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Research article

Power properties of classical test statistics in Weibull regression models with censoring and their applications to sample size calculation

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Abstract: Regression models for time-to-event data are widely used in clinical and reliability studies, particularly in the presence of censoring. In this context, the Weibull regression model provides a flexible alternative to the proportional hazards model, allowing for a fully specified survival function and interpretable measures of treatment effects. However, inference based on classical test statistics may be unreliable in small or moderate samples. In this paper, we derive closed-form approximations for the non-null asymptotic distributions of the likelihood ratio, Wald, score, and gradient tests under Pitman alternatives in Weibull regression models for censored data. These results facilitate analytical evaluation of local power and provide a basis for comparing the performance of the four tests. The proposed approximations are assessed through simulation studies, which highlight their accuracy in moderate-to-large samples and illustrate the impact of censoring and model complexity. An application is presented to the design of Phase II clinical trials to demonstrate how derived power functions can be used to estimate sample sizes. The results provide a computationally efficient tool for power analysis in censored Weibull regression models, although their use in practice should be complemented with simulation-based validation in small-sample or high-censoring scenarios.

Keywords: survival data; power function; sample size; gradient test; likelihood ratio test; score test; Wald test; Weibull model
Mathematics Subject Classification: 62F03, 62F12

1. Introduction

Survival regression models are widely applied in medicine and reliability for analyzing time-to-event outcomes. For instance, the survival time of a patient from the onset of a certain disease until death [1] the time until hepatocellular carcinoma's recurrence after liver transplantation [2], or the time to failure of a machine [3]. Such outcomes can be censored when the study finishes before the occurrence of the event of interest (Type I or time censoring) or when the study is completed after a specified number of failures has been observed (Type II or order statistic censoring). In both situations, the variable is said to be right-censored.

Although the conventional Cox proportional hazards (PH) model is used in medical research, the Weibull survival model is also considered as alternative in clinical trials, as its parametric structure makes it attractive for analytical derivations required in the study design and provides estimates of treatment effects in terms of the hazard ratio and the event/time ratio, which is a direct measure of the relative improvement in survival time that can be better understood by nonstatisticians [4].

In Phase II clinical trials, investigators aim to determine whether survival in the patients receiving a novel therapy improves relative to a historical control or a randomized control arm. Statisticians collaborating with those investigators need to calculate the sample size required to test the hypotheses of interest at a 5% significance level with 80% power. The Weibull distribution is an assumption that introduces more flexibility than the exponential distribution in the sample size calculations for single-arm trials [5, 6] and two-arm trials [7], while it is still simple enough to be used in a sample size calculation when often limited preliminary data from publications are available.

Accurate sample size determination critically depends on a precise characterization of the power of the test statistics, namely the likelihood ratio (LR) Wilks [8], Wald [9], score Rao [10], and gradient [11] tests. Under the null ($\mathcal{H}$) and alternative ($\mathscr{A}$) hypotheses, the distributions of each test statistic are asymptotically central and noncentral chi-squared, respectively. However, the chi-squared distribution may not be a good approximation for small or moderate sample sizes because it is asymptotic. Up to an error of order n^{-1}, where n is the sample size, the four statistics have the same size properties, but their local power differs in the $n^{-1/2}$ term. Therefore, although asymptotically equivalent, these tests may exhibit substantially different performance in practice, raising the question of which test should be preferred in applied settings.

The asymptotic properties of the Weibull regression model with censored data have been studied recently: The skewness coefficient of the maximum likelihood estimators (MLE) was obtained in Magalhães et al. [12]; improved test statistics for the LR, score, and gradient tests were presented in Magalhães and Gallardo [13]; and similar results limited to the exponential distribution were developed by several authors. The authors in [14–17], including computation of the asymptotic expansions up to order $n^{-1/2}$ of the non-null distribution functions of the four statistics and the second-order covariance matrix of the MLE [18]. Finally, Diniz et al. [19] derived refined estimators that allow statisticians to implement the multiple comparison procedures and modeling framework with small sample sizes. Nevertheless, explicit expressions for the non-null distribution functions of these test statistics are still not available, limiting the ability to perform analytical power comparisons and efficient sample size calculations.

The main goal of this work was to derive closed-form approximations for the non-null asymptotic distribution functions of the likelihood ratio, Wald, score, and gradient statistics in Weibull regression models with censored data. On the basis of these results, we derived tractable expressions for the power functions under local alternatives, enabling a detailed comparison of the four tests. Furthermore, we demonstrated how these approximations can be used to compute sample sizes in single-arm and two-arm Phase II clinical trials, providing a computationally efficient alternative to simulation-based methods such as bootstrapping.

The remainder of the paper is organized as follows. Section 2 presents the class of censored Weibull regression models and discusses the statistics for the LR, Wald, score, and gradient tests. Section 3 discusses the non-null asymptotic distribution for the four statistics under a sequence of Pitman alternatives and also discusses local power comparisons for the four statistics. In Section 4, we analytically compare the local power of the four statistics. In Section 5, we assess our findings under different scenarios. In Section 6, we show how our approximated closed-form expression for the power function can be used to calculate the sample size in single-arm and two-arm Phase II clinical trials with time-to-event as an endpoint. Finally, Section 7 discusses the main conclusions.

2. Weibull distribution

Let T be a non-negative random variable representing survival time. A common choice is to model T using the Weibull distribution [20], whose probability density function (PDF) is given by

$$f(t) = \sigma^{-1}\theta^{-1/\sigma}t^{1/\sigma-1}\exp\left\{-(t/\theta)^{1/\sigma}\right\},\ t > 0, \tag{2.1}$$

where $\sigma > 0$ and $\theta > 0$ are the shape and scale parameters, respectively. The exponential and Rayleigh distributions are special cases of (2.1), corresponding to $\sigma = 1$ and $\sigma = 1/2$, respectively. The assumption of a known shape parameter is common in the literature; see, for instance, [21, 22]. This assumption simplifies the derivations and allows us to obtain closed-form expressions for the non-null asymptotic distributions. In practice, σ is typically unknown, and it can be consistently estimated using the limited information available in publications [23]. When σ is replaced by a consistent estimator, its effect is expected to influence only higher-order terms in the asymptotic expansions, so that the leading terms—and hence the local power ordering of the tests—remain unchanged. Nevertheless, this substitution may have an impact in small samples, and its theoretical treatment in the context of censored Weibull regression models remains an open problem. The survival function for the Weibull model is as follows:

$$S(t) = \exp\left\{-(t/\theta)^{1/\sigma}\right\},\ t > 0. \tag{2.2}$$

The lifetime model is defined by assuming independent observations $T_1, \ldots, T_n$ from (2.1) and $\log(\theta_i) = \boldsymbol{x}_i^\top\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is a p-vector of unknown parameters and $\boldsymbol{x}_i$ is a vector of predictors related to the ith observation, $i = 1, \ldots, n$. The log link function is invertible and differentiable. From the former, $\theta_i = \exp\left(\boldsymbol{x}_i^\top\boldsymbol{\beta}\right) > 0$.

In lifetime data, the censoring restriction is common, and we observe $T_i^\star = (T_i, \delta_i)$ instead of T_i, where $T_i^\star = \min(T_i, L_i)$ under right censoring, L_i is the censoring time that is independent of T_i, and δ_i is the failure indicator, $\delta_i = 1$ if $T_i \leq L_i$, or $\delta_i = 0$ otherwise for $i = 1, \ldots, n$. Moreover, we assume that the distribution of the random variables, L_i, does not depend on θ_i for $i = 1, \ldots, n$, i.e., a noninformative censoring.

We consider a hybrid censoring scheme where the study is finished when a prefixed number $r \leq n$ out of n observations have failed or when a prefixed time, say $L_1 = \cdots = L_n = L$, has been reached. Type I censoring is a particular case for $r = n$, and Type II censoring appears when $L_1, \ldots, L_n = +\infty$.

Therefore, the theoretical framework considered in this paper accommodates general right-censoring mechanisms. In what follows, the results are derived under the assumption of independent right censoring.

When analyzing data, it is often convenient to work with log lifetimes [24], i.e., $Y_i = \log(T_i)$ instead of T_i, and then the Weibull regression model can be written as

$$y_i = \mu_i + \sigma^{-1}\varepsilon_i, \tag{2.3}$$

where $\mu_i = \log(\theta_i) = \boldsymbol{x}_i^\top\boldsymbol{\beta}$ and the error term has a standard extreme value distribution with the PDF $f(\varepsilon_i) = \exp\{\varepsilon_i - \exp(\varepsilon_i)\}$. The location-scale regression model form in (2.3) is known as the accelerated lifetime model [25]. For the sake of brevity, the log-likelihood function $\ell(\beta)$, the score vector $\boldsymbol{U}_{\boldsymbol{\beta}}$, and the Fisher information matrix $\boldsymbol{K}_{\boldsymbol{\beta\beta}}$ are presented in the Appendix.

The MLE of $\boldsymbol{\beta}$, say $\widehat{\boldsymbol{\beta}}$, is obtained from the solution of $\boldsymbol{U}_{\beta} = \boldsymbol{0}$. The estimator $\widehat{\boldsymbol{\beta}}$ does not have a closed-form expression and can be obtained through numerical maximization of the log-likelihood function using, for instance, a Newton or quasi-Newton nonlinear optimization algorithm. For a sufficiently large sample size, $\widehat{\boldsymbol{\beta}} \sim \mathrm{N}_p\left(\boldsymbol{\beta}, \boldsymbol{K}_{\boldsymbol{\beta\beta}}^{-1}\right)$, approximately.

Considering the null hypothesis $\mathcal{H} : \boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^{(0)}$ tested against the alternative hypothesis $\mathcal{A} : \boldsymbol{\beta}_2 \neq \boldsymbol{\beta}_2^{(0)}$, a partition of the parameter vector is induced as $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top)^\top$, and we can denote the unrestricted and restricted MLEs of $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ as $\left(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2\right)$ and $\left(\widetilde{\boldsymbol{\beta}}_1, \boldsymbol{\beta}_2^{(0)}\right)$, respectively. The LR ($S_1$), Wald ($S_2$), score ($S_3$) and gradient ($S_4$) statistics for testing $\mathcal{H}$ can be expressed, respectively, as

$$\begin{aligned}
S_1 &= 2\sum_{i=1}^{n}\left\{\delta_i\,(\tilde{\mu}_i - \hat{\mu}_i) - e^{y_i/\sigma}\left(e^{-\hat{\mu}_i/\sigma} - e^{-\tilde{\mu}_i/\sigma}\right)\right\},\\
S_2 &= \left(\widehat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2^{(0)}\right)^\top \left(\widehat{\boldsymbol{R}}^\top \widehat{\boldsymbol{W}}\widehat{\boldsymbol{R}}\right)\left(\widehat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2^{(0)}\right),\\
S_3 &= \widetilde{\boldsymbol{v}}^\top \widetilde{\boldsymbol{W}}^{1/2}\boldsymbol{X}_2\left(\widetilde{\boldsymbol{R}}^\top \widetilde{\boldsymbol{W}}\widetilde{\boldsymbol{R}}\right)^{-1}\boldsymbol{X}_2^\top \widetilde{\boldsymbol{W}}^{1/2}\tilde{\boldsymbol{v}},\\
S_4 &= \sigma^{-1}\widetilde{\boldsymbol{v}}^\top \widetilde{\boldsymbol{W}}^{1/2}\boldsymbol{X}_2\left(\widehat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2^{(0)}\right),
\end{aligned} \tag{2.4}$$

where $\boldsymbol{R} = \boldsymbol{X}_2 - \boldsymbol{X}_1\boldsymbol{C}$, $\boldsymbol{C} = \left(\boldsymbol{X}_1^\top\boldsymbol{W}\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}_1^\top\boldsymbol{W}\boldsymbol{X}_2$ and $\boldsymbol{X} = [\boldsymbol{X}_1\ \boldsymbol{X}_2]$, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ of dimensions $n \times q$ and $n \times (p - q)$, respectively. If $\mathcal{H}$ is true, $S_j \sim \chi^2_{p-q}$, $j = 1, 2, 3, 4$; with an approximation error of the order n^{-1}.

Given a nominal significance level, say α, $\mathcal{H}$ is rejected if the respective test statistic is greater than the upper $100(1-\alpha)\%$ quantile of a chi-squared distribution with $p-q$ degrees of freedom. Montoril and Souza [26] presented the properties and some comparisons of these four statistics, including graphical illustrations, as an extension of the classical work of Buse [27]. In contrast to [26], which focuses on the general properties and graphical comparisons of the classical test statistics, our work derives explicit non-null asymptotic distribution functions in the context of Weibull regression models with censored data. This distinction is crucial, as it enables an analytical evaluation of power and has a direct application to sample size determination, which are not addressed in the aforementioned studies.

3. Non-null distribution functions

One approach to assess the asymptotic properties of test statistics when $\mathcal{H}$ is false is to consider an hypothetical situation in which the process generating the data varies systematically as the sample size increases. In the local power analysis, the distance between $\mathcal{H}$ and the data-generating process should diminish as $n^{-1/2}$ as the sample size tends to infinity. If this distance were kept fixed, then the power of all consistent tests would tend to 1 as n tends to infinity. A hypothesis test describing this hypothetical situation can be written using local alternatives, i.e.

$$\begin{cases} \mathcal{H}: & \boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^{(0)} \\ \mathcal{A}_n: & \boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^{(0)} + \boldsymbol{\epsilon}, \end{cases} \tag{3.1}$$

where $\boldsymbol{\epsilon} = (\epsilon_{q+1}, \ldots, \epsilon_p)^\top$ with $\epsilon_s = O(n^{-1/2})$, for $s = q+1, \ldots, p$. Local power analysis is the evaluation of the behavior of the power function of a hypothesis test in a neighborhood of the null hypothesis. More details on the development of local power can be found in McManus [28].

The non-null distribution functions of the LR, Wald, score, and gradient statistics for testing (3.1) can be expressed as

$$\mathbb{P}(S_i \le x) = G_{p-q,\lambda}(x) + \sum_{k=0}^{3} a_{ik} G_{p-q+2k,\lambda}(x) + O(n^{-1}), \tag{3.2}$$

where, without losing generality, $G_{m,\lambda}(\cdot)$ is the cumulative distribution function of a noncentral chi-squared distribution with m degrees of freedom and noncentrality parameter $\lambda = \mathrm{tr}\left\{\boldsymbol{K}_{\beta\beta}\boldsymbol{\epsilon}^*\boldsymbol{\epsilon}^{*\top}\right\}/2$. The geometric interpretation of λ can be seen in Davidson and MacKinnon [29], $\boldsymbol{\epsilon}^* = \left(\boldsymbol{K}_{\beta_2\beta_1}\boldsymbol{K}_{\beta_1\beta_1}^{-1} - \boldsymbol{I}_{p-q}\right)^\top \boldsymbol{\epsilon}$, $\boldsymbol{I}_{p-q}$ denotes an identity matrix of order $p-q$, $\mathrm{tr}\{\cdot\}$ denotes the trace operator, and $i = 1, 2, 3, 4$ refers to the four test statistics in (2.4). It is worth emphasizing that, unlike in previous works such as [26], which do not provide explicit non-null distributional expressions, the result in (3.2) enables an approximation of the power function of each test statistic without relying on simulation-based methods. This feature is particularly relevant in censored Weibull regression models, where bootstrap procedures may be computationally intensive.

The quantities a_{ik} are of order $O(n^{-1/2})$ such that a_{1k} and a_{2k} were obtained by Hayakana [30], and a_{3k} and a_{4k} were obtained by Harris and Peers [31] and Lemonte and Ferrari [32], respectively, as functions of the derivatives of the log-likelihood function. They are very general, and it might not be possible to particularize their formulas for specific regression models, as happens for the Weibull distribution when σ is unknown. For this reason, we derive the quantities a_{ik} under the assumption that the parameter σ is fixed as previously discussed. Leveraging the general results of [30–32], and following extensive algebra involving derivatives and expectations of the log-likelihood, we derived the specific quantities a_{ik} for censored Weibull data; for conciseness, these are reported in the Appendix, with full derivations provided in the Supplementary Material.

Remark 1. *For a fixed $\boldsymbol{\epsilon}$ in the alternative hypothesis, the power of the test can be computed using Eq* (3.2) *as $\mathbb{P}(S_i > \chi^2_{p-q;1-\alpha})$, where $\chi^2_{\nu;1-\alpha}$ denotes the quantile $1-\alpha$ of the central chi-squared random variable with ν degrees of freedom.*

4. Local power comparisons

In this section, we derive the power function Π_i for the statistic S_i, $i = 1, 2, 3, 4$. The quantities a_{jk} and a_{ik}, given in (7.12) to (7.26), and we can perform a local power comparison as follows:

$$\Pi_i - \Pi_j = \sum_{k=0}^{3} \left(a_{jk} - a_{ik} \right) G_{p-q+2k,\lambda}(x), \text{ for } i \neq j.$$

Denoting the PDF of the random variable X as $g_{\nu,\lambda}(x)$, where X follows a noncentral chi-squared with ν degrees of freedom and a noncentrality parameter λ. An important result that can be used is $G_{m,\lambda}(x) - G_{m+2,\lambda}(x) = 2g_{m+2,\lambda}(x)$. After some algebra,

$$\begin{aligned}
&\Pi_1 - \Pi_2 = k_1 g_{p-q+4,\lambda} + k_2 g_{p-q+6,\lambda}, \quad \Pi_1 - \Pi_3 = k_3 g_{p-q+4,\lambda} + k_4 g_{p-q+6,\lambda}, \\
&\Pi_1 - \Pi_4 = k_5 g_{p-q+4,\lambda} + k_6 g_{p-q+6,\lambda}, \quad \Pi_2 - \Pi_3 = k_7 g_{p-q+4,\lambda} + k_8 g_{p-q+6,\lambda}, \\
&\Pi_2 - \Pi_4 = k_9 g_{p-q+4,\lambda} + k_{10} g_{p-q+6,\lambda}, \quad \Pi_3 - \Pi_4 = k_{11} g_{p-q+4,\lambda} + k_{12} g_{p-q+6,\lambda}.
\end{aligned}$$

Theorem 1. *When observations are not censored, the score test exhibits higher local asymptotic power under Pitman alternatives.*

Proof. For the specific case without censored observations, $\boldsymbol{W} = \boldsymbol{I}_n$ (the identity matrix with dimension n) and $\boldsymbol{W}' = \boldsymbol{0}_n$ (a zero matrix with dimension $n \times n$) the expressions above reduce to

$$\begin{aligned}
&k_1 = -\frac{\sigma}{2} k_3 = -2\sigma k_5 = -\left(\frac{\sigma}{\sigma+2}\right) k_7 = -\left(\frac{1+2\sigma}{2\sigma^3}\right) k_9 = \frac{2}{3\sigma} k_{11} = -\frac{1}{\sigma^2} \text{tr}\left\{(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d})\boldsymbol{B}\right\}, \\
&k_2 = -\frac{1}{2} k_4 = -2k_6 = -\frac{1}{3} k_8 = -\frac{2}{3} k_{10} = \frac{2}{3} k_{12} = -\frac{1}{3\sigma^3} \text{tr}\left\{\boldsymbol{B}^{(3)}\right\}.
\end{aligned}$$

Note the following.

- If $k_1 < 0$ and $k_2 < 0 \Rightarrow k_7, k_8, k_9, k_{10} > 0$. In this case, the score test (S_3) exhibits higher local asymptotic power under Pitman alternatives.
- If $k_1 > 0$ and $k_2 > 0 \Rightarrow k_3, k_4 < 0$. Therefore, if the LR test (S_1) is more powerful than the Wald test (S_2), then the score test (S_3) is more powerful than the LR test (S_1) (i.e., S_1 cannot exhibit higher local asymptotic power under Pitman alternatives).
- If $k_3 < 0$ and $k_4 < 0 \Rightarrow k_7, k_8 < 0$. Therefore, if the score test (S_3) is more powerful than the LR test (S_1), then the Wald test (S_2) is more powerful than the score test (S_3) (i.e., S_3 cannot exhibit higher local asymptotic power under Pitman alternatives).
- If $k_5 < 0$ and $k_6 < 0 \Rightarrow k_9, k_{10} < 0$. Therefore, if the gradient test (S_4) is more powerful than the LR test (S_1), then the Wald test (S_2) is more powerful than the gradient test (S_4), i.e., S_4 cannot exhibit higher local asymptotic power under Pitman alternatives, completing the proof.

□

Remark 2. *When the observations are not censored, the score test exhibits higher local asymptotic power under Pitman alternatives for the exponential regression model ($\sigma = 1$), which is not discussed in [17].*

5. Simulation study

In this section, we compared the non-null asymptotic distribution of the four statistics for the hypothesis given in Eq (3.1), where $\boldsymbol{\epsilon} = (\psi/\sqrt{n})\mathbf{1}_{p-q}^{\top}$ and $\mathbf{1}_r$ denotes a vector of ones with dimension r. Note that the components of $\boldsymbol{\epsilon}$ are of order $O(n^{-1/2})$. In addition, we compare the test's estimated power using the same approximation. Let C be the percentage of censoring in the sample. We consider the following four parameter combinations for (p, q, C, σ), namely $(3, 1, 0.1, 0.5), (5, 3, 0.1, 0.5), (5, 1, 0.25, 1)$, and $(5, 3, 0.1, 1)$, along with four sample sizes: 20, 30, 60, and 100. The design matrix $\boldsymbol{X}$ has the dimension 100×10 (the maximum sample size n for rows and the maximum number of covariates p for columns). It was drawn once from the standard uniform distribution, and then the first n rows and p columns of $\boldsymbol{X}$ for each corresponding (p, q, C, σ) combination were fixed across 10,000 replicates. This choice reflects the real-world situation where the design matrix is observed and known; varying it across replicates would represent a different scenario than our model assumes. The censoring times were generated from an exponential distribution with a mean ξ_i, where ξ_i was chosen numerically to achieve a desired censoring proportion C, by solving $\int_0^1 e^{-t/\xi_i} f_T(t)dt = 1 - C$, which can be solved numerically for $i = 1, \ldots, n$. In this simulation study, right censoring is generated via an independent censoring variable, thereby inducing random (noninformative) censoring. This setup differs from Type I or Type II censoring and is chosen to assess the performance of the proposed methods within a flexible, commonly used framework. We also chose three values of ψ to obtain the corresponding value for λ closest to 1, 3, and 7. These values are used to validate the approximation across different scenarios, where a larger λ indicates that the alternative hypothesis is further from the null hypothesis. All computations were performed using the `R` software [33].

5.1. Assessing the non-null asymptotic distribution

Figure 1 illustrates the empirical distributions of the test statistics S_1 to S_4 for the scenario defined by $p = 5$, $q = 3$, $C = 0.1$, $\sigma = 0.5$, and $\psi = 1.1$ across the four considered sample sizes, while additional simulation results for alternative parameter configurations are provided in Section E of the Supplementary Material, see Figures S1 to S12. Overall, the results indicate that the asymptotic approximations to the distributions of the four statistics align with their theoretical distributions. The agreement with the empirical distributions increases as the sample size increases, which supports the validity of the asymptotic framework in moderate to large samples.

Nevertheless, systematic discrepancies are observed in finite samples: The approximating distributions tend to underestimate the true distribution of the test statistics. This is particularly expected for smaller sample sizes where deviations from the asymptotic distribution are more pronounced; consequently, the resulting critical values may be too liberal, leading to an inflation in the estimated power of the tests. This finding suggests that caution is warranted when applying these approximations in small samples, since the apparent gain in power may not accurately reflect the true performance of the procedures, which, in practice, could result in overly optimistic conclusions about the sensitivity of the tests.

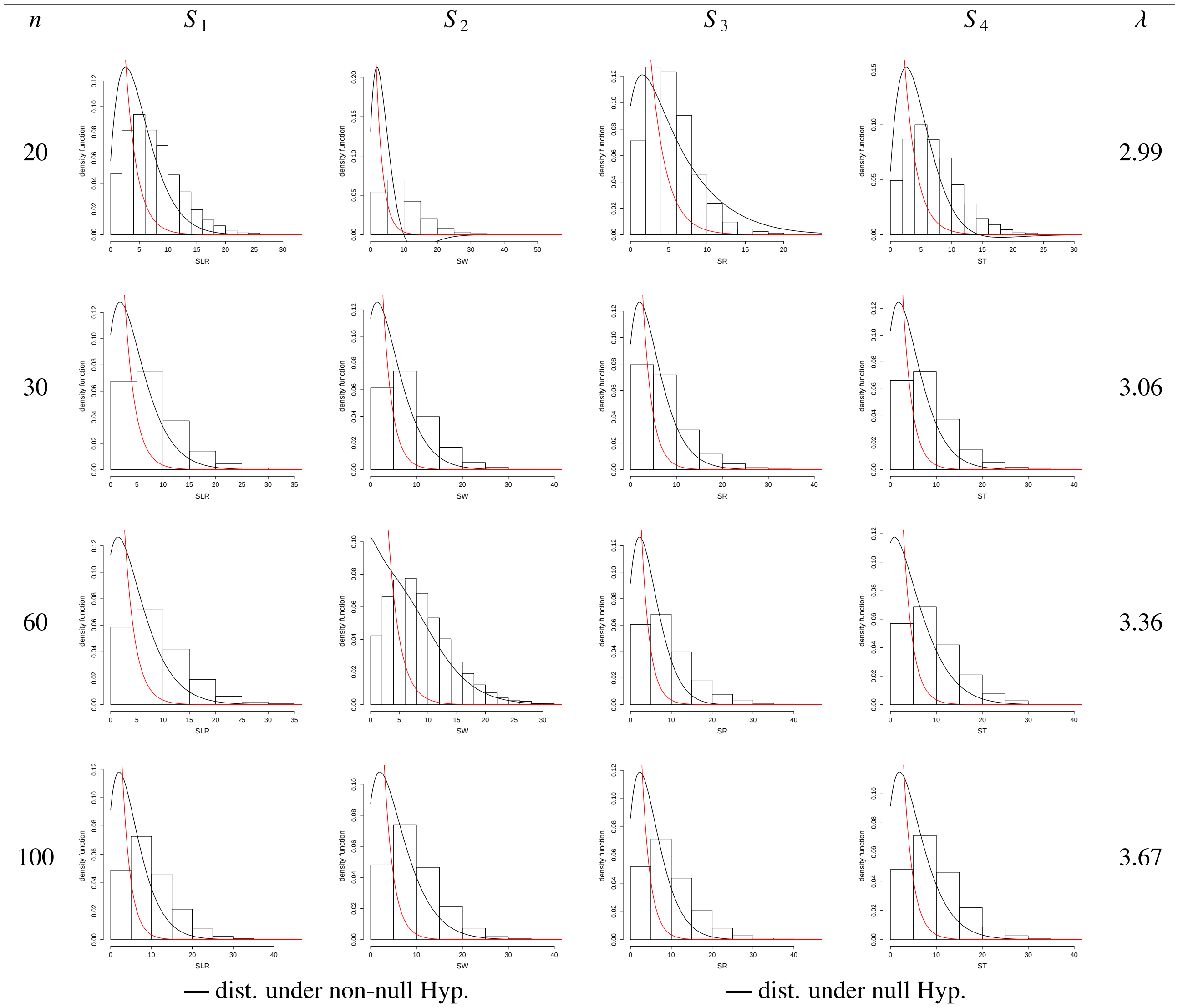

Figure 1. Non-null empirical and asymptotic distributions for tests S_1 to S_4. In the case, $p = 5$, $q = 3$, censoring=10%, $\sigma = 0.5$, and $\psi = 1.1$.

5.2. *Estimating the power of the test*

The gold standard for estimating the power of a test statistic is a bootstrap simulation, which provides a flexible and largely assumption-free framework to approximate the sampling distribution under a given alternative hypothesis. In this procedure, a grid $B \times 1$ of values $(\psi_1, \ldots, \psi_B)$ is fixed for the parameter ψ, and for each point in the grid, a set of sample replicates is generated, say M replicates, in order to estimate the corresponding statistic $S_i(\psi)$ for $i = 1, 2, 3, 4$. The probability described in Remark 1 is then approximated using the empirical distribution obtained from these M bootstrap samples. This approach, while accurate, is computationally demanding, as it requires repeatedly estimating the model parameters and computing the associated test statistics for a total of $M \times B$ times.

In the scenarios considered in this study, we set $M = 10{,}000$ and use a grid of $B = 251$ values for ψ, resulting in a total of 2,510,000 simulated samples. Such a computational burden can be substantial, particularly in more complex models or larger-scale studies, and may limit the practical applicability of the bootstrap approach in routine sample size calculations. Figure 2 presents the estimated power of the test statistics obtained via bootstrapping alongside the corresponding asymptotic approximations for the case $(p, q, C, \sigma) = (5, 3, 0.1, 0.5)$ and $\psi = 1.1$, across the four sample sizes considered. Additional results for other parameter configurations are reported in Section F of the Supplementary Material , see Figures S13 to S16.

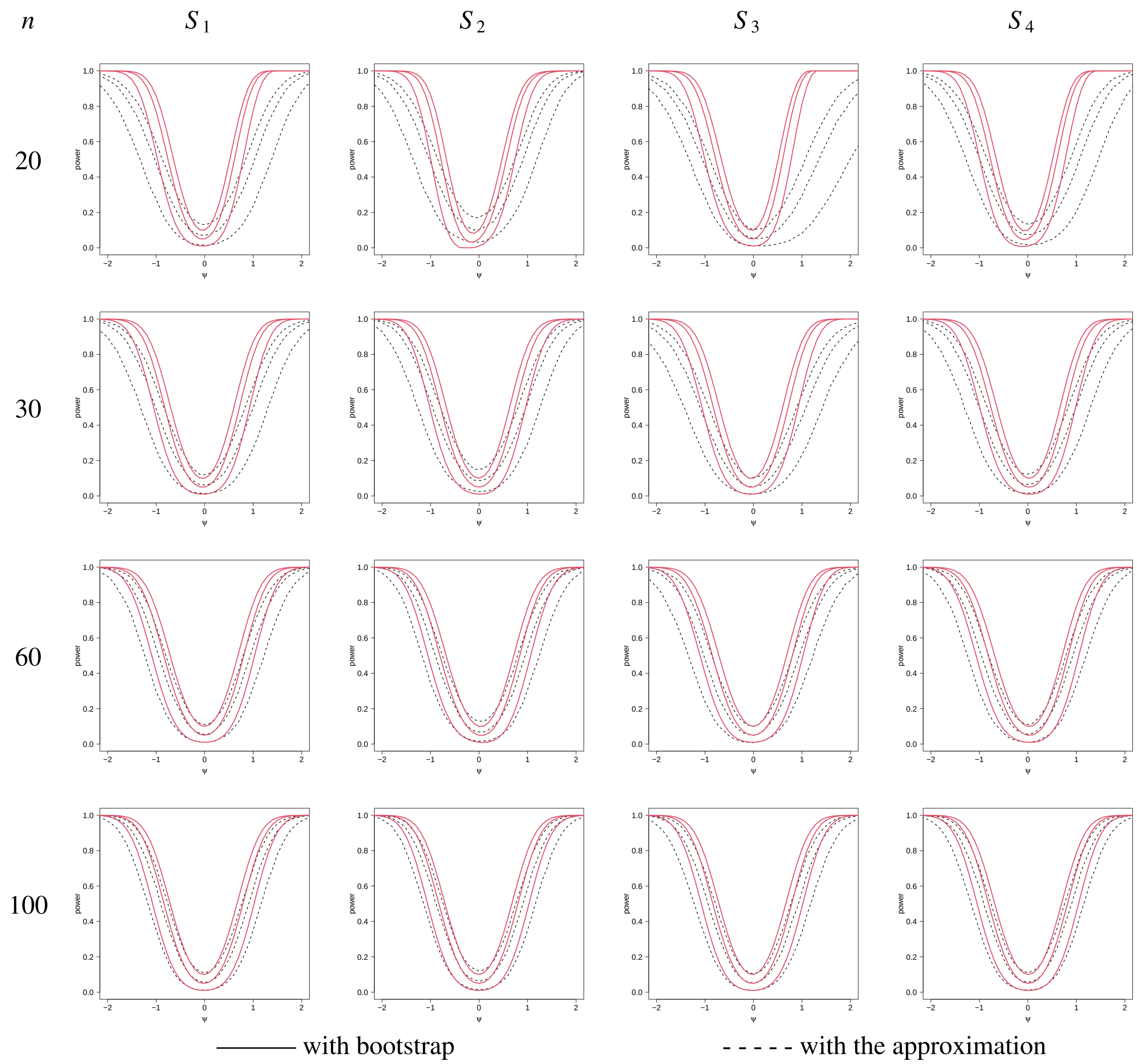

Figure 2. Power estimation via bootstrapping (dashed line) and theoretical result (continuous line). Case $p = 5$, $q = 3$, censoring = 10%, and $\sigma = 0.5$ for three levels of significance: 1%, 5%, and 10%.

Overall, the approximation closely matches the bootstrap-based estimates in most scenarios, indicating that it provides a reliable and computationally efficient alternative. However, noticeable discrepancies arise in small samples, particularly when $n = 20$, where the approximation deviates more substantially from the bootstrapping results. Furthermore, the accuracy of the approximation improves as the sample size increases, which is consistent with its asymptotic nature, while it tends to deteriorate as the number of covariates increases or as the percentage of censoring becomes larger. These patterns highlight the trade-off between computational efficiency and finite-sample accuracy, and suggest that bootstrap methods may still be preferable in settings with limited sample sizes or more challenging data structures.

The results indicate that the proposed asymptotic approximation performs well for moderate to large sample sizes. However, for small samples (e.g., $n = 20$), higher censoring levels, or more complex models, the approximation tends to underestimate the true distribution, which may lead to an overestimation of power. These findings highlight the importance of exercising caution when applying the method in such scenarios.

6. Applications

In this section, we illustrate the calculation of sample size for a two-arm Phase II clinical trial. As an illustration, three possible values of the shape parameter σ were considered. In practice, the shape parameter can be estimated from preliminary data when patient level data are available, or the sample size is calculated for a reasonable grid of values for σ and the largest sample size is adopted as a conservative strategy. Furthermore, Type I censoring is considered as expected in Phase II trials, where the study is conducted up to a fixed follow-up time.

6.1. Two-arm Phase II clinical trial

Investigators are interested in conducting a two-arm randomized Phase II trial using a pick-the-winner design to identify the most promising combination with the standard of care, (A + SOC, B + SOC), for women with residual hormone receptor-negative, HER2-negative (triple-negative) resectable breast cancer after taxane-based neoadjuvant chemotherapy. Event-free survival (EFS) at 36 months is the primary endpoint, and the 36-month EFS for SOC is 60%.

The hypotheses of interest can be translated into $\mathcal{H} : S_A(36)-S_B(36) = 0$ and $\mathcal{A} : S_A(36)-S_B(36) \neq 0$, where $S(t)$ is the survival function of the Weibull distribution defined in (2.2). For a survival of $s\%$ at time $t*$, the linear predictor can be calculated as follows:

$$\mu = \log\left\{\frac{t*}{[-\log(s)]^{\sigma}}\right\}$$

Considering a regression model with $\mu_A = \beta_0$ modeling the survival of patients receiving Treatment A and $\mu_B = \beta_0 + \beta_1$ modeling the survival of patients receiving Treatment B, the hypotheses can be rewritten as $\mathcal{H} : \beta_1 = 0$ and $\mathcal{A} : \beta_1 \neq 0$.

The sample size is calculated to achieve 80% power to detect a minimum difference in EFS of 15%, which can be translated as $\beta_1 = \epsilon = |\mu_B - \mu_A|$. For example, $\epsilon = 0.29$ when either $S_A(36) = 0.6$ or $S_B(36) = 0.6$ and $\sigma = 0.5$. Figure 3 shows the power of the four tests estimated by the approximation as a function of sample size for different values of σ at a 5% significance level. Table 1 shows the sample sizes required to reach a power of 80%.

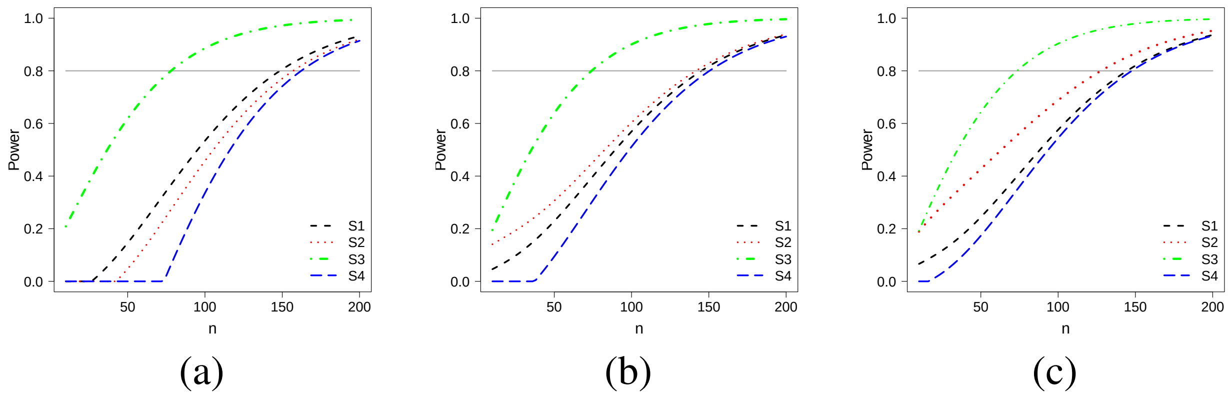

Figure 3. Power estimation for β_1 considering three values of σ: (a) 0.5 (Rayleigh or Weibull distribution with increasing hazard function), (b) 1 (Exponential or Weibull distribution with constant hazard ratio), and (c) 1.5 (Weibull distribution with decreasing hazard function).

Table 1. Sample sizes required to test $\mathcal{H}$ and $\mathcal{A}$ for β_0 with 80% of power at the 5% significance level as function of σ and the test statistic.

σ	S_1	S_2	S_3	S_4
0.5	119	102	58	122
1.0	119	125	52	118
1.5	119	137	117	117

The score test tends to require smaller sample sizes under the proposed asymptotic approximations. As discussed in the simulation study, these approximations are more accurate for moderate to large sample sizes, but may overestimate the power in small samples or under high levels of censoring. Therefore, the resulting sample sizes should be interpreted as initial estimates. In addition, differences between tests should be interpreted cautiously, as the approximations do not explicitly account for potential discrepancies in Type I error control.

7. Conclusions

In this paper, we obtained closed-form expressions for the non-null distribution functions of the likelihood ratio, Wald, score, and gradient tests under Pitman's alternatives when the response variable is censored and follows a Weibull distribution. In contrast to previous comparative studies, which focus primarily on general properties of these statistics, our results provide explicit non-null distributional expressions for censored Weibull regression models, enabling a direct analytical evaluation of power. Although the four statistics are asymptotically equivalent, selecting one without criteria may lead to inappropriate conclusions for small sample sizes. From the derived expressions, we discussed situations in which one of these tests can exhibit higher local asymptotic power under Pitman alternatives than the others. Finally, although the proposed approach provides a computationally efficient alternative to simulation-based methods, its accuracy may be affected in small samples or under high censoring. In such cases, the method can be used as an initial tool for sample size determination and complemented with more precise simulation-based techniques when necessary.

Author contributions

Conceptualization: T. M. Magalhães and M. A. Diniz; methodology: T. M. Magalhães, M. A. Diniz, Y. M. Gómez, and O. Venegas; software: M. A. Diniz and D.I. Gallardo; validation: D. I. Gallardo, Y. M. Gómez, and O. Venegas; formal analysis: T. M. Magalhães and M. A. Diniz; investigation: T. M. Magalhães, M. A. Diniz, and Y. M. Gómez; writing—original draft preparation: T. M. Magalhães, M. A. Diniz, and D. I. Gallardo; writing—review and editing: Y. M. Gómez and O. Venegas. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools only in the language editing

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Conflict of interest

The authors declare no conflict of interest.

References

1. R. S. Pinheiro, W. Andraus, D. R. Waisberg, L. S. Nacif, L. Ducatti, V. Rocha-Santos, et al., Abdominal hernias in cirrhotic patients: Surgery or conservative treatment? Results of a prospective cohort study in a high volume center: Cohort study, *Ann. Med. Surg.*, **49** (2020), 9–13. https://doi.org/10.1016/j.amsu.2019.11.009
2. R. O. Bezerra, R. S. Pinheiro, T. M. Magalhães, A. C. Fortunato, L. Tambelini, G. Conti, et al., Myosteatosis as a risk factor for hepatocelular carcinoma recurrence after deceased donor liver transplantation, *Med. Res. Arch.*, **12** (2024), 2375–1924. https://doi.org/10.18103/mra.v12i8.5733
3. E. Rojas, E. Meneses, T. Jones, D. Maxwell, Understanding failures through the lifetime of a top-level supercomputer, *J. Parallel Distr. Com.*, **154** (2021), 27–41. https://doi.org/10.1016/j.jpdc.2021.04.001
4. K. J. Carroll, On the use and utility of the Weibull model in the analysis of survival data, *Control. Clin. Trials*, **24** (2003), 682–701. https://doi.org/10.1016/S0197-2456(03)00072-2

5. J. Wu, Sample size calculation for the one-sample log-rank test, *Pharm. Stat.*, **14** (2015), 26–33. https://doi.org/10.1002/pst.1654
6. M. A. Phadnis, Sample size calculation for small sample single-arm trials for time-to-event data: Logrank test with normal approximation or test statistic based on exact chi-square distribution? *Cont. Clin. trial. Comm.*, **15** (2019), 100360. https://doi.org/10.1016/j.conctc.2019.100360
7. M. A. Phadnis, J. B. Wetmore, M. S. Mayo, A clinical trial design using the concept of proportional time using the generalized gamma ratio distribution, *Stat. Med.*, **36** (2017), 4121–4140. https://doi.org/10.1002/sim.7421
8. S. S. Wilks, The large-sample distribution of likelihood ratio for testing composite hypotheses, *Ann. Math. Stat.*, **9** (1938), 60–62. https://doi.org/10.1214/aoms/1177732360
9. A. Wald, Test of statistical hypotheses concerning several parameter when the number of observations is large, *Trans. Amer. Math. Soc.*, **54** (1943), 426–482. https://doi.org/10.1090/S0002-9947-1943-0012401-3
10. C. R. Rao, Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation, *Math. Proc. Cambridge*, **44** (1948), 50–57. https://doi.org/10.1017/S0305004100023987
11. G. R. Terrell, The gradient statistic, *Comput. Sci. Sta.*, **34** (2002), 206–215.
12. T. M. Magalhães, D. I. Gallardo, H. W. Gómez, Skewness of maximum likelihood estimators in the Weibull censored data, *Symmetry*, **11** (2019), 1351. https://doi.org/10.3390/sym11111351
13. T. M. Magalhães, D. I. Gallardo, Bartlett and Bartlett-type corrections for censored data from a Weibull distribution, *SORT*, **44** (2020), 127–140. https://doi.org/10.2436/20.8080.02.97
14. G. M. Cordeiro, E. A. Colosimo, Improved likelihood ratio tests for exponential censored data, *J. Stat. Comput. Simul.*, **56** (1997), 303–315. https://doi.org/10.1080/00949659708811796
15. G. M. Cordeiro, E. A. Colosimo, Corrected score tests for exponential censored data, *Stat. Probab. Lett.*, **44** (1999), 365–373. https://doi.org/10.1016/S0167-7152(99)00028-0
16. F. M. C. Medeiros, A. J. Lemonte, Likelihood-based inference in censored exponential regression models, *Commun. Stat. Theor. M.*, **50** (2021), 3214–3233. https://doi.org/10.1080/03610926.2019.1700281
17. A. J. Lemonte, Non null asymptotic distribution of some criteria in censored exponential regression, *Commun. Stat. Theor. M.*, **43** (2014), 3314–3328. https://doi.org/10.1080/03610926.2012.698780
18. A. J. Lemonte, Covariance matrix of maximum likelihood estimators in censored exponential regression models, *Commun. Stat. Theor. M.*, **51** (2022), 1765–1777. https://doi.org/10.1080/03610926.2020.1767142
19. M. A. Diniz, D. I. Gallardo, T. M. Magalhães, Improved inference for MCP-Mod approach using time-to-event endpoints with small sample sizes, *Phar. Stat.*, **22** (2023), 760–772. https://doi.org/10.1002/pst.2303
20. W. Weibull, A statistical distribution function of wide applicability, *J. Appl. Mech.*, **18** (1951), 293–297. https://doi.org/10.1115/1.4010337

21. K. O. Bowman, L. R. Shenton, Weibull distributions when the shape parameter is defined, *Comput. Stat. Data Anal.*, **36** (2001), 299–310. https://doi.org/10.1016/S0167-9473(00)00048-7

22. J. Yu, Q. Wang, C. Wu, Online monitoring of the Weibull distributed process based on progressive Type II censoring scheme, *J. Comput. Appl. Math.*, **443** (2024), 115744. https://doi.org/10.1016/j.cam.2023.115744

23. M. A. Phadnis, P. Sharma, N. Thewarapperuma, P. Chalise, Assessing accuracy of Weibull shape parameter estimate from historical studies for subsequent sample size calculation in clinical trials with time-to-event outcome, *Cont. Clin. Trial. Comm.*, **17** (2020), 100548. https://doi.org/10.1016/j.conctc.2020.100548

24. J. F. Lawless, *Statistical models and methods for lifetime data*, John Wiley & Sons, 2003.

25. J. D. Kalbfleisch, R. L. Prentice, *The statistical analysis of failure time data*, John Wiley & Sons, 2002.

26. M. H. Montoril, E. A. Souza, Estatística gradiente: propriedades e aplicações, *Rev. Bras. Biometr.*, **31** (2013), 43–60.

27. A. Buse, The likelihood ratio, Wald and Lagrange multiplier tests: an expository note, *Am. Stat.*, **36** (1982), 153–157. https://doi.org/10.2307/2683166

28. D. A. McManus, Who invented local power analysis? *Economet. Theor.*, **7** (1991), 265–268. https://doi.org/10.1017/S026646660000445X

29. R. Davidson, J. G. MacKinnon, Implicit alternatives and the local power of test statistics, *Econometrica*, **55** (1987), 1305–1329. https://doi.org/10.2307/1913558

30. T. Hayakawa, The likelihood ratio criterion for a composite hypothesis under a local alternative, *Biometrika*, **62** (1975), 451–460. https://doi.org/10.1093/biomet/62.2.451

31. P. Harris, H. W. Peers, The local power of the efficient scores test statistics, *Biometrika*, **67** (1980), 525–529. https://doi.org/10.1093/biomet/67.3.525

32. A. J. Lemonte, S. L. P. Ferrari, The local power of the gradient test, *Ann. Inst. Stat. Math.*, **64** (2012), 373–381. https://doi.org/10.1007/s10463-010-0315-4

33. R C. Team, *R: A language and environment for statistical computing, R foundation for statistical*, 2020.

Appendix

The logarithm of the likelihood function for the model is given by

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \left[\delta_i \left(-n \log \sigma + \frac{y_i - \mu_i}{\sigma} \right) - \exp\left(\frac{y_i - \mu_i}{\sigma} \right) \right].$$

The score vector is $\boldsymbol{U}_{\boldsymbol{\beta}} = \sigma^{-1} \boldsymbol{X}^\top \boldsymbol{W}^{1/2} \boldsymbol{v}$ and the Fisher information matrix is $\boldsymbol{K}_{\boldsymbol{\beta\beta}} = \sigma^{-2} \boldsymbol{X}^\top \boldsymbol{W} \boldsymbol{X}$, where $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)^\top$ is the design matrix, which has the rank p, $\boldsymbol{W} = \mathrm{diag}(w_1, \ldots, w_n)$ with $w_i = \mathbb{E}\left[\exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right]$ and $\boldsymbol{v} = (v_1, \ldots, v_n)^\top$ with $v_i = \left\{-\delta_i + \exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right\} w_i^{-1/2}$. Evidently, the value of w_i depends on the censoring mechanism. Specifically, we have

$w_i = q \times \left\{1 - \exp\left[-L_i^{1/\sigma}\exp(-\mu_i/\sigma)\right]\right\} + (1-q)\times(r/n)$, where $q = \mathbb{P}\left(W_{(r)} \le \log L_i\right)$ and $W_{(r)}$ denotes the rth smallest value from the values of W_i. Moreover, $q = 1$ and $q = 0$ represent Types I and II censoring, respectively. The proof is presented in the Appendix of [12].

The partition of the parameter vector $\beta = (\beta_1^\top, \beta_2^\top)^\top$ induces

$$\boldsymbol{U}_\beta = \begin{pmatrix} \boldsymbol{U}_{\beta_1} \\ \boldsymbol{U}_{\beta_2} \end{pmatrix} = \sigma^{-2}\begin{pmatrix} \sigma^{-1}\boldsymbol{X}_1^\top \boldsymbol{W}^{1/2}\boldsymbol{v} \\ \sigma^{-1}\boldsymbol{X}_2^\top \boldsymbol{W}^{1/2}\boldsymbol{v} \end{pmatrix},$$

$$\boldsymbol{K}_{\beta\beta} = \begin{pmatrix} \boldsymbol{K}_{\beta_1\beta_1} & \boldsymbol{K}_{\beta_1\beta_2} \\ \boldsymbol{K}_{\beta_2\beta_1} & \boldsymbol{K}_{\beta_2\beta_2} \end{pmatrix} = \sigma^{-2}\begin{pmatrix} \boldsymbol{X}_1^\top \boldsymbol{W} X_1 & \boldsymbol{X}_1^\top \boldsymbol{W}\boldsymbol{X}_2 \\ \boldsymbol{X}_2^\top \boldsymbol{W} X_1 & \boldsymbol{X}_2^\top \boldsymbol{W}\boldsymbol{X}_2 \end{pmatrix}.$$

Defining the following matrices may also helpful: $\boldsymbol{Z} = \boldsymbol{X}\left(\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^\top$ and, consequently, $\boldsymbol{Z}_1 = \boldsymbol{X}_1\left(\boldsymbol{X}_1^\top\boldsymbol{W}\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}_1^\top$, $\boldsymbol{Z}_d$, and $\boldsymbol{Z}_{1d}$ are diagonal matrices with the diagonal given by the diagonal of $\boldsymbol{Z}$ and $\boldsymbol{Z}_1$, respectively; $\boldsymbol{W}' = \text{diag}(w_1', \ldots, w_n')$ with ith element given by $w_i' = -\sigma^{-1}L_i^{1/\sigma}\exp\{-L_i^{1/\sigma}\exp(-\mu_i/\sigma) - \mu_i/\sigma\}$; $\boldsymbol{B}$ and $\boldsymbol{E}$ are diagonal matrices with the diagonal given by the vectors $\boldsymbol{X}\boldsymbol{\epsilon}^*$ and $\boldsymbol{X}_2\boldsymbol{\epsilon}$, respectively; and $\boldsymbol{Z}^{(2)} = \boldsymbol{Z}\odot\boldsymbol{Z}$, $\boldsymbol{Z}^{(3)} = \boldsymbol{Z}^{(2)}\odot\boldsymbol{Z}$, where $\odot$ denotes the Hadamard product. From the general results of [30–32], after an extensive algebra involving derivatives of the logarithm of the likelihood function, we can express main result of this paper. We obtained the specific quantities a_{ik} for censored data from a Weibull distribution, which can be written as follows.

For $i = 1$, the likelihood ratio test

$$\begin{aligned} a_{11} &= \frac{1}{2\sigma^2}\text{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_{1d}\boldsymbol{B}\right)\right\}, \\ a_{12} &= -\frac{1}{6\sigma^2}\text{tr}\left\{\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}, \quad a_{13} = 0. \end{aligned} \tag{7.1}$$

For $i = 2$, the Wald test,

$$\begin{aligned} a_{21} &= \frac{1}{2\sigma^2}\text{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_d\boldsymbol{B}\right) - 2\left(\frac{1}{\sigma}\boldsymbol{W} + \boldsymbol{W}'\right)\left(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\}, \\ a_{22} &= -\frac{1}{2\sigma^2}\text{tr}\left\{\left(\frac{1}{\sigma}\boldsymbol{W} + \boldsymbol{W}'\right)\boldsymbol{B}^{(3)} - \boldsymbol{W}\left(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\}, \quad a_{23} = \frac{1}{6\sigma^3}\text{tr}\left\{\boldsymbol{W}\boldsymbol{B}^{(3)}\right\}. \end{aligned} \tag{7.2}$$

For $i = 3$, the score test,

$$\begin{aligned} a_{31} &= \frac{1}{2\sigma^2}\text{tr}\left\{-\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\left(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d}\right)\boldsymbol{B} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_{1d}\boldsymbol{B}\right)\right\}, \\ a_{32} &= -\frac{1}{2\sigma^2}\text{tr}\left\{\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\left(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\}, \quad a_{33} = -\frac{1}{6\sigma^2}\text{tr}\left\{\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}. \end{aligned} \tag{7.3}$$

For $i = 4$, the gradient test,

$$a_{41} = \frac{1}{2\sigma^2}\text{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\} + \frac{1}{4\sigma^2}\text{tr}\left\{\frac{1}{\sigma}\boldsymbol{W}\boldsymbol{Z}_d\boldsymbol{B} + \left(\frac{1}{\sigma}\boldsymbol{W} + 4\boldsymbol{W}'\right)\boldsymbol{Z}_{1d}\boldsymbol{B}\right\},$$

$$a_{42} = -\frac{1}{4\sigma^2}\text{tr}\left\{\frac{1}{\sigma}\boldsymbol{W}\left(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d}\right)\boldsymbol{B} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}, \quad a_{43} = -\frac{1}{12\sigma^3}\text{tr}\left\{\boldsymbol{W}\boldsymbol{B}^{(3)}\right\}, \tag{7.4}$$

with $a_{i0} = -(a_{i1} + a_{i2} + a_{i3})$, $i = 1, 2, 3, 4$. We also present the walk-through for deriving Eqs (7.12) to (7.26) in the Supplementary Material.

Remark 3. *In [17], where the specific case $\sigma = 1$ is approached, there is a typo (the "−" is missing) in the first term within the trace in a_{31}.*

Supplementary material

Section A

We compute $w_i = \mathbb{E}\left[\exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right]$ for Type I censoring. Note that

$$\exp\left(\frac{y_i - \mu_i}{\sigma}\right) = \begin{cases} \exp\left(\frac{y_i-\mu_i}{\sigma}\right), & \text{if } y_i \leq \log L_i \\ \exp\left(\frac{\log L_i-\mu_i}{\sigma}\right), & \text{otherwise} \end{cases}$$

Therefore

$$\begin{aligned} w_i &= \int_{-\infty}^{\log L_i} \frac{1}{\sigma}\exp\left(\frac{2(y_i - \mu_i)}{\sigma} - \exp\left(\frac{y_i - \mu_i}{\sigma}\right)\right)dy_i + \exp\left(\frac{\log L_i - \mu_i}{\sigma}\right)\mathbb{P}(T_i > L_i) \\ &= 1 - \exp\left(-L_i^{1/\sigma}e^{-\mu_i/\sigma}\right)\left(1 + L_i^{1/\sigma}e^{-\mu_i/\sigma}\right) + L_i^{1/\sigma}e^{-\mu_i/\sigma}\exp\left(-L_i^{1/\sigma}e^{-\mu_i/\sigma}\right) \\ &= 1 - \exp\left(-L_i^{1/\sigma}e^{-\mu_i/\sigma}\right). \end{aligned}$$

Direct computation also shows that

$$\begin{aligned} v_i &= \mathbb{E}\left[\exp\left(\frac{2(y_i - \mu_i)}{\sigma}\right)\right] \\ &= \int_{-\infty}^{\log L_i} \frac{1}{\sigma}\exp\left(\frac{3(y_i - \mu_i)}{\sigma} - \exp\left(\frac{y_i - \mu_i}{\sigma}\right)\right)dy_i + \exp\left(\frac{2(\log L_i - \mu_i)}{\sigma}\right)\mathbb{P}(T_i > L_i) \\ &= 2 - \exp\left(-L_i^{1/\sigma}e^{-\mu_i/\sigma}\right)\left[2 + 2L_i^{1/\sigma}e^{-\mu_i/\sigma} + L_i^{2/\sigma}e^{-2\mu_i/\sigma}\right] + L_i^{2/\sigma}e^{-2\mu_i/\sigma}\exp\left(-L_i^{1/\sigma}e^{-\mu_i/\sigma}\right) \\ &= 2\left\{1 - \exp\left(-L_i^{1/\sigma}e^{-\mu_i/\sigma}\right)\left[1 + L_i^{1/\sigma}e^{-\mu_i/\sigma}\right]\right\} \\ &= 2\left\{w_i + \sigma w_i'\right\}. \end{aligned}$$

On the other hand, for Type II censoring, note that $W_i = \exp\left(\frac{y_i-\mu_i}{\sigma}\right) \sim E(1)$. By the order statistics' properties, it is possible show that $\mathbb{E}(W_{(j)}) = \text{Var}(W_{(j)}) = \sum_{k=1}^{j}(n - k + 1)^{-1}$ and

$$W_i = \begin{cases} W_{(1)}, & \text{with probability } 1/n \\ \vdots & \\ W_{(r-1)}, & \text{with probability } 1/n \\ W_{(r)}, & \text{with probability } (n - r + 1)/n \end{cases}$$

Therefore

$$w_i = \mathbb{E}(W_i) = \frac{1}{n}\sum_{j=1}^{r-1}\sum_{k=1}^{j}(n-k+1)^{-1} + \frac{(n-r+1)}{n}\sum_{k=1}^{r}(n-k+1)^{-1}.$$

With some manipulations, we find that $\mathbb{E}(W_i) = r/n$. Moreover, we have $\mathbb{E}(W_{(j)}^2) = \mathbb{E}(W_{(j)}) + \mathbb{E}^2(W_{(j)})$ and

$$V_i = \begin{cases} W_{(1)}^2, & \text{with probability } 1/n \\ \vdots & \\ W_{(r-1)}^2, & \text{with probability } 1/n \\ W_{(r)}^2, & \text{with probability } (n-r+1)/n \end{cases}$$

Therefore

$$v_i = w_i + \frac{1}{n}\sum_{j=1}^{r-1}\left[\sum_{k=1}^{j}(n-k+1)^{-1}\right]^2 + \frac{(n-r+1)}{n}\left[\sum_{k=1}^{r}(n-k+1)^{-1}\right]^2.$$

Algebraic manipulations show that

$$v_i = \frac{1}{n}\left[r + \sum_{k=1}^{r}\frac{2(r-k)+1}{n-k+1}\right].$$

Section B: Derivatives and cumulants

Let $Y_1, \ldots, Y_n$ be a random sample from Weibull censored data. The logarithm of the likelihood function is given by

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n}\left\{\delta_i\left[-n\log\sigma + \frac{y_i-\mu_i}{\sigma}\right] - \exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right\}. \tag{7.5}$$

The first three derivatives of (7.5) can be expressed, respectively, for

$$\frac{\partial\ell(\boldsymbol{\beta})}{\partial\beta_r} = \frac{1}{\sigma}\sum_{i=1}^{n}\left\{-\delta_i + \exp\left(\frac{y_i-\mu_i}{\sigma}\right)\right\}x_{ri};$$

$$\frac{\partial^2\ell(\boldsymbol{\beta})}{\partial\beta_r\partial\beta_s} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\exp\left(\frac{y_i-\mu_i}{\sigma}\right)x_{ri}x_{si};$$

$$\frac{\partial^3\ell(\boldsymbol{\beta})}{\partial\beta_r\partial\beta_s\partial\beta_t} = \frac{1}{\sigma^3}\sum_{i=1}^{n}\exp\left(\frac{y_i-\mu_i}{\sigma}\right)x_{ri}x_{si}x_{ti}.$$

The second-order cumulants are

$$\kappa_{rs} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}w_i x_{ri}x_{si}; \quad \kappa_{r,s} = -\kappa_{rs} = \frac{1}{\sigma^2}\sum_{i=1}^{n}w_i x_{ri}x_{si};$$

and the third-order cumulants are

$$\kappa_{rst} = \frac{1}{\sigma^3}\sum_{i=1}^{n} w_i x_{ri} x_{si} x_{ti}; \;\; \kappa_{rs}^{(t)} = -\frac{1}{\sigma^2}\sum_{i=1}^{n} w_i' x_{ri} x_{si} x_{ti}; \tag{7.6}$$

$$\kappa_{rs,t} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(w_i' + \frac{1}{\sigma}w_i\right) x_{ri} x_{si} x_{ti}; \;\; \kappa_{r,s,t} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(3w_i' + \frac{2}{\sigma}w_i\right) x_{ri} x_{si} x_{ti};$$

where

$$w_i = 1 - \exp\left\{-L_i^{1/\sigma}\exp(-\mu_i/\sigma)\right\},$$

$$w_i' = -\frac{1}{\sigma}L_i^{1/\sigma}\exp\{-L_i^{1/\sigma}\exp(-\mu_i/\sigma) - \mu_i/\sigma\}.$$

It can be observed that $w_i' = 0$ for Type II censoring.

Section C: Non-null asymptotic expansions

The asymptotic expansions up to order $n^{-1/2}$ for the non-null distribution functions of the LR, Wald, score, and gradient statistics for testing the null hypothesis and local alternative

$$\begin{cases} \mathcal{H}: & \boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^{(0)} \\ \mathcal{A}_n: & \boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^{(0)} + \boldsymbol{\epsilon} \quad \text{(Pitman's alternatives)}, \end{cases}$$

where $\boldsymbol{\epsilon} = (\epsilon_{q+1}, \ldots, \epsilon_p)^\top$ with $\epsilon_s = O(n^{-1/2})$, for $s = q+1, \ldots, p$, can be expressed as

$$\mathbb{P}(S_i \le x) = G_{p-q,\lambda}(x) + \sum_{k=0}^{3} a_{ik} G_{p-q+2k,\lambda}(x) + O(n^{-1}), \tag{7.7}$$

where $G_{m,\lambda}(x)$ is the cumulative function of a noncentral chi-square variate with m degrees of freedom and the noncentrality parameter λ, $\lambda = \mathrm{tr}\{\boldsymbol{K}\boldsymbol{\epsilon}^*\boldsymbol{\epsilon}^{*\top}\}/2$ and $\mathrm{tr}\{\cdot\}$ denotes the trace operator, $a_{i0} = -(a_{i1} + a_{i2} + a_{i3})$, $i = 1, 2, 3, 4$, ϵ_r^* is the ith element of the vector $\boldsymbol{\epsilon}^*$, and a_{rs} and m_{rs} are, respectively, the (i, j)th element of the matrices $\boldsymbol{A}$ and $\boldsymbol{M}$, with

$$\boldsymbol{\epsilon}^* = \begin{pmatrix} \boldsymbol{K}_{\beta_2\beta_1}\boldsymbol{K}_{\beta_1\beta_1}^{-1} \\ -\boldsymbol{I}_{p-q} \end{pmatrix}\boldsymbol{\epsilon}, \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{K}_{\beta_1\beta_1}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{M} = \boldsymbol{K}_{\beta\beta}^{-1} - \boldsymbol{A}.$$

C.1. Loglikelihood ratio test

From [30], the quantities a_{ik} in the loglikelihood ratio test ($i = 1$) are given by

$$a_{11} = -\frac{1}{6}\sum_{r,s,t}^{p}(\kappa_{rst} - 2\kappa_{r,s,t})\epsilon_r^*\epsilon_s^*\epsilon_t^* - \frac{1}{6}\sum_{r,s,t}^{p}(3\kappa_{rst} + 6\kappa_{r,st}) a_{rs}\epsilon_t^* - \frac{1}{6}\sum_{r=q+1}^{p}\sum_{s,t}^{p}(3\kappa_{rst} + 3\kappa_{r,st})\epsilon_r\epsilon_s^*\epsilon_t^*,$$

$$a_{12} = -\frac{1}{6}\sum_{r,s,t}^{p}\kappa_{r,s,t}\epsilon_r^*\epsilon_s^*\epsilon_t^*, \quad a_{13} = 0. \tag{7.8}$$

Replacing the cumulants of the Weibull censored data presented in (7.6) in the equation (7.8), we have

$$\begin{aligned} a_{11} &= \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)\sum_{r=1}^{p}(x_{ri}\epsilon_r^*)\sum_{s=1}^{p}(x_{si}\epsilon_s^*)\sum_{t=1}^{p}(x_{ti}\epsilon_t^*) \\ &\quad + \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)\sum_{r,s=1}^{p}(x_{ri}a_{rs}x_{si})\sum_{t=1}^{p}(x_{ti}\epsilon_t^*) \\ &\quad + \frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'\sum_{r=q+1}^{p}(x_{ri}\epsilon_r)\sum_{s=1}^{p}(x_{si}\epsilon_s^*)\sum_{t=1}^{p}(x_{ti}\epsilon_t^*), \qquad (7.9) \\ a_{12} &= -\frac{1}{6\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i + 3w_i'\right)\sum_{r=1}^{p}(x_{ri}\epsilon_r^*)\sum_{s=1}^{p}(x_{si}\epsilon_s^*)\sum_{t=1}^{p}(x_{ti}\epsilon_t^*), \\ a_{13} &= 0. \end{aligned}$$

Let $b_i = \sum_{r=1}^{p}(x_{ri}\epsilon_r^*) = \sum_{s=1}^{p}(x_{si}\epsilon_s^*) = \sum_{t=1}^{p}(x_{ti}\epsilon_t^*)$, $z_{1ii} = \sum_{r,s=1}^{p}(x_{ri}a_{rs}x_{si})$, and $e_i = \sum_{r=q+1}^{p}(x_{ri}\epsilon_r)$, we rewrite (7.9) as

$$\begin{aligned} a_{11} &= \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)b_i^3 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)z_{1ii}b_i + \frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'e_ib_i^2, \qquad (7.10) \\ a_{12} &= -\frac{1}{6\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i + 3w_i'\right)b_i^3, \quad a_{13} = 0. \end{aligned}$$

With a few manipulations in (7.10), we have

$$\begin{aligned} a_{11} &= \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{w_i'e_ib_i^2 + \left(\frac{1}{\sigma}w_i + 2w_i'\right)\left(b_i^3 + z_{1ii}b_i\right)\right\}, \qquad (7.11) \\ a_{12} &= -\frac{1}{6\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i + 3w_i'\right)b_i^3, \quad a_{13} = 0. \end{aligned}$$

Finally, in matrix notation, the expression (7.11) can be written as

$$\begin{aligned} a_{11} &= \frac{1}{2\sigma^2}\mathrm{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_{1d}\boldsymbol{B}\right)\right\}, \qquad (7.12) \\ a_{12} &= -\frac{1}{6\sigma^2}\mathrm{tr}\left\{\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}, \quad a_{13} = 0. \end{aligned}$$

As $a_{10} = -(a_{11} + a_{12} + a_{13})$ and from (7.12), we have

$$a_{10} = -\frac{1}{2\sigma^2}\mathrm{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_{1d}\boldsymbol{B}\right) - \frac{1}{3}\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}.$$

C.2. Wald test

From [30], the quantities a_{ik} in the Wald test ($i = 2$) are given by

$$a_{21} = -\frac{1}{2}\sum_{r,s,t}^{p}(\kappa_{rst} + 2\kappa_{r,st})\,\epsilon_r^*\epsilon_s^*\epsilon_t^* + \sum_{r,s,t}^{p}\kappa_{r,st}m_{rs}\epsilon_t^*$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{r,s,t}^{p}\left(\kappa_{rst}+2\kappa_{r,st}\right)\kappa^{r,s}\epsilon_t^* - \frac{1}{2}\sum_{r=q+1}^{p}\sum_{s,t}^{p}\left(\kappa_{rst}+\kappa_{r,st}\right)\epsilon_r\epsilon_s^*\epsilon_t^*, \qquad (7.13)\\
a_{22} &= \frac{1}{2}\sum_{r,s,t}^{p}\kappa_{r,st}\epsilon_r^*\epsilon_s^*\epsilon_t^* + \frac{1}{2}\sum_{r,s,t}^{p}\kappa_{rst}m_{rs}\epsilon_t^*,\\
a_{23} &= \frac{1}{6}\sum_{r,s,t}^{p}\kappa_{rst}\epsilon_r^*\epsilon_s^*\epsilon_t^*.
\end{aligned}
$$

Replacing the cumulants of the Weibull censored data, presented in (7.6) in Eq (7.13), we have

$$
\begin{aligned}
a_{21} &= \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right)\\
&-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+w_i'\right)\sum_{r,s=1}^{p}\left(x_{ri}m_{rs}x_{si}\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right)\\
&+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)\sum_{r,s=1}^{p}\left(x_{ri}\kappa^{r,s}x_{si}\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right)\\
&+\frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'\sum_{r=q+1}^{p}\left(x_{ri}\epsilon_r\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right), \qquad (7.14)\\
a_{22} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+w_i'\right)\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right)\\
&+\frac{1}{2\sigma^3}\sum_{i=1}^{n}w_i\sum_{r,s=1}^{p}\left(x_{ri}m_{rs}x_{si}\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right),\\
a_{23} &= \frac{1}{6\sigma^3}\sum_{i=1}^{n}w_i\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right).
\end{aligned}
$$

In addition to the quantities defined in (7.10), let $z_{ii} - z_{1ii} = \sum_{r,s=1}^{p}\left(x_{ri}m_{rs}x_{si}\right)$ and $z_{ii} = \sum_{r,s=1}^{p}\left(x_{ri}\kappa^{r,s}x_{si}\right)$, we rewrite (7.14) as

$$
\begin{aligned}
a_{21} &= \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)b_i^3 - \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+w_i'\right)(z_{ii}-z_{1ii})b_i\\
&+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)z_{ii}b_i + \frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'e_ib_i^2, \qquad (7.15)\\
a_{22} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+w_i'\right)b_i^3 + \frac{1}{2\sigma^3}\sum_{i=1}^{n}w_i(z_{ii}-z_{1ii})b_i,\\
a_{23} &= \frac{1}{6\sigma^3}\sum_{i=1}^{n}w_ib_i^3.
\end{aligned}
$$

With a few manipulations in (7.15), we have

$$
\begin{aligned}
a_{21} &= \frac{1}{2\sigma^2}\sum_{i=1}^{n} w_i' e_i b_i^2 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)\left(b_i^3 + z_{ii}b_i\right) - \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + w_i'\right)(z_{ii} - z_{1ii})b_i \\
a_{22} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + w_i'\right)b_i^3 + \frac{1}{2\sigma^3}\sum_{i=1}^{n} w_i(z_{ii} - z_{1ii})b_i, \qquad (7.16)\\
a_{23} &= \frac{1}{6\sigma^3}\sum_{i=1}^{n} w_i b_i^3.
\end{aligned}
$$

Finally, in matrix notation, the expression (7.16) can be written as

$$
\begin{aligned}
a_{21} &= \frac{1}{2\sigma^2}\mathrm{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_d\boldsymbol{B}\right) - 2\left(\frac{1}{\sigma}\boldsymbol{W} + \boldsymbol{W}'\right)(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d})\,\boldsymbol{B}\right\}, \\
a_{22} &= -\frac{1}{2\sigma^2}\mathrm{tr}\left\{\left(\frac{1}{\sigma}\boldsymbol{W} + \boldsymbol{W}'\right)\boldsymbol{B}^{(3)} - \boldsymbol{W}(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d})\,\boldsymbol{B}\right\}, \qquad (7.17)\\
a_{23} &= \frac{1}{6\sigma^3}\mathrm{tr}\left\{\boldsymbol{W}\boldsymbol{B}^{(3)}\right\}.
\end{aligned}
$$

As $a_{20} = -(a_{21} + a_{22} + a_{23})$, from (7.17), we have

$$
\begin{aligned}
a_{20} = -\frac{1}{2\sigma^2}\mathrm{tr}\Bigg\{&\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_d\boldsymbol{B}\right) - 2\left(\frac{1}{\sigma}\boldsymbol{W} + \boldsymbol{W}'\right)(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d})\,\boldsymbol{B}\ , \\
&- \left(\frac{2}{3\sigma}\boldsymbol{W} + \boldsymbol{W}'\right)\boldsymbol{B}^{(3)} + \boldsymbol{W}(\boldsymbol{Z}_d - \boldsymbol{Z}_{1d})\,\boldsymbol{B}\Bigg\}.
\end{aligned}
$$

C.3. Score test

From [31], the quantities a_{ik} in the score test ($i = 3$) are given by

$$
\begin{aligned}
a_{31} &= -\frac{1}{6}\sum_{r,s,t}^{p}\left(\kappa_{rst} - 2\kappa_{r,s,t}\right)\epsilon_r^*\epsilon_s^*\epsilon_t^* + \frac{1}{2}\sum_{r,s,t}^{p}\kappa_{r,s,t}m_{rs}\epsilon_t^* \\
&\quad - \frac{1}{2}\sum_{r,s,t}^{p}\left(\kappa_{rst} + 2\kappa_{r,st}\right)a_{rs}\epsilon_t^* + \frac{1}{2}\sum_{r=q+1}^{p}\sum_{s,t}^{p}\left(\kappa_{rst} + \kappa_{r,st}\right)\epsilon_r\epsilon_s^*\epsilon_t^*, \qquad (7.18)\\
a_{32} &= -\frac{1}{2}\sum_{r,s,t}^{p}\kappa_{r,s,t}m_{rs}\epsilon_t^*, \\
a_{33} &= -\frac{1}{6}\sum_{r,s,t}^{p}\kappa_{r,s,t}\epsilon_r^*\epsilon_s^*\epsilon_t^*.
\end{aligned}
$$

Replacing the cumulants of the Weibull censored data presented in (7.6) in Eq (7.18), we have

$$
a_{31} = \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right)
$$

$$
\begin{aligned}
&+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)\sum_{r,s=1}^{p}(x_{ri}m_{rs}x_{si})\sum_{t=1}^{p}(x_{ti}\epsilon_t^*)\\
&+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)\sum_{r,s=1}^{p}(x_{ri}a_{rs}x_{si})\sum_{t=1}^{p}(x_{ti}\epsilon_t^*)\\
&-\frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'\sum_{r=q+1}^{p}(x_{ri}\epsilon_r)\sum_{s=1}^{p}(x_{si}\epsilon_s^*)\sum_{t=1}^{p}(x_{ti}\epsilon_t^*),\\
a_{32}&=-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)\sum_{r,s=1}^{p}(x_{ri}m_{rs}x_{si})\sum_{t=1}^{p}(x_{ti}\epsilon_t^*),\\
a_{33}&=-\frac{1}{6\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)\sum_{r=1}^{p}(x_{ri}\epsilon_r^*)\sum_{s=1}^{p}(x_{si}\epsilon_s^*)\sum_{t=1}^{p}(x_{ti}\epsilon_t^*).
\end{aligned}
\tag{7.19}
$$

From the quantities defined in (7.15), we have

$$
\begin{aligned}
a_{31}&=\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)b_i^3+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)(z_{ii}-z_{1ii})b_i\\
&+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)z_{1ii}b_i-\frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'e_ib_i^2,\\
a_{32}&=-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)(z_{ii}-z_{1ii})b_i,\\
a_{33}&=-\frac{1}{6\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)b_i^3.
\end{aligned}
\tag{7.20}
$$

With a few manipulations in (7.20), we have

$$
\begin{aligned}
a_{31}&=\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)\left(b_i^3+z_{1ii}b_i\right)+\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)(z_{ii}-z_{1ii})b_i\\
&-\frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'e_ib_i^2,\\
a_{32}&=-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)(z_{ii}-z_{1ii})b_i,\\
a_{33}&=-\frac{1}{6\sigma^2}\sum_{i=1}^{n}\left(\frac{2}{\sigma}w_i+3w_i'\right)b_i^3.
\end{aligned}
\tag{7.21}
$$

Finally, in matrix notation, the expression (7.21) can be written as

$$
\begin{aligned}
a_{31}&=\frac{1}{2\sigma^2}\mathrm{tr}\left\{-\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)}+\left(\frac{2}{\sigma}\boldsymbol{W}+3\boldsymbol{W}'\right)(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d})\,\boldsymbol{B}+\left(\frac{1}{\sigma}\boldsymbol{W}+2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)}+\boldsymbol{Z}_{1d}\boldsymbol{B}\right)\right\},\\
a_{32}&=-\frac{1}{2\sigma^2}\mathrm{tr}\left\{\left(\frac{2}{\sigma}\boldsymbol{W}+3\boldsymbol{W}'\right)(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d})\,\boldsymbol{B}\right\},
\end{aligned}
\tag{7.22}
$$

$$a_{33} = -\frac{1}{6\sigma^2}\text{tr}\left\{\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}.$$

As $a_{30} = -(a_{31} + a_{32} + a_{33})$, from (7.22), we have

$$a_{30} = -\frac{1}{2\sigma^2}\text{tr}\left\{-\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)} + \left(\frac{1}{\sigma}\boldsymbol{W} + 2\boldsymbol{W}'\right)\left(\boldsymbol{B}^{(3)} + \boldsymbol{Z}_{1d}\boldsymbol{B}\right) - \frac{1}{3}\left(\frac{2}{\sigma}\boldsymbol{W} + 3\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}.$$

C.4. Gradient test

From [32], the quantities a_{ik} in the gradient test ($i = 4$) are given by

$$\begin{aligned} a_{41} &= \frac{1}{4}\sum_{r,s,t}^{p}\kappa_{rst}\kappa^{r,s}\epsilon_t^* - \frac{1}{2}\sum_{r,s,t}^{p}\left(\kappa_{rst} + 2\kappa_{r,st}\right)\epsilon_r^*\epsilon_s^*\epsilon_t^* \\ &\quad - \frac{1}{4}\sum_{r,s,t}^{p}\left(4\kappa_{r,st} + 3\kappa_{rst}\right)a_{rs}\epsilon_t^* - \frac{1}{2}\sum_{r=q+1}^{p}\sum_{s,t}^{p}\left(\kappa_{rst} + \kappa_{r,st}\right)\epsilon_r\epsilon_s^*\epsilon_t^*, \qquad (7.23) \\ a_{42} &= -\frac{1}{4}\sum_{r,s,t}^{p}\kappa_{rst}m_{rs}\epsilon_t^* + \frac{1}{4}\sum_{r,s,t}^{p}\left(\kappa_{rst} + 2\kappa_{r,st}\right)\epsilon_r^*\epsilon_s^*\epsilon_t^*, \\ a_{43} &= -\frac{1}{12}\sum_{r,s,t}^{p}\kappa_{rst}\epsilon_r^*\epsilon_s^*\epsilon_t^*. \end{aligned}$$

Replacing the cumulants of the Weibull censored data presented in (7.6) the Eq (7.23), we have

$$\begin{aligned} a_{41} &= \frac{1}{4\sigma^3}\sum_{i=1}^{n}w_i\sum_{r,s=1}^{p}\left(x_{ri}\kappa^{r,s}x_{si}\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right) \\ &\quad + \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right) \\ &\quad + \frac{1}{4\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 4w_i'\right)\sum_{r,s=1}^{p}\left(x_{ri}a_{rs}x_{si}\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right) \\ &\quad + \frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'\sum_{r=q+1}^{p}\left(x_{ri}\epsilon_r\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right), \qquad (7.24) \\ a_{42} &= -\frac{1}{4\sigma^3}\sum_{i=1}^{n}w_i\sum_{r,s=1}^{p}\left(x_{ri}m_{rs}x_{si}\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right) \\ &\quad - \frac{1}{4\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right), \\ a_{43} &= -\frac{1}{12\sigma^3}\sum_{i=1}^{n}w_i\sum_{r=1}^{p}\left(x_{ri}\epsilon_r^*\right)\sum_{s=1}^{p}\left(x_{si}\epsilon_s^*\right)\sum_{t=1}^{p}\left(x_{ti}\epsilon_t^*\right). \end{aligned}$$

From the quantities defined in (7.15), we have

$$a_{41} = \frac{1}{4\sigma^3}\sum_{i=1}^{n}w_iz_{ii}b_i + \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i + 2w_i'\right)b_i^3$$

$$+\frac{1}{4\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+4w_i'\right)z_{1ii}b_i+\frac{1}{2\sigma^2}\sum_{i=1}^{n}w_i'e_ib_i^2, \tag{7.25}$$

$$a_{42}=-\frac{1}{4\sigma^3}\sum_{i=1}^{n}w_i(z_{ii}-z_{1ii})b_i-\frac{1}{4\sigma^2}\sum_{i=1}^{n}\left(\frac{1}{\sigma}w_i+2w_i'\right)b_i^3,$$

$$a_{43}=-\frac{1}{12\sigma^3}\sum_{i=1}^{n}w_ib_i^3.$$

Finally, in matrix notation, the expression (7.21) can be written as

$$\begin{aligned}a_{41}&=\frac{1}{2\sigma^2}\mathrm{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)}+\left(\frac{1}{\sigma}\boldsymbol{W}+2\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\}\\&\quad+\frac{1}{4\sigma^2}\mathrm{tr}\left\{\frac{1}{\sigma}\boldsymbol{W}\boldsymbol{Z}_d\boldsymbol{B}+\left(\frac{1}{\sigma}\boldsymbol{W}+4\boldsymbol{W}'\right)\boldsymbol{Z}_{1d}\boldsymbol{B}\right\},\\a_{42}&=-\frac{1}{4\sigma^2}\mathrm{tr}\left\{\frac{1}{\sigma}\boldsymbol{W}\left(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d}\right)\boldsymbol{B}+\left(\frac{1}{\sigma}\boldsymbol{W}+2\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}\right\},\\a_{43}&=-\frac{1}{12\sigma^3}\mathrm{tr}\left\{\boldsymbol{W}\boldsymbol{B}^{(3)}\right\}.\end{aligned}\tag{7.26}$$

As $a_{40}=-(a_{41}+a_{42}+a_{43})$ and from (7.26), we have

$$a_{40}=\frac{1}{2\sigma^2}\mathrm{tr}\left\{\boldsymbol{W}'\boldsymbol{E}\boldsymbol{B}^{(2)}+\left(\frac{1}{3\sigma}\boldsymbol{W}+\boldsymbol{W}'\right)\boldsymbol{B}^{(3)}+\left(\frac{1}{\sigma}\boldsymbol{W}+2\boldsymbol{W}'\right)\boldsymbol{Z}_{1d}\boldsymbol{B}\right\}.$$

C.5. Particular cases

C.5.1. No censoring

$\boldsymbol{W}=\boldsymbol{I}_n$ and $\boldsymbol{W}'=\boldsymbol{0}_n$

$$\begin{aligned}a_{11}&=\frac{1}{2\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}+\boldsymbol{Z}_{1d}\boldsymbol{B}\right\},\quad a_{12}=-2a_{23}=a_{33}=4a_{43}=-\frac{1}{3\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}\right\},\quad a_{13}=0,\\a_{21}&=\frac{1}{2\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}-\left(\boldsymbol{Z}_d-2\boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\},\quad a_{22}=-\frac{1}{2\sigma^2}\mathrm{tr}\left\{\frac{1}{\sigma}\boldsymbol{B}^{(3)}-\left(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\},\\a_{31}&=\frac{1}{2\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}+\left(2\boldsymbol{Z}_d-\boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\},\quad a_{32}=-\frac{1}{\sigma^3}\mathrm{tr}\left\{\left(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\},\\a_{41}&=\frac{1}{2\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}\right\}+\frac{1}{4\sigma^3}\mathrm{tr}\left\{\left(\boldsymbol{Z}_d+\boldsymbol{Z}_{1d}\right)\boldsymbol{B}\right\},\\a_{42}&=-\frac{1}{4\sigma^3}\mathrm{tr}\left\{\left(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d}\right)\boldsymbol{B}+\boldsymbol{B}^{(3)}\right\}.\end{aligned}$$

When $q=0$

$$\begin{aligned}a_{12}&=a_{33}=-\frac{2}{3}a_{11}=-2a_{23}=4a_{43}=-\frac{1}{3\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}\right\},\quad a_{13}=0,\\a_{21}&=-\frac{1}{\sigma}a_{22}=\frac{1}{2\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}-\boldsymbol{Z}_d\boldsymbol{B}\right\},\quad a_{31}=\frac{1}{2\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}+2\boldsymbol{Z}_d\boldsymbol{B}\right\},\end{aligned}$$

$$a_{32}=-\frac{1}{\sigma^3}\mathrm{tr}\left\{\boldsymbol{Z}_d\boldsymbol{B}\right\},\quad a_{41}=\frac{1}{4\sigma^3}\mathrm{tr}\left\{2\boldsymbol{B}^{(3)}+\boldsymbol{Z}_d\boldsymbol{B}\right\},\quad a_{42}=-\frac{1}{4\sigma^3}\mathrm{tr}\left\{\boldsymbol{Z}_d\boldsymbol{B}+\boldsymbol{B}^{(3)}\right\}.$$

For $y_i=\beta x_i+\varepsilon_i$

$$\begin{aligned}
&a_{12}=a_{33}=-\frac{2}{3}a_{11}=-2a_{23}=4a_{43}=\frac{\epsilon h_2}{3\sigma^3},\quad a_{13}=0,\\
&a_{21}=-\frac{1}{\sigma}a_{22}=-\frac{\epsilon^3h_2}{2\sigma^3}+\frac{\epsilon h_2}{2\sigma^3h_1},\quad a_{31}=-\frac{\epsilon^3h_2}{2\sigma^3}-\frac{\epsilon h_2}{\sigma^3h_1},\\
&a_{32}=\frac{\epsilon h_2}{\sigma^3h_1},\quad a_{41}=-\frac{\epsilon^3h_2}{2\sigma^3}-\frac{\epsilon h_2}{4\sigma^3h_1},\quad a_{42}=\frac{\epsilon h_2}{4\sigma^3h_1}+\frac{\epsilon^3h_2}{4\sigma^3}.
\end{aligned}$$

C.5.2. Type II censoring

$$\begin{aligned}
a_{11}&=\frac{r}{2n\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}+\boldsymbol{Z}_{1d}\boldsymbol{B}\right\},\quad a_{12}=-2a_{23}=a_{33}=4a_{43}=-\frac{r}{3n\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}\right\},\quad a_{13}=0,\\
a_{21}&=\frac{r}{2n\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}-(\boldsymbol{Z}_d-2\boldsymbol{Z}_{1d})\boldsymbol{B}\right\},\quad a_{22}=-\frac{r}{2n\sigma^2}\mathrm{tr}\left\{\frac{1}{\sigma}\boldsymbol{B}^{(3)}-(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d})\boldsymbol{B}\right\},\\
a_{31}&=\frac{r}{2n\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}+(2\boldsymbol{Z}_d-\boldsymbol{Z}_{1d})\boldsymbol{B}\right\},\quad a_{32}=-\frac{r}{n\sigma^3}\mathrm{tr}\left\{(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d})\boldsymbol{B}\right\},\\
a_{41}&=\frac{r}{2n\sigma^3}\mathrm{tr}\left\{\boldsymbol{B}^{(3)}\right\}+\frac{r}{4n\sigma^3}\mathrm{tr}\left\{(\boldsymbol{Z}_d+\boldsymbol{Z}_{1d})\boldsymbol{B}\right\},\\
a_{42}&=-\frac{r}{4n\sigma^3}\mathrm{tr}\left\{(\boldsymbol{Z}_d-\boldsymbol{Z}_{1d})\boldsymbol{B}+\boldsymbol{B}^{(3)}\right\}.
\end{aligned}$$

Section D: Local power comparisons

Let Π_i the power function, up to order $n^{-1/2}$, of the statistic S_i, $i=1,\ldots,4$. We can write

$$\Pi_i-\Pi_j=\sum_{k=0}^{3}\left(a_{jk}-a_{ik}\right)G_{p-q+2k,\lambda}(x),$$

for $i\neq j$. Regarding $G_{m,\lambda}(x)-G_{m+2,\lambda}(x)=2g_{m+2,\lambda}(x)$

$$\begin{aligned}
\Pi_i-\Pi_j&=\left(a_{j0}-a_{i0}\right)G_{p-q,\lambda}(x)\\
&+\left(a_{j1}-a_{i1}\right)G_{p-q+2,\lambda}(x)+\left(a_{j2}-a_{i2}\right)G_{p-q+4,\lambda}(x)+\left(a_{j3}-a_{i3}\right)G_{p-q+6,\lambda}(x)\\
&=-\left[(a_{j1}+a_{j2}+a_{j3})-(a_{i1}+a_{i2}+a_{i3})\right]G_{p-q,\lambda}(x)\\
&+\left(a_{j1}-a_{i1}\right)G_{p-q+2,\lambda}(x)+\left(a_{j2}-a_{i2}\right)G_{p-q+4,\lambda}(x)+\left(a_{j3}-a_{i3}\right)G_{p-q+6,\lambda}(x)\\
&=-\left[(a_{j1}-a_{i1})+(a_{j2}-a_{i2})+(a_{j3}-a_{i3})\right]G_{p-q,\lambda}(x)\\
&+\left(a_{j1}-a_{i1}\right)G_{p-q+2,\lambda}(x)+\left(a_{j2}-a_{i2}\right)G_{p-q+4,\lambda}(x)+\left(a_{j3}-a_{i3}\right)G_{p-q+6,\lambda}(x)\\
&=-(a_{j1}-a_{i1})\left[G_{p-q,\lambda}(x)-G_{p-q+2,\lambda}(x)\right]\\
&-(a_{j2}-a_{i2})\left[G_{p-q,\lambda}(x)-G_{p-q+4,\lambda}(x)\right]\\
&-(a_{j3}-a_{i3})\left[G_{p-q,\lambda}(x)-G_{p-q+6,\lambda}(x)\right]
\end{aligned}$$

$$
\begin{aligned}
&= -(a_{j1} - a_{i1}) \times 2g_{p-q+2,\lambda}(x) \\
&- (a_{j2} - a_{i2})\left[2g_{p-q+2,\lambda}(x) + 2g_{p-q+4,\lambda}(x)\right] \\
&- (a_{j3} - a_{i3})\left[2g_{p-q+2,\lambda}(x) + 2_{p-q+4,\lambda}(x) + 2g_{p-q+6,\lambda}(x)\right] \\
&= -2\left[(a_{j1} - a_{i1}) + (a_{j2} - a_{i2}) + (a_{j3} - a_{i3})\right] g_{p-q+2,\lambda}(x) \\
&- 2\left[(a_{j2} - a_{i2}) + (a_{j3} - a_{i3})\right] g_{p-q+4,\lambda}(x) - 2(a_{j3} - a_{i3}) g_{p-q+6,\lambda}(x) \\
&= -2(a_{j0} - a_{i0}) g_{p-q+2,\lambda}(x) - 2\left[(a_{j2} - a_{i2}) + (a_{j3} - a_{i3})\right] g_{p-q+4,\lambda}(x) - 2(a_{j3} - a_{i3}) g_{p-q+6,\lambda}(x) \\
&= -2\left[(a_{j2} - a_{i2}) + (a_{j3} - a_{i3})\right] g_{p-q+4,\lambda}(x) - 2(a_{j3} - a_{i3}) g_{p-q+6,\lambda}(x).
\end{aligned}
$$

Section E: Non-null asymptotic distribution

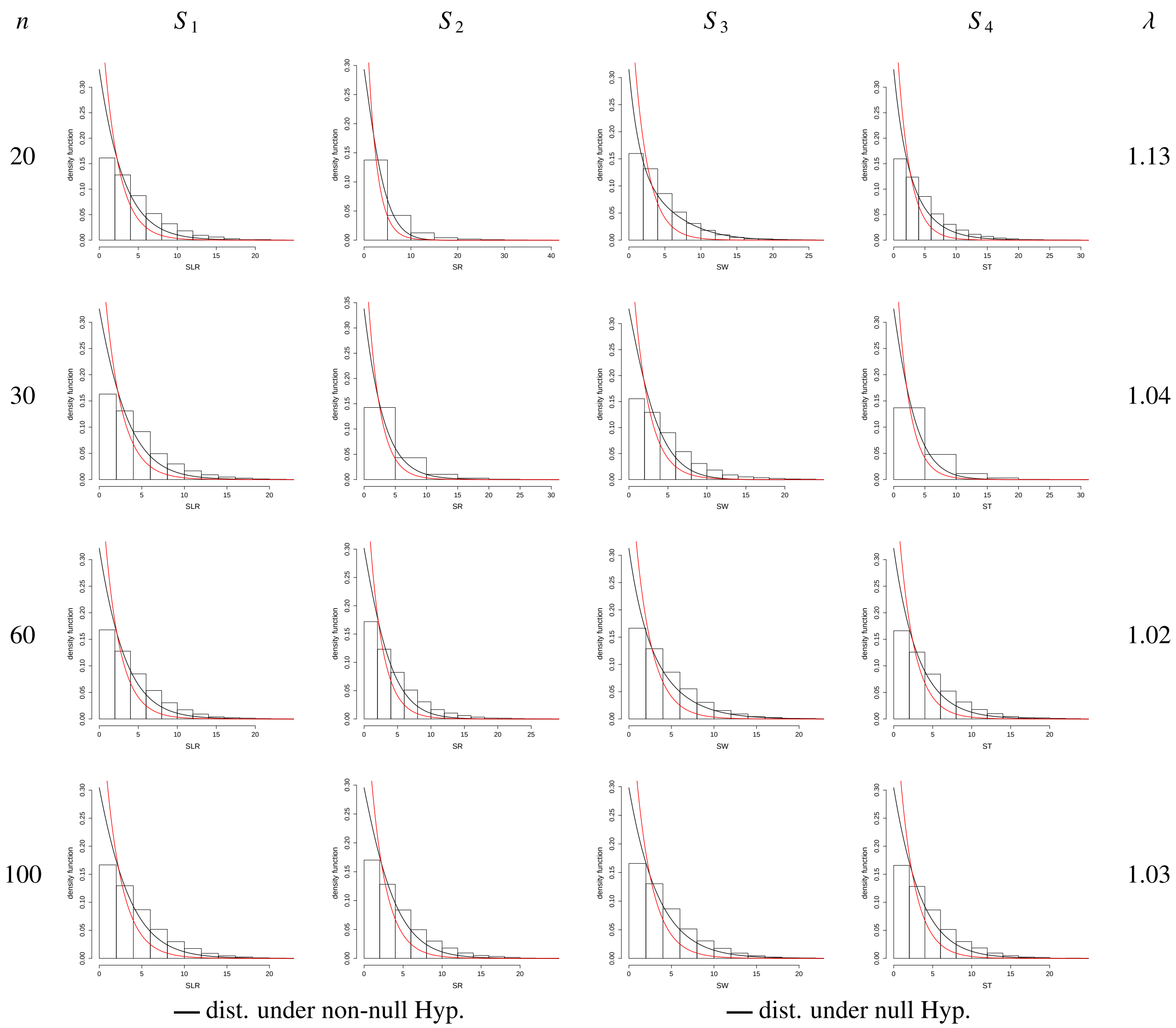


Figure S1. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 3$, $q = 1$, censoring = 10%, $\sigma = 0.5$, and $\psi = 0.575$.

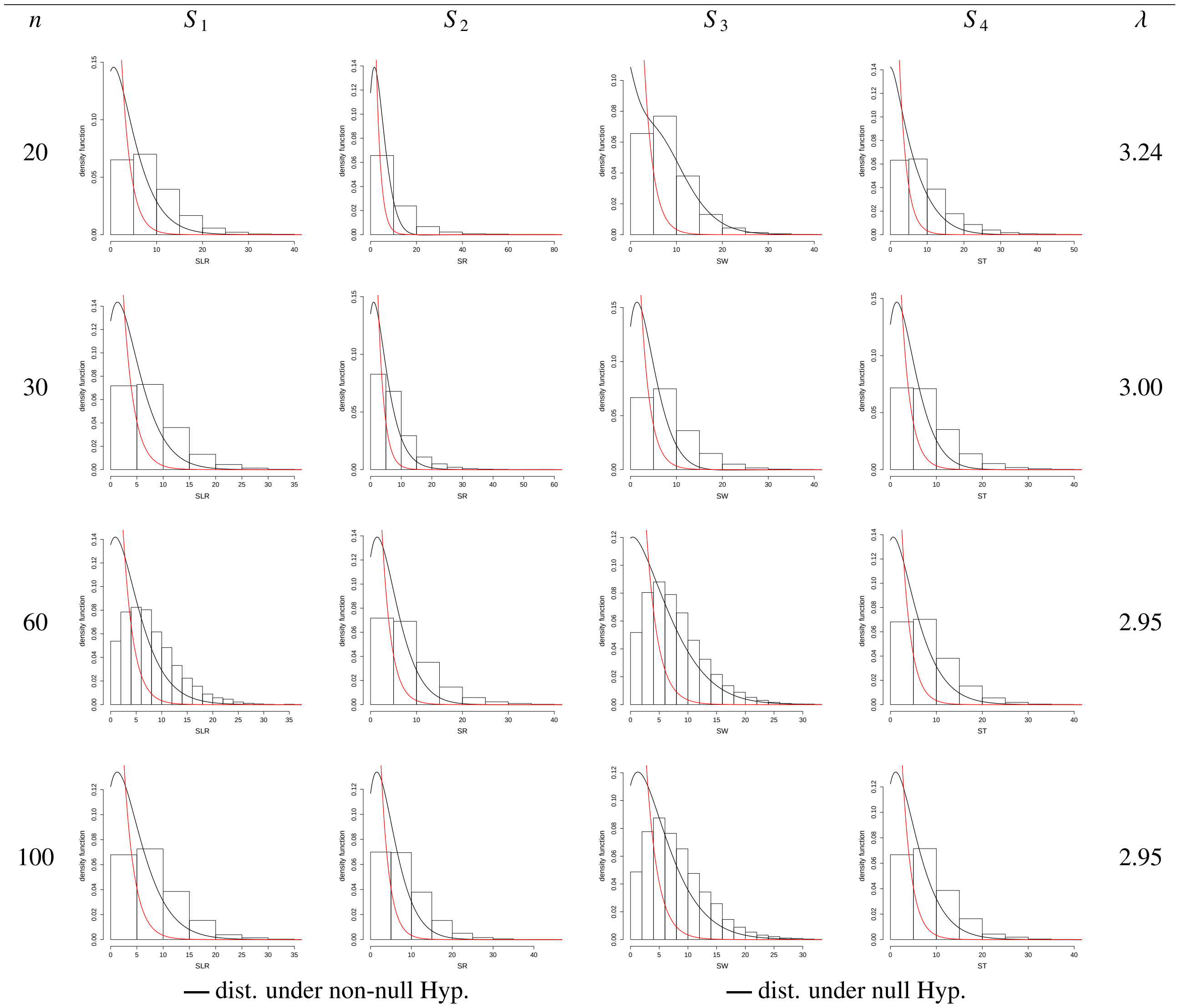


Figure S2. Non-null empirical and asymptotic distributions for SLR, SR, SW, and ST tests for the case $p = 3$, $q = 1$, censoring = 10%, $\sigma = 0.5$, and $\psi = 0.975$.

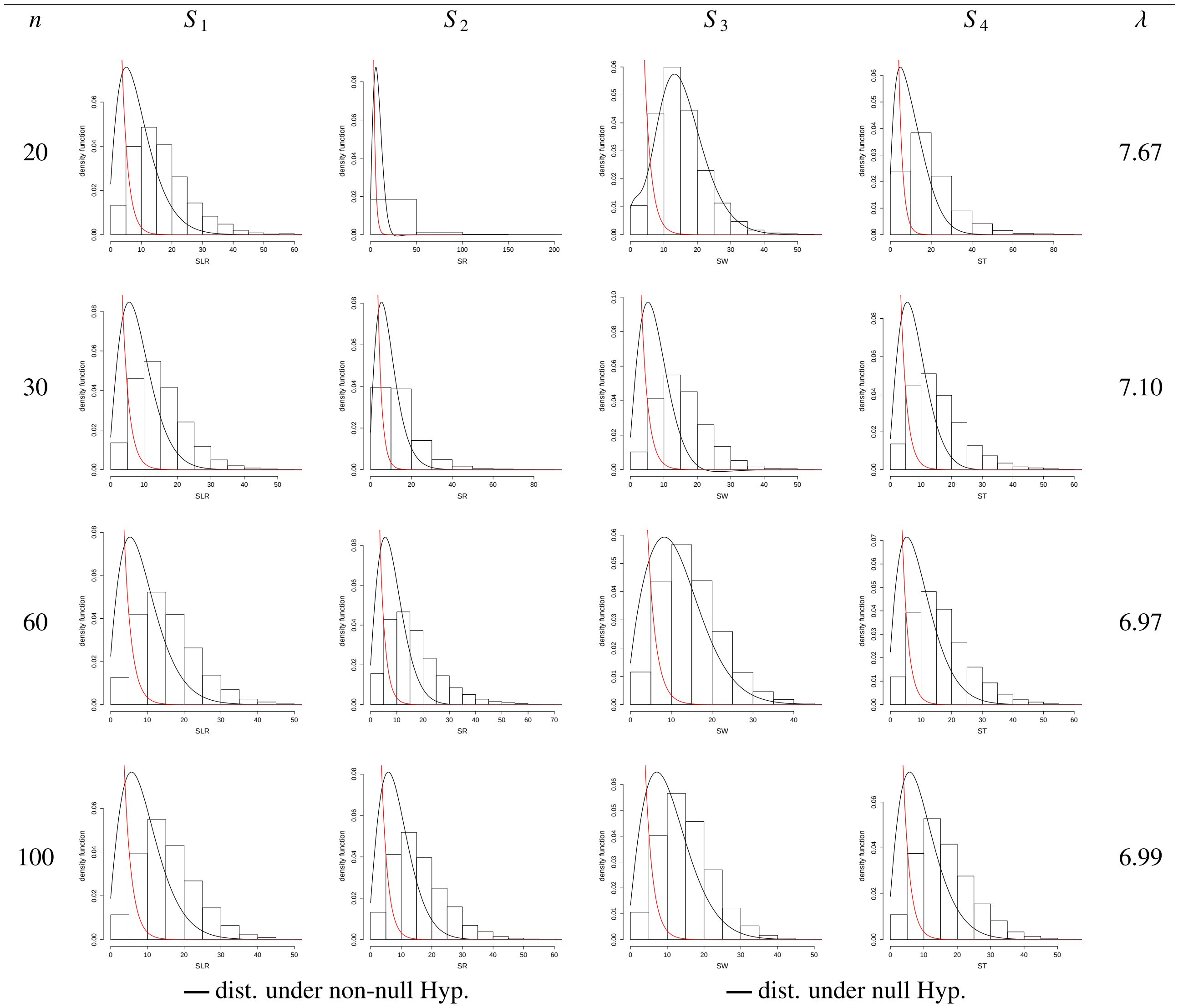
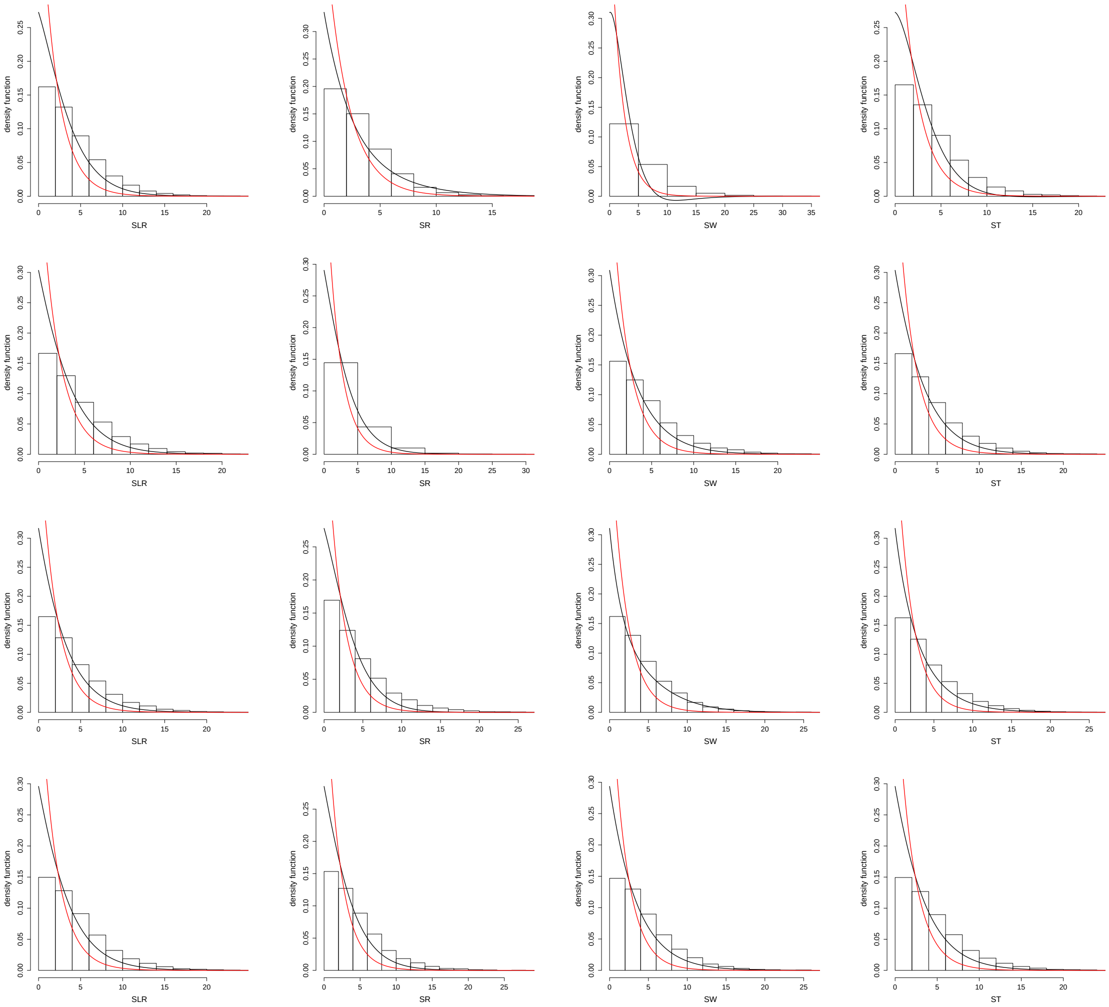


Figure S3. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 3$, $q = 1$, censoring = 10%, $\sigma = 0.5$, and $\psi = 1.5$.

n	S_1	S_2	S_3	S_4	λ
20					0.97
30					0.99
60					1.08
100					1.19



— dist. under non-null Hyp.   — dist. under null Hyp.

Figure S4. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 3$, censoring = 10%, $\sigma = 0.5$, and $\psi = 0.625$.

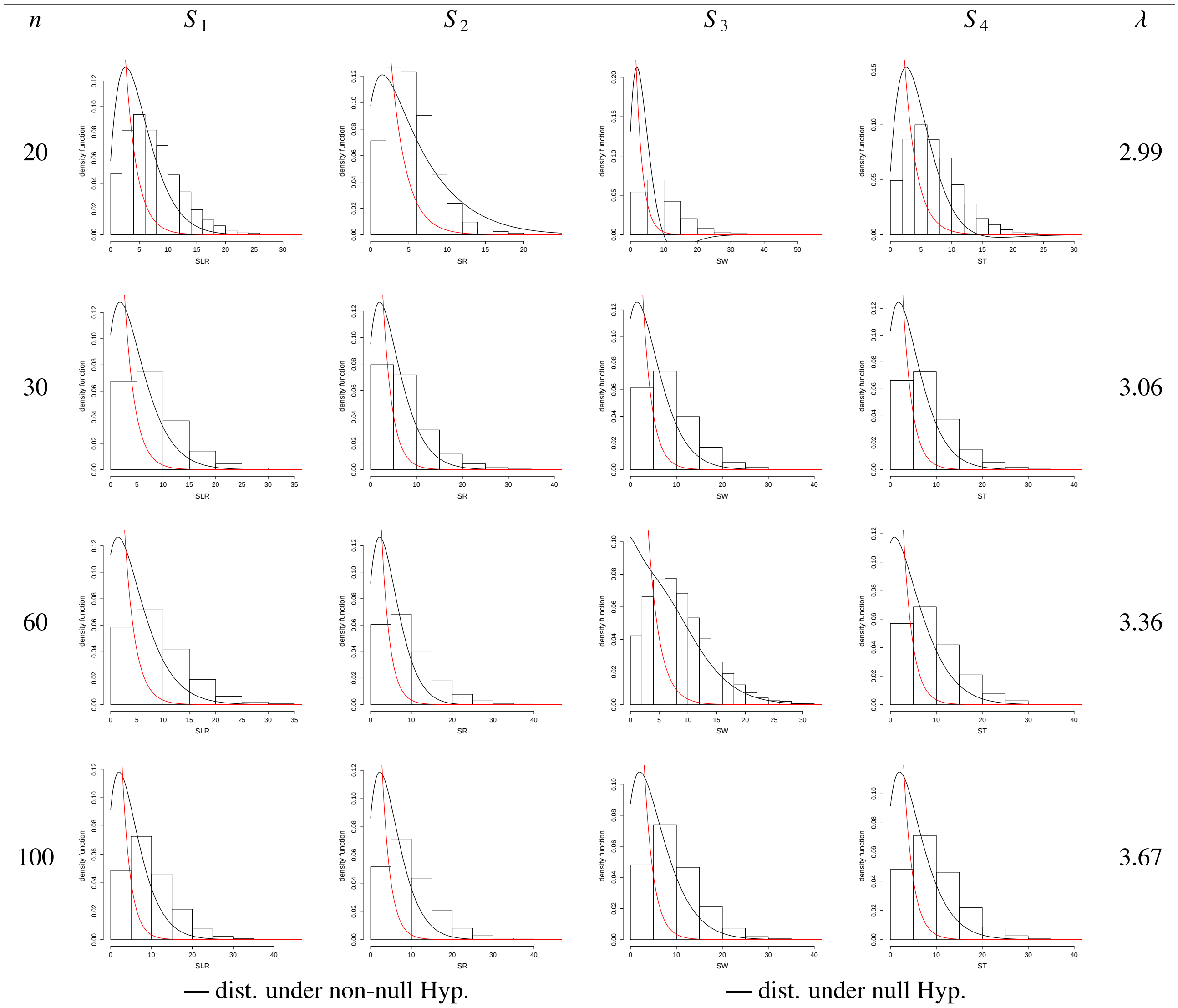


Figure S5. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 3$, censoring = 10%, $\sigma = 0.5$, and $\psi = 1.1$.

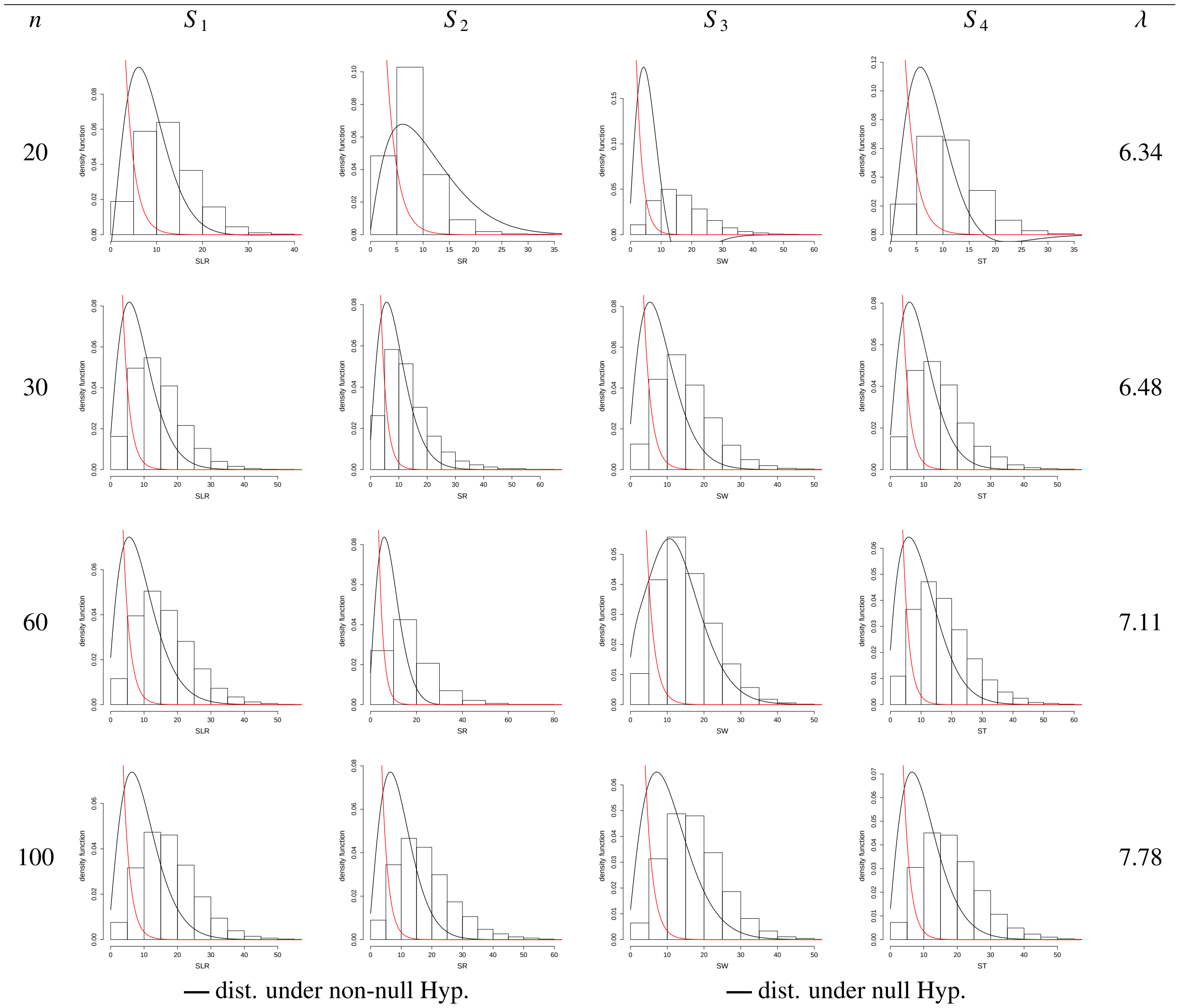


Figure S6. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 3$, censoring = 10%, $\sigma = 0.5$, and $\psi = 1.6$.

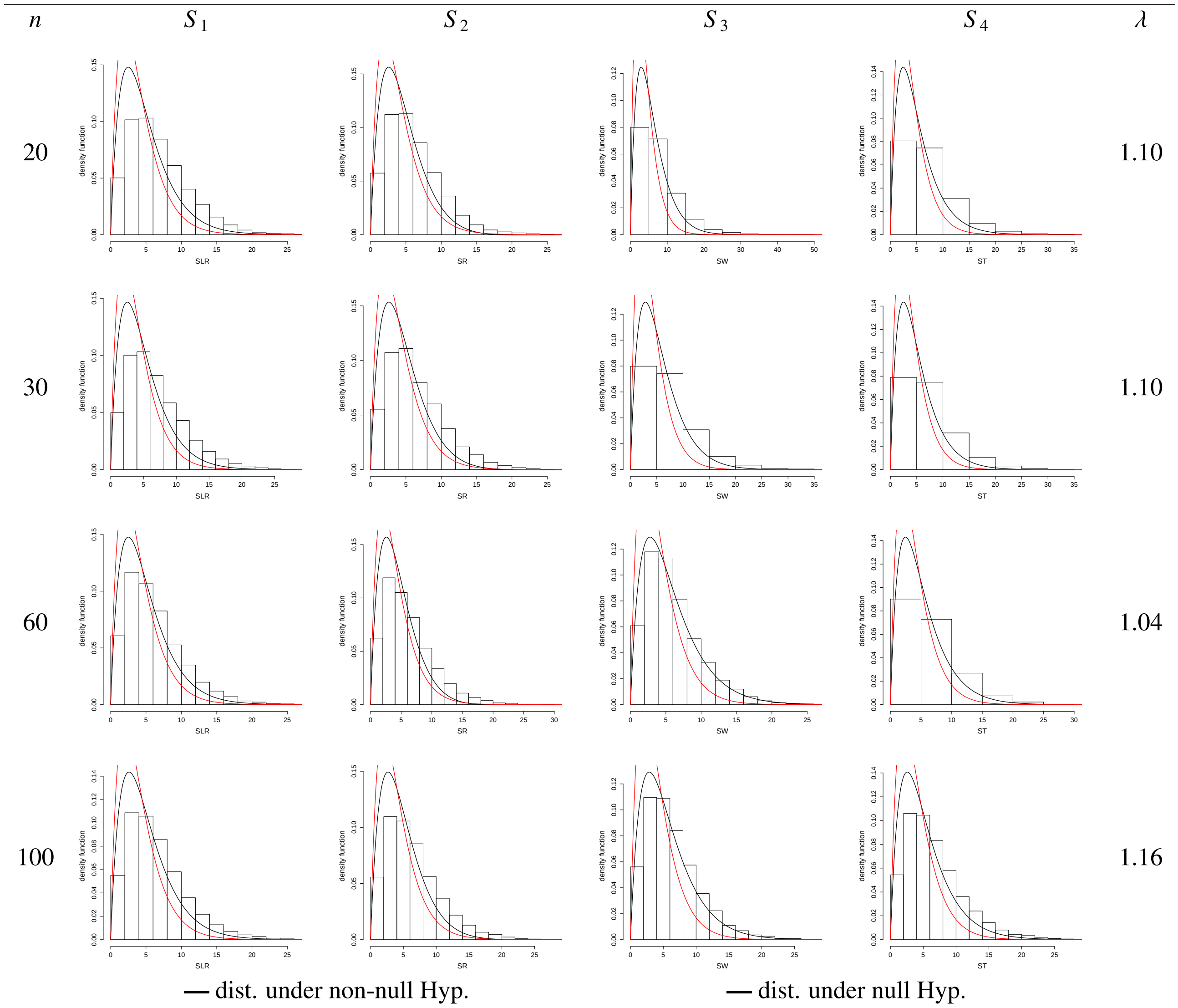


Figure S7. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 1$, censoring = 25%, $\sigma = 1$, and $\psi = 1$.

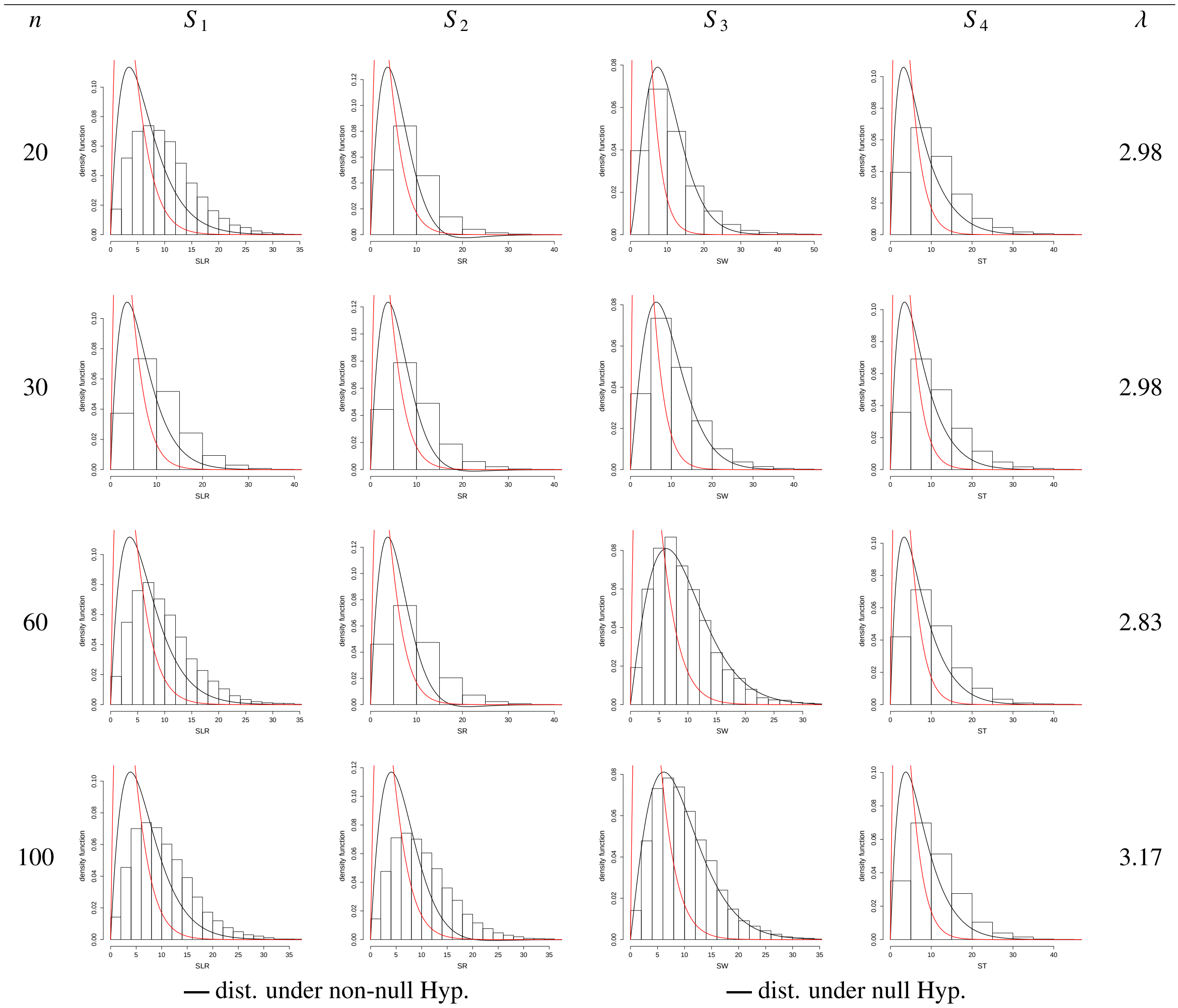


Figure S8. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 1$, censoring = 25%, $\sigma = 1$, and $\psi = 1.65$.

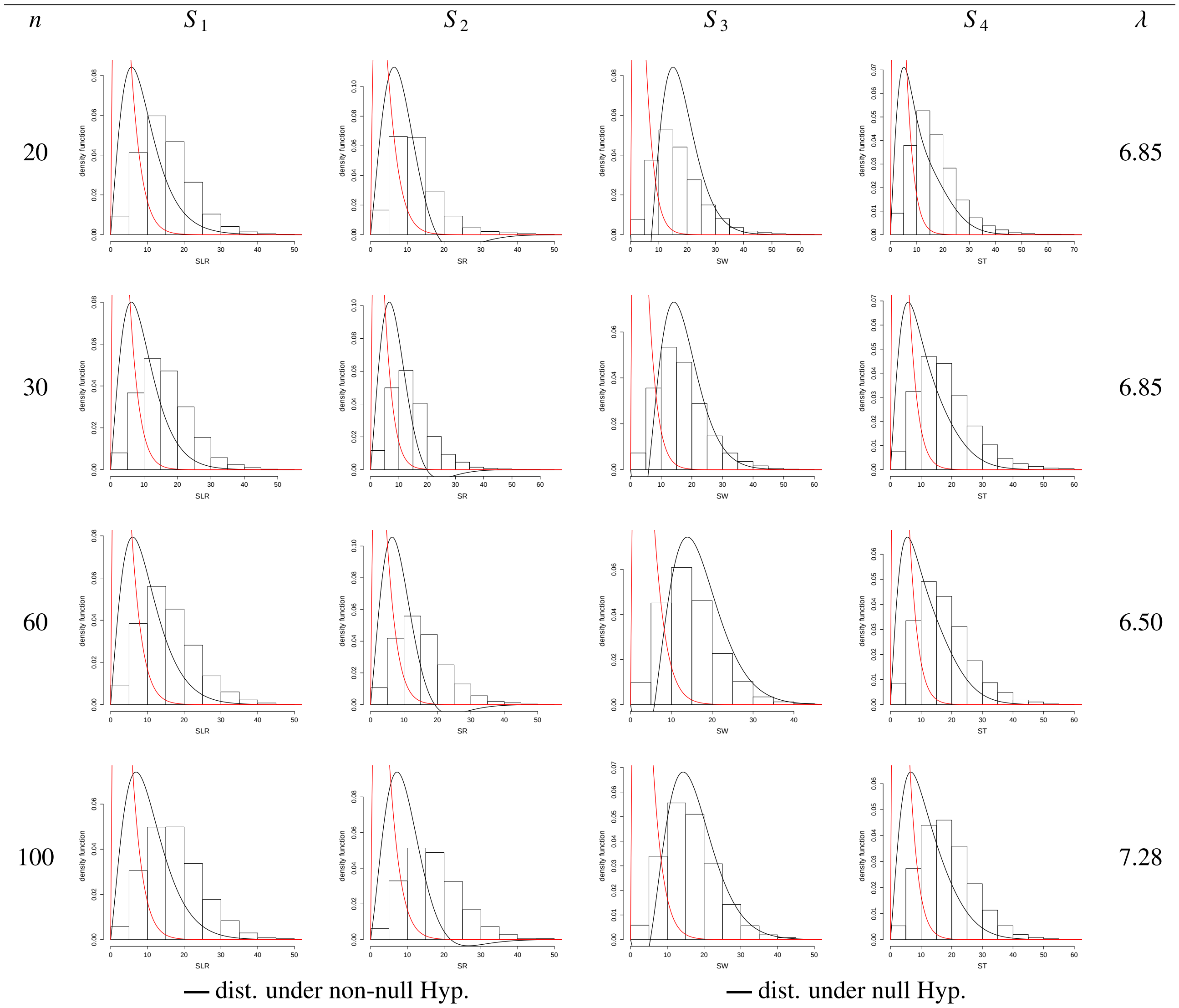


Figure S9. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 1$, censoring = 25%, $\sigma = 1$, and $\psi = 2.5$.

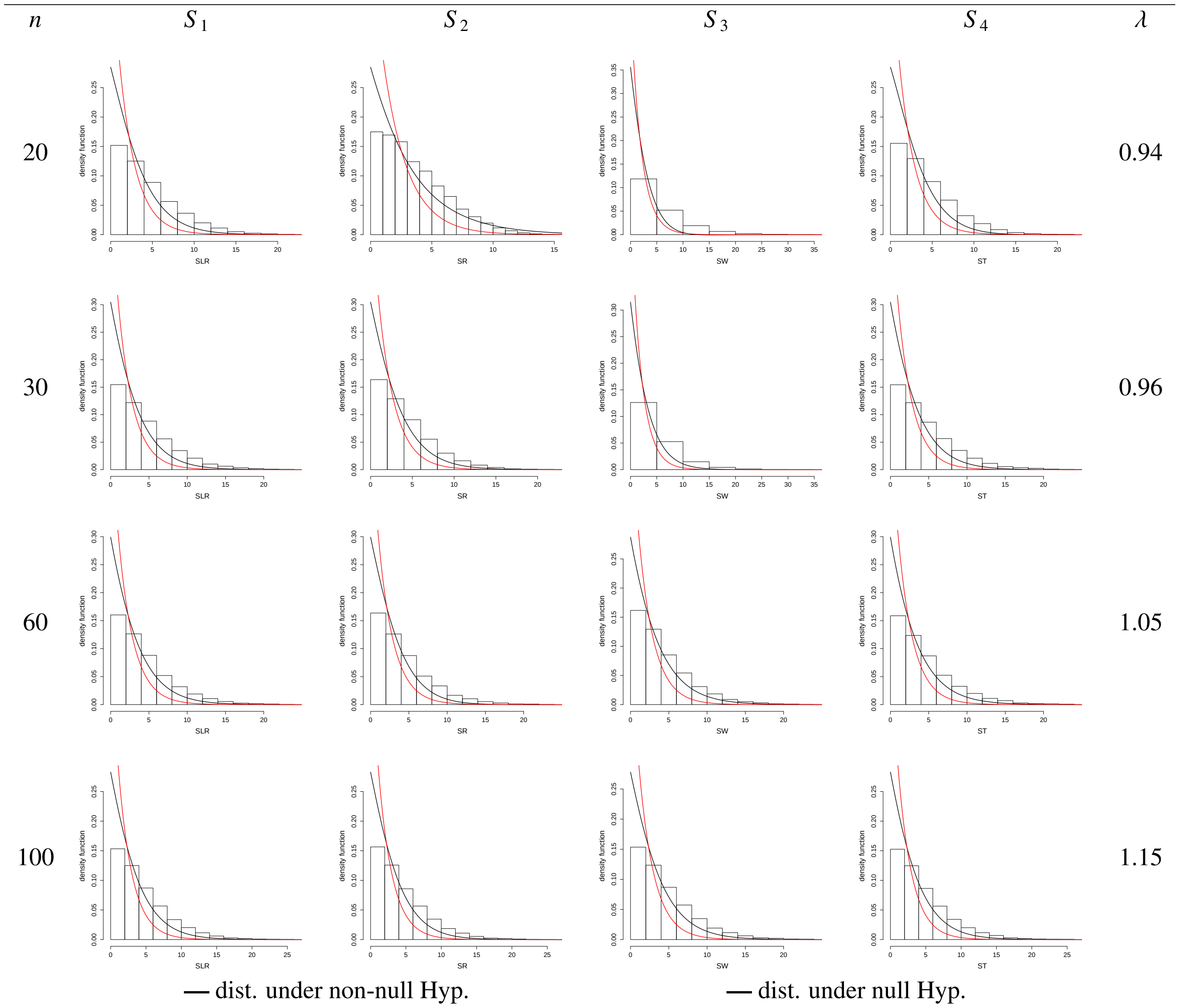


Figure S10. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 3$, censoring = 25%, $\sigma = 1$, and $\psi = 1.35$.

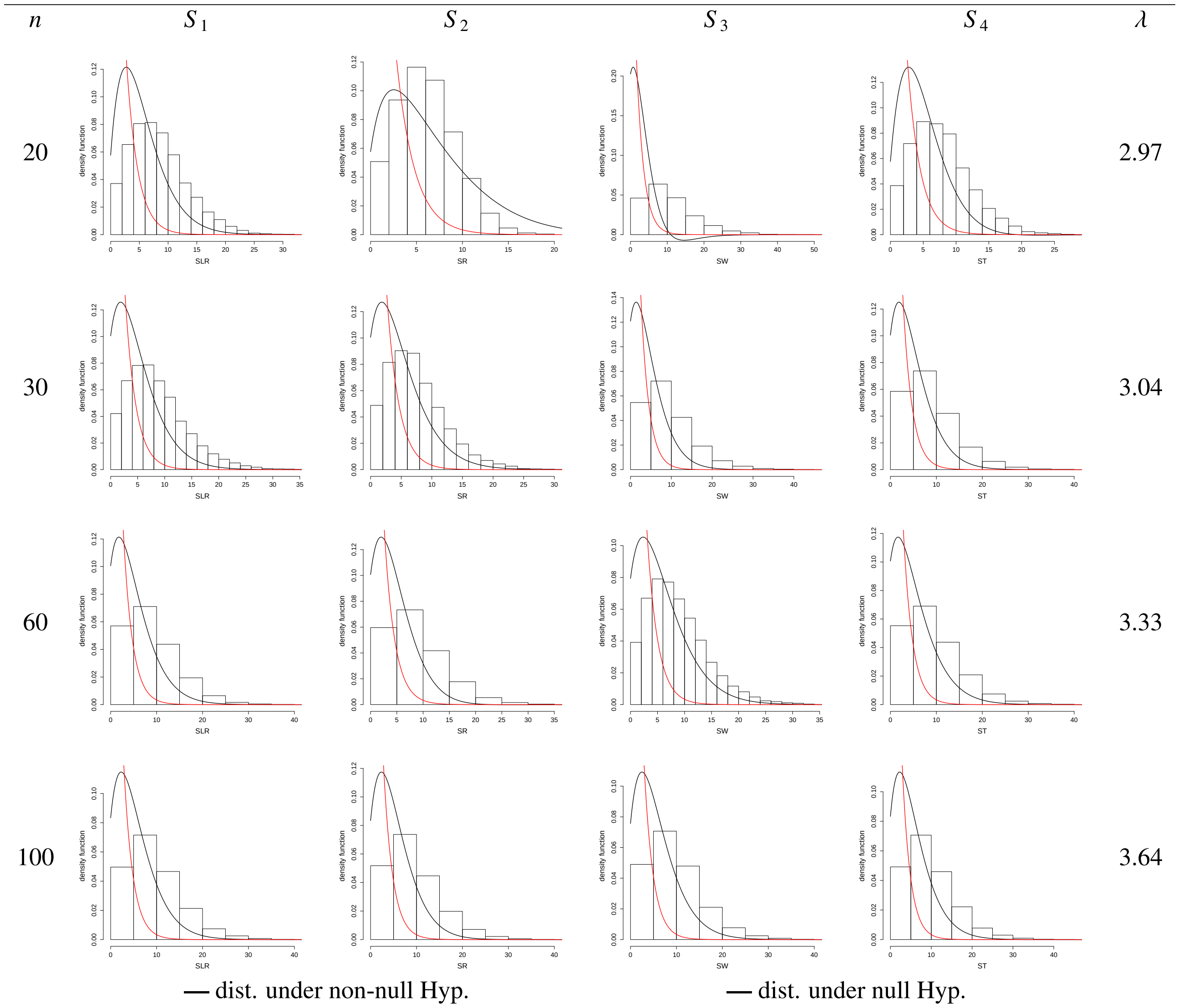


Figure S11. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 3$, censoring = 25%, $\sigma = 1$, and $\psi = 2.4$.

n	S_1	S_2	S_3	S_4	λ
20					6.68
30					6.83
60					7.48
100					8.20

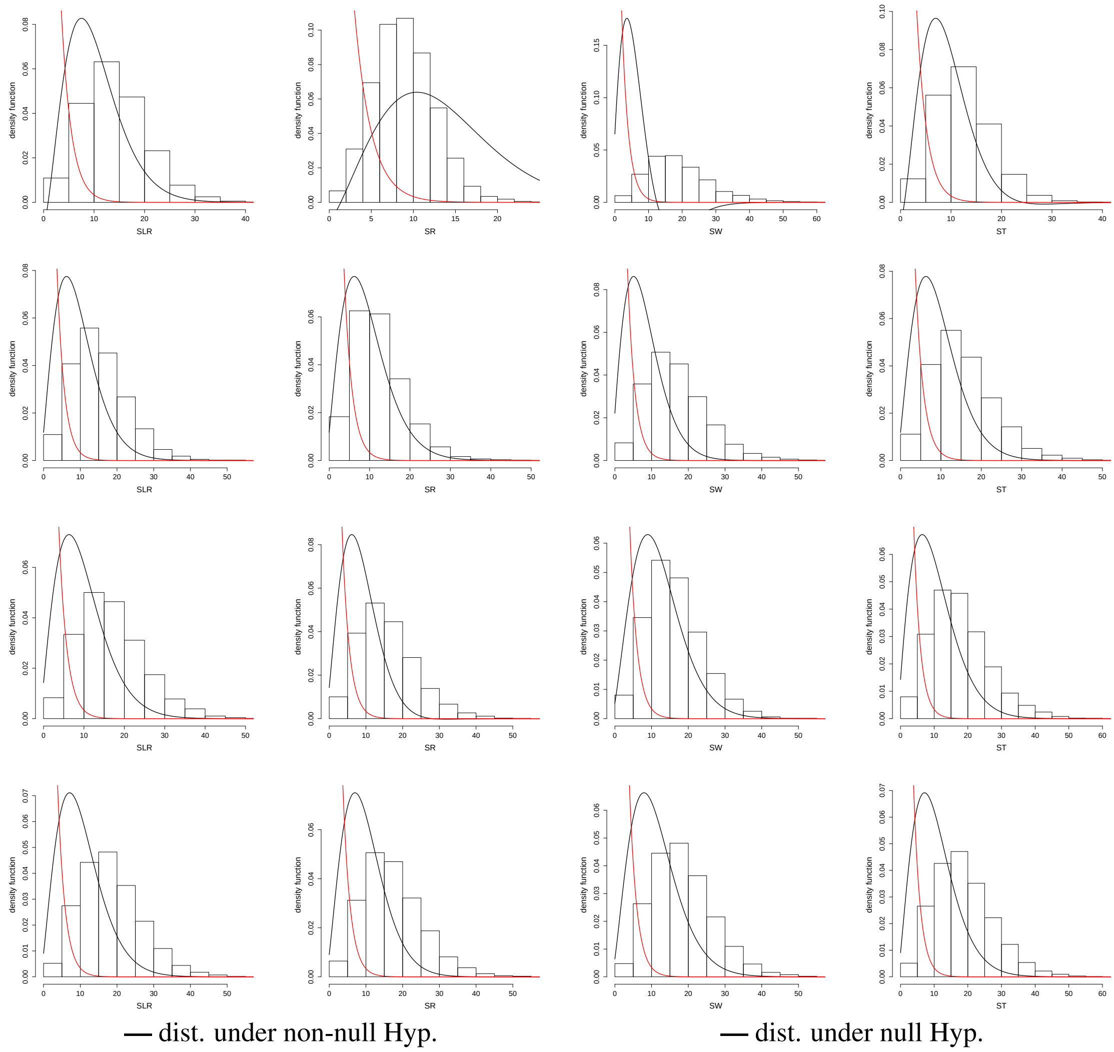


Figure S12. Non-null empirical and asymptotic distributions for the SLR, SR, SW, and ST tests for the case $p = 5$, $q = 3$, censoring = 25%, $\sigma = 1$, and $\psi = 3.6$.

Section F:Assessing the estimation of the power of the test

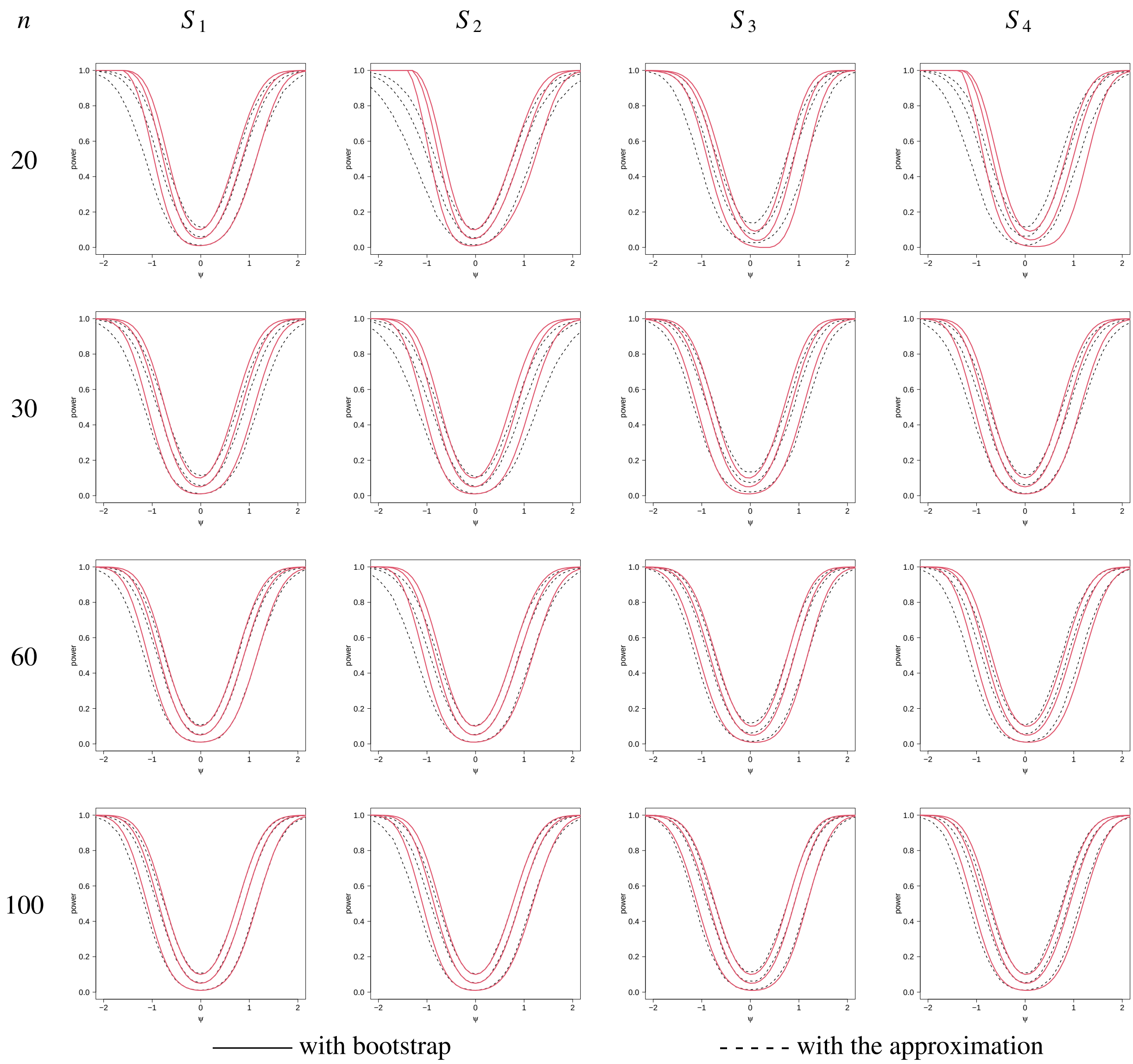


Figure S13. Power estimation via bootstrap (dashed line) and theoretical result (continuous line) for the case $p = 3$, $q = 1$, censoring = 10%, and $\sigma = 0.5$ for three levels of significance: 1%, 5%, and 10%.

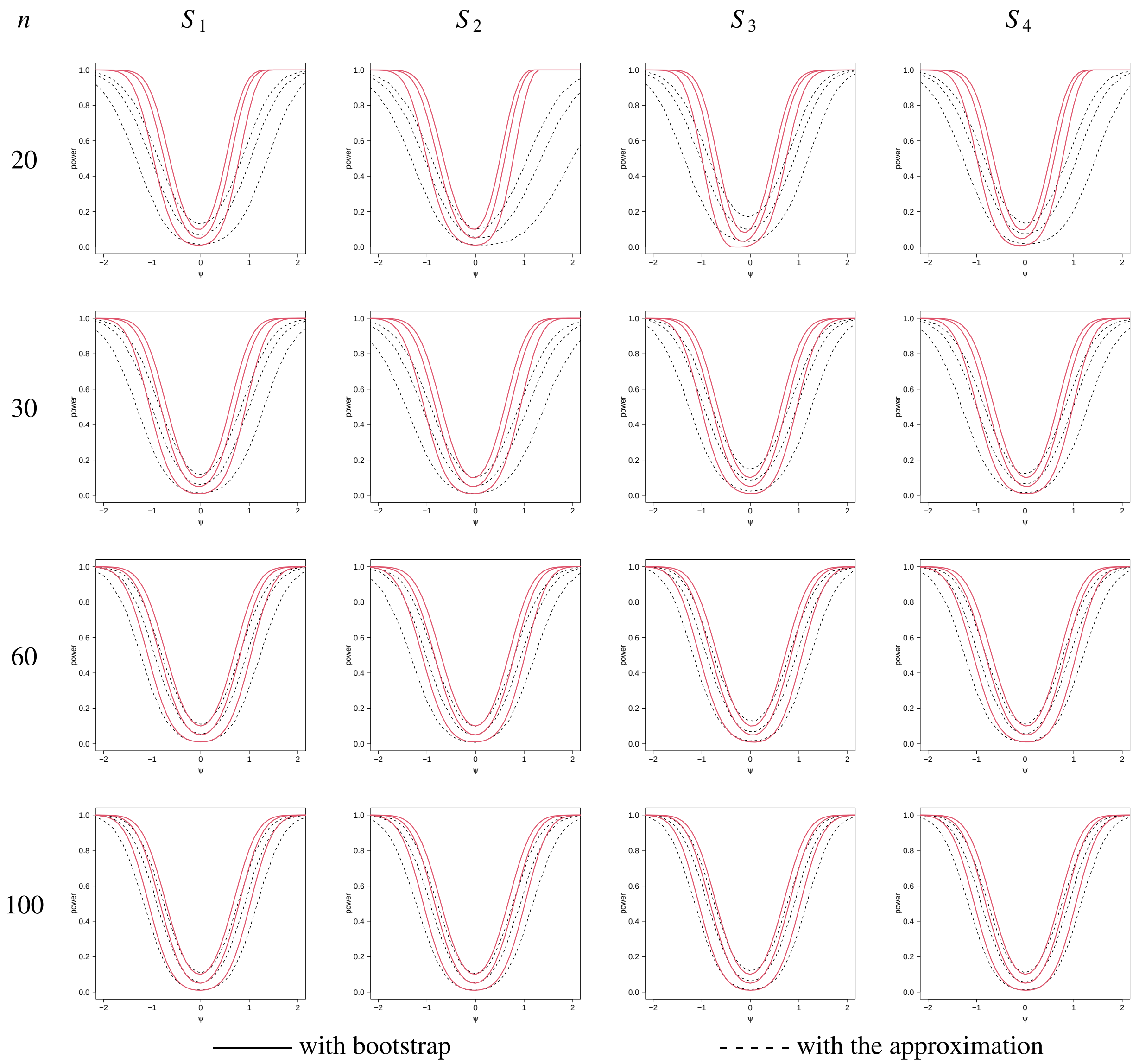


Figure S14. Power estimation via bootstrap (dashed line) and theoretical result (continuous line) for the case $p = 5$, $q = 3$, censoring = 10%, and $\sigma = 0.5$ for three levels of significance: 1%, 5%, and 10%.

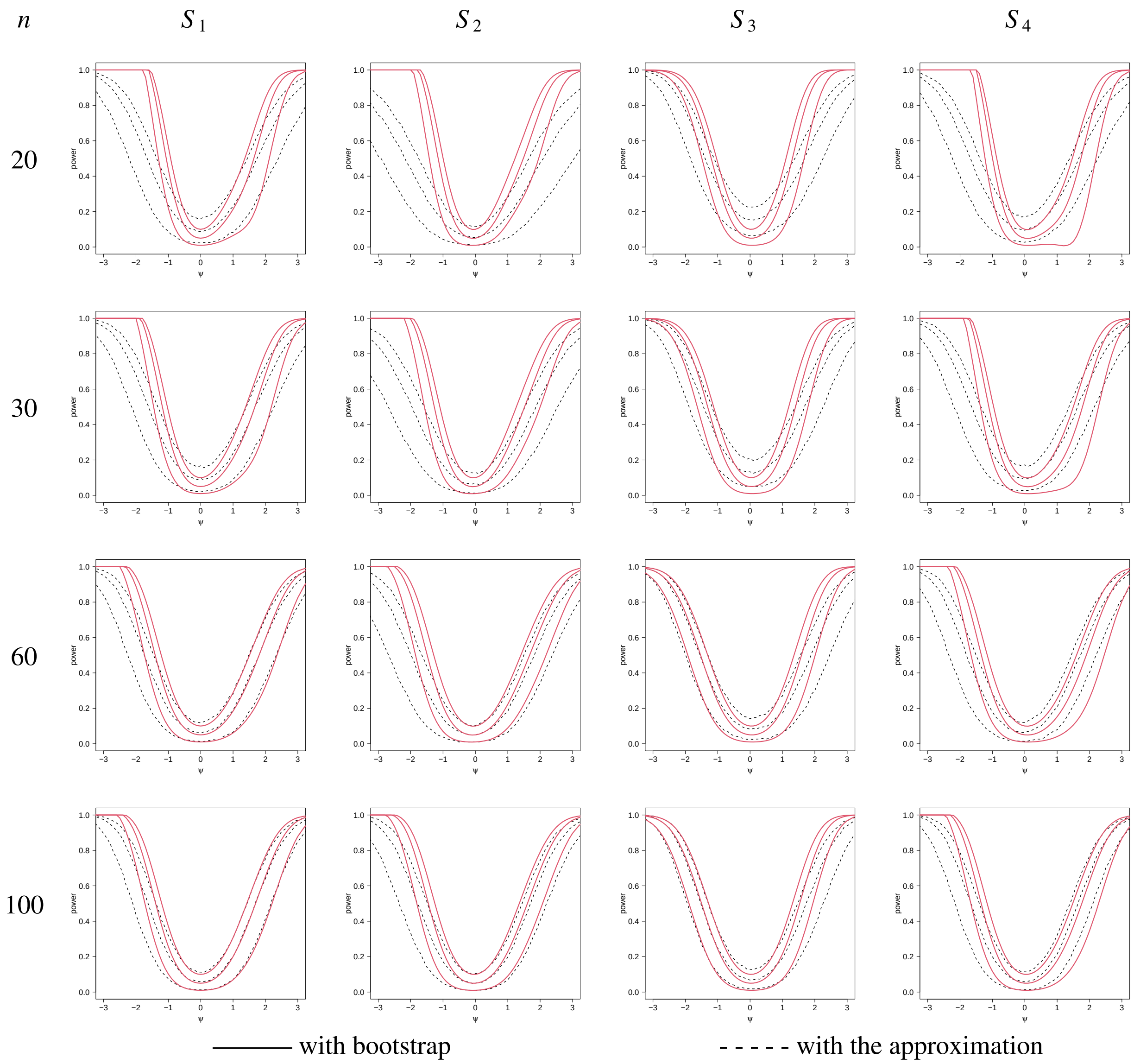


Figure S15. Power estimation via bootstrap (dashed line) and theoretical result (continuous line) for the case $p = 5$, $q = 1$, censoring = 25%, and $\sigma = 1$ for three levels of significance: 1%, 5%, and 10%.

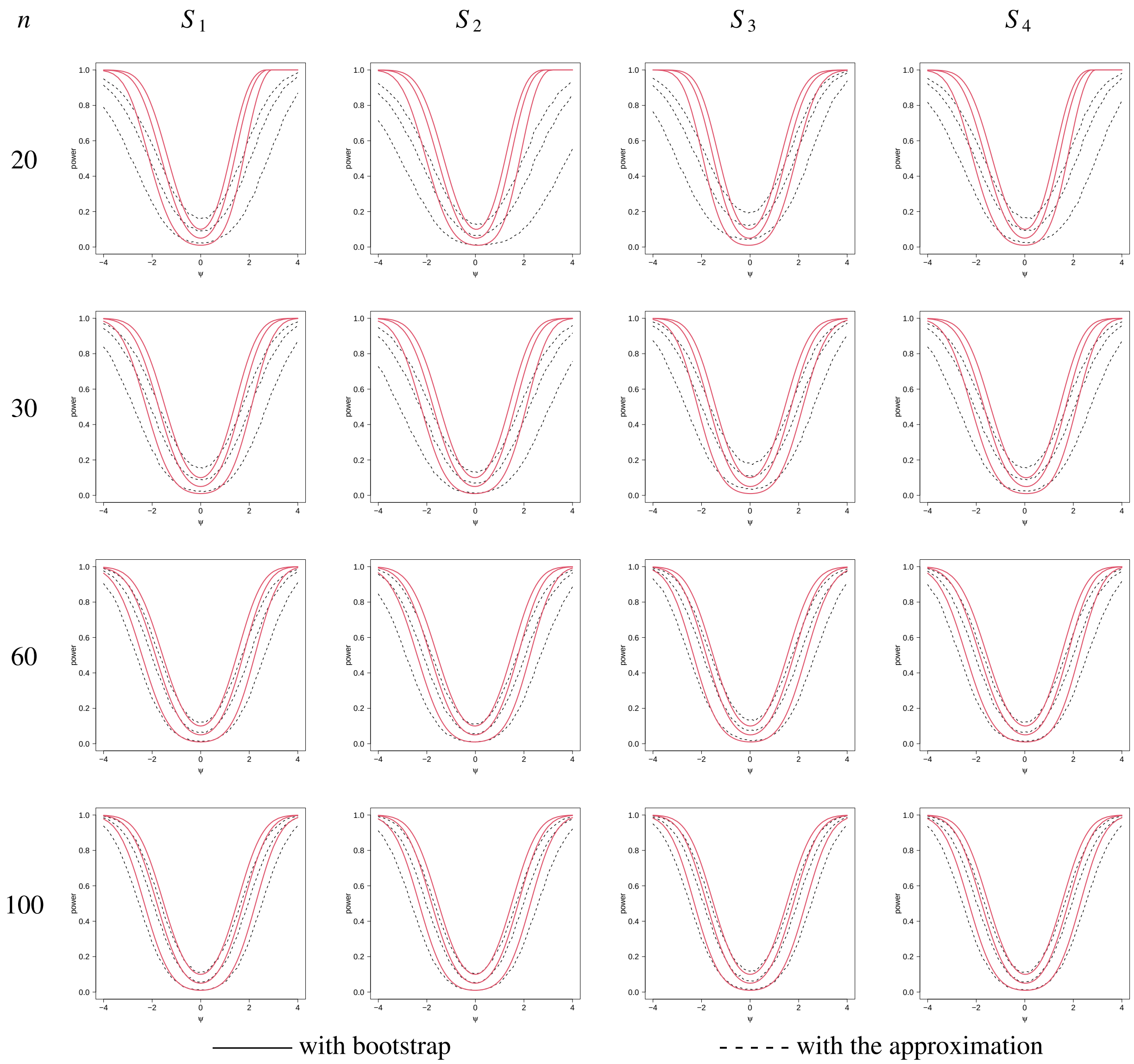


Figure S16. Power estimation via bootstrap (dashed line) and theoretical result (continuous line) for the case $p = 5$, $q = 3$, censoring = 25%, and $\sigma = 1$ for three levels of significance: 1%, 5%, and 10%.