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Research article

On the 3-coloring of planar graphs without cycles of length from 4 to 6

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Abstract: In 1976, Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable. This conjecture was proved false by Cohen-Addad et al in 2017. Erdős raised the following question: Is there an integer k such that every planar graph without cycles of length from 4 to k is 3-colorable? Borodin et al. proved that every planar graph without cycles of length from 4 to 7 is 3-colorable [Planar graphs without cycles of length from 4 to 7 are 3-colorable, J. Combin.Theory Ser. B, **93** (2005), 303–311]. However, the question whether every planar graph without cycles of length from 4 to 6 is 3-colorable is not answered yet and full of challenges. A 7-cycle is called a special 7-cycle if it shares an edge with another 7-cycle or 9-cycle. In this paper, we prove that every planar graph without cycles of length from 4 to 6 and without special 7-cycles is 3-colorable which is an improvement of Borodin's result.

Keywords: planar graphs; 3-coloring; Steinberg's conjecture
Mathematics Subject Classification: 05C15, 05C69

1. Introduction

The three-color problem of planar graphs is to ask whether a planar graph can be colored with three colors such that no two adjacent vertices are assigned the same color. Based on Steinberg's conjecture, Erdős raised the following question: Is there an integer k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] proved that such a k exists and $k \leq 11$. This result was later improved to $k \leq 10$ by Borodin [2], and to $k \leq 9$ by Borodin [3] and, independently, Sanders and Zhao [8], and to $k \leq 7$ by Borodin [4]. In 2017, Steinberg's conjecture was proved false by Cohen-Addad et al [6]. However, the question whether every planar graph without cycles of length 4 to 6 is 3-colorable is open. It was proved by Borodin et al. [5] and independently by Xu [9] that every planar graph having neither 5- and 7-cycles nor adjacent 3-cycles is 3-colorable which improved the results above. In 2017, Jin et al. proved that plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable [7].

A 7-cycle is called a *special* 7*-cycle* if it shares an edge with another 7-cycle or 9-cycle. In this paper, we prove the following Theorem 1.1, which is an improvement on the result of Borodin in [4].

Theorem 1.1. *Every planar graph without cycles of length from* 4 *to* 6 *and without special* 7*-cycles is* 3*-colorable.*

Let **G** denote the class of planar graphs without cycles of size from 4 to 6 and special 7-cycles. Let C be a cycle of a plane graph G. The vertices lying inside and outside of C are denoted by $Int(C)$ and $Out(C)$, respectively. If $Int(C) \neq \emptyset$ and $Out(C) \neq \emptyset$, then C is called a separating cycle of G. A chord of C is an edge inside C which connects two nonconsecutive vertices on C. Specially, if this edge connects two vertices of distance two on C, it is called a triangular chord. If there are two chords on C, we call them bichord of C. If $Int(C)$ has a vertex v with three neighbors v_1, v_2, v_3 on C, then $G[vv_1, vv_2, vv_3]$ is called a claw of C. If $Int(C)$ has two adjacent vertices u and v such that u has two neighbors u_1, u_2 on C and v has two neighbors v_1, v_2 on C, then $G[uv, uu_1, uu_2, vv_1, vv_2]$ is called a biclaw of C. If $Int(C)$ has five vertices x, y, z, v, w in $Int(C)$ and five vertices u, v', v'', w', w'' on C such that $T = [vv', vv'', ww', ww'', xv, yw, zu, xy, yz, zx] \subseteq E(G)$, then $G[T]$ is called a triclaw of C. If $Int(C)$ has six vertices x, y, z, u, v, w in $Int(C)$ and six vertices $u', u'', v', v'', w', w''$ on C such that

$$T = [uu', uu'', vv', vv'', ww', ww'', xv, yw, zu, xy, yz, zx] \subseteq E(G),$$

then $G[T]$ is called a combclaw of C. For example, a special chord, a special bichord, a special claw, a special biclaw, and a special combclaw can be seen in Figure 1. A *good cycle* is a 13^--cycle that has no claws, biclaws, triclaws and combclaws. A *bad cycle* is a 13^--cycle that has a claw or a biclaw or a triclaw or a combclaw.

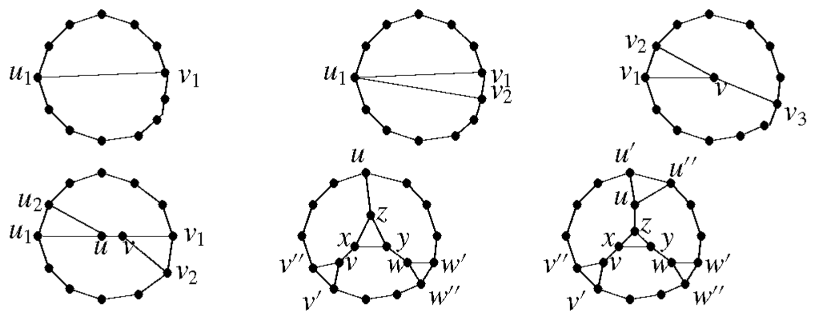

Figure 1. A special chord, a special bichord, a special claw, a special biclaw, a special triclaw, and a special combclaw.

To obtain Theorem 1.1, we prove the following stronger theorem.

Theorem 1.2. *Let* G *be a connected plane graph without cycles of size from* 4 *to* 6 *and without special* 7*-cycles. If the boundary* C_0 *of the exterior face of* G *is a good cycle, then every proper* 3*-coloring of* $G[V(C_0)]$ *can be extended to a proper* 3*-coloring of* G.

By Theorem 1.2, we can easily prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Suppose that G is a counterexample to the theorem with the smallest number of vertices. Clearly, G is connected and it must have a cycle C of length 7 by [4]. By the absence of cycles of length from 4 to 6 in G, we have that $G[V(C)] = C$. Now we consider the two graphs $G_1 = G[V(C) \cup Int(C)]$ and $G_2 = G[V(C) \cup Out(C)]$. Without loss of generality, we may assume that C is also the exterior face of G_2 by giving a new embedding of G_2 on the plane. Note that $|C| = 7$. Then, C is a good cycle of both G_1 and G_2 since all the bad cycles have the length 12 or 13 by Lemma 2.4. Let φ be a proper 3-coloring of C. By Theorem 1.2, φ can be extended to a proper 3-coloring of G_1 and G_2, and thus we obtain a proper 3-coloring of G, a contradiction. □

In the remaining of this section, we define some notations used throughout the paper. In the second section we prove some properties for a possible minimum counterexample to Theorem 1.2. In the third section, we complete the proof by showing that these properties are incompatible using the discharging method.

Let C be a cycle of plane graph and T be a chord or a bichord or a claw or a biclaw or a combclaw of C. We call the plane graph H consisting of C and T a T-partition of C. Specially, if T is a claw or a biclaw or a combclaw of C, we call the T-partition a bad partition, and the internal vertices inside C on the bad partition H are called *center vertices* of H. The boundary of each interior face of H is called a cell of H. Let k_i be the length of the cell c_i. Then, a chord, bichord, a claw, a biclaw, a triclaw, and a combclaw are further called a (k_1, k_2)-chord, a (k_1, k_2, k_3)-bichord, a (k_1, k_2, k_3)-claw, a (k_1, k_2, k_3, k_4)-biclaw, a $(k_1, k_2, k_3, k_4, k_5, k_6)$-triclaw, and a $(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$-combclaw, respectively. As shown in Figure 1, they are called a $(7, 8)$-chord, a $(7, 3, 7)$-bichord, a $(3, 8, 8)$-claw, a $(3, 8, 3, 8)$-biclaw, a $(3, 8, 8, 3, 8, 3)$-triclaw, and a $(3, 3, 8, 3, 8, 3, 9)$-combclaw, respectively.

Denote the degree of a vertex v by $d(v)$ and the size of a face f by $|f|$. A k-vertex is a vertex of degree k. By a k^+-vertex (a k^--vertex), we mean a vertex of degree at least (at most) k. Similar notation is also used for faces and cycles. If two cycles share an edge, they are called adjacent cycles.

2. Basic properties of the minimal counterexample

Let G be a minimal counterexample to Theorem 1.2. Then the exterior face of G is a good cycle C_0 of G and there exists a 3-coloring c of $G[V(C_0)]$ which cannot be extended to a proper 3-coloring of G. Any face in G other than C_0 is called internal and the vertices in $G - V(C_0)$ are also called internal. A *bad vertex* is an internal 3-vertex which is incident with a 3-face. Now, we get the basic properties of the minimal counterexample G as follows.

Lemma 2.1. *G has no separating good cycle C.*

Proof. If G has a separating good cycle C, then C is not the exterior face C_0 of G. By the minimality of G, we can extend c to $G - Int(C)$. Further, we extend the 3-coloring of $G[V(C)]$ to $G - Out(C)$ using the minimality of G. Then, we get a 3-coloring of G together, a contradiction. □

Lemma 2.2. *G has no 1-vertex.*

Proof. If G has a 1-vertex v, then v is an internal vertex by our assumption on G. Then, the 3-coloring of C_0 can be extended to a proper 3-coloring of $G-v$ by the minimality and, further, a proper 3-coloring of G, a contradiction. □

Lemma 2.3. *C_0 has no chord in G. Each 2-vertex v in G belongs to C_0 and none of them is incident with a 3-face.*

Proof. If C_0 has a chord, we can partition G into two smaller graphs and we can easily get a proper 3-coloring of G by the minimality of G. If $v \notin C_0$ is a 2-vertex, we can first extend c to $G - v$ and then color v. Therefore, every 2-vertex belongs to C_0. In addition, if this 2-vertex is incident with a 3-face, then C_0 has a chord, a contradiction. □

Lemma 2.4. (*I*) *If C is a 12^--cycle, then any chord of C is a triangular chord.* (*II*) *If C is a bad cycle of G, then C has the length either* 12 *or* 13. *Furthermore, if $|C| = 12$, then C has a* $(3, 8, 3, 8)$-*biclaw or* $(3, 3, 8, 3, 8, 3, 8)$-*combclaw; if $|C| = 13$, then C has a* $(3, 8, 8)$-*claw or* $(3, 8, 3, 9)$-*biclaw or* $(3, 8, 8, 3, 8, 3)$-*triclaw or* $(3, 3, 8, 3, 8, 3, 9)$-*combclaw.*

Proof. (I) The first statement follows from the fact that G has no cycles of size from 4 to 6 and special 7-cycles. (II) If C is a bad cycle, then $|C| \le 13$ and C has a claw or a biclaw or a triclaw or a combclaw by the definition of bad cycles. In addition, if C has a (c_1, c_2, c_3)-claw, then we have that $c_i = 3$ or $c_i \ge 7$ since G has no cycles of size from 4 to 6. If $c_1 \ge 7$, $c_2 \ge 7$, and $c_3 \ge 7$, we have that $|C| > 13$, a contradiction. So we may assume that $c_1 = 3$, $c_2 \ge 7$, and $c_3 \ge 7$. Note that if an 8-cycle has exactly one public edge with a 3-cycle, we will get a 9-cycle. So, if $c_2 = 7$ and $7 \le c_3 \le 9$, we can get a special 7-cycle, a contradiction. If $c_2 = 7$ and $c_3 \ge 10$, we have that $|C| > 13$, still a contradiction. Hence, this claw is a $(3, 8, 8)$-claw and $|C| = 13$. In the same way, we can get that C can have a $(3, 8, 3, 8)$-biclaw or a $(3, 8, 3, 9)$-biclaw or a $(3, 8, 8, 3, 8, 3)$-triclaw or a $(3, 3, 8, 3, 8, 3, 8)$-combclaw or a $(3, 3, 8, 3, 8, 3, 9)$-combclaw. So, all the bad cycles have the length 12 or 13. So if $|C| = 12$, then C is a $(3, 8, 3, 8)$-biclaw or a $(3, 3, 8, 3, 8, 3, 8)$-combclaw; if $|C| = 13$, then C is a $(3, 8, 8)$-claw or a $(3, 8, 3, 9)$-biclaw or a $(3, 8, 8, 3, 8, 3)$-triclaw or a $(3, 3, 8, 3, 8, 3, 9)$-combclaw. □

A *tetrad* is a path $T = v_1v_2v_3v_4$ in $Int(C_0)$ such that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$ where $...xv_1v_2v_3v_4x^{'}...$ is on the boundary of a face and there are triangles $t^{'}v_1v_2$, tv_3v_4, where $t^{'} \neq x$, $t \neq x^{'}$ (see Figure 2 for example).

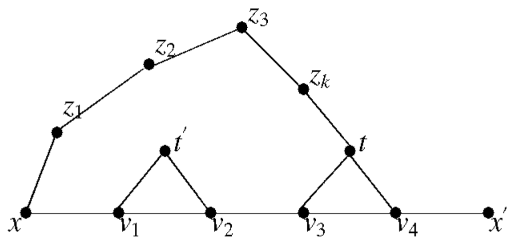

Figure 2. A tetrad.

Lemma 2.5. *G has no tetrad.*

Proof. Take a tetrad T. Let G^* be the graph obtained from G by deleting v_1, v_2, v_3, and v_4 and identifying x and t. Let v^* denote the new vertex obtained by identifying x and t. In order to prove that G^* is in **G**, we now prove that G^* has no cycles of length from 4 to 6 and special 7-cycles. If a new cycle of size from 4 to 7 occurs in G^*, we may suppose that $S^* = v^*z_1...z_kv^*$ is such a cycle, where $3 \le k \le 6$ (see Figure 2). Then, $S = xz_1...z_ktv_3v_2v_1x$ is a good cycle and separates t' and v_4 in G. Indeed, t' cannot lie on S by Lemma 2.4. Therefore, S is a separating good cycle of size from 8 to 11 in G, which contradicts Lemma 2.1.

Suppose that a new 9-cycle occurs by identifying x and t and this 9-cycle is adjacent to an original 7-cycle in G, i.e., a special 7-cycle occurs in G^*. We may suppose that $S^* = v^* z_1 ... z_8 v^*$ is such a 9-cycle. Then, $S = xz_1...z_8tv_3v_2v_1x$ is a 13-cycle in G. If t' is a vertex of S in G, then the 13-cycle will have a $(7, 3, 7)$-bichord. In this case, there exists an original 7-cycle which is adjacent to another original 7-cycle in G, a contradiction. So $t^{'}$ is not a vertex of S in G. By Lemma 2.1, S must be a bad cycle. Clearly, the bad cycle has a $(3, 8, 8)$-claw or a $(3, 8, 3, 9)$-biclaw or a $(3, 8, 8, 3, 8, 3)$-triclaw or a $(3, 3, 8, 3, 8, 3, 9)$-combclaw by Lemma 2.4. If t' is not the center vertex of the bad partition about S, then there will exist a separating good cycle in G, which contradicts Lemma 2.1. Assume that $t^{'}$ is the center vertex of the bad partition. According to the existence of $t^{'}$, there would exist a special 7-cycle in G (see Figure 3, for example, when the bad cycle S has a $(3, 8, 8)$-claw), a contradiction to our assumption. So, no 7-cycle is adjacent to a 7-cycle or a 9-cycle in G^*.

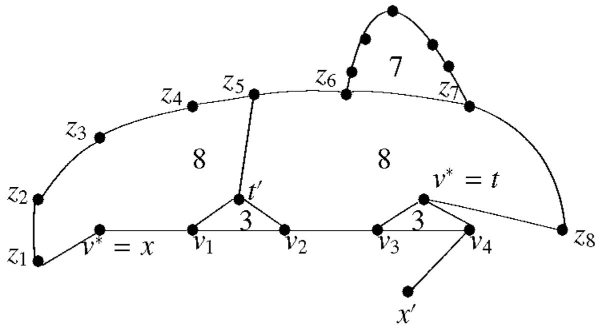

Figure 3. $|S| = 13$, and there is a $(3, 8, 8)$-claw of S. In this case there exists a special 7-cycle in G.

Now we claim that the identification does not create a chord of C_0 or identify two vertices of C_0. Suppose the identification creates a chord, then either x or t must be on C_0. Without loss of generality, assume that $x \in C_0$ and $t'' \in C_0$ is a neighbor of t different from v_3 and v_4. Then, there exists a path $P = t^{''}tv_3v_2v_1x$ that divides C_0 into two paths P_1 and P_2. Obviously, $t^{'}$ is not on C_0 since G has no 4-,5-,6-cycles and special 7-cycles. Since $|C_0| \leq 13$, $|P_1| + 5 + |P_2| + 5 \leq 23$, then either $P \cup P_1$ or $P \cup P_2$ is a cycle of length at most 11 that separates t' and v_4, a contradiction to Lemma 2.1. Suppose the identification identifies two vertices of C_0, and the discussion is the same as above. So, C_0 is still the boundary of G^* and at most one vertex in $\{x, t\}$ is on C_0.

Now, we claim that C_0 is still a good cycle of G^*. Suppose that C_0 is bad in G^*. Obviously, the bad partition is created by identifying x and t and the new vertex v^* is a vertex of the bad partition. By Lemma 2.4, $|C_0| = 12$ or $|C_0| = 13$. If $C_0 = u_1u_2...u_{13}u_1$ has a $(3, 8, 8)$-claw T and v^* is the center of $(3, 8, 8)$-claw, then the bad partition $G^*[C_0 \cup T]$ can be obtained from the graph in Figure 4 by deleting v_1, v_2, v_3, and v_4 and identifying x and t. Then, both $C_1 = xu_2...u_8tv_3v_2v_1x$ and $C_2 = xv_1v_2v_3tu_8...u_{13}u_1x$ are separating 12-cycles of G. By Lemma 2.1, they are bad cycles. So, by Lemma 2.4, C_i has a $(3, 8, 3, 8)$-biclaw or a $(3, 3, 8, 3, 8, 3, 8)$-comclaw in G for $i = 1, 2$, then G is a finite graph with at most 30 vertices since G has no separating good cycles. In these finite graphs, we can easily check that G is not a counterexample to Theorem 1.2. If v^* is the other vertex of the bad partition or C_0 has a biclaw or a triclaw or a combclaw in G^*, we can also find a contradiction although the discussion is a bit tedious.

Note that at most one vertex in $\{x, t\}$ is on C_0, the coloring c of C_0 is not damaged by identifying x and t. Therefore, $G^* \in \mathbf{G}$, and we get a 3-coloring of G^* by extending c. Next, observe that any 3-coloring c of G^* can be extended to a 3-coloring of G: We first color v_4 and v_3 (in this order), then x and v_3 get different colors. Next, we can easily color v_4 and v_3 no matter what the color of t' is. Thus, we get a 3-coloring of G, a contradiction. □

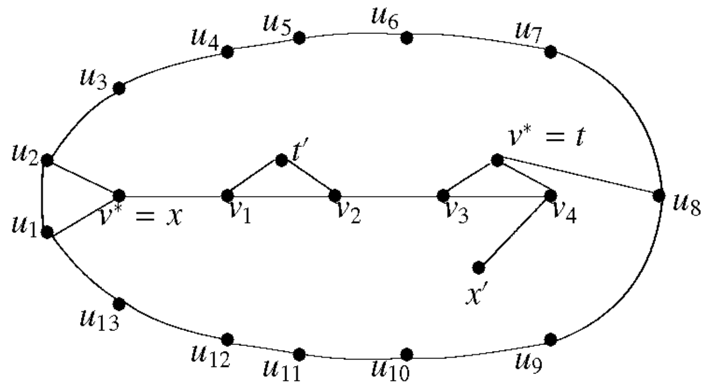

Figure 4. $|C_0| = 13$, and there will be a $(3, 8, 8)$-claw when identifying x and t.

Let f be an 8-face $v_1v_2...v_8v_1$, where $v_1, v_2, v_3, v_5, v_6, v_7$ are bad vertices and v_4, v_8 are good vertices. By the definition of bad vertices, f is internal. Assume that $v_2v_3t_{23}$, $v_5v_6t_{56}$, $v_7v_8t_{78}$, $v_1v_8t_{18}$ are 3-faces adjacent to f (see Figure 5, for example), where the degrees of v_4 and v_8 are not necessarily 4. Then, f is called an M-face.

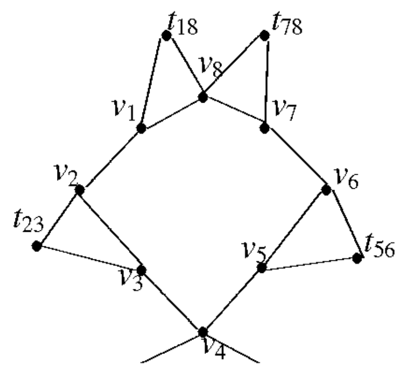

Figure 5. An M-face.

Lemma 2.6. *G cannot have an M-face.*

Proof. Suppose that f is an M-face of G as in Figure 5. We obtain G^* from G by deleting $v_1, v_2, v_3, v_5, v_6, v_7$ and identifying v_4 and v_8. Let v^* denote the new vertex obtained by identifying v_4 and v_8. To prove that G^* is in $\mathbf{G}$, we now prove that G^* has no cycles of length from 4 to 6 and special 7-cycles. If a new cycle of size from 4 to 7 occurs in G^*, we may suppose that $S^* = v^* z_1 ... z_k v^*$ is such a cycle, where $3 \leq k \leq 6$. Then, $S = v_8 z_1 ... z_k v_4 v_5 v_6 v_7 v_8$ is an 11^--cycle and separates t_{56} and v_3 in G. Indeed, t_{56} cannot lie on S by Lemma 2.4. But this means that S is a separating good cycle of size from 7 to 11 in G by Lemma 2.4, which contradicts Lemma 2.1.

Suppose that a new 9-cycle occurs by identifying v_4 and v_8 and this 9-cycle is adjacent to an original 7-cycle in G, we may suppose that $S^* = v^* z_1 ... z_8 v^*$ is such a cycle. Then, $S = v_8 z_1 ... z_8 v_4 v_5 v_6 v_7 v_8$ is a 13-cycle in G. Obviously, neither t_{56} nor t_{78} is a vertex of S. Suppose that t_{56} is a vertex of S. Since G has no 4-,5-,6-cycle and special 7-cycle, S has a $(7, 3, 7)$-bichord. In this case, there will be a 7-cycle separating t_{78} and v_3, a contradiction to Lemma 2.1. Suppose that t_{78} is a vertex of S. Then, $t_{78} = z_1$ and $v_7 z_1 ... z_8 v_4 v_5 v_6 v_7$ is a bad 12-cycle in G. At this time, we can get a 9-cycle in G which is adjacent to an original 7-cycle in G, a contradiction. Therefore, neither t_{56} nor t_{78} is a vertex of S. In addition, neither t_{18} nor t_{23} is a vertex of S by symmetry. By the planarity, we may assume that t_{56} and t_{78} are inside S and t_{18} and t_{23} are outside S. By Lemma 2.1 and Lemma 2.4, the cycle S is bad and has a bad partition. If t_{56} or t_{78} is not the center vertex of the bad partition, there will exist a separating good 9^--cycle in G, which contradicts Lemma 2.1. So both t_{56} and t_{78} are the center vertex of the bad partition, and the bad cycle can only have a $(3, 8, 3, 9)$-biclaw or a $(3, 8, 8, 3, 8, 3)$-triclaw or a $(3, 3, 8, 3, 8, 3, 9)$-combclaw. This is impossible according to the distance of the path between t_{56} and t_{78} through S. So, there is no special 7-cycles in G^*.

Now we claim that the identification does not create a chord of C_0 or identify two vertices of C_0. Suppose the identification creates a chord, then either v_4 or v_8 must be on C_0. Without loss of generality, assume that $v_8 \in C_0$ and $v' \in C_0$ is a neighbor of v_4 different from v_3 and v_5. Then, there exists a path $P = v' v_4 v_5 v_6 v_7 v_8$ that divides C_0 into two paths P_1 and P_2. Since $|C_0| \leq 13$, $|P_1| + 5 + |P_2| + 5 \leq 23$. Then either $P \cup P_1$ or $P \cup P_2$ is a cycle of length at most 11 and also a separating cycle, a contradiction to Lemma 2.1 and Lemma 2.4. Suppose the identification identifies two vertices of C_0, and the discussion is the same as above. So C_0 is still the boundary of G^* and at most one vertex in $\{v_4, v_8\}$ is on C_0.

Suppose that C_0 is bad in G^*. Obviously, the bad partition is created by identifying v_4 and v_8 and the new vertex v^* is a vertex of the bad partition. By Lemma 2.4, $|C_0| = 12$ or $|C_0| = 13$. Furthermore, if $|C_0| = 13$, then C_0 has a $(3, 8, 8)$-claw or a $(3, 8, 3, 9)$-biclaw or a $(3, 8, 8, 3, 8, 3)$-triclaw or a $(3, 3, 8, 3, 8, 3, 9)$-combclaw in G^*; if $|C_0| = 12$, then C_0 has a $(3, 8, 3, 8)$-biclaw or a $(3, 3, 8, 3, 8, 3, 8)$-combclaw in G^*. For example, a claw of C_0 in G^* is obtained from the graph in Figure 6 by deleting $v_1, v_2, v_3, v_5, v_6, v_7$ and identifying v_4 with v_8. By considering every case of the location of v^* in the bad partition of C_0 in G^*, we can always find a contradiction as the proof in Lemma 2.5.

Note that at most one vertex in $\{v_4, v_8\}$ is on C_0, and the coloring c of C_0 is not damaged by identifying v_4 and v_8. Therefore, $G^* \in \mathbf{G}$, and we get a 3-coloring of G^* by extending c. Let c be an arbitrary 3-coloring of G^* and we may assume that $c(v_4) = c(v_8) = 1$ and $c(t_{18}) = 2$. First we color v_1 and v_7. Since $c(v_4) \neq c(v_1)$ and $c(v_4) \neq c(v_7)$, we can easily extend this coloring to v_2, v_3, v_5, v_6, and get a 3-coloring of G. □

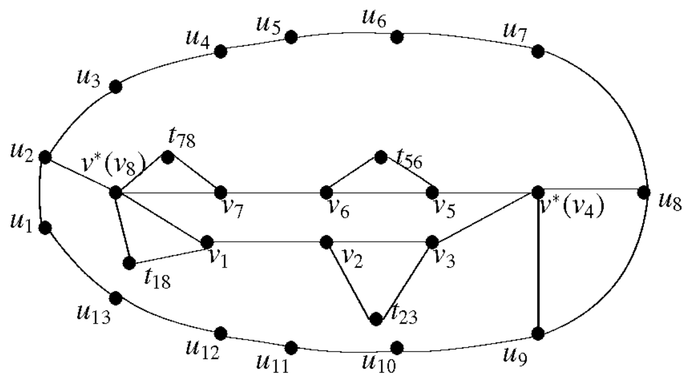

Figure 6. $|C_0| = 13$, and there is a $(3, 8, 8)$-claw when identifying v_4 and v_8.

Let f be an 8-face with boundary $v_1v_2...v_8v_1$, where $v_1, v_3, v_4, v_6, v_7, v_8$ are bad, while v_2, v_5 are internal 4-vertices. Assume that $v_2v_3t_{23}$,$v_4v_5t_{45}$, $v_5v_6t_{56}$,$v_7v_8t_{78}$, $v_1v_2t_{12}$ are 3-faces adjacent to f (see Figure 7). Then, f is called an MM-face.

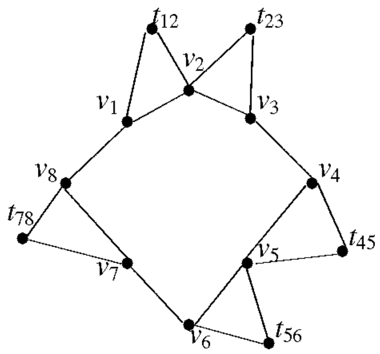

Figure 7. An MM-face.

Lemma 2.7. *G cannot have an MM-face.*

Proof. Let f be an MM-face as in Figure 7. We obtain G^* from G by deleting $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ and identifying t_{23} and t_{56}. Let v^* denote the new vertex obtained by identifying t_{23} and t_{56}. To prove that G^* is in $\mathbf{G}$, we first prove that G^* has no cycles of length from 4 to 6 and special 7-cycles. If a new cycle of size from 4 to 7 occurs in G^*, we may suppose that $S^* = v^*z_1...z_kv^*$ is such a cycle, where $3 \leq k \leq 6$. Then, $S = t_{23}z_1...z_kt_{56}v_5v_4v_3t_{23}$ is a cycle separating t_{45} and v_6 in G. Indeed, t_{45} cannot lie on S by Lemma 2.4. However, this means that S is a separating good cycle of size from 7 to 11 in G, which contradicts Lemma 2.1.

Suppose that a new 9-cycle occurs by identifying t_{23} and t_{56} and this 9-cycle is adjacent to another original 7-cycle of G, and we may suppose that $S^* = v^*z_1...z_8v^*$ is such a cycle. Then, $S = t_{23}z_1...z_8t_{56}v_5v_4v_3t_{23}$ is a 13-cycle of G. If t_{45} is a vertex of S in G, then the 13-cycle will have a

(7, 3, 7)-bichord. In this case, there exists a special 7-cycle in G, a contradiction. So t_{45} is not a vertex of S in G. By Lemma 2.1, the cycle must be a bad cycle, then the bad cycle has a (3, 8, 8)-claw or a (3, 8, 3, 9)-biclaw or a (3, 8, 8, 3, 8, 3)-triclaw or a (3, 3, 8, 3, 8, 3, 9)-combclaw. If t_{45} is not the center vertex of the bad partition, then there will exist a separating good 9^{-}-cycle in G, which contradicts Lemma 2.1. If t_{45} is the center vertex of the bad partition, there would exist a 9-cycle adjacent with a 7-cycle in G (see Figure 8, for example, when the bad cycle has a (3, 8, 8)-claw), a contradiction to our assumption. So there is no special 7-cycles in G^*.

Now we claim that the identification does not create a chord of C_0 or identify two vertices of C_0. Suppose the identification creates a chord, then either t_{23} or t_{56} must be on C_0. Without loss of generality, assume that $t_{23} \in C_0$ and $t'' \in C_0$ is a neighbor of t_{56} different from v_5 and v_6. Then, there exists a path $P = t''t_{56}v_5v_4v_3t_{23}$ that divides C_0 into two paths P_1 and P_2. Since $|C_0| \leq 13$, $|P_1| + 5 + |P_2| + 5 \leq 23$, then either $P \cup P_1$ or $P \cup P_2$ is a cycle of length at most 11 that separates t_{45} and v_6, a contradiction to Lemma 2.1. Suppose the identification identifies two vertices of C_0, and the discussion is the same as above. So, C_0 is still the boundary of G^* and at most one vertex in $\{t_{23}, t_{56}\}$ is on C_0.

Suppose C_0 is bad in G^*. Obviously, the bad partition is created by identifying t_{23} and t_{56} and the new vertex v^* is a vertex of the bad partition. By Lemma 2.4, $|C_0| = 12$ or $|C_0| = 13$. Furthermore, if $|C_0| = 13$, then C_0 has a (3, 8, 8)-claw or a (3, 8, 3, 9)-biclaw or a (3, 8, 8, 3, 8, 3)-triclaw or a (3, 3, 8, 3, 8, 3, 9)-combclaw of G^*; if $|C_0| = 12$, then C_0 has a (3, 8, 3, 8)-biclaw or a (3, 3, 8, 3, 8, 3, 8)-combclaw in G^*. For example, a claw of C_0 in G^* is obtained from the graph in Figure 9 by deleting $v_1, v_2, v_3, v_5, v_6, v_7, v_8$ and identifying t_{23} and t_{56}. By considering every case of the location of v^* in the bad partition of C_0 in G^*, we can always find a contradiction as the proof in Lemma 2.5.

Note that at most one vertex in $\{t_{23}, t_{56}\}$ is on C_0, and the coloring c of C_0 is not damaged by identifying t_{23} and t_{56}. Therefore, $G^* \in \mathbf{G}$ and we get a 3-coloring of G^* by extending c. Next, observe that any 3-coloring c of G^* can be extended to a 3-coloring of G as follows. Let c be an arbitrary 3-coloring of G^* and assume that $c(t_{23}) = c(t_{56}) = 1$. If $c(t_{45}) \neq 1$, we first color v_5, v_4, and v_6, then we color v_7, v_8, v_1 and v_2 according to the colors of t_{78} and t_{12}, and finally color v_3. If $c(t_{45}) = 1$, we put $1 \neq c(v_8) = c(v6) = c(v_4) \neq c(t_{78})$, then color v_1, v_5, v_7, and, finally, color v_2 and v_3. □

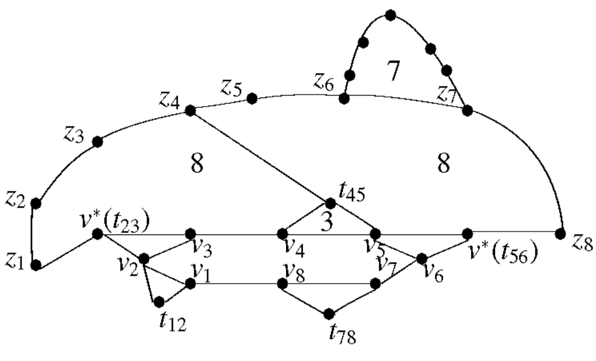

Figure 8. There is a (3, 8, 8)-claw in S and a special 7-cycle in G.

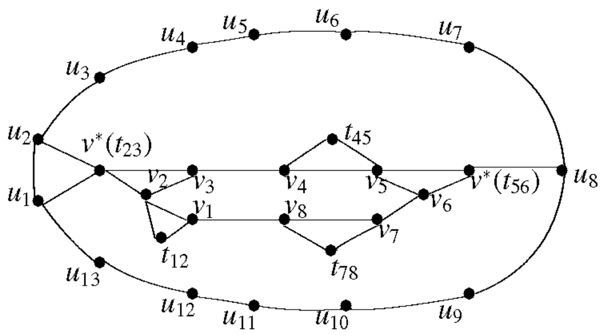

Figure 9. $|S| = 13$, and there is a $(3, 8, 8)$-claw when identifying t_{23} and t_{56}.

3. Incompatibility of the basic properties

The rest of our proof will show that the minimal counterexample G in the previous section is incompatible. Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ for G may be rewritten as $\sum_{v\in V(G)}(d(v) - 4) + \sum_{f\in F(G)}(|f| - 4) = -8$. Let $f_0 = C_0$ be the unique exterior face of G. We set the *initial charge* of every vertex v of G to be $ch(v) = d(v) - 4$, the initial charge of every face $f \neq f_0$ to be $ch(f) = |f| - 4$, and put $ch(f_0) = |f_0| + 4$. Clearly, $\sum_{x\in V(G)\cup F(G)} ch(x) = 0$.

Then, we use the discharging procedure to get a final charge ch^* by applying the following rules:

R1. Each 3-face gets $\frac{1}{3}$ from each incident vertex.

R2. Let v be an internal 3-vertex. If v is a bad vertex, then v gets $\frac{2}{3}$ from each 7^+-face containing v. Otherwise, v gets $\frac{1}{3}$ from each 7^+-face containing v.

R3. Let v be an internal 4-vertex. If v is incident with only one 3-face, then v gets $\frac{1}{3}$ from the unique face f which contains v and is not adjacent to this 3-face; if v is incident with two 3-faces, then v gets $\frac{1}{3}$ from each 7^+-face containing v.

R4. If v is an external 2-vertex, v gets $\frac{2}{3}$ from the internal face containing v.

R5. Each external 3-vertex which is not incident with any 3-face gives $\frac{1}{3}$ to the internal 7-face containing v.

R6. Let v be an external 4^+-vertex. v gives $\frac{1}{3}$ to each internal 7^+-face containing v.

R7. The outside face f_0 sends $\frac{4}{3}$ to each vertex of f_0.

After the discharging of rules from R1 to R7 is completed, the discharging of rule R8 is performed.

R8. Each internal face which is adjacent to f_0 gives the surplus charge to f_0 and each external 4^+-vertices on f_0 gives the surplus charge to f_0.

We use ch^{**} to represent the value of the face or vertex after the discharging of rules from R1 to R7 is completed and before the discharging of rule R8 is performed. We have the following lemmas.

Lemma 3.1. *If $v \in V(G)$, then $ch^*(v) \geq 0$.*

Proof. Suppose $d(v) = 2$, then by Lemma 2.3, it belongs to C_0 and it is not incident with a 3-face. Therefore, rules R4 and R7 are applied to v and $ch^*(v) = 2 - 4 + \frac{2}{3} + \frac{4}{3} = 0$.

Let $d(v) = 3$. If $v \in C_0$, then v receives $\frac{4}{3}$ from C_0 by R7. If $|C_0| = 3$, there are two 7^+-faces containing v. In this case, $ch^*(v) = 3 - 4 - \frac{1}{3} + \frac{4}{3} = 0$ by R1 and R7. Now suppose that $|C_0| \geq 7$. If v is not incident with any 3-face, then there are two internal 7^+-faces containing v. If a 7^+ face is exactly a 7-face, then this 7-face is unique since G has no special 7-cycles. In this case, $ch^*(v) = 3-4-\frac{1}{3}+\frac{4}{3} = 0$ by R5 and R7. So, we may suppose that there is no 7-face containing v. In this case, $ch^*(v) = 3-4+\frac{4}{3} = \frac{1}{3}$ by R7. If v is incident with an internal 3-face, there is one internal 7^+-faces containing v. In this case, $ch^*(v) = 3 - 4 - \frac{1}{3} + \frac{4}{3} = 0$ by R1 and R7. So we assume that $v \notin C_0$. If v is incident with a 3-face, then $ch^*(v) = 3 - 4 - \frac{1}{3} + 2 \times \frac{2}{3} = 0$ by R1 and R2. If v is not incident with any 3-face, then $ch^*(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$ by R2.

Let $d(v) = 4$. If $v \in C_0$, then v receives $\frac{4}{3}$ from f_0 by R7. In addition, if C_0 is a 3-face, then every face including v gets $\frac{1}{3}$ from v by R1 and R6. In this case, $ch^*(v)=ch^{**}(v) = 4 - 4 - 4 \times \frac{1}{3} + \frac{4}{3} = 0$ by R1, R6, R7, and R8. Now suppose that $|C_0| \geq 7$. Then, v gives $\frac{1}{3}$ to each internal face including v by R1 and R6. Thus, $ch^{**}(v) = 4 - 4 + \frac{4}{3} - 3 \times \frac{1}{3} = \frac{1}{3}$ and $ch^*(v) = 0$ by R8. So we assume that $v \notin C_0$. If v is not incident with a 3-face, then $ch^*(v) = ch(v) = 0$. If v is incident with only one 3-face, then v gets $\frac{1}{3}$ by R3 and gives $\frac{1}{3}$ by R1, then $ch^*(v) = 4 - 4 - \frac{1}{3} + \frac{1}{3} = 0$. If v is incident with two 3-faces, then v gets $\frac{1}{3} \times 2$ by R3 and gives $\frac{1}{3} \times 2$ by R1, then $ch^*(v) = 4 - 4 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 0$.

Let $d(v) = 5$ and l denote the number of 3-faces incident with v. Apparently, $l \leq 2$. If $v \in f_0$, v receives $\frac{4}{3}$ from f_0 by R7. Further, we consider the case that $|f_0| = 3$. Note that v gives $\frac{1}{3}$ to each internal 7^+-face containing v by R6 and gives $\frac{1}{3}$ to each 3-face containing v by R1, then $ch^{**}(v) = 5 - 4 - 5 \times \frac{1}{3} + \frac{4}{3} = \frac{2}{3}$. By R8, we have that $ch^*(v) = 0$. If $|C_0| \geq 7$, by R1 and R6, we have that $ch^{**}(v) = 5 - 4 - 4 \times \frac{1}{3} + \frac{4}{3} = 1$. By R8, we have that $ch^*(v) = 0$. So we assume that $v \notin C_0$. Since v gives $\frac{1}{3}$ to each 3-face containing v by R1, then $ch^*(v) \geq 5 - 4 - 2 \times \frac{1}{3} = \frac{1}{3}$.

Let $d(v) \geq 6$ and l denote the number of 3-faces incident with v. Apparently, $l \leq \lfloor \frac{d(v)}{2} \rfloor$. If $v \in C_0$, then v receives $\frac{4}{3}$ from f_0 by R7. If $|C_0| = 3$, then $ch^{**}(v) = d(v)-4-l\times\frac{1}{3}-(d(v)-l)\times\frac{1}{3}+\frac{4}{3} = \frac{2d(v)-8}{3} > 0$ by R1, R6, and R7 and $ch^*(v) = 0$ by R8. If $|C_0| \geq 7$, then there are $(d(v) - l - 1)$ internal 7^+-faces containing v. In this case, $ch^{**}(v) = d(v) - 4 + \frac{4}{3} - (d(v) - l - 1) \times \frac{1}{3} - l \times \frac{1}{3} = \frac{2d(v)-7}{3} > 0$ by R1, R6, and R7. Further, we have that $ch^*(v) = 0$ by R8. If $v \notin C_0$, then $ch^*(v) \geq d(v) - 4 - \lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{3} > 0$ by R1. □

Lemma 3.2. *If $f \in F(G)$ and $f \neq f_0$, then $ch^*(f) \geq 0$.*

Proof. If $|f| = 3$, then $ch^*(f) = |f| - 4 + 3 \times \frac{1}{3} = 0$ by R1. Suppose that $|f| \geq 12$. Since f sends at most $\frac{2}{3}$ to each incident vertex by R2, R3, and R4, we have $ch^{**}(f) = |f| - 4 - |f| \times \frac{2}{3} = \frac{|f|-12}{3} \geq 0$. If f is adjacent with f_0, $ch^*(f) = 0$. Otherwise, $ch^*(f) = ch^{**}(f)$. In the following, we may assume that $|f| \leq 11$.

If $8 \leq |f| \leq 11$ and f is incident with at least two 3^+-vertex of C_0, then the two vertices get nothing from f by the discharging rules. Therefore, $ch^{**}(f) \geq |f| - 4 - (|f| - 2) \times \frac{2}{3} \geq \frac{|f|-8}{3} \geq 0$ and $ch^*(f) = 0$ by R8. Thus, from now on, we assume that every 8^+-face is not incident with two 3^+-vertices of C_0. Hence, every 8^+-face is also not incident with a 2-vertex since f must have two external 3^+-vertex if f has a 2-vertex.

If $9 \leq |f| \leq 11$ and f has only one external vertex, this vertex must be a 4^+-vertex. Then, f receives $\frac{1}{3}$ from this vertex by R6, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{3} - (|f| - 1) \times \frac{2}{3} \geq 0$. Thus, we assume that every 9^+-face is not incident with an external vertex. Let t denote the number of bad vertices on f. When $|f| = 11$ or 10, by Lemma 2.5, we have that $t \leq 8$. Then, $ch^*(f) \geq 11 - 4 - 8 \times \frac{2}{3} - 3 \times \frac{1}{3} = \frac{2}{3}$ if $|f| = 11$, and $ch^*(f) \geq 10 - 4 - 8 \times \frac{2}{3} - 2 \times \frac{1}{3} = 0$ if $|f| = 10$. When $|f| = 9$, then $t \leq 7$ by Lemma 2.5. If there exist precisely seven bad vertices on the 9-face, those seven bad vertices must be divided into four consecutive bad vertices and three consecutive bad vertices by two good vertices by Lemma 2.5.

Thus, the unique form of the 9-face is as Figure 10. In Figure 10, v_5 takes nothing from the 9-face f by R3 no matter $d(v_5) = 4$ or $d(v_5) \geq 5$ and, thus, $ch^*(f) = ch^{**}(f) = 9 - 4 - 7 \times \frac{2}{3} - \frac{1}{3} = 0$. Apparently, when $t < 7$, $ch^*(f) > 0$.

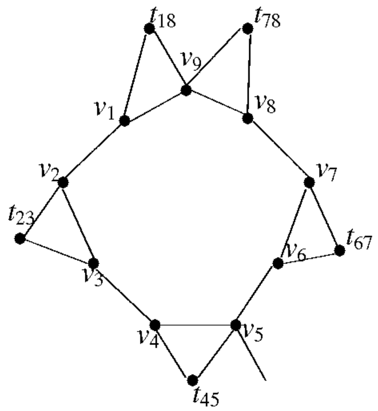

Figure 10. 9-face.

Next, suppose that $|f| = 8$. Let t denote the number of bad vertices on f. If all the vertices on f are internal, by Lemma 2.5, we have that $t \leq 6$. If $t \leq 4$, then $ch^*(f) \geq 8 - 4 - 4 \times \frac{1}{3} - 4 \times \frac{2}{3} = 0$. Suppose that f is incident with 5 bad vertices. If at least one good vertex of f fails to take $\frac{1}{3}$ from f, then $ch^*(f) \geq 8 - 4 - 2 \times \frac{1}{3} - 5 \times \frac{2}{3} = 0$. Therefore, we may suppose that each good vertex takes $\frac{1}{3}$ from f. It follows that all the good vertices are internal 4^--vertices by R2 and R3. However, it is impossible to draw out such an 8-face.

Now suppose that f is incident with 6 bad vertices. These six vertices must be divided into either four consecutive bad vertices or three consecutive bad vertices by two good vertices since five consecutive bad vertices on f will form a tetrad. Now, we consider the following two cases.

Case 1: There are four consecutive bad vertices v_1, v_2, v_3, and v_4 and two consecutive bad vertices v_6 and v_7. If v_1 and v_2 are in the same 3-face $[v_1v_2t_{12}]$, then v_3 and v_4 must be in the same 3-face $[v_3v_4t_{34}]$, then we get a tetrad, a contradiction to Lemma 2.5. So v_1 and v_8 are in the same 3-face $[v_1v_8t_{18}]$, v_2 and v_3 are in the same 3-face $[v_2v_3t_{23}]$, and v_4 and v_5 are in the same 3-face $[v_4v_5t_{45}]$. In addition, if v_6 and v_7 are in the same 3-face $[v_6v_7t_{67}]$, then the two good vertices v_5 and v_8 get nothing from f by rules. In this case, $ch^*(f) = 8 - 4 - 6 \times \frac{2}{3} = 0$ (see Figure 11 (a)). So we assume that v_5, v_6 are in the same 3-face and v_7, v_8 are in the same 3-face. Observe that $d(v_5) \geq 4$ and $d(v_8) \geq 4$. If $d(v_5) > 5$ or $d(v_8) > 5$, then $ch^*(f) = 8 - 4 - 6 \times \frac{2}{3} - 1 \times \frac{1}{3} + \frac{1}{3} = 0$ by R3 and R2 (see Figure 11 (b)). Therefore, we just assume that both v_5 and v_8 are internal 4-vertices and we have the graph in Figure 7. However, this is an MM-face, contrary to Lemma 2.7. Case 2: There are three consecutive bad vertices v_1, v_2, and v_3. Then, the other three consecutive bad vertices are v_5, v_6, and v_7. In the same way as in Case 1, we can easily get that $ch^*(f) \geq 0$ or f is an M-face (as in Figure 5).

Therefore, we suppose that f is incident with exactly one external vertex v by our assumption. Clearly, v belongs to C_0 and $d(v) \geq 4$, then f gets $\frac{1}{3}$ from v by R6 and gives at most $\frac{2}{3}$ to each internal vertex by R2. In this case, $ch^*(f) \geq 8 - 4 - 6 \times \frac{2}{3} - 1 \times \frac{1}{3} + \frac{1}{3} = 0$.

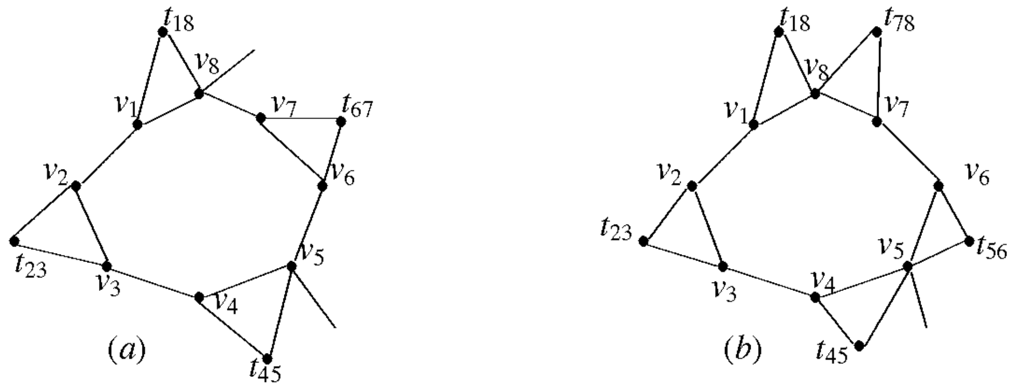

Figure 11. Two graphs (a) and(b) about the 8-face.

Finally, let $|f| = 7$. Since there are no special 7-cycles in G, the 7-face is adjacent to at most one 3-face and, thus, has at most two bad vertices. If all the vertices on f are internal vertices, then $ch^*(f) \geq 7 - 4 - 2 \times \frac{2}{3} - 5 \times \frac{1}{3} = 0$ by R2 and R3. So, we assume that f is incident with at least one external vertex. If f is incident with exactly one external vertex v, clearly, v belongs to C_0, $d(v) \geq 4$ and f gets $\frac{1}{3}$ from v by R6. In this case $ch^*(f) \geq 7 - 4 - 2 \times \frac{2}{3} - 4 \times \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Now suppose that f is incident with exactly two external vertices. Then, the two external vertices are 3^+-vertices and at most one external 3^+-vertex of f are in a 3-face since G has no special 7-cycles. In this situation, we can easily get that $ch^{**}(f) \geq \frac{4}{3}$ by R2, R3, R5, R6, and $ch^*(f) = 0$ by R8. So we suppose that f is incident with at least three external vertices. Then, there exist two external 3^+-vertices on f, and the rest external vertices are 2-vertices. In addition, there are at most five external 2-vertices since f is a 7-face, and, at most one external 3^+-vertices of f is in a 3-face since G has no special 7-cycles. Thus, we get that $ch^{**}(f) \geq 7 - 4 - 5 \times \frac{2}{3} + \frac{1}{3} = 0$ by R2, R3, R4, R5, and R6 and $ch^*(f) = 0$ by R8. □

By Lemma 3.1 and Lemma 3.2, it is sufficient for us to check that the external face $f_0 = C_0$ has positive final charge.

Lemma 3.3. $ch^*(f_0) > 0$

Proof. Recall that $ch(f_0) = |f_0| + 4$. If $|f_0| \leq 11$, $ch^*(f_0) \geq |f_0| + 4 - |f_0| \times \frac{4}{3} = \frac{12-|f_0|}{3} > 0$ by R7. It is not difficult to see that $ch^{**}(f_0) = 0$ when $|f_0|$=12 and $ch^{**}(f_0) = -\frac{1}{3}$ when $|f_0| = 13$. In the following, we may assume that $|f_0| = 12$ or 13.

Let v be a vertex on f_0. If $d(v) \geq 5$, then $ch^{**}(v) \geq \frac{2}{3}$ (see the proof of Lemma 3.1) and $ch^*(f_0) > 0$ by R8. If $d(v) = 4$, then $ch^{**}(v) \geq \frac{1}{3}$ when $|f_0| \geq 7$ (see the proof of Lemma 3.1). So, if f_0 has at least two (or one) 4-vertices when $|f_0| = 13$ (or 12), we have that $ch^*(f_0) > 0$ by R8. Hence, we may assume that f_0 has only 4^--vertices and at most one 4-vertex when $|f_0| = 13$ and f_0 has only 3^--vertices when $|f_0| = 12$ in the following. Let f be a 9^+-face which is adjacent to f_0. Then, f is incident with at least two 3^+-vertex of C_0 and the two vertices get nothing from f by the discharging rules. Therefore, $ch^{**}(f) \geq |f| - 4 - (|f| - 2) \times \frac{2}{3} \geq \frac{|f|-8}{3} \geq \frac{1}{3}$ (see the proof of Lemma 3.2). So, when $|f_0| = 12$ and f is adjacent to at least one 9^+-face or $|f_0| \geq 13$ and f is adjacent to at least two 9^+-faces, we have that $ch^*(f_0) > 0$ by R8. Thus, from now on, we assume that f_0 is adjacent to only 8^--face when $|f_0| = 12$, and, f_0 is adjacent to only 9^{-}-faces and at most one 9-face when $|f_0| = 13$.

If f is a 7-face which is adjacent to f_0, then f and f_0 share at most five edges since the external face f_0 has no chord by Lemma 2.3. In the same way as the proof in Lemma 3.2, if f and f_0 share exactly one or two or three or four or five edges, then $ch^{**}(f) \geq \frac{4}{3}$ or $\frac{1}{3}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or 0, respectively. If f is an 8-face which is adjacent to f_0, then f and f_0 share at most six edges by Lemma 2.3. As the proof in Lemma 3.2, if f and f_0 share exactly one or two or three or four or five or six edges, then $ch^{**}(f) \geq \frac{1}{3}$ or $\frac{1}{3}$ or $\frac{1}{3}$ or $\frac{1}{3}$ or 0 or 0, respectively. If f is a 9-face which is adjacent to f_0, then f and f_0 share at most seven edges by Lemma 2.3. As the proof in Lemma 3.2, if f and f_0 share exactly one or two or three or four or five or six or seven edges, then $ch^{**}(f) \geq \frac{2}{3}$ or 1 or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{3}$, respectively. So, if $|f_0| = 12$ or 13, by our assumption on f_0 and some analysis, we can easily find some faces which make $ch^{*}(f_0) > 0$. □

This completes the proof of Theorem 1.2.

4. Conclusions

About Steinberg's conjecture, it is left with the following open problem: whether every planar graph without cycles of length from 4 to 6 is 3-colorable. In this paper, we prove that every planar graph without cycles of length from 4 to 6 and without special 7-cycles is 3-colorable, which is an improvement of Borodin's result.

Author contributions

Zuosong Liang: Conceptualization, Investigation, Methodology, Writing-original draft, Writing-review, Validation; Danzhang Liao: Validation, Writing-original draft; Chunsong Bai: Conceptualization, Investigation, Validation.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All of the authors have not disclosed any competing interests.

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