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Research article

Existence, uniqueness, and stability analysis of fractional order singular boundary value problems

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Abstract: This paper is concerned with a class of singular multi-point boundary value problems (BVPs) subject to integral Riemann–Stieltjes boundary conditions, involving the ϖ-Caputo fractional derivative and a nonlinear p-Laplacian operator. The analysis is performed in the range $1 < p \leq 2$. To support the theoretical framework, suitable estimates for the Green's functions arising in the integral formulation of the problem are derived. The existence of at least one solution is further obtained through the application of Schaefer's fixed-point (FP) theorem. The uniqueness of solutions to the associated nonlinear ϖ-Caputo fractional differential equation is established by means of the Banach contraction principle. In addition, the stability of solutions is investigated in the sense of both Ulam–Hyers and Ulam–Hyers–Rassias. As a result, the study provides a comprehensive treatment of the existence, uniqueness, and stability properties for the considered singular multi-point fractional BVP with integral Riemann–Stieltjes conditions. The theoretical results are complemented by two examples that illustrate the applicability of the developed framework.

Keywords: generalized ϖ-Caputo type fractional derivative; singular boundary value problems; existence; uniqueness and stability results; nonlinear p-Laplacian operator; fixed-point methods
Mathematics Subject Classification: 26A33, 34B15, 34K10, 34K37, 39A10

1. Introduction

Boundary value problems (BVPs) have long been pivotal in the study of differential equations because of their broad applicability across scientific and engineering domains. These problems generally involve solving differential equations subject to specific boundary conditions, which define the behavior of the solution at the domain's boundaries (e.g., [1, 2]).

In recent years, fractional-order differential equations have garnered significant interest, as they provide a more accurate framework for modeling complex phenomena that traditional integer-order models cannot capture (e.g., [3–5]). In particular, the existence and uniqueness of solutions to fractional differential equations with various boundary conditions have been extensively studied (e.g., [6, 7]). Fractional calculus (FC), which generalizes the concepts of differentiation and integration to non-integer orders, has become highly useful in disciplines such as biology, physics, and engineering, as it effectively models systems exhibiting memory-dependent and hereditary dynamics. The introduction of generalized ϖ-Caputo fractional derivatives has further enriched this field, offering greater flexibility for modeling intricate real-world processes (e.g., [8–10]).

Singular BVPs, a subclass of BVPs, present distinct challenges due to the singularities in either the differential operator or the boundary conditions. These singularities complicate the analysis, particularly when proving the existence and stability of solutions (e.g., [11–13]). In addition, singular problems involving eigenvalue structures and generalized integral boundary conditions require more refined analytical tools (e.g., [14–16]). The p-Laplacian operator, a nonlinear generalization of the classical Laplacian, is commonly used in such contexts (e.g., [17, 18]). However, the presence of fractional orders and singularities makes the problem more intricate, requiring advanced analytical techniques.

Further complicating the situation is the inclusion of Riemann–Stieltjes integral boundary conditions, which generalize classical boundary conditions by incorporating the function's behavior over intervals (e.g., [19]). These conditions introduce additional complexity, allowing for more precise modeling of boundary interactions that depend not only on the function's values but also on its history.

Bai [20] examined the existence of positive solutions by utilizing the properties of Green's functions for the following p-Laplacian problem:

$$\begin{aligned}&(\varphi_p(D^{\alpha}_{0^+}\upsilon(\theta))' + f(\theta, \upsilon(\theta)) = 0,\ \theta \in (0,1),\\&\upsilon(0) = D^{\beta}_{0^+}\upsilon(0) = 0, \quad {}^{C}D^{\beta}_{0^+}\upsilon(0) = {}^{C}D^{\beta}_{0^+}\upsilon(1) = 0,\end{aligned}$$

where $0 < \beta < 1$, and $2 < \alpha < \beta + 2$. The notation $D^{\alpha}_{0^+}$ indicates the Riemann–Liouville fractional derivative of order α, and ${}^{C}D^{\beta}_{0^+}$ denotes the Caputo fractional derivative of order β. The mapping φ_p is defined as the p-Laplacian operator for $p > 1$, while f is assumed to be a continuous function on the domain $[0, 1] \times \mathbb{R}$.

Liu et al. [21] employed the method of lower and upper solutions to investigate the existence of solutions for the following BVP:

$$\begin{aligned}&D^{\alpha}_{0^+}(\varphi_p({}^{C}D^{\beta}_{0^+}\upsilon(\theta))) = f(\theta, \upsilon(\theta), {}^{C}D^{\beta}_{0^+}\upsilon(\theta)),\ \theta \in (0,1),\\&{}^{C}D^{\beta}_{0^+}\upsilon(0) = \upsilon'(0) = 0,\\&\upsilon(1) = r_1\upsilon(\eta), \quad {}^{C}D^{\beta}_{0^+}\upsilon(1) = r_2\, {}^{C}D^{\beta}_{0^+}\upsilon(\xi),\end{aligned}$$

where $\alpha > 1$, $\beta \le 2$, and $r_1, r_2 \ge 0$. The operator φ_p denotes the nonlinear p-Laplacian for $p > 1$. The symbol $D^{\alpha}_{0^+}$ represents the Riemann–Liouville fractional derivative of order α, whereas ${}^{C}D^{\beta}_{0^+}$ stands for the Caputo fractional derivative of order β. Moreover, the function f is continuous on $[0, 1] \times [0, +\infty) \times (-\infty, 0]$ with values in $[0, +\infty)$, that is, $f \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$.

Panigrahi and Kumar [19] derived sufficient conditions for the existence and multiplicity of positive solutions by applying fixed-point (FP) index theory and the Avery–Peterson FP theorem to the following class of nonlinear singular fractional differential equations:

$$\begin{aligned} &D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}y(\theta))) + f(\theta, y(\theta), \ldots, y^{(n-2)}(\theta)) = 0, \quad \theta \in (0,1), \\ &y^{(k)}(0) = 0, \quad 0 \le k \le n-2, \quad D_{0^+}^{\alpha}y(0) = 0, \quad D_{0^+}^{\alpha-1}y(1) = \int_0^1 D_{0^+}^{\alpha-1}y(\theta)dA(\theta), \end{aligned}$$

where φ_p is the p-Laplacian operator with $p > 1$, and φ_q is the inverse of φ_p with $\frac{1}{p} + \frac{1}{q} = 1$. Additionally, $0 < \beta \le 1$, $n-1 < \alpha \le n$ with $n \ge 3$, $A : [0,1] \to \mathbb{R}$ is a function of bounded variation, and f may exhibit singularities at $\theta = 0$.

Cheng and Xu [22] studied the existence of at least one, two, and three positive solutions for the following BVP for a singular fractional differential equation with a generalized Laplacian and positive parameter:

$$\begin{aligned} &D_{0^+}^{\beta}(\varphi(D_{0^+}^{\alpha}y(\theta))) = \lambda h(\theta) f(y(\theta)), \ \ \theta \in (0,1), \\ &y(0) = y'(0) = y(1) = 0, \qquad \varphi(D_{0^+}^{\alpha}y(0)) = (\varphi(D_{0^+}^{\alpha}y(1)))' = 0, \end{aligned}$$

where $1 < \beta \le 2$, $2 < \alpha \le 3$, $\lambda > 0$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ denote the standard Riemann–Liouville fractional derivatives, $h : (0,1) \to [0,\infty)$ is locally integrable, $f : [0,\infty) \to [0,\infty)$ is continuous, and Φ is an odd increasing homeomorphism.

Despite the substantial progress in the study of fractional BVPs, the existing literature remains limited in providing a unified framework that simultaneously incorporates generalized fractional operators, singular nonlinear structures, and highly nonlocal boundary conditions of integral type. Although various studies have addressed these aspects separately, their interplay remains insufficiently explored, particularly in the presence of nonlinear p-Laplacian operators.

To address this gap, the present study investigates the existence, uniqueness, and stability of solutions to a class of ϖ-Caputo fractional-order singular p-Laplacian BVPs subject to Riemann–Stieltjes integral boundary conditions. The analysis is carried out by employing FP techniques together with suitable a priori estimates, providing a rigorous theoretical framework for such problems. The results of this study contribute to the development of fractional differential equations with singularities and complex boundary conditions, offering potential insights for applications in nonlinear analysis, mathematical physics, and applied mathematics.

Motivated by the above considerations, we study the following generalized ϖ-Caputo fractional p-Laplacian BVP:

$$\begin{aligned} &{}^{C}D_{a^+}^{\alpha,\varpi}(\varphi_p(-{}^{C}D_{a^+}^{\beta,\varpi}\vartheta(\theta))) + f(\theta, \vartheta(\theta)) = 0, \ \ \theta \in (a,b), \\ &u^{(k)}(a) = 0, \qquad k = 0,1,...,n-2, \\ &{}^{C}D_{a^+}^{\delta_0,\varpi}\vartheta(b) = -\sum_{j=1}^{m}\int_a^b {}^{C}D_{a^+}^{\delta_j,\varpi}\vartheta(\eta)\, dA_j(\eta), \\ &{}^{C}D_{a^+}^{\beta,\varpi}\vartheta(a) = 0, \\ &\varphi_p({}^{C}D_{a^+}^{\beta,\varpi}\vartheta(b)) = \sum_{j=1}^{d}\lambda_j\varphi_p({}^{C}D_{a^+}^{\beta,\varpi}\vartheta(\gamma_j)), \end{aligned} \tag{1.1}$$

where $\alpha \in (1,2]$, $\beta \in (n-1,n]$, $n \in \mathbb{N}$, $n \ge 3$, $\delta_j, \lambda_j \in \mathbb{R}^+$, $\sum_{j=1}^{d}\lambda_j < 1$, $\delta_0 \le \delta_j < \beta - 1$, the integral in the boundary condition is a Riemann–Stieltjes integral with respect to A_j functions of bounded variation for $j = 0, 1, ..., m$, $\gamma_j \in (a, b)$, $\varphi_p(k) = |k|^{p-2}k$ with $p > 1$ and $\varphi_p^{-1} = \varphi_r,\ r = \frac{p}{p-1}$, the function f may be singular at $\theta = a$ and/or $\theta = b$, and ${}^C D_{a^+}^{\alpha,\varpi}, {}^C D_{a^+}^{\beta,\varpi}, {}^C D_{a^+}^{\delta_j,\varpi}$ denote the generalized ϖ-Caputo derivative of order α, β, δ_j, respectively.

To better highlight the novelty of the present work, we provide a direct comparison with some closely related studies.

In [20], Bai studied the existence of positive solutions using Green's function techniques for p-Laplacian problems involving Riemann–Liouville and Caputo derivatives. Nevertheless, the considered model does not include generalized fractional operators nor integral boundary conditions of Riemann–Stieltjes type, both of which are essential components of our formulation.

Similarly, Liu et al. [21] investigated fractional p-Laplacian BVPs by employing the method of lower and upper solutions. However, their model is based on classical Caputo derivatives and standard multi-point boundary conditions. In contrast, our work incorporates the generalized ϖ-Caputo fractional derivative together with Riemann–Stieltjes integral boundary conditions, which significantly enhance the nonlocal structure of the problem.

On the other hand, compared to existing works such as [19], the present study provides a unified framework combining singular nonlinearities, generalized fractional operators, nonlinear p-Laplacian structures, and highly nonlocal integral boundary conditions. To the best of our knowledge, such a combination has not been investigated in the literature.

Moreover, Cheng and Xu [22] focused on multiplicity results for singular fractional problems with generalized Laplacian operators. Their approach is restricted to standard fractional derivatives and does not address stability properties. In contrast, our study not only considers a more general fractional operator but also establishes Ulam–Hyers and Ulam–Hyers–Rassias stability results.

The mathematical framework presented in problem (1.1) is motivated by the increasing necessity to model complex physical phenomena that exhibit both nonlinear diffusion and multi-scale memory effects. The integration of the p-Laplacian operator φ_p is particularly significant in the study of non-Newtonian fluid mechanics and glaciology, where it characterizes power-law rheology and gas filtration in porous media, as pioneered by Leibenson [23]. By employing the ϖ-Caputo fractional derivative, this model extends the classical Caputo and Caputo–Hadamard frameworks, providing a generalized approach to describe systems in which the memory effect is governed by a non-uniform time scale or heterogeneous medium properties (e.g., [24]). The inclusion of Riemann–Stieltjes integral boundary conditions aligns with modern engineering requirements for nonlocal feedback control systems. Such conditions are essential in advanced thermostat models and smart material stabilizers, where boundary data is obtained via weighted averages across the domain rather than at discrete points (e.g., [2, 7]). This study specifically addresses the existence and uniqueness of solutions to such systems, even in the presence of singular nonlinearities f, which frequently occur in chemical kinetics and boundary-layer theory. Consequently, the results established herein bridge the gap between abstract FC and the practical requirements of nonlinear rheological modeling, offering a robust theoretical foundation for analyzing complex viscoelastic materials.

In this paper, we will always assume that the following conditions hold:

(H1) $f \in C((a, b) \times \mathbb{R}, \mathbb{R})$, there exist $q \in L^1(a, b)$ and $g \in C(\mathbb{R}, \mathbb{R}^+)$ such that $|f(\theta, \vartheta)| \leq q(\theta)g(\vartheta)$ for $(\theta, \vartheta) \in (a, b) \times \mathbb{R}$ with $q_0 := \int_a^b \varpi'(s)q(s)ds < \infty$.

(H2) There exists $r \in L^1(a, b)$ such that $|f(\theta, \vartheta_2) - f(\theta, \vartheta_1)| \leq r(\theta)|\vartheta_2 - \vartheta_1|$ for $(\theta, \vartheta_i) \in (a, b) \times \mathbb{R}$ for $i = 1, 2$ with $r_0 := \int_a^b \varpi'(s)r(s)ds < \infty$.

These assumptions ensure the well-definedness of the operator and allow the application of FP theorems.

In Section 2, we introduce the fundamental concepts, key lemmas, and important theorems that lay the foundation for subsequent analysis. This section also focuses on the formulation of Green's functions, which play a crucial role in solving the BVP at hand, and presents the main results pertinent to the problem under investigation. Section 3 is dedicated to the application of FP theorems, where we derive both existence and uniqueness results for the proposed fractional-order singular p-Laplacian BVP. In Section 4, we shift our focus to the stability analysis, exploring various forms of stability to assess the long-term behavior of the solutions. This includes an in-depth examination of how solutions behave under different parameter variations and boundary conditions, as well as a discussion on the robustness of the model in the context of real-world applications. To showcase the practical relevance of the theoretical results, Section 5 provides several numerical examples, offering concrete illustrations of how the theoretical findings can be applied to fractional-order singular p-Laplacian BVPs. Finally, the paper concludes with a concise summary of the key findings, followed by a discussion of potential avenues for future research, including open problems and possible extensions of the current work.

2. Preliminaries

In this section, we present a concise survey of the existing literature on fractional differential equations to place our study within the broader research context. For the sake of completeness and clarity, we also introduce the fundamental definitions, notations, auxiliary lemmas, and essential results concerning the ϖ-Caputo fractional derivative that will be employed throughout the paper. These preliminary materials form the theoretical basis for the subsequent analysis and ensure the self-contained nature of our exposition.

Definition 2.1. *[4,24] Let $\varrho > 0$ be the order of integration. For an integrable function $\vartheta : [a,b] \to \mathbb{R}$, the left-sided ϖ-Riemann–Liouville fractional integral with respect to another function $\varpi : [a,b] \to \mathbb{R}$ (where ϖ is increasing and differentiable with $\varpi'(\theta) \neq 0$ on J) is defined as*

$$I_{a+}^{\varrho,\varpi}\vartheta(\theta) = \frac{1}{\Gamma(\varrho)} \int_a^\theta \varpi'(s) \left[\varpi(\theta) - \varpi(s)\right]^{\varrho-1} \vartheta(s)\, ds, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the classical Euler gamma function.

It is important to note that the operator defined in Eq (2.1) serves as a generalization for several well-known fractional integrals. Specifically, this operator reduces to the Riemann–Liouville fractional integral when the kernel function is defined as $\varpi(\theta) = \theta$, and it simplifies to the Hadamard fractional integral in the case where $\varpi(\theta) = \ln\theta$. By adjusting this underlying function, the operator provides a unified framework for treating these distinct fractional types.

Definition 2.2. *[24] Let $n \in \mathbb{N}$ and consider two functions $\vartheta, \varpi \in C^n([a,b], \mathbb{R})$ such that ϖ is increasing with $\varpi'(\theta) \neq 0$ for all $\theta \in [a,b]$. The left-sided ϖ-Caputo fractional derivative of ϑ of order $\varrho > 0$ is defined as*

$$^{C}D_{a^+}^{\varrho,\varpi}\vartheta(\theta) = I_{a^+}^{n-\varrho,\varpi}\left[\vartheta_{\varpi}^{[n]}(\theta)\right], \tag{2.2}$$

where the ceiling index n is given by $n = \lceil\varrho\rceil$. To facilitate the notation, we introduce the generalized *n-th order derivative operator:*

$$\vartheta_{\varpi}^{[n]}(\theta) := \left(\frac{1}{\varpi'(\theta)}\frac{d}{d\theta}\right)^{n} \vartheta(\theta).$$

In an explicit integral form, the ϖ-Caputo derivative (2.2) *is expressed as*

$$^{C}D_{a^+}^{\varrho,\varpi}\vartheta(\theta) = \begin{cases} \dfrac{1}{\Gamma(n-\varrho)}\displaystyle\int_a^\theta \varpi'(s)\,(\varpi(\theta)-\varpi(s))^{n-\varrho-1}\,\vartheta_{\varpi}^{[n]}(s)\,ds, & \varrho \notin \mathbb{N}, \\ \vartheta_{\varpi}^{[n]}(\theta), & \varrho \in \mathbb{N}. \end{cases} \tag{2.3}$$

The flexibility of this operator allows it to generalize several standard fractional derivatives by selecting an appropriate kernel function. Specifically, it reduces to the Caputo derivative when $\varpi(\theta) = \theta$, and it yields the Caputo–Hadamard derivative when $\varpi(\theta) = \ln\theta$. This unifying property demonstrates the operator's versatility in adapting to various logarithmic or power-law behaviors within the framework of FC.

Furthermore, for a function $\vartheta \in C^n([a,b],\mathbb{R})$, the ϖ-Caputo derivative of order ϱ can be related to the ϖ-Riemann–Liouville derivative $D_{a^+}^{\varrho;\varpi}$ through the following expression

$$^{C}D_{a^+}^{\varrho,\varpi}\vartheta(\theta) = D_{a^+}^{\varrho,\varpi}\left[\vartheta(\theta) - \sum_{k=0}^{n-1}\frac{\vartheta_{\varpi}^{[k]}(a)}{k!}\,(\varpi(\theta)-\varpi(a))^k\right].$$

Lemma 2.3. *[24] Let $\varrho > 0$ be the order of fractional differentiation and $n = \lceil\varrho\rceil$. For an increasing function $\varpi(\theta)$ with continuous derivative $\varpi'(\theta) \neq 0$ in $[a,b]$, the following properties of the ϖ-fractional operators are established:*

1) *For any continuous function $\vartheta \in C([a,b],\mathbb{R})$, the ϖ-Caputo fractional derivative acts as the left-inverse of the ϖ-Riemann–Liouville fractional integral:*

$$^{C}D_{a^+}^{\varrho,\varpi} I_{a^+}^{\varrho,\varpi}\vartheta(\theta) = \vartheta(\theta), \quad \forall\theta \in [a,b].$$

2) *If $\vartheta \in C^n([a,b],\mathbb{R})$, the composition of the integral with the ϖ-Caputo derivative satisfies the following identity:*

$$I_{a^+}^{\varrho,\varpi}\,{}^{C}D_{a^+}^{\varrho,\varpi}\vartheta(\theta) = \vartheta(\theta) - \sum_{k=0}^{n-1}\frac{[\varpi(\theta)-\varpi(a)]^k}{k!}\vartheta_{\varpi}^{[k]}(a), \quad \theta \in [a,b],$$

where $\vartheta_{\varpi}^{[k]}(a)$ denotes the k-th order generalized ϖ-derivative.

3) *The operator $I_{a^+}^{\varrho;\varpi}$ is a linear and bounded mapping from $C([a,b],\mathbb{R})$ into itself.*

Lemma 2.4. *[18] Let $\varphi_p : \mathbb{R} \to \mathbb{R}$ be the p-Laplacian operator defined as $\varphi_p(\theta) = |\theta|^{p-2}\theta$. The following properties are satisfied:*

1) *Differentiation: If $1 < p < 2$ and $\theta \neq 0$, then the derivative is given by*

$$\varphi_p'(\theta) = (p-1)|\theta|^{p-2}.$$

2) *Lipschitz property: If $1 < p < 2$, $\theta z > 0$ and $|\theta|, |z| \geq l > 0$, then*

$$|\varphi_p(\theta) - \varphi_p(z)| \leq (p-1)l^{p-2}|\theta - z|.$$

3) *Lipschitz property: If $p > 2$ and $|\theta|, |z| \leq L$, we have*

$$|\varphi_p(\theta) - \varphi_p(z)| \leq (p-1)L^{p-2}|\theta - z|.$$

4) *Inversion and duality: The operator φ_p is invertible, and its inverse is the q-Laplacian operator, $\varphi_p^{-1}(\theta) = \varphi_q(\theta)$, where q is defined by the relation $\frac{1}{p} + \frac{1}{q} = 1$.*

Lemma 2.5. *[24] Let $\alpha \geq 0$, $\beta > 0$, and $\theta > a$. For an increasing function $\varpi(\theta)$ on $[a, b]$, the following identities for ϖ-fractional operators are valid:*

1) *The ϖ-Riemann–Liouville integral maps generalized ϖ- functions as follows:*

$$I_{a^+}^{\alpha,\varpi}[\varpi(\theta) - \varpi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\varpi(\theta) - \varpi(a)]^{\alpha+\beta-1}.$$

2) *Provided that $\beta > \alpha$, the ϖ-Caputo derivative satisfies:*

$${}^C D_{a^+}^{\alpha,\varpi}[\varpi(\theta) - \varpi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}[\varpi(\theta) - \varpi(a)]^{\beta-\alpha-1}.$$

3) *Let $n = \lceil \alpha \rceil$. For every $k \in \{0, 1, \ldots, n-1\}$, the derivative of the basis functions is zero:*

$${}^C D_{a^+}^{\alpha,\varpi}[\varpi(\theta) - \varpi(a)]^k = 0.$$

4) *The integral operators satisfy the index law for any suitable function $\vartheta(\theta)$:*

$$I_{a^+}^{\alpha,\varpi}\left(I_{a^+}^{\beta,\varpi}\vartheta(\theta)\right) = I_{a^+}^{\alpha+\beta,\varpi}\vartheta(\theta).$$

The following FP theorems are essential tools utilized to establish the validity of our primary results.

Theorem 2.6. *(Schaefer's FP theorem) [25] Let X be a Banach space (BS), and $T : X \to X$ be a continuous and compact mapping. Suppose that the set*

$$\mathcal{E} = \{\vartheta \in X : \vartheta = \lambda T\vartheta \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Then, the operator T has at least one FP in X.

Theorem 2.7. *(Banach FP theorem) [26] Let K be a closed subset of a BS X, and $T : K \to K$ be a contraction mapping. Then, T admits a unique FP $\vartheta^* \in K$ such that $T\vartheta^* = \vartheta^*$.*

We now proceed to characterize the Green's function for the considered problem. The following lemmas detail the explicit construction of this function and highlight its qualitative property.

Lemma 2.8. *Let $1 < \alpha \leq 2$, $n-1 < \beta \leq n$, and $h \in C([a, b], \mathbb{R})$. We consider the following p-Laplacian ϖ-Caputo fractional differential equation:*

$${}^C D_{a^+}^{\alpha,\varpi}(\varphi_p(-{}^C D_{a^+}^{\beta,\varpi}\vartheta(\theta))) + h(\theta) = 0, \quad \theta \in (a, b) \tag{2.4}$$

with boundary conditions

$$\vartheta^{(k)}(a) = 0, \quad k = 0, 1, ..., n-2, \ {}^{C}D^{\delta_0,\varpi}_{a^+}\vartheta(b) = -\sum_{j=1}^{m}\int_a^b {}^{C}D^{\delta_j,\varpi}_{a^+}\vartheta(\eta)dA_j(\eta), \tag{2.5}$$

$$ {}^{C}D^{\beta,\varpi}_{a^+}\vartheta(a) = 0, \ \ \varphi_p({}^{C}D^{\beta,\varpi}_{a^+}\vartheta(b)) = \sum_{j=1}^{d}\lambda_j\varphi_p({}^{C}D^{\beta,\varpi}_{a^+}\vartheta(\gamma_j)). \tag{2.6}$$

We denote

$$\Delta_1 := \varpi(b) - \varpi(a) - \sum_{j=1}^{d}\lambda_j(\varpi(\gamma_j) - \varpi(a)) \neq 0, \tag{2.7}$$

and

$$\Delta_2 := \frac{(n-1)!}{\Gamma(n-\delta_0)}[\varpi(b)-\varpi(a)]^{n-\delta_0-1} + \sum_{j=1}^{m}\int_a^b \frac{(n-1)!}{\Gamma(n-\delta_j)}[\varpi(\eta)-\varpi(a)]^{n-\delta_j-1}dA_j(\eta) \neq 0. \tag{2.8}$$

Then the solution $\vartheta(\theta) \in C([a,b],\mathbb{R})$ *of the problem* (2.4) − (2.6) *can be represented as follows:*

$$\vartheta(\theta) = \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\widetilde{h}(s)ds\Big)dA_j(\eta)$$
$$+ \int_a^b \varpi'(s)H(\theta,s)\widetilde{h}(s)ds, \tag{2.9}$$

where $H(\theta, s)$ *and* $\widetilde{h}(\theta)$ *are given by*

$$H(\theta,s) = \begin{cases} \dfrac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2\Gamma(\beta-\delta_0)}(\varpi(b)-\varpi(s))^{\beta-\delta_0-1} - \dfrac{(\varpi(\theta)-\varpi(s))^{\beta-1}}{\Gamma(\beta)}, & s \leq \theta, \\ \\ \dfrac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2\Gamma(\beta-\delta_0)}(\varpi(b)-\varpi(s))^{\beta-\delta_0-1}, & s \geq \theta, \end{cases} \tag{2.10}$$

and

$$\widetilde{h}(\theta) = \varphi_r\Big(\frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(s)G(\theta,s)h(s)ds\Big)$$

such that

$$G(\theta,s) = G_1(\theta,s) + \frac{\varpi(\theta)-\varpi(a)}{\Delta_1}\sum_{j=1}^{d}\lambda_j G_1(\gamma_j,s) \tag{2.11}$$

with

$$G_1(\theta,s) = \begin{cases} (\varpi(\theta)-\varpi(a))\dfrac{(\varpi(b)-\varpi(s))^{\alpha-1}}{\varpi(b)-\varpi(a)} - (\varpi(\theta)-\varpi(s))^{\alpha-1}, & s \leq \theta, \\ \\ (\varpi(\theta)-\varpi(a))\dfrac{(\varpi(b)-\varpi(s))^{\alpha-1}}{\varpi(b)-\varpi(a)}, & s \geq \theta. \end{cases} \tag{2.12}$$

Proof. Let $\widetilde{\vartheta}(\theta) := \varphi_p(-{}^C D_{a^+}^{\beta,\varpi}\vartheta(\theta))$. Then the BVP (2.4), (2.6) can be written as follows:

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\varpi}\widetilde{\vartheta}(\theta) &= -h(\theta), \quad \theta \in (a,b), \\ \widetilde{\vartheta}(a) = 0, \quad \widetilde{\vartheta}(b) &= \sum_{j=1}^{d} \lambda_j \widetilde{\vartheta}(\gamma_j). \end{aligned}$$

Applying the integral $I_{a^+}^{\alpha,\varpi}$ to both sides, we obtain

$$\widetilde{\vartheta}(\theta) = c_0 + c_1(\varpi(\theta) - \varpi(a)) - \frac{1}{\Gamma(\alpha)} \int_a^\theta \varpi'(s)(\varpi(\theta) - \varpi(s))^{\alpha-1} h(s)ds,$$

where c_0 and c_1 are constants. Using the first boundary condition $\widetilde{\vartheta}(a) = 0$, we obtain that $c_0 = 0$. Using the second boundary condition, we get the following result:

$$c_1 = \frac{1}{\Delta_1 \Gamma(\alpha)} \Big(\int_a^b \varpi'(s)(\varpi(b) - \varpi(s))^{\alpha-1} h(s)ds - \sum_{j=1}^{d} \lambda_j \int_a^{\gamma_j} \varpi'(s)(\varpi(\gamma_j) - \varpi(s))^{\alpha-1} h(s)ds \Big),$$

where Δ_1 is in (2.7). Thus, we have

$$\begin{aligned} \widetilde{\vartheta}(\theta) = \frac{1}{\Gamma(\alpha)} \Big(&\int_a^\theta \varpi'(s)[\varpi(b) - \varpi(s))^{\alpha-1} \frac{\varpi(\theta) - \varpi(a)}{\varpi(b) - \varpi(a)} - (\varpi(\theta) - \varpi(s))^{\alpha-1}] h(s)ds \\ &+ \int_\theta^b \varpi'(s)[\varpi(b) - \varpi(s))^{\alpha-1} \frac{\varpi(\theta) - \varpi(a)}{\varpi(b) - \varpi(a)} h(s)ds \Big) \\ + \frac{1}{\Delta_1 \Gamma(\alpha)} \Big(&\sum_{j=1}^{d} \lambda_j (\varpi(\gamma_j) - \varpi(a)) \int_\theta^b \varpi'(s)[\varpi(b) - \varpi(s))^{\alpha-1} \frac{\varpi(\theta) - \varpi(a)}{\varpi(b) - \varpi(a)} h(s)ds \\ &- \sum_{j=1}^{d} \lambda_j (\varpi(b) - \varpi(a)) \int_a^{\gamma_j} \varpi'(s)[\varpi(\gamma_j) - \varpi(s))^{\alpha-1} \frac{\varpi(\theta) - \varpi(a)}{\varpi(b) - \varpi(a)} h(s)ds \Big). \end{aligned}$$

Rewriting the above expression, we obtain

$$\widetilde{\vartheta}(\theta) = \frac{1}{\Gamma(\alpha)} \int_a^b \varpi'(s) G(\theta, s) h(s)ds,$$

such that

$$G(\theta, s) = G_1(\theta, s) + \frac{\varpi(\theta) - \varpi(a)}{\Delta_1} \sum_{j=1}^{d} \lambda_j G_1(\mu_j, s),$$

where $G_1(\theta, s)$ is defined in (2.12). Since $\varphi_p(-{}^C D_{a^+}^{\beta,\varpi}\vartheta(\theta)) = \widetilde{\vartheta}(\theta)$, using the inverse operator property and defining

$$\varphi_r(\widetilde{\vartheta}(\theta)) = \varphi_r\Big(\frac{1}{\Gamma(\alpha)} \int_a^b \varpi'(s) G(\theta, s) h(s)ds \Big) := \widetilde{h}(\theta),$$

with the boundary condition (2.5), the following BVP is obtained:

$$ {}^{C}D_{a^+}^{\beta,\varpi}\vartheta(\theta) = \widetilde{h}(\theta), \quad \theta \in (a,b), \tag{2.13}$$

$$ \vartheta^{(k)}(a) = 0, \quad k = 0,1,...,n-2, \tag{2.14}$$

$$ {}^{C}D_{a^+}^{\delta_0,\varpi}\vartheta(b) = -\sum_{j=1}^{m}\int_a^b {}^{C}D_{a^+}^{\delta_j,\varpi}\vartheta(\eta)dA_j(\eta). \tag{2.15}$$

Similarly, applying the integral $I_{a^+}^{\beta,\varpi}$ to both sides of the equation (2.13), we get

$$ \vartheta(\theta) = d_0 + d_1(\varpi(\theta)-\varpi(a)) + d_2(\varpi(\theta)-\varpi(a))^2 + ... + d_{n-1}(\varpi(\theta)-\varpi(a))^{n-1} \\ -\frac{1}{\Gamma(\beta)}\int_a^\theta \varpi'(s)(\varpi(\theta)-\varpi(s))^{\beta-1}\widetilde{h}(s)ds.$$

Using the boundary conditions (2.14), it yields that $d_0 = d_1 = ... = d_{n-2} = 0$. Then,

$$ \vartheta(\theta) = d_{n-1}(\varpi(\theta)-\varpi(a))^{n-1} - I_{a^+}^{\beta,\varpi}\widetilde{h}(\theta). \tag{2.16}$$

When taking the ϖ-Caputo fractional derivative of (2.16)

$$ {}^{C}D_{a^+}^{\delta_0,\varpi}\vartheta(\theta) = d_{n-1}\frac{(n-1)!}{\Gamma(n-\delta_0)}(\varpi(\theta)-\varpi(a))^{n-\delta_0-1} - \frac{1}{\Gamma(\beta-\delta_0)}\int_a^\theta \varpi'(s)(\varpi(\theta)-\varpi(s))^{\beta-\delta_0-1}\widetilde{h}(s)ds$$

and from the boundary condition (2.15), we find

$$ d_{n-1} = \frac{1}{\Delta_2}\Big(\frac{1}{\Gamma(\beta-\delta_0)}\int_a^b \varpi'(s)(\varpi(\theta)-\varpi(s))^{\beta-\delta_0-1}\widetilde{h}(s)ds \\ +\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\widetilde{h}(s)dsdA_j(\eta)\Big),$$

where Δ_2 is in (2.8). If we plug d_{n-1} in (2.16), we can represent the solution of the problem (2.4) – (2.6) as follows:

$$ \vartheta(\theta) = \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\widetilde{h}(s)ds\Big)dA_j(\eta) \\ +\int_a^b \varpi'(s)H(\theta,s)\widetilde{h}(s)ds,$$

where $H(\theta,s)$ is in (2.10). □

Corollary 2.9. $\Delta_1 > 0$ *and* $\Delta_2 > 0$ *in* (2.7) *and* (2.8)*, respectively.*

Proof. Since $1 - \sum_{j=1}^{d}\lambda_j > 0$, we can easily see that $\Delta_1 > 0$. In addition, it follows that $\Delta_2 > 0$ from the definition of Δ_2. □

Lemma 2.10. *The following properties are provided for Green's functions $G(\theta, s)$ and $H(\theta, s)$:*
(i) $G(\theta, s)$ and $H(\theta, s)$ are continuous on $[a, b] \times [a, b]$.
(ii) For all $s, \theta \in [a, b]$, $|G(\theta, s)| \leq R_1$ and $H(\theta, s) \leq R_2$ such that

$$R_1 = 2(\varpi(b) - \varpi(a))^{\alpha-1}(1 - \sum_{j=1}^{d} \lambda_j)^{-1}$$

and

$$R_2 = \frac{(\varpi(b) - \varpi(a))^{n+\beta-\delta_0-2}}{\Delta_2\Gamma(\beta - \delta_0)} + \frac{(\varpi(b) - \varpi(a))^{\beta-1}}{\Gamma(\beta)}.$$

Proof. (i) Due to the continuity of ϖ and the function $G_1(\theta, s)$ in (2.12) being continuous, $G(\theta, s)$ and $H(\theta, s)$ in (2.11) and (2.10), respectively, are continuous in $[a, b] \times [a, b]$.
(ii) For $s \leq \theta$, we know that

$$G_1(\theta, s) = (\varpi(\theta) - \varpi(a))\frac{(\varpi(b) - \varpi(s))^{\alpha-1}}{(\varpi(b) - \varpi(a))} - (\varpi(\theta) - \varpi(s))^{\alpha-1}.$$

Then, we have

$$\begin{aligned}|G_1(\theta, s)| =& \left|(\varpi(\theta) - \varpi(a))\frac{(\varpi(b) - \varpi(s))^{\alpha-1}}{(\varpi(b) - \varpi(a))} - (\varpi(\theta) - \varpi(s))^{\alpha-1}\right| \\ \leq& (\varpi(b) - \varpi(a))\frac{(\varpi(b) - \varpi(a))^{\alpha-1}}{(\varpi(b) - \varpi(a))} + (\varpi(b) - \varpi(a))^{\alpha-1} \\ =& 2(\varpi(b) - \varpi(a))^{\alpha-1}.\end{aligned}$$

Similarly, for $s \geq \theta$, we get

$$|G_1(\theta, s)| \leq (\varpi(b) - \varpi(a))^{\alpha-1} \leq 2(\varpi(b) - \varpi(a))^{\alpha-1}.$$

Thus, we have

$$|G_1(\theta, s)| \leq 2(\varpi(b) - \varpi(a))^{\alpha-1},$$

for all $s, \theta \in [a, b]$. Now, if we substitute Green's function into the equation, then we achieve

$$G(\theta, s) = G_1(\theta, s) + \frac{\varpi(\theta) - \varpi(a)}{\Delta_1} \sum_{j=1}^{d} \lambda_j G_1(\gamma_j, s).$$

By arranging the equation, we find that

$$|G(\theta, s)| \leq \frac{2(\varpi(b) - \varpi(a))^{\alpha-1}}{1 - \sum_{j=1}^{d} \lambda_j} = R_1.$$

Similarly, we proceed with the function $H(\theta, s)$. For $s \le \theta$, we get

$$\begin{aligned}|H(\theta, s)| =&\left|\frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2\Gamma(\beta-\delta_0)}(\varpi(b)-\varpi(a))^{\beta-\delta_0-1}-\frac{(\varpi(\theta)-\varpi(s))^{\beta-1}}{\Gamma(\beta)}\right|\\ \le&\frac{(\varpi(b)-\varpi(a))^{n-1}}{\Delta_2\Gamma(\beta-\delta_0)}(\varpi(b)-\varpi(a))^{\beta-\delta_0-1}+\frac{(\varpi(b)-\varpi(a))^{\beta-1}}{\Gamma(\beta)}\\ \le&\frac{(\varpi(b)-\varpi(a))^{n+\beta-\delta_0-2}}{\Delta_2\Gamma(\beta-\delta_0)}+\frac{(\varpi(b)-\varpi(a))^{\beta-1}}{\Gamma(\beta)}=R_2,\end{aligned}$$

Easily, we get the same inequality for $s \ge \theta$. Therefore, $H(\theta, s) \le R_2$, for all $s, \theta \in [a, b]$. □

3. Existence and uniqueness outcomes

In this section, we investigate the existence and uniqueness of solutions for the problem (1.1). Consider the BS $X = C([a, b], \mathbb{R})$ with supremum norm $\|\vartheta\| = \sup_{\theta\in[a,b]} |\vartheta(\theta)|$. Define an operator $T : X \to X$ by

$$\begin{aligned}T\vartheta(\theta) =& \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b\frac{1}{\Gamma(\beta-\delta_j)}\left(\int_a^\eta\varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta(s)))ds\right)dA_j(\eta)\\ &+\int_a^b\varpi'(s)H(\theta,s)\varphi_r(F(s,\vartheta(s)))ds, \end{aligned}\tag{3.1}$$

where $F(\theta, \vartheta(\theta)) := \frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(s)G(\theta, s)f(s, \vartheta(s))ds$.

We define $K = \{\vartheta \in X : \|\vartheta\| \le M\}$, where $M \ge R\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}$ such that

$$\widetilde{g} = \sup_{\vartheta\in K,\theta\in[a,b]} g(\vartheta(\theta)) \text{ and } R = \frac{1}{\Delta_2}\sum_{j=1}^{m}\frac{(A_j(b)-A_j(a))(\varpi(b)-\varpi(a))^{n+\beta-\delta_j-1}}{\Gamma(\beta-\delta_j)}+R_2(\varpi(b)-\varpi(a)). \tag{3.2}$$

Since K is bounded and g is continuous, there exists a constant $C_g > 0$ such that

$$g(\vartheta(\theta)) \le C_g, \quad \forall\vartheta \in K,\ \theta \in [a, b].$$

Thus, $\tilde{g}$ can be estimated a priori by C_g.

The set K is a closed, bounded, and convex subset of the BS X, which is sufficient for the application of the FP theorems used in this paper.

It is straightforward to verify that T is well-defined on K due to the boundedness of G, H and the assumption (H1).

Theorem 3.1. *Assume that* $1 < p \le 2$ *and (H1) holds. Then the BVP* (1.1) *has a solution in* K.

Proof. To establish that the operator T possesses at least one FP, we will apply Schaefer's FP theorem. The proof is divided into four distinct steps for clarity and systematic progression.

Step 1. Let $\vartheta_n \in K$ be a sequence of functions such that $\vartheta_n \to \vartheta$ where $\vartheta \in K$. Since f is continuous with respect to its second variable, we have

$$|f(\tau, \vartheta_n(\tau)) - f(\tau, \vartheta(\tau))| \to 0 \text{ as } n \to \infty.$$

Also, by (H1) and the continuity of ϖ', the integral

$$\int_a^b \varpi'(\tau)\,|f(\tau, \vartheta_n(\tau)) - f(\tau, \vartheta(\tau))|\,d\tau$$

is finite, guaranteeing that all subsequent integral expressions are well-defined. Using the Lebesgue dominated convergence theorem, we obtain

$$\int_a^b \varpi'(\tau)|f(\tau, \vartheta_n(\tau)) - f(\tau, \vartheta(\tau))|d\tau \to 0 \text{ as } n \to \infty.$$

For all $\theta \in [a, b]$, since

$$\begin{aligned} |F(\theta, \vartheta(\theta))| &= \left|\frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(s)G(\theta, s)f(s, \vartheta(s))ds\right| \leq \frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(s)|G(\theta, s)||f(s, \vartheta(s))|ds \\ &\leq \frac{R_1}{\Gamma(\alpha)}\int_a^b \varpi'(s)q(s)g(\vartheta(s))ds \leq \frac{R_1}{\Gamma(\alpha)}\widetilde{g}\int_a^b \varpi'(s)q(s)ds \leq \frac{R_1}{\Gamma(\alpha)}\widetilde{g}q_0, \end{aligned}$$

we have

$$|\varphi_r(F(s, \vartheta_n(s))) - \varphi_r(F(s, \vartheta(s)))| \leq (r-1)\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-2}|F(s, \vartheta_n(s)) - F(s, \vartheta(s))| \tag{3.3}$$

by property 3 of Lemma 2.4. Using inequality (3.3), we obtain

$$\begin{aligned} &|T\vartheta_n(\theta) - T\vartheta(\theta)| \\ &= \Bigg|\frac{(\varpi(\theta) - \varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^m \int_a^b \frac{1}{\Gamma(\beta - \delta_j)}\left(\int_a^\eta \varpi'(s)(\varpi(\eta) - \varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s, \vartheta_n(s)))ds\right)dA_j(\eta) \\ &\quad + \int_a^b \varpi'(s)H(\theta, s)\varphi_r(F(s, \vartheta_n(s)))ds \\ &\quad - \frac{(\varpi(\theta) - \varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^m \int_a^b \frac{1}{\Gamma(\beta - \delta_j)}\left(\int_a^\eta \varpi'(s)(\varpi(\eta) - \varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s, \vartheta(s)))ds\right)dA_j(\eta) \\ &\quad - \int_a^b \varpi'(s)H(\theta, s)\varphi_r(F(s, \vartheta(s)))ds\Bigg| \\ &\leq \frac{(\varpi(\theta) - \varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^m \int_a^b \frac{1}{\Gamma(\beta - \delta_j)}\Bigg(\int_a^\eta \varpi'(s)(\varpi(\eta) - \varpi(s))^{\beta-\delta_j-1} \\ &\qquad |\varphi_r(F(s, \vartheta_n(s))) - \varphi_r(F(s, \vartheta(s)))|ds\Bigg)dA_j(\eta) \\ &\quad + \int_a^b \varpi'(s)|H(\theta, s)||\varphi_r(F(s, \vartheta_n(s))) - \varphi_r(F(s, \vartheta(s)))|ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\\
&\qquad (r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}|F(s,\vartheta_n(s))-F(s,\vartheta(s))|ds\Big)dA_j(\eta)\\
&\quad+\int_a^b \varpi'(s)|H(\theta,s)|(r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}|F(s,\vartheta_n(s))-F(s,\vartheta(s))|ds\\
&\leq \frac{(\varpi(b)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\\
&\qquad (r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\Big|\frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(\tau)G(s,\tau)(f(\tau,\vartheta_n(\tau))-f(\tau,\vartheta(\tau)))d\tau\Big|ds\Big)dA_j(\eta)\\
&\quad+\int_a^b \varpi'(s)|H(\theta,s)|(r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\Big|\frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(\tau)G(s,\tau)(f(\tau,\vartheta_n(\tau))-f(\tau,\vartheta(\tau)))d\tau\Big|ds\\
&\leq \frac{R_1(r-1)(\varpi(b)-\varpi(a))^{n-1}}{\Delta_2\Gamma(\alpha)}\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\\
&\qquad \Big(\int_a^b \varpi'(\tau)|f(\tau,\vartheta_n(\tau))-f(\tau,\vartheta(\tau))|d\tau\Big)ds\Big)dA_j(\eta)\\
&\quad+\frac{R_1R_2(r-1)}{\Gamma(\alpha)}\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\int_a^b \varpi'(s)\Big(\int_a^b \varpi'(\tau)|f(\tau,\vartheta_n(\tau))-f(\tau,\vartheta(\tau))|d\tau\Big)ds,
\end{aligned}
$$

which implies that $\|T\vartheta_n - T\vartheta\| \to 0$ as $n \to \infty$. So the operator T is continuous.

Step 2. For $\theta \in [a,b]$ and $\vartheta \in K$, we have

$$
\begin{aligned}
|T\vartheta(\theta)| &= \Big|\frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta(s)))\,ds\Big)dA_j(\eta)\\
&\quad+\int_a^b \varpi'(s)H(\theta,s)\varphi_r(F(s,\vartheta(s)))\,ds\Big|\\
&\leq \frac{(\varpi(b)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}|\varphi_r(F(s,\vartheta(s)))|\,ds\Big)dA_j(\eta)\\
&\quad+\int_a^b \varpi'(s)|H(\theta,s)||\varphi_r(F(s,\vartheta(s)))|\,ds.
\end{aligned}
$$

Using the inequality

$$
|\varphi_r(F(s,\vartheta(s)))| \leq |F(s,\vartheta(s))|^{r-1} \leq \left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1} \quad \text{for all } s \in [a,b],
$$

we obtain

$$
|T\vartheta(\theta)| \leq \left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}\Big[\frac{(\varpi(b)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}ds\Big)dA_j(\eta)
$$

$$+\int_a^b \varpi'(s)|H(\theta,s)|ds\Big]$$
$$\leq \left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}\left[\frac{\varpi(b)-\varpi(a)}{\Delta_2}\sum_{j=1}^{m}\frac{(A_j(b)-A_j(a))(\varpi(b)-\varpi(a))^{n+\beta-\delta_j-2}}{\Gamma(\beta-\delta_j)}+\int_a^b \varpi'(s)|H(\theta,s)|ds\right]$$
$$\leq \left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}\left[\frac{1}{\Delta_2}\sum_{j=1}^{m}\frac{(A_j(b)-A_j(a))(\varpi(b)-\varpi(a))^{n+\beta-\delta_j-1}}{\Gamma(\beta-\delta_j)}+R_2(\varpi(b)-\varpi(a))\right]=\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}R,$$

where R is defined as in (3.2). Taking the supremum over $\theta\in[a,b]$ yields

$$\|T\vartheta\|\leq R\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}.$$

Consequently, by the choice of M, we have $\|T\vartheta\|\leq M$, which shows that $T:K\to K$ is uniformly bounded.

Step 3. Let $\theta_1,\theta_2\in[a,b]$ with $\theta_1<\theta_2$ be two points. Then, for any $\vartheta\in K$, we get

$$|T\vartheta(\theta_2)-T\vartheta(\theta_1)|$$
$$=\Big|\frac{(\varpi(\theta_2)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b\frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta(s)))ds\Big)dA_j(\eta)$$
$$+\int_a^b \varpi'(s)H(\theta_2,s)\varphi_r(F(s,\vartheta(s))ds$$
$$-\frac{(\varpi(\theta_1)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b\frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta(s)))ds\Big)dA_j(\eta)$$
$$-\int_a^b \varpi'(s)H(\theta_1,s)\varphi_r(F(s,\vartheta(s)))ds\Big|$$
$$=\frac{\left|(\varpi(\theta_2)-\varpi(a))^{n-1}-(\varpi(\theta_1)-\varpi(a))^{n-1}\right|}{\Delta_2}$$
$$\times\sum_{j=1}^{m}\frac{1}{\Gamma(\beta-\delta_j)}\int_a^b\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta(s)))ds\Big)dA_j(\eta)$$
$$+\int_a^b \varpi'(s)\varphi_r(F(s,\vartheta(s)))|H(\theta_2,s)-H(\theta_1,s)|ds.$$

As $\theta_2\to\theta_1$, the right-hand side of the above inequality converges to zero, due to the continuity of the functions $(\varpi(\theta)-\varpi(a))^{n-1}$ and $H(\theta,s)$. Since $\vartheta\in K$ and K is bounded, it follows from Step 2 that $F(s,\vartheta(s))$ is uniformly bounded. Hence, $\varphi_r(F(s,\vartheta(s)))$ is also uniformly bounded for all $\vartheta\in K$. Therefore, the above estimate is independent of ϑ, which ensures the uniform equicontinuity of T on K. This demonstrates that the operator T is equicontinuous on K. Combined with the previously established uniform boundedness, we have rigorously verified both key conditions required for the application of the Arzela–Ascoli theorem. Consequently, it follows that $T(K)$ is relatively compact in

X. Consequently, we conclude that the operator $T : K \to K$ is completely continuous.

Step 4. We show that $\Lambda = \{\vartheta \in K : \vartheta = \lambda T\vartheta \text{ for some } \lambda \in (0,1)\}$ is bounded. Let $\vartheta \in \Lambda$. Then, $\vartheta = \lambda T\vartheta$ for some $\lambda \in (0,1)$. So, for each $\theta \in [a,b]$, from Step 2, we arrive at

$$\|T\vartheta\| \le R\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1},$$

where $\widetilde{g}$ and R are in (3.2). Using the relation $\vartheta = \lambda T\vartheta$ and the fact that $\lambda \in (0,1)$, we derive

$$\|\vartheta\| = \|\lambda T\vartheta\| \le R\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1}.$$

It follows that the set Λ is bounded. Since K is bounded in X, any subset $\Lambda \subseteq K$ is also bounded, which justifies the application of Schaefer's FP theorem. By employing Schaefer's FP theorem, we conclude that T admits at least one FP, yielding a solution of (1.1) in K. This completes the proof. □

Theorem 3.2. *Assume* $1 < p \le 2$ *and (H1)-(H2) hold. If* $L < 1$, *then the BVP* (1.1) *has a unique solution, where*

$$L = \frac{(r-1)\, r_0\, R_1}{\Gamma(\alpha)}\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-2} R. \tag{3.4}$$

Proof. Consider the operator $T : X \to X$ in (3.1). In the proof of Theorem 3.1, we have seen that $T : K \to K$. Using the inequality $|F(s,\vartheta(s))| \le \dfrac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}$ for all $s \in [a,b]$ and Lemma 2.10, for any $\vartheta_1, \vartheta_2 \in K$, we get

$$\begin{aligned}
&|T\vartheta_2(\theta) - T\vartheta_1(\theta)| \\
&= \Bigg| \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\left(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta_2(s)))ds\right)dA_j(\eta) \\
&\quad + \int_a^b \varpi'(s)H(\theta,s)\varphi_r(F(s,\vartheta_2(s)))ds \\
&\quad - \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\left(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\vartheta_1(s)))ds\right)dA_j(\eta) \\
&\quad - \int_a^b \varpi'(s)H(\theta,s)\varphi_r(F(s,\vartheta_1(s)))ds \Bigg| \\
&\le \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\bigg(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1} \\
&\qquad\qquad |\varphi_r(F(s,\vartheta_2(s))) - \varphi_r(F(s,\vartheta_1(s)))|ds\bigg)dA_j(\eta) \\
&\quad + \int_a^b \varpi'(s)|H(\theta,s)||\varphi_r(F(s,\vartheta_2(s))) - \varphi_r(F(s,\vartheta_1(s)))|ds \\
&\le \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\bigg(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}
\end{aligned}$$

$$(r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}|F(s,\vartheta_2(s))-F(s,\vartheta_1(s))|ds\Big)dA_j(\eta)$$

$$+\int_a^b \varpi'(s)|H(\theta,s)|(r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}|F(s,\vartheta_2(s))-F(s,\vartheta_1(s))|ds$$

$$= \frac{(r-1)(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2\Gamma(\alpha)}\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}$$

$$\Big(\int_a^b \varpi'(\tau)G(s,\tau)|f(\tau,\vartheta_2(\tau))-f(\tau,\vartheta_1(\tau))|d\tau\Big)ds\Big)dA_j(\eta)$$

$$+(r-1)\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\int_a^b \varpi'(s)|H(\theta,s)|\Big(\frac{1}{\Gamma(\alpha)}\int_a^b \varpi'(\tau)G(s,\tau)|f(\tau,\vartheta_2(\tau))-f(\tau,\vartheta_1(\tau))|d\tau\Big)ds$$

$$\leq \frac{(r-1)r_0R_1}{\Gamma(\alpha)}\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2}\Big(\frac{1}{\Delta_2}\sum_{j=1}^{m}\frac{(A_j(b)-A_j(a))(\varpi(b)-\varpi(a))^{n+\beta-\delta_j-1}}{\Gamma(\beta-\delta_j)}+R_2(\varpi(b)-\varpi(a))\Big)\|\vartheta_2-\vartheta_1\|$$

$$\leq \frac{(r-1)\ r_0\ R_1}{\Gamma(\alpha)}\Big(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\Big)^{r-2} R\ \|\vartheta_2-\vartheta_1\|.$$

Thus, we have

$$\|T\vartheta_2 - T\vartheta_1\| \leq L\ \|\vartheta_2-\vartheta_1\|.$$

Hence, the operator T is a contraction with contraction constant L. Since $L < 1$, the Banach FP theorem guarantees that T admits a unique FP. This FP yields the unique solution of the problem (1.1) on $[a, b]$. The proof is therefore complete. □

4. Stability outcomes

In the study of functional and differential equations, stability concepts are essential for understanding the relationship between approximate and exact solutions. One of the most fundamental notions in this context is Ulam–Hyers stability, which examines whether an approximate solution to a given equation remains close to the exact solution. Over time, this concept has been extended to broader frameworks, including generalized Ulam–Hyers stability and Ulam–Hyers–Rassias stability, where the bounds controlling the deviation are allowed to depend on additional functions or parameters. These generalized forms provide greater flexibility and are widely used in the analysis of nonlinear and fractional models. In this section, we present stability outcomes for the problem (1.1).

Let the function $\nu \in X$ be defined as follows:

$$\nu(\theta) = \frac{(\varpi(\theta)-\varpi(a))^{n-1}}{\Delta_2}\sum_{j=1}^{m}\int_a^b \frac{1}{\Gamma(\beta-\delta_j)}\Big(\int_a^\eta \varpi'(s)(\varpi(\eta)-\varpi(s))^{\beta-\delta_j-1}\varphi_r(F(s,\nu(s)))ds\Big)dA_j(\eta)$$

$$+\int_a^b \varpi'(s)H(\theta,s)\varphi_r(F(s,\nu(s)))ds,$$

where $F(\theta,\nu(\theta)) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^b \varpi'(s)G(\theta,s)f(s,\nu(s))ds.$

We define the operator $A : K \to K$ by

$$A\nu(\theta) = D_a^{\alpha,\varpi}(\varphi_p(-D_a^{\beta,\varpi}\nu(\theta))) + f(\theta, \nu(\theta)). \tag{4.1}$$

Definition 4.1. *For each $\epsilon > 0$ and for each solution $\nu(\theta)$ satisfying*

$$\|A\nu\| \le \epsilon, \tag{4.2}$$

the problem (1.1) *is said to be Ulam–Hyers stable provided that there exists a positive constant c and a solution $\vartheta \in X$ of* (1.1) *such that the following inequality holds:*

$$\|\vartheta - \nu\| \le c\epsilon^*,$$

where $\epsilon*$ is a positive real number depending on ϵ.

Definition 4.2. *For each $\epsilon > 0$ and for each solution $\nu(\theta)$ of (4.2), if there exists a function $z \in C(\mathbb{R}^+, \mathbb{R}^+)$, such that*

$$\|\vartheta - \nu\| \le z(\epsilon), \quad \theta \in [a, b],$$

for a solution $\vartheta \in X$ of (1.1)*, the problem* (1.1) *is said to be generalized Ulam–Hyers stable.*

Definition 4.3. *Let $\varepsilon > 0$ and let $\psi : [a, b] \to \mathbb{R}^+$ be a nonnegative continuous function. The BVP (1.1) is said to be Ulam–Hyers–Rassias stable if for every function $\nu \in K$ satisfying*

$$\|A\nu\| \le \varepsilon\psi(\theta), \quad \theta \in [a, b], \tag{4.3}$$

there exist a real number $c > 0$ and a solution $\vartheta \in C([a, b], \mathbb{R})$ of the BVP (1.1) such that

$$\|\vartheta - \nu\| \le c\varepsilon^*\psi(\theta), \quad \theta \in [a, b],$$

where ε^* is a positive real number depending on ε.

Theorem 4.4. *Under Assumptions (H1) and (H2), the problem* (1.1) *is both Ulam–Hyers and generalized Ulam–Hyers stable if $L < 1$, which is given by (3.4).*

Proof. Let $\vartheta \in X$ be a solution of (1.1), and let $\nu(\theta)$ denote any solution satisfying (4.2). Consider the operator A defined by (4.1). According to Lemma 2.8, for every solution $\nu \in X$ of (1.1), the operators A and $T - I$ (where I is the identity operator) are equivalent. By construction of the operator T via the integral representation using the Green's function, it is clear that

$$A\vartheta = 0 \iff T\vartheta = \vartheta,$$

so that the zeros of A correspond exactly to the FPs of T. This equivalence establishes a fundamental connection between the FP formulation and the stability analysis framework. Consequently, by invoking the FP property of the operator T, we obtain

$$\begin{aligned} |\vartheta(\theta) - \nu(\theta)| &= |\vartheta(\theta) - T\nu(\theta) + T\nu(\theta) - \nu(\theta)| = |T\vartheta(\theta) - T\nu(\theta) + T\nu(\theta) - \nu(\theta)| \\ &\le |T\nu(\theta) - T\vartheta(\theta)| + |T\nu(\theta) - \nu(\theta)| \le \|T\nu - T\vartheta\| + \|T\nu - \nu\| \\ &< L\|\nu - \vartheta\| + \|(T - I)\nu\| = L\|\nu - \vartheta\| + \|A\nu\| \end{aligned}$$

$$\leq \quad L\|\nu - \vartheta\| + \epsilon.$$

Because $L < 1$ and $\epsilon > 0$, it follows that

$$\|\nu - \vartheta\| \leq \frac{\epsilon}{1-L} = z(\epsilon).$$

This shows that the stability constants depend explicitly on the contraction constant L, and hence on the parameters R_1, R_2, and the nonlinear term. Letting $z(\epsilon) = \frac{\epsilon}{1-L}$, we deduce the generalized Ulam–Hyers stability of the problem. Furthermore, by fixing $\epsilon^* = \frac{\epsilon}{1-L}$ and taking $c = 1$, the conclusion is obtained. □

Theorem 4.5. *Under Assumptions (H1) and (H2) with the inequality $L < 1$, and since there exists a function $\theta \in C([a,b], \mathbb{R}^+)$ satisfying the condition (4.3), the problem (1.1) is Ulam–Hyers–Rassias stable with respect to ψ.*

Proof. From the proof of Theorem 4.4, we have

$$|\vartheta(\theta) - \nu(\theta)| \leq L\|\nu - \vartheta\| + \epsilon\psi(\theta)$$

with $\epsilon^* = \frac{\epsilon}{1-L}$ and $c = 1$. This establishes the desired result and concludes the proof. □

5. Examples

In this section, to validate the theoretical framework established in the previous sections and illustrate its practical utility, we present two numerical examples.

Example 5.1. *We consider a specific ϖ-Caputo fractional-order singular p-Laplacian problem on the interval $[0,1]$:*

$$\begin{cases} {}^{C}D_{0^+}^{1.5,\varpi}\Big(\varphi_3\big(-\,{}^{C}D_{0^+}^{2.5,\varpi}\vartheta(\theta)\big)\Big) + f(\theta, \vartheta(\theta)) = 0, \quad \theta \in (0,1), \\ \vartheta(0) = 0, \quad \vartheta'(0) = 0, \quad {}^{C}D_{0^+}^{2.5,\varpi}\vartheta(0) = 0, \\ {}^{C}D_{0^+}^{0.5,\varpi}\vartheta(1) = -\int_0^1 {}^{C}D_{0^+}^{0.8,\varpi}\vartheta(\eta)\,d\eta^2, \\ \varphi_3({}^{C}D_{0^+}^{2.5,\varpi}\vartheta(1)) = \frac{1}{2}\varphi_3({}^{C}D_{0^+}^{2.5,\varpi}\vartheta(1/2)), \end{cases} \tag{5.1}$$

where $\varphi_3(s) = |s|s$, $\varpi(\theta) = (\theta+1)^2$, $f(\theta,\vartheta) = \frac{\vartheta}{\sqrt{\theta}}$, and $A_0(\theta) = \theta^2$. For all $\theta \in [0,1]$, it is easy to see that $\varpi'(\theta) = 2(\theta+1) \neq 0$ and $\varpi(\theta) = (\theta+1)^2$ is an increasing function.

Now, we verify that the function f satisfies the condition (H1). $f(\theta,\vartheta) = \frac{\vartheta}{\sqrt{\theta}}$ is continuous on $(0,1] \times \mathbb{R}$. Since $|f(\theta,\vartheta)| = \left|\frac{\vartheta}{\sqrt{\theta}}\right| = \frac{1}{\sqrt{\theta}}|\vartheta|$ for $\theta \in (0,1]$, we obtain $g(\vartheta) = |\vartheta|$, $q(\theta) = \frac{1}{\sqrt{\theta}}$ and $q_0 = \int_0^1 \frac{1}{\sqrt{s}} 2s ds = \frac{4}{3}$. Consequently, the condition (H1) holds for the function f.

Thus, the fractional BVP (5.1) has a solution on $[0,1]$ by Theorem 3.1.

Now, let us proceed to derive the existence and uniqueness outcome for a more complex example.

Example 5.2. *Consider the following ϖ-Caputo fractional-order singular p-Laplacian problem:*

$$\begin{cases} {}^C D_{0^+}^{\frac{4}{3},e^\theta}(\varphi_{\frac{3}{2}}(-{}^C D_{0^+}^{\frac{13}{3},e^\theta}\vartheta(\theta))) + f(\theta,\vartheta(\theta)) = 0, \quad \theta\in(0,2), \\ \vartheta^{(k)}(0) = 0, \qquad k = 0,1,2,3, \\ {}^C D_{0^+}^{\frac{1}{3},e^\theta}\vartheta(2) = -\Big\{ \int_0^2 {}^C D_{0^+}^{\frac{2}{3},e^\theta}\vartheta(\eta)\,dA_0(\eta) + \int_0^2 {}^C D_{0^+}^{\frac{4}{3},e^\theta}\vartheta(\eta)\,dA_1(\eta) + \int_0^2 {}^C D_{0^+}^{\frac{7}{3},e^\theta}\vartheta(\eta)\,dA_2(\eta)\Big\} \\ {}^C D_{0^+}^{\frac{13}{3},e^\theta}\vartheta(0) = 0, \\ \varphi_{\frac{3}{2}}({}^C D_{0^+}^{\frac{13}{3},e^\theta}\vartheta(2)) = \frac{1}{3}\varphi_{\frac{3}{2}}({}^C D_{0^+}^{\frac{7}{2},e^\theta}\vartheta(\frac{1}{2})) + \frac{1}{5}\varphi_{\frac{3}{2}}({}^C D_{0^+}^{\frac{7}{2},e^\theta}\vartheta(1)) + \frac{1}{10}\varphi_{\frac{3}{2}}({}^C D_{0^+}^{\frac{7}{2},e^\theta}\vartheta(\frac{3}{2})), \end{cases} \tag{5.2}$$

where $f(\theta,\vartheta) = \dfrac{e^{-\theta}\arctan\left(\frac{\vartheta^2}{1+\vartheta^2}\right)}{3000\sqrt{\theta}\sqrt{2-\theta}}$, *and* $A_j(\theta) = \theta^2$, $j = 0,1,2$. *For all* $\theta\in[0,2]$, *it is easy to see that* $\varpi'(\theta) = e^\theta \neq 0$ *and* $\varpi(\theta) = e^\theta$ *is an increasing function.*

Now, we verify that the function f *satisfies (H1) and (H2).* $f(\theta,\vartheta) = \dfrac{e^{-\theta}\arctan\left(\frac{\vartheta^2}{1+\vartheta^2}\right)}{3000\sqrt{\theta}\sqrt{2-\theta}}$ *is continuous on* $(0,2)\times\mathbb{R}$. *Since for* $\theta\in(0,2)$

$$|f(\theta,\vartheta)| = \left|\frac{\arctan\left(\frac{\vartheta^2}{1+\vartheta^2}\right)e^{-\theta}}{3\sqrt{\theta}\sqrt{2-\theta}}10^{-3}\right| \leq \frac{e^{-\theta}}{3\sqrt{\theta}\sqrt{2-\theta}}10^{-3}\vartheta^2,$$

we find $g(\vartheta) = \vartheta^2$ *and* $q(\theta) = \dfrac{e^{-\theta}}{3\sqrt{\theta}\sqrt{2-\theta}}10^{-3}$ *with* $q_0 = \displaystyle\int_0^2 \frac{1}{3\sqrt{s}\sqrt{2-s}}10^{-3}ds = \frac{\pi}{3}10^{-3}$. *Also, we obtain*

$$|f(\theta,\vartheta_2) - f(\theta,\vartheta_1)| = \frac{e^{-\theta}10^{-3}}{3\sqrt{\theta}\sqrt{2-\theta}}\left|\arctan\left(\frac{\vartheta_2^2}{1+\vartheta_2^2}\right) - \arctan\left(\frac{\vartheta_1^2}{1+\vartheta_1^2}\right)\right| \leq \frac{2e^{-\theta}10^{-3}}{3\sqrt{\theta}\sqrt{2-\theta}}|\vartheta_2-\vartheta_1|$$

with $r(\theta) = \dfrac{2e^{-\theta}10^{-3}}{3\sqrt{\theta}\sqrt{2-\theta}}$ *for* $\theta\in(0,2)$, *such that* $r_0 = \displaystyle\int_0^2 \frac{2}{3\sqrt{s}\sqrt{2-s}}10^{-3}ds = \frac{2\pi}{3}10^{-3}$.

We can easily calculate that

$$R_1 = 2(\varpi(b)-\varpi(a))^{\alpha-1}(1-\sum_{j=1}^d \lambda_j)^{-1} = 2(e^2-1)^{\frac{1}{3}}(1-\frac{19}{30})^{-1} \approx 10.14,$$

$$\begin{aligned}\Delta_2 =& \frac{(n-1)!}{\Gamma(n-\delta_0)}[\varpi(b)-\varpi(a)]^{n-\delta_0-1} + \sum_{j=1}^m \int_a^b \frac{(n-1)!}{\Gamma(n-\delta_j)}[\varpi(\eta)-\varpi(a)]^{n-\delta_j-1}dA_j(\eta) \\ =& \frac{24}{\Gamma(\frac{14}{3})}[e^2-1]^{\frac{11}{3}} + \int_0^2 \frac{24}{\Gamma(\frac{13}{3})}[e^\eta-1]^{\frac{10}{3}}2\eta d\eta\end{aligned}$$

$$+\int_0^2 \frac{24}{\Gamma(\frac{10}{3})}[e^\eta - 1]^{\frac{7}{3}} 2\eta d\eta + \int_0^2 \frac{24}{\Gamma(\frac{8}{3})}[e^\eta - 1]^{\frac{5}{3}} 2\eta d\eta$$
$$\approx 4909,$$

$$R_2 = \frac{(\varpi(b) - \varpi(a))^{n+\beta-\delta_0-2}}{\Delta_2\Gamma(\beta - \delta_0)} + \frac{(\varpi(b) - \varpi(a))^{\beta-1}}{\Gamma(\beta)} = \frac{(e^2 - 1)^7}{\Delta_2\Gamma(\frac{10}{3})} + \frac{(e^2 - 1)^{\frac{10}{3}}}{\Gamma(\frac{13}{3})} \approx 118,$$

and

$$\begin{aligned} R =& \frac{1}{\Delta_2}\sum_{j=1}^{m} \frac{(A_j(b) - A_j(a))(\varpi(b) - \varpi(a))^{n+\beta-\delta_j-1}}{\Gamma(\beta - \delta_j)} + R_2(\varpi(b) - \varpi(a)) \\ =& \frac{1}{\Delta_2}\sum_{j=1}^{3} \frac{4(e^2 - 1)^{\frac{25}{3}-\delta_j}}{\Gamma(\frac{13}{3} - \delta_j)} + R_2(e^2 - 1) \\ =& \frac{4}{\Delta_2}\left\{\frac{(e^2 - 1)^{\frac{23}{3}}}{\Gamma(\frac{11}{3})} + \frac{(e^2 - 1)^{\frac{20}{3}}}{\Gamma(\frac{8}{3})} + \frac{(e^2 - 1)^{\frac{18}{3}}}{\Gamma(2)}\right\} + R_2(e^2 - 1) \approx 1245. \end{aligned}$$

Thus, the inequality $\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1} R \leq M$ *is satisfied for* $M = 1$, *since* $\left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-1} R \approx 0.175.$

Noting that $L = \frac{(r-1)\, r_0\, R_1}{\Gamma(\alpha)} \left(\frac{R_1\widetilde{g}q_0}{\Gamma(\alpha)}\right)^{r-2} R \approx 0.701 < 1$, *an application of Theorem 3.2 guarantees the existence and uniqueness of a solution to the fractional BVP (5.2) on* $[0, 2]$. *In addition, by Theorem 4.4, the same problem satisfies both the Ulam–Hyers stability and the generalized Ulam–Hyers stability criteria.*

6. Conclusions

In this work, we have thoroughly investigated the existence and stability of solutions for a class of singular p-Laplacian fractional-order BVPs that involve generalized Caputo derivatives and integral boundary conditions formulated in the sense of the Riemann–Stieltjes integral. By establishing an appropriate functional framework and reformulating the original problem as an equivalent integral equation, we derived sufficient conditions guaranteeing the existence of at least one solution and, in some cases, multiple solutions. These outcomes were obtained under suitable assumptions regarding the growth, continuity, and singular behavior of the nonlinear terms, ensuring that the theoretical analysis accommodates a wide range of nonlinearities commonly encountered in applications.

The singular and nonlinear nature of the problem, combined with the nonlocal features of the generalized Caputo fractional derivative and the presence of integral boundary conditions, posed significant analytical challenges. To address these difficulties, we employed FP theorems in carefully chosen BSs, supported by compactness arguments and appropriate a priori estimates. In particular, the use of Schaefer's and Banach FP theorems proved to be effective in establishing solvability despite the complications arising from singularities and the p-Laplacian operator's nonlinear structure. These methods enabled rigorous demonstrations of both existence and multiplicity outcomes, thereby extending the scope of classical fractional BVP theory.

Beyond existence, we provided a detailed analysis of the stability of solutions within the framework of Ulam–Hyers and, when appropriate, Ulam–Hyers–Rassias stability. This investigation confirmed that the solutions obtained are robust to small perturbations in the fractional differential equations, ensuring that the model remains reliable in practical situations where approximations, measurement errors, or uncertainties are unavoidable. This stability perspective is particularly relevant for applications in physics, engineering, and biology, where fractional-order models often capture memory effects and hereditary properties.

The results obtained in this study contribute to the ongoing development of the qualitative theory of nonlinear fractional differential equations, particularly by integrating several complex features simultaneously: (i) the nonlinear p-Laplacian operator, (ii) singular nonlinear terms, (iii) generalized Caputo-type fractional derivatives, and (iv) Riemann–Stieltjes integral boundary conditions. This combination not only broadens the class of problems for which existence and stability can be rigorously established but also offers a solid foundation for future theoretical and computational studies.

Looking ahead, several research directions emerge naturally from this work. One promising avenue is the development and analysis of numerical approximation schemes tailored to such singular and nonlocal fractional problems. Additionally, extending the investigation to models with variable-order fractional derivatives, impulsive effects, or coupled systems of fractional differential equations could provide further insights and applications. Moreover, the study of partial fractional differential equations with p-Laplacian structures and more general nonlocal boundary conditions offers an opportunity to deepen both the theoretical understanding and the practical applicability of FC in modeling complex dynamical systems. Overall, this study lays a comprehensive foundation for both the mathematical theory and computational exploration of nonlinear singular fractional BVPs, emphasizing their existence, stability, and robustness under real-world conditions.

Author contributions

İzel Nüzket: Writing–original draft; Ebru Aslan: Writing–original draft; Erbil Çetin: Writing–review and editing, Supervision; Aynur Şahin: Writing–review and editing; Fatma Serap Topal: Writing–review and editing. All authors have read and agreed to the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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