\|$, so that $\|gu_0 - u\| \le k$. Assume that $\|gu_m - u\| \le k$ for some $m \in \mathbb{W}$. Using (2.6), we get

$$\begin{aligned}\|gu_{m+1} - u\| &= \|\alpha_m w + (1-\alpha_m)z_m - u\|\\ &\le \alpha_m\|w - u\| + (1-\alpha_m)\|z_m - u\|\\ &\le \alpha_m\|w - u\| + (1-\alpha_m)\|gu_m - u\|\\ &\le \alpha_m k + (1-\alpha_m)k = k.\end{aligned}$$

By induction, $\|gu_n - u\| \le k$ for all $n \in \mathbb{W}$. Hence, $\{gu_n\}$ is bounded, and so $\{f_{g,\lambda}u_n\}$ is also bounded.

By Lemma 7 Part (ii), we get that $A^g(f_{g,\lambda})$ is closed and convex. Thus, $a \in A^g(f_{g,\lambda})$ exists such that $a = P_{A^g(f_{g,\lambda})}w$. Using (2.6), we obtain

$$\begin{aligned}\|gu_{n+1} - a\|^2 &= \|\alpha_n(w - a) + (1-\alpha_n)(z_n - a)\|^2\\ &\le \alpha_n\|w - a\|^2 + (1-\alpha_n)\|z_n - a\|^2\\ &\le \alpha_n\|w - a\|^2 + \|\beta_n(gu_n - a) + (1-\beta_n)(f_{g,\lambda}u_n - a)\|^2\\ &\le \alpha_n\|w - a\|^2 + \|gu_n - a\|^2 - \beta_n(1-\beta_n)\|f_{g,\lambda}u_n - gu_n\|^2.\end{aligned}$$

Therefore

$$\beta_n(1-\beta_n)\|f_{g,\lambda}u_n - gu_n\|^2 \le \alpha_n\|w - a\|^2 + \|gu_n - a\|^2 - \|gu_{n+1} - a\|^2. \tag{2.7}$$

From

$$gu_{n+1} - gu_n = \alpha_n w + (1-\alpha_n)(\beta_n gu_n + (1-\beta_n)f_{g,\lambda}u_n) - gu_n,$$

it follows that

$$\|gu_{n+1} - gu_n\| \le \alpha_n\|w - gu_n\| + (1-\alpha_n)(1-\beta_n)\|f_{g,\lambda}u_n - gu_n\|. \tag{2.8}$$

Case A. Define $\Gamma_n = \|gu_n - a\|^2$, for all $n \in \mathbb{W}$. Assume that there exists $n_0 \in \mathbb{W}$ such that

$$\Gamma_{n+1} \le \Gamma_n, \qquad n \ge n_0.$$

Then $\{\Gamma_n\}_{n\ge n_0}$ is nonincreasing, hence convergent. Thus, $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. It follows from $\lim_{n\to\infty}\alpha_n = 0, \liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and (2.7) that

$$\lim_{n\to\infty}\|gu_n - f_{g,\lambda}u_n\| = 0. \tag{2.9}$$

As $\lim_{n\to\infty}\alpha_n = 0$ and (2.9), we obtain, from (2.8), $\lim_{n\to\infty}\|gu_{n+1} - gu_n\| = 0$.

Since $\{gu_n\}$ is bounded and $\mathcal{H}$ is reflexive, we may choose a subsequence, still denoted by $\{gu_{n_m}\}$, such that

$$gu_{n_m} \rightharpoonup v$$

and

$$\limsup_{n\to\infty}\langle w-a, gu_n - a\rangle = \lim_{m\to\infty}\langle w-a, gu_{n_m}-a\rangle. \tag{2.10}$$

Since $f_{g,\lambda}$ is g-nonexpansive and $gu_n - f_{g,\lambda}u_n \to 0$, Lemma 8 gives $v \in A^g(f_{g,\lambda})$.

Then, using (2.10) and Lemma 1, we obtain

$$\limsup_{n\to\infty}\langle w-a, gu_n-a\rangle = \langle w-a, v-a\rangle \leq 0. \tag{2.11}$$

From the g-averaged Halpern-type iteration (2.5), (2.6), and Lemma 4 Part (i), we obtain

$$\begin{aligned}\|gu_{n+1}-a\|^2 &= \|(1-\alpha_n)(z_n-a)+\alpha_n(w-a)\|^2\\ &\leq (1-\alpha_n)\|z_n-a\|^2 + 2\alpha_n\langle w-a, gu_{n+1}-a\rangle\\ &\leq (1-\alpha_n)\|gu_n-a\|^2 + 2\alpha_n\langle w-a, gu_{n+1}-a\rangle.\end{aligned} \tag{2.12}$$

Therefore

$$\begin{aligned}\|gu_{n+1}-a\|^2 &\leq (1-\alpha_n)\|gu_n-a\|^2 + 2\alpha_n\langle w-a, gu_{n+1}-gu_n\rangle + 2\alpha_n\langle w-a, gu_n-a\rangle\\ &\leq (1-\alpha_n)\|gu_n-a\|^2 + 2\alpha_n\|w-a\|\|gu_{n+1}-gu_n\| + 2\alpha_n\langle w-a, gu_n-a\rangle.\end{aligned} \tag{2.13}$$

Since (2.11), $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\lim_{n\to\infty}\|gu_{n+1}-gu_n\| = 0$ hold, Lemma 5 implies that

$$gu_n \to a.$$

Case B. Assume that for every $n_0 \in \mathbb{W}$, there exists $m \geq n_0$ such that

$$\Gamma_m < \Gamma_{m+1}.$$

Define

$$\tau(n) = \max\{m \leq n : \Gamma_m < \Gamma_{m+1}\}.$$

By Lemma 6, $\tau(n) \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \qquad \Gamma_n \leq \Gamma_{\tau(n)+1}$$

for all $n \geq n_0$.

Replacing n with $\tau(n)$ in (2.7) yields

$$\begin{aligned}\beta_{\tau(n)}(1-\beta_{\tau(n)})\|f_{g,\lambda}u_{\tau(n)}-gu_{\tau(n)}\|^2 &\leq \alpha_{\tau(n)}\|w-a\|^2 + \|gu_{\tau(n)}-a\|^2 - \|gu_{\tau(n)+1}-a\|^2\\ &\leq \alpha_{\tau(n)}\|w-a\|^2.\end{aligned} \tag{2.14}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ are satisfied, we have $\lim_{n\to\infty}\|f_{g,\lambda}u_{\tau(n)} - gu_{\tau(n)}\| = 0$.

Replacing n with $\tau(n)$ in (2.13), we find, for all $n \geq n_0$, that

$$\left\|gu_{\tau(n)+1}-a\right\|^2 \leq (1-\alpha_{\tau(n)})\|gu_{\tau(n)}-a\|^2 + 2\alpha_{\tau(n)}\langle w-a, gu_{\tau(n)+1}-gu_{\tau(n)}\rangle + 2\alpha_{\tau(n)}\langle w-a, gu_{\tau(n)}-a\rangle.$$

Since $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have

$$\alpha_{\tau(n)}\|gu_{\tau(n)} - a\|^2 \leq 2\alpha_{\tau(n)} \langle w - a, gu_{\tau(n)+1} - gu_{\tau(n)}\rangle + 2\alpha_{\tau(n)} \langle w - a, gu_{\tau(n)} - a\rangle .$$

Since $\alpha_{\tau(n)} > 0$, we obtain

$$\|gu_{\tau(n)} - a\|^2 \leq 2 \langle w - a, gu_{\tau(n)+1} - gu_{\tau(n)}\rangle + 2 \langle w - a, gu_{\tau(n)} - a\rangle . \tag{2.15}$$

Replacing n with $\tau(n)$ in (2.8), we have

$$\|gu_{\tau(n)+1} - gu_{\tau(n)}\| \leq \alpha_{\tau(n)}\|w - gu_{\tau(n)}\| + (1 - \alpha_{\tau(n)})(1 - \beta_{\tau(n)})\|f_{g,\lambda}u_{\tau(n)} - gu_{\tau(n)}\|. \tag{2.16}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, (2.16), and $\lim_{n\to\infty} \|f_{g,\lambda}u_{\tau(n)} - gu_{\tau(n)}\| = 0$, we have

$$\lim_{n\to\infty} \|gu_{\tau(n)+1} - gu_{\tau(n)}\| = 0. \tag{2.17}$$

Since $\{gu_{\tau(n)}\}$ is a bounded sequence, a subsequence $\{gu_{\tau(n_m)}\}$ of $\{gu_{\tau(n)}\}$ exists such that

$$\limsup_{n\to\infty} \langle w - a, gu_{\tau(n)} - a\rangle = \lim_{m\to\infty} \langle w - a, gu_{\tau(n_m)} - a\rangle .$$

Repeating the argument of Case A for $\{gu_{\tau(n_m)}\}$, we have

$$\limsup_{n\to\infty} \langle w - a, gu_{\tau(n)} - a\rangle \leq 0. \tag{2.18}$$

Using (2.15), (2.17), and (2.18), we have $\lim_{n\to\infty} \|gu_{\tau(n)} - a\| = 0$. From (2.17), we have

$$\lim_{n\to\infty} \|gu_{\tau(n)+1} - a\| = 0.$$

By Lemma 6, we have $\Gamma_n \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0$. Therefore, we conclude that

$$\lim_{n\to\infty} gu_n = a.$$

□

Remark 7. *The assumption* $A^g(f_{g,\lambda}) \neq \emptyset$ *is the natural nonemptiness assumption for Theorem 3. By Lemma 10, it is automatically satisfied whenever* $A^g(f) \neq \emptyset$.

Corollary 1. *Let the assumptions of Theorem 3 hold. Assume in addition that* $A^g(f_{g,\lambda}) \subseteq A^g(f)$. *Then* $\{gu_n\}$ *converges strongly to* $a = P_{A^g(f)}w$.

The following example illustrates Theorem 3 under the natural assumption $A^g(f_{g,\lambda}) \neq \emptyset$. It also shows why this assumption is better than requiring $A^g(f) \neq \emptyset$, because in the example below, $A^g(f) = \emptyset$. Numerical results, including tables and graphs, have been obtained using Wolfram Mathematica (Wolfram Alpha Notebook Edition).

Example 2. *Let $\mathcal{H} = \mathbb{R}^2$ be endowed with the Euclidean norm $\|(u_1, u_2)\| = \sqrt{|u_1|^2 + |u_2|^2}, (u_1, u_2) \in \mathcal{H}$. For each $(u_1, u_2) \in \mathcal{H}$, define self-mappings f and g on $\mathcal{H}$ by*

$$f(u_1, u_2) = \begin{cases} (\frac{1}{u_1}, 0), & \text{if } (u_1, u_2) \in \mathcal{H},\ u_1 \neq 0, \\ (0, 0), & \text{if } u_1 = 0, \end{cases}$$

and

$$g(u_1, u_2) = (u_1, 0).$$

Consider the convex set $\mathcal{M} = \left[\frac{1}{2}, 2\right] \times \{0\}$. Then $f(\mathcal{M}) = g(\mathcal{M}) = \mathcal{M}$. Hence, both f and g are self-mappings on $\mathcal{M}$. Moreover, $g(\mathcal{M})$ is convex.

We observe that f is not g-nonexpansive on $\mathcal{M}$. Take, for instance, $u = (\frac{1}{2}, 0)$ and $v = (1, 0)$, we have $\|fu - fv\| > \|gu - gv\|$. However, f is b-enriched g-nonexpansive on $\mathcal{M}$, where $b \geq \frac{3}{2}$ and hence, $\lambda \leq 0.4$.

Notice that

$$A^g(f) = \emptyset, \qquad A^g(f_{g,\lambda}) = \{(1, z_2) : z_2 \in \mathbb{R}\},$$

for $\lambda = 0.4, 0.2, 0.1$. The metric projection of $w = (w_1, w_2) \in \mathcal{H}$ onto this closed affine line is

$$P_{A^g(f_{g,\lambda})}(w_1, w_2) = (1, w_2).$$

For the numerical calculation, we use a reindexing of (2.5) starting at $n = 1$. Namely, we take

$$u_1 = (1.5, 0), \qquad w = (2, 0), \qquad \alpha_n = \frac{0.4}{n+1}, \qquad \beta_n = \frac{1}{20}.$$

This is only an index shift of the recurrence in Definition 9. The generated sequence $\{gu_n\}$ converges strongly to

$$(1, 0) = P_{A^g(f_{g,\lambda})}(2, 0).$$

Since $gu_n = (u_{n,1}, 0)$, the numerical results are presented in terms of the first coordinates $\{(gu_n)_1\}$.

Table 1 shows the results computed up to $N = 10{,}000$ steps to illustrate the convergence behavior. Using the stopping criterion $|(gu_n)_1 - 1| \leq 10^{-3}$, the stopping indices are $n = 527$, 1055, and 2109 for $\lambda = 0.4$, 0.2, and 0.1, respectively. The results indicate that the sequence $\{(gu_n)_1\}$ converges to 1, with faster convergence for larger values of λ.

Table 1. Comparison of the g-averaged Halpern-type iteration for the first coordinate $(gu_n)_1$ for $\lambda = 0.4, 0.2, 0.1$.

n	1	2	5	10	50	100	1000	10000
$\lambda = 0.1$	1.5	1.53667	1.46603	1.31028	1.04605	1.0219	1.00211	1.00021
$\lambda = 0.2$	1.5	1.47333	1.29609	1.13745	1.02175	1.01069	1.00105	1.00011
$\lambda = 0.4$	1.5	1.34667	1.12491	1.05574	1.01063	1.00529	1.00053	1.00005

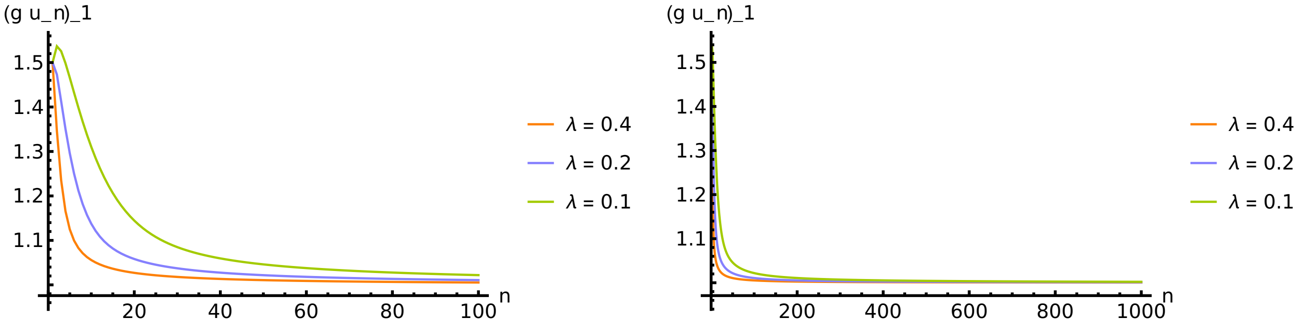

Figure 1. Comparison of the g-averaged Halpern-type iteration for the first coordinate $(gu_n)_1$ for $\lambda = 0.4, 0.2, 0.1$.

Figure 1 clearly show that larger values of λ *accelerate the convergence of the sequence* $\{(gu_n)_1\}$ *to* 1.

Theorem 4. *Let* $\mathcal{M}$ *be a closed and convex subset of* $\mathcal{H}$ *and* $f : \mathcal{M} \to \mathcal{M}$. *Assume that* f *is a* b-*enriched nonexpansive mapping with* $A(f_\lambda) \neq \emptyset$ *for some*

$$0 < \lambda \leq \frac{1}{b+1}.$$

Fix $w \in \mathcal{M}$ *and let* $\{u_n\}$ *be a sequence in* $\mathcal{M}$ *defined by* $u_0 \in \mathcal{M}$ *and the averaged Halpern-type iteration*

$$u_{n+1} = \alpha_n w + (1-\alpha_n)(\beta_n u_n + (1-\beta_n)[(1-\lambda)u_n + \lambda f u_n]), \qquad n \in \mathbb{W}, \tag{2.19}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *such that (C1), (C2), and (C5) hold. Then* $\{u_n\}$ *converges strongly to* $a = P_{\mathrm{Fix}(f)} w$.

Proof. Apply Theorem 3 with $g = I$, it follows that the sequence $\{u_n\}$ converges strongly to $a = P_{A(f_\lambda)} w$. Since $\mathcal{M}$ is closed and $\{u_n\} \subset \mathcal{M}$ with $u_n \to a$, we deduce that $a \in \mathcal{M}$. Moreover, since $a \in A(f_\lambda) \cap \mathcal{M}$, we have

$$\|f_\lambda a - a\| \leq \|a - a\| = 0.$$

Thus, $a \in \mathrm{Fix}(f_\lambda)$. Because $0 < \lambda \leq 1$, we have

$$\mathrm{Fix}(f_\lambda) = \mathrm{Fix}(f).$$

Moreover, if $p \in \mathrm{Fix}(f)$, then $p \in A(f_\lambda)$, since

$$\|f_\lambda u - p\| \leq \|u - p\|, \qquad u \in \mathcal{M}.$$

Therefore,

$$\mathrm{Fix}(f) \subseteq A(f_\lambda).$$

Since $a = P_{A(f_\lambda)} w$ and $a \in \mathrm{Fix}(f)$, we get

$$\mathrm{dist}(w, A(f_\lambda)) = \|w - a\| \geq \mathrm{dist}(w, \mathrm{Fix}(f)).$$

The reverse inequality follows from $\mathrm{Fix}(f) \subseteq A(f_\lambda)$. Hence, we have

$$\|w - a\| = \mathrm{dist}(w, \mathrm{Fix}(f)).$$

Since $\mathrm{Fix}(f)$ is closed and convex, the metric projection is single-valued. Therefore

$$a = P_{\mathrm{Fix}(f)} w.$$

□

3. Conclusions

We introduced the set of g-attractive points of a mapping f and proved its basic structural properties, including closedness, convexity, and its relation to the set of common fixed points of the mappings f and g. We then studied a g-averaged Halpern-type iteration for b-enriched g-nonexpansive mappings in real Hilbert spaces. For $0 < \lambda \leq 1/(b+1)$, Theorem 3 proves the strong convergence of $\{gu_n\}$ to the metric projection of the anchor w onto $A^g(f_{g,\lambda})$, under the natural assumption $A^g(f_{g,\lambda}) \neq \emptyset$. This assumption is weaker than $A^g(f) \neq \emptyset$, as shown by the numerical example. In the special case $g = I$, the method gives an averaged Halpern-type convergence theorem for b-enriched nonexpansive mappings. The computations suggest that larger admissible values of λ may improve the convergence speed. A natural further problem is to determine whether Condition (C5) can be replaced by a weaker asymptotic condition on $\{\beta_n\}$, for example, by using [27, Lemma 2.3].

Author contributions

All authors contributed equally and significantly to the writing of this article.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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