AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 12961–12976.
DOI: 10.3934/math.2026533
Received: 01 April 2026
Revised: 26 April 2026
Accepted: 30 April 2026
Published: 11 May 2026

Research article

New advances in fixed point theory for contractive mappings on composed S-metric spaces

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Abstract: Motivated by recent developments of Kil et al., who proposed a new class of composed S-metric type spaces extending the notion of controlled S-metric spaces, this paper investigated fixed point phenomena within this broader setting. We derived existence and uniqueness theorems for fixed points corresponding to various contractive conditions by making essential use of the structural features of the proposed framework. The presented theorems encompassed and extended a number of well-established results in the existing literature. To demonstrate the practical relevance of the theory, a concrete example was included, illustrating the effectiveness of composed S-metric spaces in addressing the solvability of nth-degree polynomial equations.

Keywords: composed S-metric spaces; fixed point theory; S-metric spaces; polynomial equations
Mathematics Subject Classification: 54H25, 47H10

1. Introduction and preliminaries

Fixed point (FP) theory, a cornerstone of modern mathematical analysis, plays a crucial role in a wide range of disciplines, including topology, nonlinear analysis, and optimization [1]. Results concerning FPs have profound implications in the study of dynamical systems, differential equations, and even economics. Over the years, FP results in metric spaces have led to fundamental advances, with the classical Banach FP theorem [2] being one of the most significant contributions to the field. While classical FP results, such as those in metric spaces (MS) and more specialized spaces like Banach and Hilbert spaces, provide a solid foundation, they do not always cover more complex or generalized settings where the distance between points might be more nuanced. This gap has led to the development of generalized metric spaces, such as S-metric spaces (S-MS). As an extension of traditional MS, S-MS have gained significant attention in recent years due to their ability to provide a more generalized framework for analyzing metric functions. An early investigation into S-MS can be found in the work of Sedghi et al. [3], who first outlined the foundational properties of these spaces and explored their basic structure. Also, a common FP theorem for self-mappings on complete S-MS is subsequently obtained.

Definition 1.1. *[3] Given a nonempty set $\mathcal{A}$, a function $S_m : \mathcal{A}^3 \to [0, \infty)$ is called an S-metric on $\mathcal{A}$ whenever it fulfills the following hypothesis for every $q, h, r, u \in \mathcal{A}$:*

(i) $S_m(q, h, r) = 0$ *if, and only if,* $q = h = r$.
(ii) $S_m(q, h, r) \le S_m(q, q, u) + S_m(h, h, u) + S_m(r, r, u)$.

The pair $(\mathcal{A}, S_m)$ is called an S-MS.

Example 1.1. *[3] Let $\mathcal{A} = \mathbb{R}$ and $\|.\|$ be a norm on $\mathcal{A}$. Then,*

$$S_m(q, h, z) = \|q + h - 2z\| + \|h - z\|$$

is an S-MS on $\mathcal{A}$.

Subsequently, numerous researchers have advanced this field by proposing various generalizations and investigating the properties, existence, and uniqueness of FP. Shahraki et al. [4] investigated a generalized Suzuki-type function defined on an S-MS to establish the existence of an FP. In the framework of the S-MS, Saluja et al. [5] developed FP results for integral-type contractive and proposed an application to a fractional integral problem based on Riemann–Liouville calculus, accompanied by a numerical illustration. The study of S-MS continues to attract researchers through various generalizations. In this direction, Taş et Özgür [6] introduced the notion of the parametric S-MS and gave some basic facts. Also they proved some FP results under various expansive mappings. Subsequently, the authors refined their earlier approach and, in [7], presented the notion of a parametric S_b-metric space. Following [8], the notion of an S_b-metric space, due to Souayah and Mlaiki, generalizes b-MS by combining features of both S-metrics and b-metrics, in particular, a relaxed triangular-type inequality controlled by a constant $b \ge 1$.

Definition 1.2. *[8] Let $\mathcal{A} \neq \emptyset$ and let $b \ge 1$. A mapping $S_b : \mathcal{A}^3 \to [0, \infty)$ is said to define an S_b-metric on $\mathcal{A}$ if, for every $q, h, r, u \in \mathcal{A}$: it fulfills the following properties:*

(i) $S_b(q, h, r) = 0$ *if, and only if,* $q = h = r$.
(ii) $S_b(q, h, r) \le b[S_b(q, q, u) + S_b(h, h, u) + S_b(r, r, u)]$.

The pair $(\mathcal{A}, S_b)$ is called an S_b-metric space.

Example 1.2. *Let $\mathcal{A} \neq \emptyset$ and $card(\mathcal{A}) \ge 5$. Suppose $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a partition of $\mathcal{A}$ such that $card(\mathcal{A}_1) \ge 4$. Let $b \ge 1$. Then,*

$$S_b(q, h, r) = \begin{cases} 0, \ \textit{if} & q = h = r = 0, \\ 3b, \ \textit{if} & (q, h, r) \in \mathcal{A}_1^3, \\ 1, \ \textit{if} & (q, h, r) \notin \mathcal{A}_1^3, \end{cases}$$

S_b is an S_b-metric on $\mathcal{A}$ with coefficient $b \ge 1$ for all $q, h, r \in \mathcal{A}$.

Also, Mlaiki in [9] introduced the notion of an extended S_b-metric space by substituting the constant b in the triangle-type inequality of Definition 1.2 by a function θ. Some FP results were presented. Building on this work and on [7], Mani al. proposed in [10] the framework of the extended parametric S_b-MS. They established an enhanced form of the Banach contraction principle specifically for extended parametric S_b-MS, utilizing an auxiliary function to achieve the result. To demonstrate the applicability of their approach, they provided concrete examples and applied it to solving Fredholm integral equations. To set the context, we begin by reviewing some fundamental definitions of some generalizations of S-MS pertinent for our work. Motivated by the concept of the S_b-MS, Mustafa et al. [11] established the notion of S_p-MS as follows:

Definition 1.3. *[11] Let $\mathcal{A} \neq \emptyset$ and $\Gamma : [0,\infty) \to [0,\infty)$ be a strictly increasing function and continuous, where $a \leq \Gamma(a)$ for all $a > 0$ and $\Gamma(0) = 0$. A function $S_p : \mathcal{A}^3 \to [0,\infty)$ is said to define an S_p-metric on $\mathcal{A}$ provided that, for all $q, h, r, u \in \mathcal{A}$, the following properties hold:*

(i) $S_p(q,h,r) = 0$ *if, and only if,* $q = h = r$.
(iii) $S_p(q,h,r) \leq \Gamma\Big(S_p(q,q,u) + S_p(h,h,u) + S_p(r,r,u)\Big)$.

The pair $(\mathcal{A}, S_p)$ is called a S_p-MS.

Remark 1. *Each S-MS is an S_p-MS with $\Gamma(a) = a$. Moreover, every S_b-MS with $b \geq 1$ induces an S_p-MS with $\Gamma(a) = b.a$.*

Example 1.3. *[11] Let $(\mathcal{A}, S)$ be an S-MS with coefficient $b \geq 1$. Then,*

1. $S_p(q,h,z) = \exp(S(q,h,z)) - 1$ *is an S_p-MS with* $\Gamma(a) = \exp(a) - 1$.
2. $S_p(q,h,z) = \exp(S(q,h,z)) \ln(1 + S(q,h,z))$ *is an S_p-MS with* $\Gamma(a) = \exp(b.a)\ln(1 + b.a)$.

The concept of S-MS was further expanded to include controlled S-metric-like spaces [12] by applying a control function into the triangle inequality, as showed in the following definition:

Definition 1.4. *[12] Let $\mathcal{A} \neq \emptyset$ and let $\alpha : \mathcal{A}^2 \to [1,\infty)$. A mapping $S : \mathcal{A}^3 \to [0,\infty)$ is said to induce a controlled S-MS structure on $\mathcal{A}$ if, for every $q, h, z, u \in \mathcal{A}$, the following conditions hold:*

- $S(q,h,z) = 0 \Longleftrightarrow q = h = z$;
- $S(q,h,z) \leq \alpha(q,u)S(q,q,u) + \alpha(h,u)S(h,h,u) + \alpha(z,u)S(z,z,u)$.

Following these developments, numerous researchers have further advanced the field by proposing various generalizations and rigorously studying the properties, existence, and uniqueness of FP. Some relevant references in this context include [13]. Recently, Kil et al. [14] extended the concept of a controlled S-MS to a new metric space called composed S-MS by incorporating composed functions into the righthand side of the triangle inequality. As a result, some of the findings discussed in the literature will emerge as special cases of the results derived in the composed S-MS. The precise definition and formulation of this new concept are provided in the following section. Section 3 focuses on the principal fixed point results. In Section 4, we provide an illustrative application to polynomial equations. Concluding remarks and directions for future research are given in Section 5.

2. Composed S-metric spaces

Definition 2.1. *[14] Let $\mathcal{A}$ be a nonempty set and consider the non-constant functions $\alpha : [0,\infty) \to [0,\infty)$ such that $\alpha(0) = 0$. A function $C_S : \mathcal{A}^3 \to [0,\infty)$ is said to be a composed S-metric if it satisfies, for all $q, h, w, u \in \mathcal{A}$:*

1. $C_S(q,h,w) = 0 \Longleftrightarrow q = h = w$ *for all* $q,h,w \in \mathcal{A}$,
2. $C_S(q,h,w) \leq \alpha(C_S(q,q,u)) + \alpha(C_S(h,h,u)) + \alpha(C_S(w,w,u))$.

The pair $(\mathcal{A}, C_S)$ is a called a composed S-metric space (CSM).

Remark 2. *It should be noted that Definition 2.1 does not impose any additional assumptions on the composed function α other than $\alpha(0) = 0$, in contrast with the framework introduced by Kil et al. in [14].*

Remark 3. *The use of self-distances of the form $C_S(x,x,u)$ on the righthand side of the triangle inequality is a structural feature inherited from the foundational framework of S-MS introduced by Sedghi et al. [3], where the triangle inequality takes the analogous form $S_m(q,h,z) \leq S_m(q,q,u) + S_m(h,h,u) + S_m(z,z,u)$. In this setting, the self-distance $C_S(x,x,u)$ serves as a one-sided proximity measure from x to a reference point u, and the triangle inequality bounds the trivariate distance in terms of how each individual point relates to that common reference. This is a deliberate and well-established structural choice in the S-metric literature.*

Regarding the absence of a symmetry requirement: this is intentional. The general framework of this paper does not assume symmetry, i.e., it does not require $C_S(q,q,h) = C_S(h,h,q)$. As a direct consequence, one may have $C_S(q,q,h) \neq C_S(h,h,q)$, which means convergence and Cauchy conditions must be stated carefully using the appropriate ordering of arguments, as done in Definition 2.3. Symmetry is imposed as an explicit additional hypothesis only in Theorems 2 and 3, where it is genuinely required by the proof technique. Theorem 1, by contrast, holds in the fully asymmetric setting, as noted in Remark 4.

Definition 2.2. *Let $(\mathcal{A}, C_S)$ be a CSM. $\mathcal{A}$ is said to be symmetric if $C_S(q,q,h) = C_S(h,h,q)$ for all $q,h \in \mathcal{A}$.*

Remark 4. *We observe that every CSM is an S-MS when $\alpha(q) = q$. The converse, however, is not always valid as demonstrated in Example 2.1.*

Example 2.1. *Let $\mathcal{A} = [1,+\infty)$. Define a function $C_S : \mathcal{A}^3 \longrightarrow [0,\infty)$ by*

$$C_S(q,h,w) = (q-w)^2 + (h-w)^2. \tag{2.1}$$

Define the composed function $\alpha : [1,+\infty) \longrightarrow [e,\infty)$ by $\alpha(q) = e^q$.
Then, $(\mathcal{A}, C_S)$ is a CSM but is not an S-MS in the usual sense.

Proof. We begin by proving that the function C_S is not an S-MS in the usual sense, and the triangle inequality fails to hold. Indeed, for $q = 4$, $h = 5$, $w = 1$, and $t = 4$ we have $C_S(q,h,w) = (q-w)^2 + (h-w)^2 = 25 > C_S(q,q,t) + C_S(h,h,t) + C_S(w,w,t) = 20$.
Verifying the first condition 1 of Definition 2.1 is straightforward. We should prove condition 2.
For all $q,h,w,u \in [1,\infty)$, we have

$$C_S(q,h,w) = (q-w)^2 + (h-w)^2 = (q-u+u-w)^2 + (h-u+u-w)^2.$$

By using the classical inequality $(p + p')^2 \leq 2(p^2 + p'^2)$, we obtain

$$\begin{aligned} C_S(q,h,w) = (q-s+s-w)^2 + (h-s+s-w)^2 &\leq 2(q-s)^2 + 2(s-w)^2 + 2(h-s)^2 \\ &+ 2(w-s)^2 \qquad (2.2) \\ &\leq 2(q-s)^2 + 2(h-s)^2 \\ &+ 4(w-s)^2. \qquad (2.3) \end{aligned}$$

Knowing that for all $q \geq 0$, $q \leq e^q$ and $4q \leq e^{2q}$, we get from (2.3)

$$\begin{aligned} C_S(q,h,w) &\leq e^{2(q-s)^2} + e^{2(h-s)^2} + e^{2(w-s)^2} \\ &= \alpha(2(q-s)^2) + \alpha(2(h-s)^2) + \alpha(2(w-s)^2) \\ &= \alpha(C_S(q,q,s)) + \alpha(C_S(h,h,s)) + \alpha(C_S(w,w,s)). \end{aligned}$$

This validates the triangle inequality. □

Example 2.2. *Suppose $\mathcal{A} = \mathbb{N}$ and let $C_S : \mathcal{A}^3 \longrightarrow [0,\infty)$ be defined by*

$$\begin{cases} C_S(d,d,d) = 0, & \forall d \in \mathbb{N}, \\ C_S(d,d,e) = d + e, & \forall d, e \in \mathbb{N}, \\ C_S(c,h,e) = 2(c+h+e), & \textit{for all distinct } d, h, e \in \mathbb{N}. \end{cases}$$

Define the composed function $\alpha : [0,+\infty) \longrightarrow [0,\infty)$ by $\alpha(t) = 2t + 1$.
Then, $(\mathcal{A}, C_S)$ is a CSM but is not an S-MS.

Proof. It is clear that the condition referred to the self distance is valid. Let $c, d, k, t \in \mathbb{N}$, and we have

$$\begin{aligned} \alpha(C_S(d,d,t)) + \alpha(C_S(h,h,t)) + \alpha(C_S(e,e,t)) &= \alpha(d+t) + \alpha(h+t) + \alpha(e+t) \\ &= 2(d+t) + 1 + 2(h+t) + 1 + 2(e+t) + 1 \\ &\geq 2(d+h+e) = C_S(d,h,e). \end{aligned}$$

Therefore, $(\mathcal{A}, C_S)$ is a CSM. However, it is not an S-MS in the usual sense. Indeed, the triangle inequality does not hold for every $d, h, e, t \in \mathbb{N}$ since

$$\begin{aligned} C_S(d,d,t) + C_S(h,h,t) + C_S(e,e,t) &= d + h + e + 2t \\ &\ngeq 2 \quad (d+h+e) = C_S(d,h,e) \text{ for } d = h = e = t = 1. \end{aligned}$$

□

Example 2.3. *[14] Let $\mathcal{A} = \mathbb{R}$. Define $C_S : \mathcal{A}^3 \to [0,\infty)$ as*

$$C_S(c,d,t) = e^{|d+t-2c|+|d-t|} - 1, \quad forall\, c, d, t \in \mathcal{A}.$$

Then, $(\mathcal{A}, C_S)$ is a CSM with $\alpha : [0,\infty) \to [0,\infty)$ such that

$$\alpha(z) = \frac{z^3 + 3z^2 + 3z}{3}, \quad forall\, z \geq 0.$$

To clarify the hierarchy and inclusions between the different generalizations of S-MS, we present the following diagram in Figure 1.

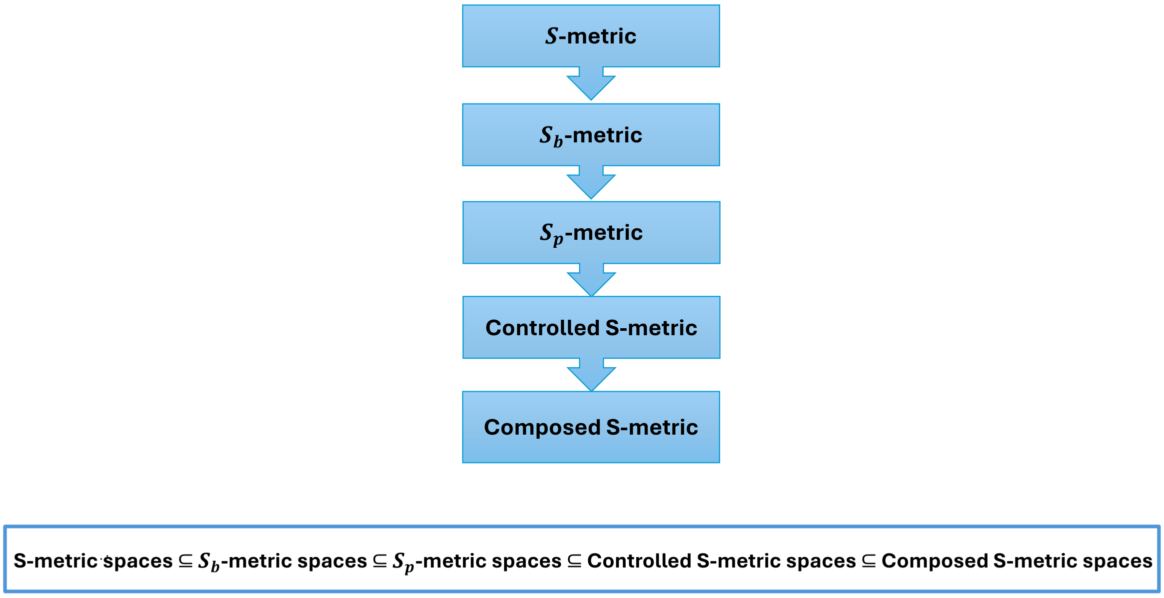

Figure 1. Hierarchical inclusion of S-MS and their generalizations.

Definition 2.3. *Let $(\mathcal{A}, C_S)$ be a CSM and $\{q_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$. Then,*

(i) A sequence $\{q_n\}_{n\in\mathbb{N}}$ is is said to converge to $q \in \mathcal{A}$ if, and only if, $C_S(q_n, q_n, q) \longrightarrow 0$ as $n \longrightarrow \infty$. Equivalently, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $C_S(q_n, q_n, q) < \epsilon$, we write $\lim_{n\longrightarrow\infty} q_n = q$.

(ii) The sequence $\{q_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence if, and only if, $C_S(q_n, q_n, q_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $C_S(q_n, q_n, q_m) < \epsilon$.

(iii) $(\mathcal{A}, C_S)$ is considered a complete CSM if every Cauchy sequence within it converges.

In the following section we explore several FP theorems and an application within this extended framework.

3. FP results

The first principal result presented in this paper is the establishment of a generalized FP theorem for CSM, which builds upon the Banach contraction principle.

Remark 5. *The function α is kept minimally constrained at the definitional level by design: Only $\alpha(0) = 0$ is required in Definition 2.1, so as to define the broadest possible class of CSM. Stronger regularity conditions on α are imposed as explicit hypotheses at the theorem level, only where they are genuinely needed for the estimates to go through. In particular, the subadditivity* *condition $\alpha(ks + t) \leq k\alpha(s) + \alpha(t)$ appearing in Theorems 1–3 is needed to control the iterated applications of α that arise from repeated use of the triangle inequality. Under this condition, one has $\alpha^k(t) \leq c^k \cdot t$ whenever $\alpha(t) \leq ct$, so that each iterated application of α contributes at most a factor of c, keeping all intermediate estimates finite and well-controlled. This separation between the definitional framework and the theorem-level assumptions is analogous to how completeness is a property assumed at the theorem level rather than built into the definition of a metric space.*

Theorem 1. *Let $(\mathcal{A}, C_S)$ be a complete CSM with a control function $\alpha : [0,\infty) \longrightarrow [0,\infty)$. Let $F : \mathcal{A} \to \mathcal{A}$ be a mapping such that:*

$$C_S(Fq, Fh, Fw) \leq rC_S(q,h,w) \quad \text{for all } q,h,w \in \mathcal{A} \text{ and } r \in (0,1). \tag{3.1}$$

For $\varpi_0 \in \mathcal{A}$, define a sequence $\{\varpi_n\}$ by $\varpi_n = F^n\varpi_0$. Suppose that the following assumptions hold:

1. *$\alpha(ks + t) \leq k\alpha(s) + \alpha(t)$ for all positive numbers s, t, and k.*
2. *There exists a constant $c > 0$ such that $\alpha(t) \leq ct$ for all $t \geq 0$, and the joint condition $2cr < 1$ holds.*

Then, F has a unique FP.

Proof. Consider the sequence $\{\varpi_n\} \subset \mathcal{A}$ defined in the statement of the theorem. We have

$$C_S(\varpi_n, \varpi_n, \varpi_{n+1}) = C_S(F^n\varpi_0, F^n\varpi_0, F^n\varpi_1) \leq r^n C_S(\varpi_0, \varpi_0, \varpi_1) \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Let $n, m \in \mathbb{N}$ with $n < m$. Since $\alpha(t) \leq ct$ for all $t \geq 0$, we have $\alpha^k(t) \leq c^k \cdot t$ for all $k \geq 1$, so each iterated application of α is controlled. Repeated application of the triangle inequality yields

$$\begin{aligned}
C_S(\varpi_n, \varpi_n, \varpi_m) &\leq \sum_{j=0}^{m-n-1} 2^j \cdot \alpha^{j+1}\Big(C_S(\varpi_{n+j}, \varpi_{n+j}, \varpi_{n+j+1})\Big) \\
&\leq \sum_{j=0}^{m-n-1} 2^j \cdot c^{j+1} \cdot r^{n+j} C_S(\varpi_0, \varpi_0, \varpi_1) \\
&= c \cdot r^n \cdot C_S(\varpi_0, \varpi_0, \varpi_1) \sum_{j=0}^{\infty} (2cr)^j \\
&= \frac{c \cdot r^n}{1 - 2cr} \cdot C_S(\varpi_0, \varpi_0, \varpi_1),
\end{aligned}$$

where the geometric series converges because $2cr < 1$ by hypothesis. Since $r^n \to 0$ as $n \to \infty$, we conclude that $\lim_{n,m\to\infty} C_S(\varpi_n, \varpi_n, \varpi_m) = 0$, and therefore $\{\varpi_n\}$ is a Cauchy sequence. By condition (2) of the theorem and knowing that $0 < r < 1$ and $\alpha(0) = 0$, we can affirm that $\lim_{n,m\to\infty} C_S(\varpi_n, \varpi_n, \varpi_m) = 0$. Therefore, $\{\varpi_n\}$ is a Cauchy sequence. As $(\mathcal{A}, C_S)$ is a complete CSM, it follows that $\varpi_n \to \varpi \in \mathcal{A}$ that is,

$$\lim_{n\to\infty} C_S(\varpi_n, \varpi_n, \varpi) = 0. \tag{3.3}$$

We will establish that ϖ is an FP of F. From the triangle inequality we deduce

$$C_S(\varpi, \varpi, F\varpi) \leq 2\alpha(C_S(\varpi, \varpi, \varpi_{n+1})) + \alpha(C_S(F\varpi, F\varpi, \varpi_{n+1}))$$

$$
\begin{aligned}
&= 2\alpha(C_S(\varpi,\varpi,\varpi_{n+1})) + \alpha(C_S(F\varpi,F\varpi,F\varpi_n)) \\
&\leq 2\alpha(C_S(\varpi,\varpi,\varpi_{n+1})) + \alpha(rC_S(\varpi,\varpi,\varpi_n)). \qquad (3.4)
\end{aligned}
$$

Taking the limit in inequality (3.4), when $n \longrightarrow \infty$, gives us

$$C_S(\varpi,\varpi,F\varpi) \leq 2\alpha(0) + \alpha(0).$$

Knowing that $\alpha(0) = 0$, we deduce that $C_S(\varpi,\varpi,F\varpi) = 0$, namely, $F\varpi = \varpi$, therefore ϖ is a FP of F.
Let ϖ_1, ϖ_2 be two FPs of F, and we have

$$C_S(\varpi_1,\varpi_1,\varpi_2) = C_S(F\varpi_1,F\varpi_1,F\varpi_2) \leq rC_S(\varpi_1,\varpi_1,\varpi_2).$$

Since $r < 1$, $C_S(\varpi_1,\varpi_1,\varpi_2) = 0$, therefore $\varpi_1 = \varpi_2$. □

Remark 6. *It is worth noting that Theorem 1 is established without assuming the symmetry condition of the composed S-metric.*

Following the preceding results, we extend our analysis to the symmetric case and present the next FP theorem. We emphasize that the proof of the following theorem does not rely on the triangle inequality, which is commonly employed in a large number of related works.

Theorem 2. *Let $(\mathcal{A}, C_S)$ be a complete symmetric CSM with a control function $\alpha : [0,\infty) \longrightarrow [0,\infty)$. Let $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ be a continuous mapping such that:*

$$C_S(\mathcal{T}q,\mathcal{T}h,\mathcal{T}w) \leq \psi(C_S(q,h,w)) \quad \textit{for all } q,h,w \in \mathcal{A}, \qquad (3.5)$$

where $\psi : [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing function such that

$$\lim_{n\to\infty} \psi^n(t) = 0 \quad \textit{for all } t > 0. \qquad (3.6)$$

Moreover, suppose that the following conditions are satisfied:

1. *$\alpha(ks+t) \leq k\alpha(s) + \alpha(t)$ for all positive numbers s, t, and k.*
2. *$C_S(\mathcal{T}^i q,\mathcal{T}^i q,\mathcal{T}^{i+n}q) < \epsilon$ then $\alpha(C_S(\mathcal{T}^i q,\mathcal{T}^i q,\mathcal{T}^{i+n}q)) < \epsilon$ for all $i,n \in \mathbb{N}$ and $q \in \mathcal{A}$.*
3. *$C_S(\mathcal{T}^i w,\mathcal{T}^i w,\mathcal{T}^{(k+1)i}h) < \epsilon$ then $\alpha(C_S(\mathcal{T}^i w,\mathcal{T}^i w,\mathcal{T}^{(k+1)i}h)) < \epsilon$ for all $i,k \in \mathbb{N}$ and $w,h \in \mathcal{A}$.*

Then, $\mathcal{T}$ has a unique FP.

Proof. Let $q \in \mathcal{A}$ and $\epsilon > 0$. By (3.6), let n be a natural number such that $\psi^n(t) \leq \dfrac{\epsilon}{4}$.
Let $F = \mathcal{T}^n$ and $q_k = F^k(q)$ for $k \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
C_S(q_{k+1},q_{k+1},q_k) &= C_S(F^{k+1}q,F^{k+1}q,F^k q) \\
&= C_S(\mathcal{T}^{n(k+1)}q,\mathcal{T}^{n(k+1)}q,\mathcal{T}^{nk}q) \\
&\leq \psi^{nk}(C_S(\mathcal{T}^n q,\mathcal{T}^n q,q)). \qquad (3.7)
\end{aligned}
$$

Since $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, then $\lim_{n\to\infty} \psi^{nk}(C_S(\mathcal{T}^n q,\mathcal{T}^n q,q)) = 0$. Therefore, we obtain that

$$\lim_{k\longrightarrow\infty} C_S(q_{k+1},q_{k+1},q_k) = 0.$$

Therefore, let k be such that

$$C_S(q_{k+1}, q_{k+1}, q_k) < \frac{\epsilon}{4}.$$

Then, using condition 2 of the theorem, we obtain

$$\alpha(C_S(q_{k+1}, q_{k+1}, q_k)) < \frac{\epsilon}{4}. \tag{3.8}$$

Let us define the ball $B(q_k, \epsilon)$ as follows:

$$B(q_k, \epsilon) := \{h \in \mathcal{A} | C_S(q_k, q_k, h) \leq \epsilon\}.$$

Note that $q_k \in B(q_k, \epsilon)$, therefore $B(q_k, \epsilon) \neq \emptyset$. Hence, for all $w \in B(q_k, \epsilon)$, using the properties of ψ, we get

$$\begin{array}{rcl} C_S(Fw, Fw, q_{k+1}) = C_S(Fw, Fw, Fq_k) & = & C_S(Fq_k, Fq_k, Fw) \\ & \leq & C_S(\mathcal{T}^n q_k, \mathcal{T}^n q_k, \mathcal{T}^n w) \\ & \leq & \psi^n(C_S(q_k, q_k, w)). \end{array}$$

From (3.6), there exists $n_0 \in \mathbb{N}$ and $\epsilon > 0$ such that for all $n \geq 0$, we have

$$C_S(Fw, Fw, q_{k+1}) \leq \frac{\epsilon}{2}.$$

Then, by condition 3 of the theorem we obtain

$$\alpha(C_S(Fw, Fw, q_{k+1})) \leq \frac{\epsilon}{2}. \tag{3.9}$$

Using the triangle inequality, we get

$$\begin{array}{rcl} C_S(q_k, q_k, F_w) & \leq & 2\alpha\big(C_S(q_k, q_k, q_{k+1})\big) + \alpha(C_S(Fw, Fw, q_{k+1})) \\ & = & 2\alpha\big(C_S(q_{k+1}, q_{k+1}, q_k)\big) + \alpha(C_S(Fw, Fw, q_{k+1})). \end{array}$$

Using the inequalities (3.8) and (3.9), we obtain

$$C_S(q_k, q_k, F_w) \leq 2\frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

Then, F maps $B(q_k, \epsilon)$ to itself. Since $q_k \in B(q_k, \epsilon)$, we have $Fq_k \in B(q_k, \epsilon)$. By repeating this process, we get

$$F^m q_k \in B(q_k, \epsilon) \text{ for all } m \in \mathbb{N}.$$

That is, for all $l \geq k$, $q_l \in B(q_k, \epsilon)$. Hence,

$$C_S(q_m, q_m, q_l) < \epsilon \text{ for all } m, l > k.$$

Consequently, $\{q_k\}$ is a Cauchy sequence, and by virtue of the completeness of $\mathcal{A}$, there exists $q^* \in \mathcal{A}$ such that $q_k \longrightarrow q^*$ as $k \longrightarrow \infty$. Furthermore, $q^* = \lim_{k\to\infty} q_{k+1} = \lim_{k\to\infty} q_k = F(q^*)$. Thus, F has q as an FP. It remains to establish the uniqueness of the FP for F. Consider q_1 and q_2 to be two fixed points of F.

$$C_S(q_1, q_1, q_2) \quad = \quad C_S(Fq_1, Fq_1, Fq_2)$$

$$\begin{aligned} &= C_S(q_1, q_1, q_2) \\ &\leq \psi^n(q_1, q_1, q_2) \\ &< C_S(q_1, q_1, q_2). \end{aligned}$$

Then, $C_S(q_1, q_1, q_2) = 0 \Longrightarrow q_1 = q_2$ and, hence, F has a unique FP in $\mathcal{A}$.
On the other hand, $\mathcal{T}^{nk+r}(q) = F^k(\mathcal{T}^r(q)) \longrightarrow q^*$ as $k \longrightarrow \infty$. Hence, $\mathcal{T}^m q \longrightarrow q^*$ as $m \longrightarrow \infty$ for every q.
That is, $u = \lim_{m\to\infty} \mathcal{T} q_m = \mathcal{T}(q^*)$. Thereby, $\mathcal{T}$ has a unique FP. □

To derive further FP results in the setting of CSM, we introduce the following class of functions. Let $\mathcal{M}_f$ be the set of all continuous functions $m_f : \mathbb{R}_+^5 \longrightarrow \mathbb{R}_+$ meeting the following conditions for some $r \in [0, 1)$:

(M_1) for all $k, h, w \in \mathbb{R}_+$, if $h \leq m_f(k, k, 0, w, h)$ with $w \leq 2k + h$, then $h \leq r.k$,
(M_2) for all $h \in \mathbb{R}_+$, if $h \leq m_f(h, 0, h, h, 0)$ then $h = 0$.

Theorem 3. *Let $(\mathcal{A}, C_S)$ be a symmetric complete CSM with a control function $\alpha : [0, \infty) \longrightarrow [0, \infty)$. Let $F : \mathcal{A} \to \mathcal{A}$ be a mapping such that for all $k, h \in \mathcal{A}$ and $m_f \in \mathcal{M}_f$:*

$$C_S(Fk, Fk, Fh) \leq m_f(C_S(k, k, h), C_S(Fk, Fk, k), C_S(Fk, Fk, h), C_S(Fh, Fh, k), C_S(Fh, Fh, h)). \quad (3.10)$$

For $\varpi_0 \in \mathcal{A}$, take $\varpi_n = F^n\varpi_0$. Let the following conditions be fulfilled:

i) $\alpha(ks + t) \leq k\alpha(s) + \alpha(t)$ *for all positive numbers s, t, and k.*
ii) $\alpha(C_S(\varpi_n, \varpi_n, \varpi_{n+1})) \leq C_S(\varpi_n, \varpi_n, \varpi_{n+1})$ *for all $n \in \mathbb{N}$.*
iii) $\lim_{n\to\infty} \sum_{k=n+3}^{m-2} 2^{k-n-1}\alpha^{k-n+1}\Big(r^k C_S(\varpi_0, \varpi_0, \varpi_1)\Big) + 2^{m-n-2}\alpha^{m-n-1}\Big(r^m C_S(\varpi_0, \varpi_0, \varpi_1)\Big) = 0.$

Hence, F has a unique FP.

Proof. Let ϖ_n be the sequence defined in the theorem. Using (3.10) and the symmetry of the metric, we get:

$$\begin{aligned} C_S(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+2}) &= C_S(F\varpi_n, F\varpi_n, F\varpi_n) \\ &\leq m_f\Big(C_S(\varpi_n, \varpi_n, \varpi_{n+1}), C_S(\varpi_{n+1}, \varpi_{n+1}, \varpi_n), C_S(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+1}), \\ &\quad C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_n), C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_{n+1})\Big) \\ &= m_f\Big(C_S(\varpi_n, \varpi_n, \varpi_{n+1}), C_S(\varpi_{n+1}, \varpi_{n+1}, \varpi_n), 0, C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_n), \\ &\quad C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_{n+1})\Big). \end{aligned} \quad (3.11)$$

On the other hand, an application of the triangle inequality yields

$$\begin{aligned} C_S(\varpi_n, \varpi_n, \varpi_{n+2}) &\leq \alpha(C_S(\varpi_n, \varpi_n, \varpi_{n+1})) + \alpha(C_S(\varpi_n, \varpi_n, \varpi_{n+1})) + \alpha(C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_{n+1})) \\ &= 2\alpha(C_S(\varpi_n, \varpi_n, \varpi_{n+1})) + \alpha(C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_{n+1})). \end{aligned} \quad (3.12)$$

By applying the property of α to the Eq (3.12), we get

$$C_S(\varpi_n, \varpi_n, \varpi_{n+2}) \leq 2C_S(\varpi_n, \varpi_n, \varpi_{n+1}) + C_S(\varpi_{n+2}, \varpi_{n+2}, \varpi_{n+1}). \quad (3.13)$$

Then, using (3.11) and (3.13), we can affirm that the function m_f satisfies the condition (M_1), that is

$$\begin{aligned} C_S(\varpi_{n+1},\varpi_{n+1},\varpi_{n+2}) &\leq rC_S(\varpi_n,\varpi_n,\varpi_{n+1}) \\ &\leq r^nC_S(\varpi_0,\varpi_0,\varpi_1), \quad \text{for } r\in(0,1). \end{aligned} \tag{3.14}$$

Now, let $n,m\in\mathbb{N}$, $n<m$ by using the triangle inequality similarly to the Theorem 1, and we obtain

$$\begin{aligned} C_S(\varpi_n,\varpi_n,\varpi_m) &\leq \alpha(2C_S(\varpi_n,\varpi_n,\varpi_{n+1}))+\alpha(C_S(\varpi_m,\varpi_m,\varpi_{n+1})) \\ &\leq \alpha(2C_S(\varpi_n,\varpi_n,\varpi_{n+1}))+\alpha\Big(2\alpha(C_S(\varpi_m,\varpi_m,\varpi_{n+2}))+\alpha(C_S(\varpi_{n+1},\varpi_{n+1},\varpi_{n+2}))\Big) \\ &\leq 2\alpha(C_S(\varpi_n,\varpi_n,\varpi_{n+1}))+\alpha^2(C_S(\varpi_{n+1},\varpi_{n+1},\varpi_{n+2}))+2\alpha^2\Big((C_S(\varpi_m,\varpi_m,\varpi_{n+2})\Big) \\ &\leq 2\alpha(C_S(\varpi_n,\varpi_n,\varpi_{n+1}))+\alpha^2(C_S(\varpi_{n+1},\varpi_{n+1},\varpi_{n+2}))+2\alpha^2\Big(2\alpha(C_S(\varpi_m,\varpi_m,\varpi_{n+3})) \\ &+ \alpha(C_S(\varpi_{n+2},\varpi_{n+2},\varpi_{n+3}))\Big) \\ &\leq 2\alpha(C_S(\varpi_n,\varpi_n,\varpi_{n+1}))+\alpha^2(C_S(\varpi_{n+1},\varpi_{n+1},\varpi_{n+2}))+2\alpha^3(C_S(\varpi_{n+2},\varpi_{n+2},\varpi_{n+3})) \\ &+ 2^2\alpha^3\Big(2\alpha(C_S(\varpi_m,\varpi_m,\varpi_{n+3})) \\ &\vdots \\ &\leq 2\alpha(C_S(\varpi_n,\varpi_n,\varpi_{n+1}))+\alpha^2(C_S(\varpi_{n+1},\varpi_{n+1},\varpi_{n+2}))+2\alpha^3(C_S(\varpi_{n+2},\varpi_{n+2},\varpi_{n+3})) \\ &+ \sum_{k=n+3}^{m-2}2^{k-n-2}\alpha^{k-n+1}\Big(C_S(\varpi_k,\varpi_k,\varpi_{k+1})\Big)+2^{m-n-3}\alpha^{m-n-1}\Big(C_S(\varpi_m,\varpi_m,\varpi_{m-1})\Big). \end{aligned}$$

Using (3.14), we get

$$\begin{aligned} C_S(\varpi_n,\varpi_n,\varpi_m) &\leq 2\alpha(r^nC_S(\varpi_0,\varpi_0,\varpi_1))+\alpha^2(r^{n+1}C_S(\varpi_0,\varpi_0,\varpi_1))+2\alpha^3(r^{n+2}C_S(\varpi_0,\varpi_0,\varpi_1)) \\ &+ \sum_{k=n+3}^{m-2}2^{k-n-2}\alpha^{k-n+1}\Big(r^kC_S(\varpi_0,\varpi_k,0_1)\Big)+2^{m-n-3}\alpha^{m-n-1}\Big(r^{m-1}C_S(\varpi_0,\varpi_0,\varpi_1)\Big). \end{aligned}$$

By conditions *iii*) and $\alpha(0)=0$, we can observe that $\lim_{n,m\to\infty} C_S(\varpi_n,\varpi_n,\varpi_m)=0$. Therefore, $\{\varpi_n\}$ is a Cauchy sequence. By virtue of completeness, there is a $\varpi\in\mathcal{A}$ such that $\varpi_n\to\varpi$ when n goes to ∞. Let us prove that ϖ is an FP of F.

$$\begin{aligned} C_S(\varpi_{n+1},\varpi_{n+1},F\varpi) &= C_S(F\varpi_n,F\varpi_n,F\varpi) \\ &\leq m_f\Big(C_S(\varpi_n,\varpi_n,\varpi),C_S(F\varpi_n,F\varpi_n,\varpi_n),C_S(F\varpi_n,F\varpi_n,\varpi), \\ &\quad C_S(F\varpi,F\varpi,\varpi_n),C_S(F\varpi,F\varpi,\varpi)\Big) \\ &\leq m_f\Big(C_S(\varpi_n,\varpi_n,\varpi),C_S(\varpi_{n+1},\varpi_{n+1},\varpi_n),C_S(\varpi_{n+1},\varpi_{n+1},\varpi), \\ &\quad C_S(F\varpi,F\varpi,\varpi_n),C_S(F\varpi,F\varpi,\varpi)\Big). \end{aligned}$$

As $n\longrightarrow\infty$, taking the limit gives us

$$\begin{aligned} C_S(\varpi,\varpi,F\varpi) &\leq m_f\Big(0,0,0,C_S(F\varpi,F\varpi,\varpi),C_S(F\varpi,F\varpi,\varpi)\Big) \\ &= m_f\Big(0,0,0,C_S(\varpi,\varpi,F\varpi),C_S(\varpi,\varpi,F\varpi)\Big). \end{aligned}$$

Given that m_f satisfies the condition (M_1), we have $C_S(\varpi,\varpi,F\varpi) \leq r.0$ which gives us $C_S(\varpi,\varpi,F\varpi)=0$. Therefore, $\varpi = F\varpi$, that is, ϖ is an FP of F.
Let ϖ_1,ϖ_2 be two FPs of F.

$$\begin{aligned} C_S(\varpi_1,\varpi_1,\varpi_2) &= C_S(F\varpi_1,F\varpi_1,F\varpi_2) \qquad (3.15)\\ &\leq m_f\Big(C_S(\varpi_1,\varpi_1,\varpi_2),C_S(F\varpi_1,F\varpi_1,\varpi_1),C_S(F\varpi_1,F\varpi_1,\varpi_2),\\ &\qquad C_S(F\varpi_2,F\varpi_2,\varpi_1),C_S(F\varpi_2,F\varpi_2,\varpi_2)\Big)\\ &= m_f\Big(C_S(\varpi_1,\varpi_1,\varpi_2),C_S(\varpi_1,\varpi_1,\varpi_1),C_S(\varpi_1,\varpi_1,\varpi_2),\\ &\qquad C_S(\varpi_2,\varpi_2,\varpi_1),C_S(\varpi_2,\varpi_2,\varpi_2)\Big)\\ &= m_f\Big(C_S(\varpi_1,\varpi_1,\varpi_2),0,C_S(\varpi_1,\varpi_1,\varpi_2),C_S(\varpi_2,\varpi_2,\varpi_1),0\Big)\\ &= m_f\Big(C_S(\varpi_1,\varpi_1,\varpi_2),0,C_S(\varpi_1,\varpi_1,\varpi_2),C_S(\varpi_1,\varpi_1,\varpi_2),0\Big). \qquad (3.16)\end{aligned}$$

Since m_f satisfies the condition (M_2) and giving (3.16), we obtain that $C_S(\varpi_1,\varpi_1,\varpi_2)=0$ then $\varpi_1=\varpi_2$. □

Corollary 3.1. *By choosing $M(t_1,t_2,t_3,t_4,t_5)=rt_1$ in Theorem 3, we obtain Theorem 1.*

The following corollary is an FP result for Kannan contraction in a CSM.

Corollary 3.2. *Let $(\mathcal{A},C_S)$ be a symmetric complete CSM with a control function $\alpha:[0,\infty)\longrightarrow[0,\infty)$. Let $F:\mathcal{A}\to\mathcal{A}$ be a mapping such that for all $k,h\in\mathcal{A}$ and some $a\in[0,\frac{1}{2})$,*

$$C_S(Fk,Fk,Fh)\leq a[C_S(Fk,Fk,k)+C_S(Fh,Fh,h)]. \qquad (3.17)$$

Hence, F has a unique FP.

Proof. The key idea is using the result of Theorem 3 by selecting the appropriate function m_f. We choose $m_f(k,h,w,s,t)=a.(h+t)$ for some $0<a\leq\frac{1}{2}$ and all $k,h,w,s,t\in\mathbb{R}_+$. Indeed, by choosing for $m_f(k,h,w,s,t)=a.(h+t)$ in the contraction (3.10), we obtain

$$C_S(Fk,Fk,Fh)\leq a[C_S(Fk,Fk,k)+C_S(Fh,Fh,h)].$$

It is easy to see that m_f is continuous. Let us verify that m_f satisfies the conditions (M_1) and (M_2).
For all $k,h,w\in\mathbb{R}_+$ we have $m_f(k,k,0,w,h)=a.(k+h)$. So if $h\leq m_f(k,k,0,w,h)$ with $w\leq 2k+h$, then $h\leq a(k+h)$, which implies that $h\leq\frac{a}{1-a}k$ where $\frac{a}{1-a}<1$.
Therefore, m_f satisfies the condition (M_1).
Next, if $h\leq m_f(h,0,h,h,0)=a.(0+0)=0$, then $h=0$ and (M_2) holds.
Finally, the chosen function $m_f(k,h,w,s,t)=a.(h+t)\in\mathcal{M}_f$, and by applying Theorem 3 it follows that F possesses a unique FP. □

Corollary 3.3. *Let $(\mathcal{A},C_S)$ be a symmetric complete CSM with a control function $\alpha:[0,\infty)\longrightarrow[0,\infty)$. Let $F:\mathcal{A}\to\mathcal{A}$ be a mapping such that for all $k,h\in\mathcal{A}$ and some $h\in[0,1)$,*

$$C_S(Fk,Fk,Fh)\leq a\max[C_S(Fk,Fk,k)+C_S(Fh,Fh,h)]. \qquad (3.18)$$

Hence, F has a unique FP.

Proof. The claim is a direct consequence of Theorem 3 with $m_f(k,h,w,s,t) = h\max\{h,t\}$ for some $h \in [0,1)$ and all $k,h,w,s,t \in \mathbb{R}_+$. Indeed, m_f is continuous.
Let us check the condition (M_1). We have $m_f(k,k,0,w,h) = h\max\{k,h\}$. So, if $h \leq m_f(k,k,0,w,h)$ with $w \leq k+2h$, then $h \leq he$ or $h \leq hh$.
Thus, $h \leq hk$ and F fulfills the condition (M_1).
Next, if $h \leq m_f(h,0,h,h,0) = h\max\{0,0\} = 0$, then $h = 0$ and the condition (M_2) holds.
Finally, $m_f(k,h,w,s,t) = h\max\{h,t\} \in \mathcal{M}_f$ and by Theorem 3, F has a unique FP. □

4. Application

Our application consists in solving the following equation by applying the first FP theorem introduced in this paper.

$$-(m^6-1)v^{m+1} + v^m - m^6 v - 2 = 0, \quad m \in \mathbb{N}, \quad m \geq 3.$$

The equation is reformulated as an equivalent FP problem, and by verifying that the corresponding operator satisfies all the assumptions of the Theorem 1, we deduce the existence and uniqueness of the solution.

Theorem 4. *Let $m \in \mathbb{N}$, for $m \geq 3$, and the following equation*

$$-(m^6-1)v^{m+1} + v^m - m^6 v - 2 = 0 \tag{4.1}$$

has a unique solution in the interval $[0,1]$.

Proof. Let $\mathcal{A} = [0,1]$, and for any $p,s,q \in \mathcal{A}$, define the function $C_S : \mathcal{A}^3 \longrightarrow [0,\infty)$ by

$$C_S(p,s,q) = |p-s| + |s-q|.$$

The composed function $\alpha : [0,\infty) \longrightarrow [0,\infty)$ is given by $\alpha(p) = 2\sqrt{p}$.
First of all, let us showed that $(\mathcal{A}, C_S)$ is a complete CSM. We will show that the triangle inequality holds.
Let $p,s,q,c \in [0,1]$, and we have

$$\begin{aligned} C_S(p,s,q) = |p-s| + |s-q| &= |p-c+c-s| + |s-c+c-q| \\ &\leq |p-c| + 2|s-c| + |q-c|. \end{aligned}$$

Knowing that $p \leq \sqrt{p}$ for all $0 \leq p \leq 1$ and $C_S(p,p,c) = 2|p-c|$, we get

$$\begin{aligned} C_S(p,s,q) &\leq \sqrt{|p-c|} + 2\sqrt{|s-c|} + \sqrt{|q-c|} \\ &\leq 2\sqrt{2|p-c|} + 2\sqrt{2|s-c|} + 2\sqrt{2|q-c|} \\ &= \alpha(2|p-c|) + \alpha(2|s-c|) + \alpha(2|q-c|) \\ &= \alpha(C_S(p,p,c)) + \alpha(C_S(s,s,c)) + \alpha(C_S(q,q,c)). \end{aligned}$$

Then, $([0,1], C_S)$ is a complete CSM.
Define $F : \mathcal{A} \longrightarrow \mathcal{A}$ such that for all $p \in \mathcal{A}$,

$$Fp = \frac{p^m - 2}{(m^6-1)p^m + m^6}. \tag{4.2}$$

Given that $m \geq 3$, we will set $m = 3$ to simplify the computation. However, using this approach, it can be demonstrated that the results are valid for any $m \geq 3$. Therefore, Eq (4.2) takes the form

$$Fp = \frac{p^3 - 2}{728p^3 + 729}. \tag{4.3}$$

We will now show that F meets the contraction (3.5) used in Theorem 1. Indeed,

$$\begin{aligned}
C_S(Fp, Fs, Fq) &= |Fp - Fs| + |Fs - Fq| \\
&= \left|\frac{p^3 - 2}{728p^3 + 729} - \frac{s^3 - 2}{728s^3 + 729}\right| + \left|\frac{s^3 - 2}{728s^3 + 729} - \frac{q^3 - 2}{728q^3 + 729}\right| \\
&= \left|\frac{2185(p^3 - s^3)}{(728p^3 + 729)(728s^3 + 729)}\right| + \left|\frac{2185(s^3 - q^3)}{(728s^3 + 729)(728q^3 + 729)}\right| \\
&\leq \left|\frac{2185(p - s)(p^2 + ps + s^2)}{10 \times 729}\right| + \left|\frac{2185(s - q)(s^2 + sq + q^2)}{10 \times 729}\right| \\
&\leq \left|\frac{2185 \times 3(p - s)}{10 \times 729}\right| + \left|\frac{2185 \times 3(s - q)}{10 \times 729}\right| \\
&\leq 0.9\big(|p - s| + |s - q|\big) \\
C_S(Fp, Fs, Fq) &\leq 0.9\big(C_S(p, s, q)\big).
\end{aligned} \tag{4.4}$$

Thus, F satisfies (3.5) with $r = 0.9$.
Alternatively, the composed function $\alpha(p) = 2\sqrt{p}$ satisfies the condition

$$\alpha(kp + s) \leq k\alpha(p) + \alpha(s) \quad \text{for all } p \in [0, 1]. \tag{4.5}$$

Now, we are left with the second condition of the Theorem 1.

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=n+3}^{m-2} 2^{k-n-1}\alpha^{k-n+1}\Big(r^k C_S(p_0, p_0, p_1)\Big) + 2^{m-n-2}\alpha^{m-n-1}\Big(r^m C_S(p_0, p_0, p_1)\Big) = \\
&\lim_{n\to\infty} \sum_{k=n+3}^{m-2} 2^{k-n-1}\Big(\sqrt{r^k C_S(p_0, p_0, p_1)}\Big)^{k-n+1} + 2^{m-n-2}\Big(\sqrt{r^m C_S(p_0, p_0, p_1)}\Big)^{m-n-1} = \\
&\lim_{n\to\infty} \sum_{k=n+3}^{m-2} 2^{k-n-1}\Big(\sqrt{0.9^k 2|p_0 - p_1|}\Big)^{k-n+1} + 2^{m-n-2}\Big(\sqrt{0.9^m 2|p_0 - p_1|}\Big)^{m-n-1} = 0.
\end{aligned} \tag{4.6}$$

The Eq (4.6) proves that the second hypothesis of Theorem 1 is satisfied.
Finally, thanks to the results (4.4)–(4.6), all the hypotheses of Theorem 1 are satisfied. Therefore, F has a unique FP in $\mathcal{A}$, which is a unique real solution of Eq (4.1). □

5. Conclusions and perspectives

In this paper, we have established the existence and uniqueness of FP in the CSM under appropriate conditions. Our results encompass and generalize some existing results in the literature, offering new insights and tools for research in metric and topological spaces. Through illustrative examples, we have shown the practical relevance and applicability of CSM in solving n-th degree polynomial equations.

The concept of CSM opens up several avenues for future research. First, it would be valuable to explore additional FP theorems within this framework. Another promising direction, in line with recent developments, is to apply the proposed approach to establish the existence of solutions to integral equations, thereby obtaining solutions to the Helmholtz problem, as in [15]. Furthermore, future work could investigate the relationship between CSM and other generalized metric spaces, such as fuzzy or probabilistic MS, to develop a broader unified theory of metrics and their applications.

Use of Generative-AI tools declaration

The author declares that he has not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank Ongoing Research Funding program, (ORF-2026-1726), King Saud University, Riyadh, Saudi Arabia for financial support.

Conflict of interest

No conflict of interest.

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