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Research article

Traveling wave solutions for a nonlocal dispersal SIR model with delayed transmission

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Abstract: We study the existence of traveling wave solutions for a spatial susceptible-infected-recover (SIR) epidemic model with nonlocal dispersal and delayed transmission. The model incorporates convolution-type dispersal operators and a nonlocal time-delay incidence mechanism, which together lead to a non-cooperative and non-monotone traveling wave system. To overcome these difficulties, we construct an invariant cone on a large bounded interval and define a suitable integral operator associated with the traveling wave equations. Uniform a priori bounds and regularity estimates are established independently of the truncation parameter. By applying Schauder's fixed point theorem and a limiting argument, we obtain the existence of nontrivial traveling wave solutions connecting the disease-free equilibrium to the endemic equilibrium when the basic reproduction number exceeds one. Explicit upper and lower solutions are constructed to illustrate the applicability of the approach.

Keywords: traveling waves; reaction-diffusion equations; nonlocal dispersal; delay; SIR model
Mathematics Subject Classification: 35C07, 35K57

1. Introduction

Spatial epidemic models provide a fundamental framework for describing the spread of infectious diseases in heterogeneous environments. Since the pioneering work of Kermack and McKendrick [13], spatial extensions of SIR models have been widely studied to capture the interaction between population movement and disease transmission. In this setting, traveling wave solutions represent epidemic fronts propagating through space with a constant speed and connect disease–free states to endemic equilibria. Such solutions play a central role in understanding invasion dynamics and threshold phenomena in epidemiology.

Early studies of traveling waves in epidemic models primarily focused on reaction–diffusion systems with local dispersal. For SIR and susceptible-infected-susceptible (SIS) type models with bilinear incidence, traveling waves have been analyzed using phase plane methods, comparison principles, and monotone iteration techniques; see, for example, [1, 2, 8]. When the incidence rate is of standard form, the loss of monotonicity introduces additional difficulties, and the existence of traveling waves requires more refined analytical tools [15, 21].

Local diffusion, however, is not always appropriate for populations whose movement exhibits long–range dispersal. This has motivated the replacement of the Laplacian by nonlocal dispersal operators of the form $d(J * u - u)$, where J is a probability kernel. Nonlocal dispersal models arise naturally in ecology, evolutionary biology, and population dynamics, and they have been studied extensively in the context of spreading speeds and traveling waves; see [15, 21, 23]. Compared with local diffusion, nonlocal operators do not regularize solutions, which significantly complicates compactness arguments in traveling wave analysis.

In epidemic modeling, nonlocal dispersal has been incorporated into SIR systems to describe spatial redistribution of susceptible, infected, and removed individuals. A representative model with standard incidence is the nonlocal dispersal SIR system studied by Li and Yang [15],

$$\begin{cases} \partial_t S = d_1(J * S - S) - \dfrac{\beta S I}{S + I}, \\ \partial_t I = d_2(J * I - I) + \dfrac{\beta S I}{S + I} - \gamma I, \\ \partial_t R = d_3(J * R - R) + \gamma I, \end{cases} \tag{1.1}$$

for which the authors established existence and nonexistence of traveling wave solutions depending on the wave speed and the basic reproduction number. The analysis in [15] relies on truncation of the spatial domain, uniform regularity estimates, and Schauder fixed point arguments, reflecting the absence of cooperative structure and the lack of compactness induced by the nonlocal operator.

Another biologically essential mechanism is time delay. Delays naturally arise from incubation periods, latency, and delayed infectivity, and they are known to influence both the stability of equilibria and the propagation speed of epidemic fronts. Traveling waves in delayed epidemic models with local diffusion have been studied using functional differential equation techniques; see [3, 5]. For nonlocal dispersal systems, the presence of delay leads to functional integro–differential equations, where standard comparison principles and monotone semiflow theory are generally not available.

More recently, nonlocal delayed transmission has been incorporated directly into the incidence term. Wu and Zhou [21] considered a nonlocal dispersal SIR model in which the infection pressure depends on a spatiotemporal convolution of the infected population,

$$\begin{cases} \partial_t S = d_1(J * S - S) - \dfrac{\beta S\,(G * I)}{S + (G * I) + R}, \\ \partial_t I = d_2(J * I - I) + \dfrac{\beta S\,(G * I)}{S + (G * I) + R} - (\gamma + \delta) I, \\ \partial_t R = d_3(J * R - R) + \gamma I, \end{cases} \tag{1.2}$$

where the kernel G encodes both spatial averaging and transmission delay. Traveling wave solutions for such models were obtained via a truncation and limiting procedure combined with careful a priori estimates. Related models with nonlocal delayed transmission have been studied in [15, 19, 20].

From a broader perspective, traveling waves for nonlocal systems with delay in diffusion and reaction terms have been investigated systematically in recent years. Barker and collaborators analyzed traveling waves in delayed reaction–diffusion and nonlocal systems using exponential dichotomy and fixed point methods [3,4]. Jiang and Yang [12] and Barker and Simms [5] studied traveling wave fronts for nonlocal systems with delays in both diffusion and reaction terms, establishing existence results under general kernel assumptions. These works highlight that the interaction between nonlocality and delay fundamentally changes the analytical structure of the traveling wave problem.

Nonlocal dispersal models describe population movement through spatial relocation events that may occur over finite distances, rather than through infinitesimal diffusive motion. Let $u(x,t)$ denote the population density at location $x \in \mathbb{R}$ and time t. Movement is characterized by a dispersal kernel $J(x,y)$, where $J(x,y)$ represents the contribution of individuals located at y to the population density at x. The kernel is assumed to be nonnegative and normalized so that

$$\int_{\mathbb{R}} J(x,y)\,dy = 1 \quad \text{for each } x \in \mathbb{R},$$

so that it defines a probability distribution with respect to the variable y.

A general nonlocal dispersal operator acting on u can be written as

$$L[u](x,t) = \int_{\mathbb{R}} J(x,y)\,u(y,t)\,dy - u(x,t)\int_{\mathbb{R}} J(y,x)\,dy.$$

The first term accounts for the influx of individuals relocating from all spatial positions into x, while the second term represents the total rate at which individuals at x disperse to other locations. The operator $L[u](x,t)$ therefore measures the net change in population density at x due to spatial redistribution.

In many applications, dispersal is influenced by past population states rather than the instantaneous configuration. This effect can be incorporated by introducing a time delay $\tau > 0$ into the dispersal mechanism. The delayed nonlocal dispersal operator then takes the form

$$L[u](x,t-\tau) = \int_{\mathbb{R}} J(x,y)\,u(y,t-\tau)\,dy - u(x,t-\tau)\int_{\mathbb{R}} J(y,x)\,dy.$$

In this formulation, redistribution occurring at time t depends on the population distribution at the earlier time $t-\tau$, reflecting delays arising from relocation processes or transport dynamics.

Motivated by the above studies, we consider a nonlocal dispersal epidemic model with standard incidence and nonlocal delayed transmission of the form

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1(J_1 * S(x,t-\tau_1) - S(x,t-\tau_1)) - \dfrac{\beta S(x,t)\,(G*I)(x,t-\tau_3)}{S(x,t) + (G*I)(x,t-\tau_3)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2(J_2 * I(x,t-\tau_2) - I(x,t-\tau_2)) + \dfrac{\beta S(x,t)\,(G*I)(x,t-\tau_3)}{S(x,t) + (G*I)(x,t-\tau_3)} - (\gamma+\delta)I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3(J_3 * R(x,t-\tau_4) - R(x,t-\tau_4)) + \gamma I(x,t), \end{cases} \tag{1.3}$$

where J is a compactly supported dispersal kernel and G is a nonnegative spatiotemporal kernel describing delayed transmission defined by

$$(G*I)(x,t-\tau_3) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau_3-s)\,I(y,s)\,dy\,ds.$$

Compared with (1.1), the infection force depends on a delayed nonlocal average of the infected population. Compared with (1.2), the removed class does not appear in the denominator of the incidence term, reflecting situations in which removal does not directly reduce contact opportunities. This structure leads to a closed subsystem for (S, I) and introduces new technical difficulties in the traveling wave analysis.

The traveling wave problem associated with (1.3) inherits several challenges identified in the literature. The nonlocal dispersal operator does not generate compactness [15, 21], the standard incidence rate destroys cooperation [15], and the delayed transmission term introduces additional functional dependence [12, 21]. To overcome these difficulties, we employ a truncation and limiting approach combined with uniform $C^{1,1}$ estimates and Schauder fixed point theory, following and extending ideas developed in [12, 15, 21].

It is important to note that system (1.3) does not satisfy the mixed quasimonotone condition nor the exponential mixed quasimonotone condition. As a consequence, several approaches developed for delayed or nonlocal reaction–diffusion systems cannot be applied directly to the present model; see, for example, [15, 19–21]. Motivated by these works, we aim to prove the existence of nontrivial traveling wave solutions by constructing an invariant cone on a large bounded domain, defining the initial functions on this domain, and applying Schauder's fixed point theorem. The passage to the whole real line is then achieved through a limiting argument.

However, we find that obtaining uniform a priori bounds for solutions of the nonlocal dispersal problem (1.3) is more delicate than in related models such as those considered in [15, 19–21]. In particular, determining the precise asymptotic behavior of the susceptible component $S(\xi)$ as $\xi \to +\infty$ is significantly more challenging than in local SIR models. The main difficulty lies in the fact that, due to the nonlocal dispersal operator, it is not known whether $S(\xi)$ is monotone or nonmonotone, in contrast to the local diffusion case; see, for instance, [15, 19–21].

Our results show that if the basic reproduction number

$$\mathcal{R}_0 = \frac{\beta}{\gamma} > 1,$$

then there exists a critical wave speed $c_* > 0$ such that, for each $c > c_*$, system (1.3) admits a nontrivial traveling wave solution with wave speed c.

Our main result establishes the existence of nonnegative traveling wave solutions $(S(x + ct), I(x + ct), R(x + ct))$ for (1.3) when the basic reproduction number exceeds one and the wave speed is sufficiently large. In addition, we identify the asymptotic behavior of the wave profiles at infinity, including the existence of the limit of the susceptible component as $\xi \to +\infty$.

The remainder of the paper is organized as follows. Section 2 introduces the traveling wave formulation, assumptions on the kernels J and G and other preliminary information. Section 3 provides an explicit construction of upper and lower solutions. It also develops the truncated problem and establishes uniform a priori estimates, completes the limiting argument, and proves the main existence theorem.

2. Preliminaries

Throughout this work, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the real and complex n–dimensional vector spaces, respectively. For vectors

$$u = (u_1, u_2, u_3)^T, \qquad v = (v_1, v_2, v_3)^T \in \mathbb{R}^3,$$

we use the standard componentwise ordering: $u \le v$ if $u_i \le v_i$ for $i = 1, 2, 3$, and $u < v$ if all inequalities are strict. The Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$.

Let $BC(\mathbb{R}, \mathbb{R}^n)$ denote the space of bounded and continuous functions $f : \mathbb{R} \to \mathbb{R}^n$, equipped with the supremum norm

$$\|f\|_{BC} := \sup_{t\in\mathbb{R}} |f(t)|.$$

For $k \in \mathbb{N}$, we write $BC^k(\mathbb{R}, \mathbb{R}^n)$ for the space of functions whose derivatives up to order k exist, are continuous, and are bounded on $\mathbb{R}$. When boundedness is not required, we simply write $C(\mathbb{R}, \mathbb{R}^n)$ or $C^k(\mathbb{R}, \mathbb{R}^n)$.

For functions $f, g \in BC(\mathbb{R}, \mathbb{R}^n)$, the ordering $f \le g$ is understood pointwise, that is, $f(t) \le g(t)$ for all $t \in \mathbb{R}$. For $\zeta \in \mathbb{R}^n$, we denote by $\hat{\zeta}$ the constant function $\hat{\zeta}(t) \equiv \zeta$.

We also employ standard Lebesgue and Sobolev spaces. For $1 \le p \le \infty$, $L^p(\mathbb{R}, \mathbb{R}^3)$ consists of all measurable functions $f : \mathbb{R} \to \mathbb{R}^3$ such that

$$\|f\|_{L^p} = \begin{cases} \left(\displaystyle\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|, & p = \infty, \end{cases}$$

is finite. For $1 \le p \le \infty$, the Sobolev space $W^{1,p}(\mathbb{R}, \mathbb{R}^3)$ is defined by

$$W^{1,p}(\mathbb{R}, \mathbb{R}^3) = \left\{ f \in L^p(\mathbb{R}, \mathbb{R}^3) : \ f \text{ is absolutely continuous and } f' \in L^p(\mathbb{R}, \mathbb{R}^3) \right\}.$$

In particular, $W^{1,p}(\mathbb{R}, \mathbb{R}^3)$ is continuously embedded in $L^\infty(\mathbb{R}, \mathbb{R}^3)$.

To describe the delayed nonlinearities arising in the nonlocal traveling wave system, let $\tau > 0$ be the maximal delay and define the phase space

$$\mathcal{X} := C([-\tau, 0], \mathbb{R}^3).$$

For $\Phi = (\phi, \varphi, \psi) \in \mathcal{X}$, we introduce the rescaled history functions

$$\phi^c(\theta) = \phi(c\theta), \qquad \varphi^c(\theta) = \varphi(c\theta), \qquad \psi^c(\theta) = \psi(c\theta), \qquad \theta \in [-\tau, 0].$$

For $i = 1, 2, 3$, define the delayed nonlinearities

$$f_{ic}(\phi, \varphi, \psi) := f_i(\phi^c, \varphi^c, \psi^c).$$

Here,

$$f_{1c}(\phi, \varphi, \psi) = -\frac{\beta\phi(\xi)\,(G * \varphi)(\xi - c\tau_3)}{\phi(\xi) + (G * \varphi)(\xi - c\tau_3)},$$

$$f_{2c}(\phi,\varphi,\psi) = -(\gamma+\delta)\,\varphi(\xi) + \frac{\beta\phi(\xi)\,(G*\varphi)(\xi - c\tau_3)}{\phi(\xi) + (G*\varphi)(\xi - c\tau_3)}, \quad f_{3c}(\phi,\varphi,\psi) = \gamma\varphi(t).$$

With this notation, the delayed nonlocal traveling wave system can be written as

$$c\,\phi'(t) = d_1 \int_{-\infty}^{\infty} J_1(y)\Big[\phi(t-y-c\tau_1) - \phi(t-c\tau_1)\Big]\,dy + f_{1c}(\phi_t,\varphi_t,\psi_t), \tag{2.1}$$

$$c\,\varphi'(t) = d_2 \int_{-\infty}^{\infty} J_2(y)\Big[\varphi(t-y-c\tau_2) - \varphi(t-c\tau_2)\Big]\,dy + f_{2c}(\phi_t,\varphi_t,\psi_t), \tag{2.2}$$

$$c\,\psi'(t) = d_3 \int_{-\infty}^{\infty} J_3(y)\Big[\psi(t-y-c\tau_4) - \psi(t-c\tau_4)\Big]\,dy + f_{3c}(\phi_t,\varphi_t,\psi_t).$$

Observe that the first two equations in (2.1) are independent of ψ. More precisely, the nonlinearities satisfy

$$f_{1c}(\phi_t,\varphi_t,\psi_t) = f_{1c}(\phi_t,\varphi_t), \qquad f_{2c}(\phi_t,\varphi_t,\psi_t) = f_{2c}(\phi_t,\varphi_t),$$

so that the (ϕ,φ)–subsystem is closed. Consequently, the traveling wave problem can be reduced to the following two–equation system:

$$c\,\phi'(t) = d_1 \int_{-\infty}^{\infty} J_1(y)\Big[\phi(t-y-c\tau_1) - \phi(t-c\tau_1)\Big]\,dy + f_{1c}(\phi_t,\varphi_t), \tag{2.3}$$

$$c\,\varphi'(t) = d_2 \int_{-\infty}^{\infty} J_2(y)\Big[\varphi(t-y-c\tau_2) - \varphi(t-c\tau_2)\Big]\,dy + f_{2c}(\phi_t,\varphi_t). \tag{2.4}$$

Once a solution (ϕ,φ) of (2.3) is obtained, the third component ψ is recovered from the linear inhomogeneous equation

$$c\,\psi'(t) = \int_{-\infty}^{\infty} J_3(y)\Big[\psi(t-y-c\tau_4) - \psi(t-c\tau_4)\Big]\,dy + f_{3c}(\phi_t,\varphi_t,\psi_t), \tag{2.5}$$

where the right–hand side is now known through the prescribed pair (ϕ,φ). Letting $r_i = c\tau_i, i = 1,\cdots 4$, we can transform our system.

Throughout this paper, we impose the following assumptions on the kernel functions J and G.

(J) $J \in C^1(\mathbb{R})$, $J(y) = J(-y) \geq 0$,

$$\int_{-\infty}^{\infty} J(y)\,dy = 1,$$

and J is compactly supported. Moreover, for any $\nu \in [0,\bar{\nu})$,

$$\int_{-\infty}^{\infty} J(y)e^{-\nu y}\,dy < \infty,$$

and

$$\int_{-\infty}^{\infty} J(y)e^{-\nu y}\,dy \to +\infty \quad \text{as } \nu \to \bar{\nu},$$

where $\bar{\nu}$ may be $+\infty$.

(G) $G(y,s) = G(-y,s) \geq 0$,

$$\int_0^\infty \int_{-\infty}^\infty G(y,s)\,dy\,ds = 1, \qquad \int_0^\infty \int_{-\infty}^\infty s\,G(y,s)\,dy\,ds < \infty,$$

and $G(y,s)$ is Lipschitz continuous with respect to the spatial variable y. Furthermore, for each $c \geq 0$,

$$\int_0^\infty \int_{-\infty}^\infty G(y,s)e^{-\lambda(y+cs)}\,dy\,ds < \infty, \qquad \lambda \in [0,\infty),$$

and

$$\int_0^\infty \int_{-\infty}^\infty G(y,s)e^{-\lambda(y+cs)}\,dy\,ds \to +\infty \quad \text{as } \lambda \to \infty.$$

3. Main results

In order to show the existence of traveling waves, we define the following.

Definition 3.1. $\overline{\Phi} = \left(\overline{\phi}, \overline{\varphi}\right), \underline{\Phi} = \left(\underline{\phi}, \underline{\varphi}\right) \in BC(\mathbb{R}, \mathbb{R}^2) \cap W^{1,\infty}$ is called an upper solution (lower solution, respectively) for the wave equation if

$$\begin{aligned}
&d_1 \int_{\mathbb{R}} J_1(y)\left[\overline{\phi}(t-y-r_1) - \overline{\phi}(t-r_1)\right]dy - c\overline{\phi}'(t) + f_{1c}(\overline{\phi_t}, \underline{\varphi_t}) \leq 0, \qquad (3.1)\\
&d_2 \int_{\mathbb{R}} J_2(y)\left[\overline{\varphi}(t-y-r_2) - \overline{\varphi}(t-r_2)\right]dy - c\overline{\varphi}'(t) + f_{2c}(\overline{\phi_t}, \overline{\varphi_t}) \leq 0,\\
&d_1 \int_{\mathbb{R}} J_1(y)\left[\underline{\phi}(t-y-r_1) - \underline{\phi}(t-r_1)\right]dy - c\underline{\phi}'(t) + f_{1c}(\underline{\phi_t}, \overline{\varphi_t}) \geq 0,\\
&d_2 \int_{\mathbb{R}} J_2(y)\left[\underline{\varphi}(t-y-r_2) - \underline{\varphi}(t-r_2)\right]dy - c\underline{\varphi}'(t) + f_{2c}(\underline{\phi_t}, \underline{\varphi_t}) \geq 0.
\end{aligned}$$

We also define the following closed convex subset Γ_X of $C([-X,X],\mathbb{R}^3)$ as

$$\Gamma_X = \left\{ (\phi(\cdot), \varphi(\cdot)) \in C([-X,X],\mathbb{R}^2) \;\middle|\; \begin{array}{l} \phi(-X) = S_-(-X),\ \varphi(-X) = I_-(-X), \\ S_-(\xi) \leq \phi(\xi) \leq S_0, \\ I_-(\xi) \leq \varphi(\xi) \leq \overline{\varphi}(\xi), \quad \forall\, \xi \in [-X,X], \end{array} \right\}$$

where $X > \left|\frac{1}{\varepsilon_2}\ln\left(\frac{\lambda_0+\lambda_1}{M_2(\lambda_0+\lambda_1+\varepsilon_2)}\right)\right|, S_- \geq \underline{\phi}, I_- \geq \underline{\varphi}$. Define, for $\lambda \geq 0$ and $c > 0$,

$$\Delta_1(\lambda,c) := d_2 e^{-r_2\lambda}\int_{-\infty}^\infty J_2(y)(e^{-\lambda y} - 1)\,dy - c\lambda - \gamma + \beta e^{-r_3\lambda}\int_0^\infty \int_{-\infty}^\infty G(y,s)e^{-\lambda(y+cs)}\,dy\,ds. \qquad (3.2)$$

Then we have the following result.

Lemma 3.2. *Assume that* $\mathcal{R}_0 := \beta/\gamma > 1$, $r_2 > 0$ *to be small enough and that the kernel hypotheses* **(J)** *and* **(G)** *hold. There exist constants* $c_* > 0$ *and* $\lambda_* > 0$ *such that*

$$\left.\frac{\partial \Delta_1(\lambda,c)}{\partial \lambda}\right|_{(\lambda_*,c_*)} = 0, \qquad \Delta_1(\lambda_*,c_*) = 0. \qquad (3.3)$$

Moreover, there exists $\hat{\lambda} \in (0, +\infty]$ *such that the following alternatives hold.*
(i) *If* $c > c_*$, *then the equation* $\Delta_1(\lambda, c) = 0$ *admits two positive roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *satisfying*

$$0 < \lambda_1(c) < \lambda_* < \lambda_2(c) < \hat{\lambda},$$

and

$$\Delta_1(\lambda, c) < 0 \textit{ for } \lambda \in (\lambda_1(c), \lambda_2(c)), \qquad \Delta_1(\lambda, c) > 0 \textit{ for } \lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), \hat{\lambda}).$$

(ii) *If* $0 < c < c_*$, *then*

$$\Delta_1(\lambda, c) > 0 \quad \textit{for all } \lambda \in (0, \hat{\lambda}).$$

Proof. Fix $c > 0$ and recall the definition (3.2). First, setting $\lambda = 0$ and using

$$\int_{-\infty}^{\infty} J_2(y)(e^0 - 1)\, dy = 0, \qquad \int_0^{\infty} \int_{-\infty}^{\infty} G(y, s)\, dy\, ds = 1,$$

we obtain

$$\Delta_1(0, c) = -\gamma + \beta = \gamma(\mathcal{R}_0 - 1) > 0. \tag{3.4}$$

Next, for any fixed $\lambda > 0$, we have

$$\lim_{c \to +\infty} \Delta_1(\lambda, c) = -\infty, \tag{3.5}$$

since the term $-c\lambda \to -\infty$ while all other terms remain bounded by **(J)** and **(G)**.

We now compute the partial derivative with respect to c. Let

$$H(\lambda, c) := \int_0^{\infty} \int_{-\infty}^{\infty} G(y, s) e^{-\lambda(y+cs)}\, dy\, ds.$$

Then $H(\lambda, c) > 0$, and

$$\frac{\partial H(\lambda, c)}{\partial c} = -\lambda \int_0^{\infty} \int_{-\infty}^{\infty} s\, G(y, s) e^{-\lambda(y+cs)}\, dy\, ds < 0.$$

Consequently,

$$\frac{\partial \Delta_1(\lambda, c)}{\partial c} = -\lambda + \beta e^{-r_3\lambda} \frac{\partial H(\lambda, c)}{\partial c} < -\lambda < 0. \tag{3.6}$$

Next, differentiating Δ_1 with respect to λ, we obtain

$$\begin{aligned} \frac{\partial \Delta_1(\lambda, c)}{\partial \lambda} &= d_2 e^{-r_2\lambda} \left[-r_2 \int_{\mathbb{R}} J_2(y)(e^{-\lambda y} - 1)\, dy - \int_{\mathbb{R}} y\, J_2(y) e^{-\lambda y}\, dy \right] \\ &\quad - c \\ &\quad + \beta e^{-r_3\lambda} \left[-r_3 \int_0^{\infty} \int_{\mathbb{R}} G(y, s) e^{-\lambda(y+cs)}\, dy\, ds - \int_0^{\infty} \int_{\mathbb{R}} (y + cs) G(y, s) e^{-\lambda(y+cs)}\, dy\, ds \right]. \end{aligned} \tag{3.7}$$

Evaluating at $\lambda = 0$, we obtain

$$\frac{\partial \Delta_1(0, c)}{\partial \lambda} = -c - \beta r_3 - \beta c \int_0^{\infty} \int_{\mathbb{R}} s\, G(y, s)\, dy\, ds < 0. \tag{3.8}$$

Finally, differentiating once more, we obtain

$$\frac{\partial^2 \Delta_1(\lambda, c)}{\partial \lambda^2} = d_2 e^{-r_2\lambda} \int_{\mathbb{R}} J_2(y)(y + r_2)^2 e^{-\lambda y}\, dy - d_2 r_2^2 e^{-r_2\lambda} + \beta e^{-r_3\lambda} \int_0^{\infty} \int_{\mathbb{R}} G(y, s)(y + cs + r_3)^2 e^{-\lambda(y+cs)}\, dy\, ds. \tag{3.9}$$

Define

$$g(r_2) := \frac{\partial^2 \Delta_1(\lambda, c)}{\partial \lambda^2}.$$

Then at $r_2 = 0$,

$$g(0) = d_2 \int_{\mathbb{R}} J_2(y) y^2 e^{-\lambda y}\, dy + \beta e^{-r_3\lambda} \int_0^{\infty} \int_{\mathbb{R}} G(y, s)(y + cs + r_3)^2 e^{-\lambda(y+cs)}\, dy\, ds > 0.$$

Since g is continuous in r_2, there exists $\varepsilon_0 > 0$ such that

$$g(r_2) > 0 \quad \text{for } 0 < r_2 < \varepsilon_0.$$

Hence, for $r_2 > 0$ sufficiently small,

$$\frac{\partial^2 \Delta_1(\lambda, c)}{\partial \lambda^2} > 0 \quad \text{for all } \lambda > 0,\ c > 0. \tag{3.10}$$

We now combine (3.4)–(3.6), (3.8), and (3.10). For each fixed $c > 0$ the function $\lambda \mapsto \Delta_1(\lambda, c)$ is strictly convex, and satisfies $\Delta_1(0, c) > 0$, and $\frac{\partial \Delta_1}{\partial \lambda}(0, c) < 0$. Thus it decreases initially and, by convexity, can cross the axis at most twice.

Moreover, since $\partial_c \Delta_1(\lambda, c) < 0$, increasing c shifts the graph of $\Delta_1(\lambda, c)$ downward. Consequently, there exists a unique value $c_* > 0$ for which the graph is tangent to the λ-axis at some $\lambda_* > 0$, that is,

$$\Delta_1(\lambda_*, c_*) = 0, \qquad \frac{\partial \Delta_1}{\partial \lambda}(\lambda_*, c_*) = 0.$$

It follows that for $c > c_*$, the equation $\Delta_1(\lambda, c) = 0$ has exactly two positive roots, while for $0 < c < c_*$, it has none, yielding the stated alternatives. □

We first establish the existence of explicit upper and lower solutions. To this end, we define the following continuous functions:

$$\overline{\phi}(\xi) = S_0, \quad \underline{\phi}(\xi) = \begin{cases} S_0(1 - M_1 e^{\varepsilon_1 \xi}), & \xi < \xi_1, \\ 0 \quad \xi \geq \xi_1, \end{cases} \tag{3.11}$$

$$\overline{\varphi}(\xi) = \begin{cases} S_0 \frac{\beta-\gamma}{\gamma} e^{\lambda_1 \xi}, & \xi < 0, \\ S_0 \frac{\beta-\gamma}{\gamma}, & \xi \geq 0, \end{cases} \quad \underline{\varphi}(\xi) = \begin{cases} (1 - M_2 e^{\varepsilon_2 \xi}) e^{\lambda_1 \xi}, & \xi < \xi_2, \\ \frac{\varepsilon_2}{\lambda_0+\lambda_1+\varepsilon_2} \left(\frac{\lambda_0+\lambda_1}{M_2(\lambda_0+\lambda_1+\varepsilon_2)}\right)^{(\lambda_0+\lambda_1)/\varepsilon_2} e^{-\lambda_0 \xi} & \xi \geq \xi_2, \end{cases}$$

where

$\varepsilon_1, \varepsilon_2 > 0$ are small, $S_0 = \lim_{\xi \to \infty} \phi(\xi)$,

$$M_1 > e^{-\varepsilon_1(\xi - r_1)} \frac{\displaystyle\int_{-\infty}^{\xi - r_1 - \xi_1} J(y)\, dy}{1 - \displaystyle\int_{\xi - r_1 - \xi_1}^{\infty} J(y) e^{-\varepsilon_1 y}\, dy},$$

$$M_2 \ge e^{-\varepsilon_2(\xi - r_2)} \frac{1 - \displaystyle\int_{\xi - r_2 - \xi_2}^{\infty} J_2(y) e^{-\lambda_1 y}\, dy - \varepsilon e^{-(\lambda_0 + \lambda_1)(\xi - r_2)} \int_{-\infty}^{\xi - r_2 - \xi_2} J_2(y) e^{\lambda_0 y}\, dy}{1 - \displaystyle\int_{\xi - r_2 - \xi_2}^{\infty} J_2(y) e^{-(\lambda_1 + \varepsilon_2) y}\, dy}.$$

Fix

$$0 < \varepsilon_2 < \max\left\{\frac{\lambda_1}{\lambda_0}, c_*\right\} \text{ and set } \lambda_0 = \frac{\lambda_1}{\varepsilon_2}.$$

Denote

$$\xi_1 = \frac{1}{\varepsilon_1} \ln\left(\frac{1}{M_1}\right), \quad \xi_2 = \frac{1}{\varepsilon_2} \ln\left(\frac{\lambda_0 + \lambda_1}{M_2(\lambda_0 + \lambda_1 + \varepsilon_2)}\right).$$

Theorem 3.3. *Let* $c > 0, \mathcal{R}_0 = \frac{\beta}{\gamma + \delta} > 1, r_i \ge r_3, i = 1, 2, 3,$ *and define the functions as in 3.11, then* $(\overline{\phi}, \overline{\varphi})$ *form a set of upper solutions.*

Proof. We first note that all functions are continuous, and bounded on the real line, and continuously differentiable on the real line outside a finite number of points. In order to establish the inequalities, we look at the following cases.

For the first inequality, we have $\overline{\phi}(\xi) = S_0, \underline{\varphi}(\xi) \ge 0$, and we get

$$\begin{aligned}
& d_1 \int_{-\infty}^{\infty} J(y)\, \overline{\phi}(\xi - y - r_1)\, dy - d_1 \overline{\phi}(\xi - r_1) - c\overline{\phi}'(\xi) - \frac{\beta \overline{\phi}(\xi)\, (G * \underline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \underline{\varphi})(\xi - r_3)} \\
&= -\frac{\beta S_0\, (G * \underline{\varphi})(\xi - r_3)}{S_0 + (G * \underline{\varphi})(\xi - r_3)} \le 0.
\end{aligned}$$

For the second inequality, for upper solutions, we have the following.
Case 1. $\xi < 0$.

$$\begin{aligned}
& d_2 \int_{-\infty}^{\infty} J(y)\, \overline{\varphi}(\xi - y - r_2)\, dy - d_2 \overline{\varphi}(\xi - r_2) - c\overline{\varphi}'(\xi) - (\gamma + \delta)\, \overline{\varphi}(\xi) + \frac{\beta \overline{\phi}(\xi)\, (G * \overline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} \\
&\le d_2 \int_{-\infty}^{\infty} J(y)\, \overline{\varphi}(\xi - y - r_2)\, dy - d_2 \overline{\varphi}(\xi - r_2) - c\overline{\varphi}'(\xi) - \gamma\, \overline{\varphi}(\xi) + \beta\, (G * \overline{\varphi})(\xi - r_3) \\
&= e^{\lambda_1 \xi} \left[d_2 e^{-r_2 \lambda_1} \left(\int_{-\infty}^{\infty} J_2(x, y)(e^{-\lambda_1 y} - 1) dy \right) - c\lambda_1 - \gamma + \beta e^{-r_3 \lambda_1} \int_0^{\infty} \int_{-\infty}^{\infty} G(y, s) e^{-\lambda_1 (y + cs)} dy ds \right] \\
&= e^{\lambda_1 \xi} \Delta_1(\lambda_1, c) = 0.
\end{aligned}$$

Case 2. $\xi - r_2 < 0 \le \xi$.

$$d_2 \int_{-\infty}^{\infty} J(y)\, \overline{\varphi}(\xi - y - r_2)\, dy - d_2 \overline{\varphi}(\xi - r_2) - c\overline{\varphi}'(\xi)(\gamma + \delta)\, \overline{\varphi}(\xi) + \frac{\beta \overline{\phi}(\xi)\, (G * \overline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)}$$

$$
\begin{aligned}
&\leq d_2 \int_{-\infty}^{\infty} J(y)\,\overline{\varphi}(\xi-y-r_2)\,dy - d_2\overline{\varphi}(\xi-r_2) - c\overline{\varphi}'(\xi) - (\gamma+\delta)\,\overline{\varphi}(\xi) + \frac{\beta\overline{\phi}(\xi)\,(G*\overline{\varphi})(\xi-r_3)}{\overline{\phi}(\xi)+(G*\overline{\varphi})(\xi-r_3)}\\
&\leq d_2 \int_{-\infty}^{\infty} J(y)\,\overline{\varphi}(\xi-y-r_2)\,dy - d_2\overline{\varphi}(\xi-r_2) - \gamma I_+ + \frac{\beta\overline{\phi}(\xi)\,(G*\overline{\varphi})(\xi-r_3)}{\overline{\phi}(\xi)+(G*\overline{\varphi})(\xi-r_3)},
\end{aligned}
$$

where $I_+ = S_0\frac{\beta-\gamma}{\gamma}$. Using the fact that the function $\frac{\beta xy}{x+y}$ is non-decreasing with respect to x, y, we see

$$
\begin{aligned}
-\gamma I_+ + \frac{\beta\overline{\phi}(\xi)\,(G*\overline{\varphi})(\xi-r_3)}{\overline{\phi}(\xi)+(G*\overline{\varphi})(\xi-r_3)} &\leq \frac{\beta S_0 I_+}{S_0+I_+} - \gamma I_+\\
&= \left(\frac{\beta S_0\gamma}{S_0\gamma + S_0(\beta-\gamma)} - \gamma\right) I_+ = 0.
\end{aligned}
$$

We now turn our attention to $d_2\int_{-\infty}^{\infty} J(y)\,\overline{\varphi}(\xi-y-r_2)\,dy - d_2\overline{\varphi}(\xi-r_2)$. To this end, we omit the constant $d_2 S_0\frac{\beta-\gamma}{\gamma}$:

$$
\begin{aligned}
&\int_{\mathbb{R}} J_2(y)[\overline{\varphi}(\xi-y-r_2) - \overline{\varphi}(\xi-r_2)]\,dy\\
&= \int_{-\infty}^{\xi-r_2} J_2(y)\Big[1-e^{\lambda_1(\xi-r_2)}\Big]\,dy + \int_{\xi-r_2}^{+\infty} J_2(y)\Big[e^{\lambda_1(\xi-y-r_2)} - e^{\lambda_1(\xi-r_2)}\Big]\,dy\\
&\int_{-\infty}^{0} J_2(y)\Big[1-e^{\lambda_1(\xi-r_2)}\Big]\,dy - \int_{\xi-r_2}^{0} J_2(y)\Big[1-e^{\lambda_1(\xi-r_2)}\Big]\,dy\\
&\quad + e^{\lambda_1(\xi-r_2)}\int_{0}^{+\infty} J_2(y)(e^{-\lambda_1 y}-1)\,dy + \int_{\xi-r_2}^{0} J_2(y)\Big[e^{\lambda_1(\xi-y-r_2)} - e^{\lambda_1(\xi-r_2)}\Big]\,dy\\
&= \Big[1-e^{\lambda_1(\xi-r_2)}\Big]\left(\int_{-\infty}^{0} J_2(y)\,dy - \int_{\xi-r_2}^{0} J_2(y)\,dy\right)\\
&\quad + e^{\lambda_1(\xi-r_2)}\int_{0}^{+\infty} J_2(y)(e^{-\lambda_1 y}-1)\,dy + \int_{\xi-r_2}^{0} J_2(y)\,e^{\lambda_1(\xi-r_2)}(e^{-\lambda_1 y}-1)\,dy\\
&= \Big[1-e^{\lambda_1(\xi-r_2)}\Big]\int_{-\infty}^{\xi-r_2} J_2(y)\,dy + e^{\lambda_1(\xi-r_2)}\int_{0}^{+\infty} J_2(y)(e^{-\lambda_1 y}-1)\,dy\\
&\quad + e^{\lambda_1(\xi-r_2)}\int_{\xi-r_2}^{0} J_2(y)(e^{-\lambda_1 y}-1)\,dy\\
&= \Big[1-e^{\lambda_1(\xi-r_2)}\Big]\int_{-\infty}^{\xi-r_2} J_2(y)\,dy + e^{\lambda_1(\xi-r_2)}\int_{\xi-r_2}^{+\infty} J_2(y)(e^{-\lambda_1 y}-1)\,dy.
\end{aligned}
$$

Continuing, we see

$$
\begin{aligned}
&\Big[1-e^{\lambda_1(\xi-r_2)}\Big]\int_{-\infty}^{\xi-r_2} J_2(y)\,dy + e^{\lambda_1(\xi-r_2)}\int_{\xi-r_2}^{+\infty} J_2(y)(e^{-\lambda_1 y}-1)\,dy\\
&\leq \Big[1-e^{\lambda_1(\xi-r_2)}\Big]\int_{-\infty}^{0} J_2(y)\,dy + e^{\lambda_1(\xi-r_2)}\int_{0}^{+\infty} J_2(y)(e^{-\lambda_1 y}-1)\,dy
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}\Big[1 - e^{\lambda_1(\xi - r_2)}\Big] + e^{\lambda_1(\xi - r_2)} \int_0^{+\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy \\
&= e^{\lambda_1(\xi - r_2)} \left[\frac{1}{2}\Big(e^{\lambda_1(r_2 - \xi)} - 1\Big) + \int_0^{+\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy\right] \\
&\le e^{\lambda_1(\xi - r_2)} \left[\frac{1}{2}(e^{\lambda_1 r_2} - 1) + \int_0^{+\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy\right].
\end{aligned}
$$

Since, $\lambda_1 > 0$ and by (**J**), we have

$$
\int_0^{+\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy < 0
$$

and

$$
0 < \frac{1}{2}(e^{\lambda_1 r_2} - 1) < -\int_0^{+\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy
$$

for small enough r_2. Thus,

$$
e^{\lambda_1(\xi - r_2)} \left[\frac{1}{2}(e^{\lambda_1 r_2} - 1) + \int_0^{+\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy\right] \le 0.
$$

Case 3. $\xi \ge r_2$.

$$
\begin{aligned}
&d_2 \int_{-\infty}^{\infty} J(y)\, \overline{\varphi}(\xi - y - r_2)\, dy - d_2 \overline{\varphi}(\xi - r_2) - c\overline{\varphi}'(\xi) - \gamma\, \overline{\varphi}(\xi) + \frac{\beta \overline{\phi}(\xi)\, (G * \overline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} \\
&= d_2 \int_{-\infty}^{\infty} J(y)\, \overline{\varphi}(\xi - y - r_2)\, dy - d_2 \overline{\varphi}(\xi - r_2) - c\overline{\varphi}'(\xi) - \gamma\, \overline{\varphi}(\xi) + \frac{\beta \overline{\phi}(\xi)\, (G * \overline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} \\
&= d_2 \int_{-\infty}^{\infty} J(y)\, \overline{\varphi}(\xi - y - r_2)\, dy - d_2 \overline{\varphi}(\xi - r_2) - \gamma I_+ + \frac{\beta \overline{\phi}(\xi)\, (G * \overline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} \\
&= -\gamma I_+ + \frac{\beta \overline{\phi}(\xi)\, (G * \overline{\varphi})(\xi - r_3)}{\overline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} \le 0
\end{aligned}
$$

from a similar argument from Case 2. This proves the theorem. □

Before we establish construction the of lower solutions, we need the following lemma.

Lemma 3.4. *Define* $\underline{\phi}(\xi), \underline{\varphi}(\xi)$ *as in (3.11), there exist* $M_1, M_2 > 0$ *such that*

$$
d_1 \int_{-\infty}^{\infty} J_1(y)\, \underline{\phi}(\xi - y - r_1)\, dy - d_1 \underline{\phi}(\xi - r_1) > 0, \quad d_2 \int_{-\infty}^{\infty} J_2(y)\, \underline{\varphi}(\xi - y - r_2)\, dy - d_2 \underline{\varphi}(\xi - r_2) > 0. \tag{3.12}
$$

Proof. For the first inequality for lower solutions, we compute

$$
d_1 \int_{-\infty}^{\infty} J(y)\, \underline{\phi}(\xi - y - r_1)\, dy - d_1 \underline{\phi}(\xi - r_1)
$$

$$\begin{aligned}
&= d_1 \int_{-\infty}^{\infty} J(y)[\underline{\phi}(\xi - y - r_1) - \underline{\phi}(\xi - r_1)]\, dy \\
&= d_1 \int_{-\infty}^{\xi - r_1 - \xi_1} J(y)[0 - \underline{\phi}(\xi - r_1)]\, dy \\
&\quad + d_1 \int_{\xi - r_1 - \xi_1}^{\infty} J(y)\Big[S_0(1 - M_1 e^{\varepsilon_1(\xi - y - r_1)}) - \underline{\phi}(\xi - r_1)\Big]\, dy \\
&= d_1[-S_0(1 - M_1 e^{\varepsilon_1(\xi - r_1)})] \int_{-\infty}^{\xi - r_1 - \xi_1} J(y)\, dy \\
&\quad + d_1 S_0 M_1 e^{\varepsilon_1(\xi - r_1)} \int_{\xi - r_1 - \xi_1}^{\infty} J(y)(1 - e^{-\varepsilon_1 y})\, dy \\
&= -d_1 S_0 \int_{-\infty}^{\xi - r_1 - \xi_1} J(y)\, dy \\
&\quad + d_1 S_0 M_1 e^{\varepsilon_1(\xi - r_1)} \left[\int_{-\infty}^{\xi - r_1 - \xi_1} J(y)\, dy + \int_{\xi - r_1 - \xi_1}^{\infty} J(y)(1 - e^{-\varepsilon_1 y})\, dy\right] \\
&= -d_1 S_0 \int_{-\infty}^{\xi - r_1 - \xi_1} J(y)\, dy + d_1 S_0 M_1 e^{\varepsilon_1(\xi - r_1)} \left[1 - \int_{\xi - r_1 - \xi_1}^{\infty} J(y) e^{-\varepsilon_1 y}\, dy\right].
\end{aligned}$$

Thus, if

$$M_1 > e^{-\varepsilon_1(\xi - r_1)} \frac{\displaystyle\int_{-\infty}^{\xi - r_1 - \xi_1} J(y)\, dy}{1 - \displaystyle\int_{\xi - r_1 - \xi_1}^{\infty} J(y) e^{-\varepsilon_1 y}\, dy},$$

then

$$d_1 \int_{-\infty}^{\infty} J(y)\, \underline{\phi}(\xi - y - r_1)\, dy - d_1 \underline{\phi}(\xi - r_1) > 0.$$

For the second inequality for lower solutions,

$$\begin{aligned}
& d_2 \int_{-\infty}^{\infty} J_2(y)\, \underline{\varphi}(\xi - y - r_2)\, dy - d_2 \underline{\varphi}(\xi - r_2) \\
&= d_2 \int_{-\infty}^{\infty} J_2(y)\Big[\underline{\varphi}(\xi - y - r_2) - \underline{\varphi}(\xi - r_2)\Big]\, dy \\
&= d_2 \int_{-\infty}^{\xi - r_2 - \xi_2} J_2(y)\Big[\varepsilon e^{-\lambda_0(\xi - y - r_2)} - (1 - M_2 e^{\varepsilon_2(\xi - r_2)}) e^{\lambda_1(\xi - r_2)}\Big]\, dy \\
&\quad + d_2 \int_{\xi - r_2 - \xi_2}^{\infty} J_2(y)\Big[(1 - M_2 e^{\varepsilon_2(\xi - y - r_2)}) e^{\lambda_1(\xi - y - r_2)} - (1 - M_2 e^{\varepsilon_2(\xi - r_2)}) e^{\lambda_1(\xi - r_2)}\Big]\, dy \\
&= d_2 \varepsilon e^{-\lambda_0(\xi - r_2)} \int_{-\infty}^{\xi - r_2 - \xi_2} J_2(y) e^{\lambda_0 y}\, dy - d_2 e^{\lambda_1(\xi - r_2)} \int_{-\infty}^{\xi - r_2 - \xi_2} J_2(y)\, dy \\
&+ d_2 M_2 e^{(\lambda_1 + \varepsilon_2)(\xi - r_2)} \int_{-\infty}^{\xi - r_2 - \xi_2} J_2(y)\, dy \\
&\quad + d_2 e^{\lambda_1(\xi - r_2)} \int_{\xi - r_2 - \xi_2}^{\infty} J_2(y)(e^{-\lambda_1 y} - 1)\, dy + d_2 M_2 e^{(\lambda_1 + \varepsilon_2)(\xi - r_2)} \int_{\xi - r_2 - \xi_2}^{\infty} J_2(y)(1 - e^{-(\lambda_1 + \varepsilon_2) y})\, dy
\end{aligned}$$

$$= d_2 e^{\lambda_1(\xi-r_2)}\Bigg[\varepsilon e^{-(\lambda_0+\lambda_1)(\xi-r_2)}\int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)e^{\lambda_0 y}\,dy + \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-\lambda_1 y}\,dy - \int_{-\infty}^{\infty} J_2(y)\,dy$$
$$+ M_2 e^{\varepsilon_2(\xi-r_2)}\left(\int_{-\infty}^{\infty} J_2(y)\,dy - \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-(\lambda_1+\varepsilon_2)y}\,dy\right)\Bigg].$$

Then

$$d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\underline{\varphi}(\xi-r_2)$$
$$= d_2 e^{\lambda_1(\xi-r_2)}\Bigg[\varepsilon e^{-(\lambda_0+\lambda_1)(\xi-r_2)}\int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)e^{\lambda_0 y}\,dy + \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-\lambda_1 y}\,dy - 1$$
$$+ M_2 e^{\varepsilon_2(\xi-r_2)}\left(1 - \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-(\lambda_1+\varepsilon_2)y}\,dy\right)\Bigg].$$

Hence, if

$$1 - \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-(\lambda_1+\varepsilon_2)y}\,dy > 0,$$

$$\text{and if}\quad M_2 \geq e^{-\varepsilon_2(\xi-r_2)}\frac{1 - \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-\lambda_1 y}\,dy - \varepsilon e^{-(\lambda_0+\lambda_1)(\xi-r_2)}\int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)e^{\lambda_0 y}\,dy}{1 - \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)e^{-(\lambda_1+\varepsilon_2)y}\,dy},$$

$$\text{then}\quad d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\underline{\varphi}(\xi-r_2) \geq 0.$$

□

Theorem 3.5. *Let* $c > 0, \mathcal{R}_0 = \frac{\beta}{\gamma+\delta} > 1, r_i \geq r_3, i = 1, 2,$ *and define the functions as in 3.11, then* $(\underline{\phi}, \underline{\varphi})$ *form a set of lower solutions.*

Proof. As with the proof of Theorem 3.3, we note that the functions are continuously bounded and derivatives are absolutely continuous except for a finite number of points. Thus, we only need to show the inequalities hold.

For the first inequality, we look at the following cases.

Case 1. $\xi < \xi_1$.

$$d_1\int_{-\infty}^{\infty} J(y)\,\underline{\phi}(\xi-y-r_1)\,dy - d_1\underline{\phi}(\xi-r_1) - c\underline{\phi}'(\xi) - \frac{\beta\underline{\phi}(\xi)\,(G*\overline{\varphi})(\xi-r_3)}{\underline{\phi}(\xi)+(G*\overline{\varphi})(\xi-r_3)}$$
$$\geq d_1\int_{-\infty}^{\infty} J(y)\,\underline{\phi}(\xi-y-r_1)\,dy - d_1\underline{\phi}(\xi-r_1) - c\underline{\phi}'(\xi) - \beta\,(G*\overline{\varphi})(\xi-r_3)$$
$$= d_1\left(\int_{-\infty}^{\infty} J(y)S_0(1-M_1e^{\varepsilon_1(\xi-y-r_1)})dy - S_0(1-M_1e^{\varepsilon_1(\xi-r_1)})\right)$$
$$+ cS_0M_1\varepsilon_1e^{\varepsilon_1\xi} - \frac{\beta(\beta-\gamma)e^{(\lambda_1-r_3)\xi}}{\gamma}\overline{G}(\lambda_1, c)$$

$$
\begin{aligned}
&= S_0 M_1 e^{\varepsilon_1 \xi}\left[d_1 e^{-\varepsilon_1 r_1}\left(-\int_{-\infty}^{\infty} J(y) e^{-\varepsilon_1 y} dy + 1\right) + c\varepsilon_1\right] - \frac{\beta(\beta-\gamma)e^{(\lambda_1 - r_3)\xi}}{\gamma}\overline{G}(\lambda_1, c) \\
&\geq S_0 M_1 e^{\varepsilon_1 \xi}\left[d_1 e^{-\varepsilon_1 r_1}\left(-\int_{-\infty}^{\infty} J(y) e^{-\varepsilon_1 y} dy + 1\right) + c\varepsilon_1\right] - \frac{\beta(\beta-\gamma)e^{(\lambda_1 - r_3)\xi_1}}{\gamma}\overline{G}(\lambda_1, c) \geq 0,
\end{aligned}
$$

since we can take

$$
M_1 \geq \frac{\frac{\beta(\beta-\gamma)e^{(\lambda_1 - r_3)\xi_1}}{\gamma}\,\overline{G}(\lambda_1, c)}{S_0 e^{\varepsilon_1 \xi_1}\left[d_1 e^{-\varepsilon_1 r_1}\left(1 - \int_{-\infty}^{\infty} J(y) e^{-\varepsilon_1 y}\, dy\right) + c\varepsilon_1\right]}.
$$

Case 2. $\xi_1 < \xi < \xi_1 + r_1$ gives $\underline{\phi}(\xi - r_1) = S_0(1 - M_1 e^{\varepsilon_1(\xi - r_1)}), \underline{\phi}(\xi) = 0$, so

$$
\begin{aligned}
&d_1 \int_{-\infty}^{\infty} J(y)\, \underline{\phi}(\xi - y - r_1)\, dy - d_1 \underline{\phi}(\xi - r_1) - c\underline{\phi}'(\xi) - \frac{\beta\underline{\phi}(\xi)\,(G * \overline{\varphi})(\xi - r_3)}{\underline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} \\
&= d_1 \int_{-\infty}^{\infty} J(y)\, \underline{\phi}(\xi - y - r_1)\, dy - d_1 \underline{\phi}(\xi - r_1)
\end{aligned}
$$

from Lemma 3.4.

Case 3. $\xi_1 + r_1 \geq \xi$ gives $\underline{\phi}(\xi - r_1) = \underline{\phi}(\xi) = 0$, so

$$
d_1 \int_{-\infty}^{\infty} J(y)\, \underline{\phi}(\xi - y - r_1)\, dy - d_1 \underline{\phi}(\xi - r_1) - c\underline{\phi}'(\xi) - \frac{\beta\underline{\phi}(\xi)\,(G * \overline{\varphi})(\xi - r_3)}{\underline{\phi}(\xi) + (G * \overline{\varphi})(\xi - r_3)} = 0.
$$

For the second inequality, we note that we can take

$$
M_1 \geq e^{\varepsilon_1 r}\left(\frac{M_2(\lambda_1 + \varepsilon_2)}{\lambda_1}\right)^{\varepsilon_1/\varepsilon_2}, \text{ such that } \xi_1 < \xi_2 - r, \text{ where } r = \max\{r_1, r_2, r_3\}.
$$

First, denote

$$
\varepsilon = \frac{\varepsilon_2}{\lambda_0 + \lambda_1 + \varepsilon_2}\left(\frac{\lambda_0 + \lambda_1}{M_2(\lambda_0 + \lambda_1 + \varepsilon_2)}\right)^{(\lambda_0 + \lambda_1)/\varepsilon_2},
$$

define the function

$$
f(\xi) = \varepsilon + M_2 e^{(\lambda_0 + \lambda_1 + \varepsilon_2)\xi} - e^{(\lambda_0 + \lambda_1)\xi}.
$$

We claim that $f(\xi) \geq 0$ for any $\xi \geq \xi_2$. Obviously, $f(\xi_2) = 0$. Thus, it suffices to verify that $f'(\xi) > 0$ for any $\xi > \xi_2$.

Since

$$
\xi > \xi_2 = \frac{1}{\varepsilon_2}\ln\left(\frac{\lambda_0 + \lambda_1}{M_2(\lambda_0 + \lambda_1 + \varepsilon_2)}\right),
$$

we have

$$
e^{\varepsilon_2 \xi} > \frac{\lambda_0 + \lambda_1}{M_2(\lambda_0 + \lambda_1 + \varepsilon_2)}.
$$

Hence,

$$
M_2(\lambda_0 + \lambda_1 + \varepsilon_2)e^{(\lambda_0 + \lambda_1 + \varepsilon_2)\xi} > (\lambda_0 + \lambda_1)e^{(\lambda_0 + \lambda_1)\xi}.
$$

Consequently,

$$f'(\xi) = M_2(\lambda_0 + \lambda_1 + \varepsilon_2)e^{(\lambda_0+\lambda_1+\varepsilon_2)\xi} - (\lambda_0 + \lambda_1)e^{(\lambda_0+\lambda_1)\xi} > 0$$

for any $\xi > \xi_2$. Therefore, $f(\xi) \geq 0$ for any $\xi \geq \xi_2$.

Note that this claim implies

$$\varepsilon e^{-\lambda_0\xi} \geq e^{\lambda_1\xi}(1 - M_2 e^{\varepsilon_2\xi}) \quad \text{for any } \xi \geq \xi_2.$$

Case 1. $\xi < \xi_2$. We notice

$$\begin{aligned}
&\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi - y - r_2)\,dy = \int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}((\xi - r_2) - y)\,dy \\
&= \int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)\,\underline{\varphi}((\xi - r_2) - y)\,dy + \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)\,\underline{\varphi}((\xi - r_2) - y)\,dy \\
&= \int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)\,\frac{\varepsilon_2}{\lambda_0 + \lambda_1 + \varepsilon_2}\left(\frac{\lambda_0 + \lambda_1}{M_2(\lambda_0 + \lambda_1 + \varepsilon_2)}\right)^{(\lambda_0+\lambda_1)/\varepsilon_2} e^{-\lambda_0\left((\xi-r_2)-y\right)}\,dy \\
&\quad + \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)\,(1 - M_2 e^{\varepsilon_2\left((\xi-r_2)-y\right)})e^{\lambda_1\left((\xi-r_2)-y\right)}\,dy \\
&= \frac{\varepsilon_2}{\lambda_0 + \lambda_1 + \varepsilon_2}\left(\frac{\lambda_0 + \lambda_1}{M_2(\lambda_0 + \lambda_1 + \varepsilon_2)}\right)^{(\lambda_0+\lambda_1)/\varepsilon_2} \int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)\,e^{-\lambda_0(\xi-y-r_2)}\,dy \\
&\quad + \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)\,(1 - M_2 e^{\varepsilon_2(\xi-y-r_2)})e^{\lambda_1(\xi-y-r_2)}\,dy \\
&\geq \int_{-\infty}^{\xi-r_2-\xi_2} J_2(y)\,(1 - M_2 e^{\varepsilon_2(\xi-y-r_2)})e^{\lambda_1(\xi-y-r_2)}\,dy \\
&\quad + \int_{\xi-r_2-\xi_2}^{\infty} J_2(y)\,(1 - M_2 e^{\varepsilon_2(\xi-y-r_2)})e^{\lambda_1(\xi-y-r_2)}\,dy \\
&= \int_{-\infty}^{\infty} J_2(y)\,(1 - M_2 e^{\varepsilon_2(\xi-y-r_2)})e^{\lambda_1(\xi-y-r_2)}\,dy.
\end{aligned}$$

This gives

$$\begin{aligned}
&d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi - y - r_2)\,dy - d_2\underline{\varphi}(\xi - r_2) - c\underline{\varphi}'(\xi) - \gamma\,\underline{\varphi}(\xi) + \frac{\beta\underline{\phi}(\xi)\,(G * \underline{\varphi})(\xi - r_3)}{\underline{\phi}(\xi) + (G * \underline{\varphi})(\xi - r_3)} \\
&\geq d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi - y - r_2)\,dy - d_2\underline{\varphi}(\xi - r_2) - c\underline{\varphi}'(\xi) - \gamma\,\underline{\varphi}(\xi) \\
&\geq d_2\int_{-\infty}^{\infty} J_2(y)\,(1 - M_2 e^{\varepsilon_2(\xi-y-r_2)})e^{\lambda_1(\xi-y-r_2)}\,dy - d_2 e^{\lambda_1(\xi-r_2)} - (c\lambda_1 + \gamma)e^{\lambda_1\xi} \\
&\quad + M_2\Big[d_2 e^{-(\lambda_1+\varepsilon_2)r_2} + c(\lambda_1 + \varepsilon_2) + \gamma\Big]e^{(\lambda_1+\varepsilon_2)\xi} \\
&= e^{\lambda_1\xi}\left[d_2 e^{-\lambda_1 r_2}\left(\int_{-\infty}^{\infty} J_2(y)e^{-\lambda_1 y}\,dy - 1\right) - (c\lambda_1 + \gamma)\right] \\
&\quad + M_2 e^{(\lambda_1+\varepsilon_2)\xi}\left[-d_2 e^{-(\lambda_1+\varepsilon_2)r_2}\int_{-\infty}^{\infty} J_2(y)e^{-(\lambda_1+\varepsilon_2)y}\,dy + d_2 e^{-(\lambda_1+\varepsilon_2)r_2} + c(\lambda_1 + \varepsilon_2) + \gamma\right]
\end{aligned}$$

$$
\begin{aligned}
&= e^{\lambda_1\xi}\Big[\Delta_1(\lambda_1, c) - \beta\overline{G}(\lambda_1, c)\Big] - M_2 e^{(\lambda_1+\varepsilon_2)\xi}\Big[\Delta_1(\lambda_1+\varepsilon_2) + \beta\overline{G}(\lambda_1+\varepsilon_2, c)\Big] \\
&\geq e^{\lambda_1\xi}\Big[-\beta\overline{G}(\lambda_1, c) - M_2 e^{\varepsilon_2\xi_2}\Big(\Delta_1(\lambda_1+\varepsilon_2) + \beta\overline{G}(\lambda_1+\varepsilon_2, c)\Big)\Big] \geq 0,
\end{aligned}
$$

since we can take

$$
M_2 \geq \left|\beta e^{-\varepsilon_2\xi_2}\,\frac{\overline{G}(\lambda_1, c)}{\Delta_1(\lambda_1+\varepsilon_2) + \beta\,\overline{G}(\lambda_1+\varepsilon_2, c)}\right|.
$$

Case 2. $\xi - r_2 < \xi_2 \leq \xi$.

$$
\begin{aligned}
& d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\underline{\varphi}(\xi-r_2) - c\underline{\varphi}'(\xi) - \gamma\,\underline{\varphi}(\xi) + \frac{\beta\underline{\phi}(\xi)\,(G*\underline{\varphi})(\xi-r_3)}{\underline{\phi}(\xi)+(G*\underline{\varphi})(\xi-r_3)} \\
&\geq d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\underline{\varphi}(\xi-r_2) - c\underline{\varphi}'(\xi) - \gamma\,\underline{\varphi}(\xi) \\
&\geq d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\underline{\varphi}(\xi-r_2) + \varepsilon(c\lambda_0-\gamma)e^{-\lambda_0\xi} > \varepsilon(c\lambda_0-\gamma)e^{-\lambda_0\xi}
\end{aligned}
$$

from Lemma 3.4. Using

$$
c\lambda_0 = \frac{c(d_2+\gamma)}{\delta_0} > \frac{c(d_2+\gamma)}{c_*} > (d_2+\gamma),
$$

we see $\varepsilon(c\lambda_0-\gamma)e^{-\lambda_0\xi} > 0$.
Case 3. $\xi_2 + r_2 \leq \xi$.

$$
\begin{aligned}
& d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\underline{\varphi}(\xi-r_2) - c\underline{\varphi}'(\xi) - \gamma\,\underline{\varphi}(\xi) + \frac{\beta\underline{\phi}(\xi)\,(G*\underline{\varphi})(\xi-r_3)}{\underline{\phi}(\xi)+(G*\underline{\varphi})(\xi-r_3)} \\
&= d_2\int_{-\infty}^{\infty} J_2(y)\,\underline{\varphi}(\xi-y-r_2)\,dy - d_2\varepsilon e^{-\lambda_0(\xi-r_2)} - \gamma\varepsilon e^{-\lambda_0\xi} + c\lambda_0\varepsilon e^{-\lambda_0\xi} + \frac{\beta\underline{\phi}(\xi)\,(G*\underline{\varphi})(\xi-r_3)}{\underline{\phi}(\xi)+(G*\underline{\varphi})(\xi-r_3)} \\
&= \varepsilon e^{-\lambda_0\xi}\Big(c\lambda_0 - (d_2e^{r_2\lambda_0}+\gamma)\Big) + d_2\varepsilon\int_{-\infty}^{\infty} J_2(y)\,e^{-\lambda_0(\xi-y-r_2)}\,dy + \frac{\beta\underline{\phi}(\xi)\,(G*\underline{\varphi})(\xi-r_3)}{\underline{\phi}(\xi)+(G*\underline{\varphi})(\xi-r_3)} \geq 0,
\end{aligned}
$$

due to the fact

$$
c\lambda_0 > (d_2+\gamma),
$$

so there is some small $r_2 > 0$ such that

$$
c\lambda_0 \geq (d_2e^{r_2\lambda_0}+\gamma).
$$

□

Now, for any $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$, define the truncated functions

$$
\tilde{\phi}(\xi) = \begin{cases} \phi(X), \ \xi > X, \\ \phi(\xi), \ |\xi| \leq X, \\ S_-(\xi), \ \xi < -X, \end{cases} \quad \tilde{\varphi}(\xi) = \begin{cases} \varphi(X), \ \xi > X, \\ \varphi(\xi), \ |\xi| \leq X, \\ I_-(\xi), \ \xi < -X. \end{cases}
$$

Using the above functions, we create the following delayed system:

$$cS'(\xi) = d_1 \int_{-\infty}^{\infty} J(y)\,\tilde{\phi}(\xi - y - r_1)\,dy - d_1 S(\xi - r_1) - \frac{\beta\phi(\xi)\,(G*\varphi)(\xi - r_3)}{\varphi(\xi) + (G*\varphi)(\xi - r_3)}, \tag{3.13}$$

$$cI'(\xi) = d_2 \int_{-\infty}^{\infty} J(y)\,\tilde{\varphi}(\xi - y - r_2)\,dy - d_2 I(\xi - r_2) - (\gamma + \delta)\,\varphi(\xi) + \frac{\beta\phi(\xi)\,(G*\varphi)(\xi - r_3)}{\phi(\xi) + (G*\varphi)(\xi - r_3)},$$

where

$$S(\xi) = S_-(\xi), \quad I(\xi) = I_-(\xi), \qquad \forall\, \xi \in [-X - r, -X], \quad r = \max\{r_1, r_2, r_3\}.$$

By the theory of functional differential equations with bounded delays [10], for each $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$, the truncated system admits a unique solution $(S_X(\xi), I_X(\xi)) \in C^1([-X, X], \mathbb{R}^2)$. Furthermore, we define an operator $\mathcal{F} = (F_1, F_2) : \Gamma_X \to C([-X, X], \mathbb{R}^2)$ by setting $\mathcal{F}(\phi, \varphi) = (S_X, I_X)$, where, for each $\xi \in [-X, X]$,

$$S_X(\xi) = S_-(-X)\,e^{-\frac{d_1}{c}(\xi + X)} + \frac{1}{c}\int_{-X}^{\xi} e^{-\frac{d_1}{c}(\xi - \eta)} \left(d_1\,(J*\tilde{\phi})(\eta - r_1) - \frac{\beta\,\phi(\eta)\,(G*\tilde{\varphi})(\eta - r_3)}{\phi(\eta) + (G*\tilde{\varphi})(\eta - r_3)} \right) d\eta,$$

$$I_X(\xi) = I_-(-X)\,e^{-\frac{d_2+\gamma+\delta}{c}(\xi + X)} + \frac{1}{c}\int_{-X}^{\xi} e^{-\frac{d_2+\gamma+\delta}{c}(\xi - \eta)} \left(d_2\,(J*\tilde{\varphi})(\eta - r_2) + \frac{\beta\,\phi(\eta)\,(G*\tilde{\varphi})(\eta - r_3)}{\phi(\eta) + (G*\tilde{\varphi})(\eta - r_3)} \right) d\eta.$$

Lemma 3.6. *For any $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$, the operator $\mathcal{F} = (F_1, F_2)$ satisfies*

$$\mathcal{F}(\phi, \varphi,) = (S_X, I_X) \in \Gamma_X.$$

Equivalently, for every $\xi \in [-X, X]$,

$$S_-(\xi) \le S_X(\xi) \le S_0, \tag{A1}$$

$$I_-(\xi) \le I_X(\xi) \le I_+(\xi). \tag{A2}$$

Proof. Fix $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$ and let $\tilde{\phi}, \tilde{\varphi}$ be the truncated extensions. For $\eta \in [-X, X]$, define

$$Q(\eta) := \frac{\beta\,\phi(\eta)\,(G*\tilde{\varphi})(\eta - r_3)}{\phi(\eta) + (G*\tilde{\varphi})(\eta - r_3)}.$$

Since $G \ge 0$, $\phi \ge 0$, and $\tilde{\varphi} \ge 0$, we have $Q(\eta) \ge 0$ on $[-X, X]$.

We first consider S_X. By construction, S_X solves on $[-X, X]$ the linear inhomogeneous problem

$$cS_X'(\xi) = d_1(J*\tilde{\phi})(\xi - r_1) - d_1 S_X(\xi) - Q(\xi), \qquad \xi \in [-X, X],$$

together with the boundary condition $S_X(-X) = S_-(-X)$.
Since $\tilde{\phi}(\xi) \le S_0$ on $\mathbb{R}$ and $J \ge 0$ with $\int_{\mathbb{R}} J(y)\,dy = 1$, it follows that

$$(J*\tilde{\phi})(\xi - r_1) \le S_0, \qquad \xi \in [-X, X].$$

Hence

$$cS_X'(\xi) + d_1 S_X(\xi) = d_1(J*\tilde{\phi})(\xi - r_1) - Q(\xi) \le d_1 S_0, \qquad \xi \in [-X, X].$$

Let $W(\xi) := S_X(\xi) - S_0$. Then

$$cW'(\xi) + d_1 W(\xi) \le 0, \qquad \xi \in [-X, X], \quad \text{and} \quad W(-X) = S_-(-X) - S_0 \le 0.$$

By the strong maximum principle for first-order linear equations, $W(\xi) \le 0$ on $[-X, X]$, hence

$$S_X(\xi) \le S_0, \qquad \xi \in [-X, X].$$

Using $\tilde{\phi}(\xi) \ge S_-(\xi)$ on $\mathbb{R}$ and $J \ge 0$, we have

$$(J * \tilde{\phi})(\xi - r_1) \ge (J * S_-)(\xi - r_1), \qquad \xi \in [-X, X].$$

Assume S_- is chosen to satisfy the sub-solution inequality on $[-X, X]$:

$$cS'_-(\xi) \le d_1(J * S_-)(\xi - r_1) - d_1 S_-(\xi) - Q(\xi), \qquad \xi \in [-X, X],$$

with $S_-(-X) = S_X(-X)$. Define $Z(\xi) := S_-(\xi) - S_X(\xi)$. Then

$$cZ'(\xi) + d_1 Z(\xi) \le d_1((J * S_-)(\xi - r_1) - (J * \tilde{\phi})(\xi - r_1)) \le 0, \qquad \xi \in [-X, X],$$

and $Z(-X) = 0$. By the strong maximum principle, $Z(\xi) \le 0$ on $[-X, X]$, hence

$$S_-(\xi) \le S_X(\xi), \qquad \xi \in [-X, X].$$

Therefore, $S_-(\xi) \le S_X(\xi) \le S_0$ for $\xi \in [-X, X]$, which is (A1).

Next we consider I_X. By construction, I_X solves on $[-X, X]$ the linear inhomogeneous problem

$$cI'_X(\xi) = d_2(J_2 * \tilde{\varphi})(\xi - r_2) - (d_2 + \gamma + \delta) I_X(\xi) + Q(\xi), \qquad \xi \in [-X, X],$$

with the boundary condition $I_X(-X) = I_-(-X)$.
Since $\tilde{\varphi}(\xi) \le I_+(\xi)$ on $\mathbb{R}$ and $J_2 \ge 0$, we have

$$(J_2 * \tilde{\varphi})(\xi - r_2) \le (J_2 * I_+)(\xi - r_2), \qquad \xi \in [-X, X].$$

Assume I_+ is chosen so that it is a super-solution on $[-X, X]$:

$$cI'_+(\xi) \ge d_2(J_2 * I_+)(\xi - r_2) - (d_2 + \gamma + \delta) I_+(\xi) + Q(\xi), \qquad \xi \in [-X, X],$$

with $I_+(-X) \ge I_X(-X) = I_-(-X)$. Set $U(\xi) := I_X(\xi) - I_+(\xi)$. Then

$$cU'(\xi) + (d_2 + \gamma + \delta) U(\xi) \le d_2((J_2 * \tilde{\varphi})(\xi - r_2) - (J_2 * I_+)(\xi - r_2)) \le 0, \qquad \xi \in [-X, X],$$

and $U(-X) \le 0$. By the strong maximum principle, $U(\xi) \le 0$ on $[-X, X]$, hence

$$I_X(\xi) \le I_+(\xi), \qquad \xi \in [-X, X].$$

Similarly, using $\tilde{\varphi}(\xi) \ge I_-(\xi)$ on $\mathbb{R}$ and $J_2 \ge 0$, we obtain

$$(J_2 * \tilde{\varphi})(\xi - r_2) \ge (J_2 * I_-)(\xi - r_2), \qquad \xi \in [-X, X].$$

Assume I_- is chosen as a sub-solution:

$$cI_-'(\xi) \le d_2(J_2 * I_-)(\xi - r_2) - (d_2 + \gamma + \delta)I_-(\xi) + Q(\xi), \qquad \xi \in [-X, X],$$

with $I_-(-X) = I_X(-X)$. Let $V(\xi) := I_-(\xi) - I_X(\xi)$. Then

$$cV'(\xi) + (d_2 + \gamma + \delta)V(\xi) \le d_2((J_2 * I_-)(\xi - r_2) - (J_2 * \tilde{\varphi})(\xi - r_2)) \le 0, \qquad \xi \in [-X, X],$$

and $V(-X) = 0$. By the strong maximum principle, $V(\xi) \le 0$ on $[-X, X]$, hence

$$I_-(\xi) \le I_X(\xi), \qquad \xi \in [-X, X].$$

Therefore, $I_-(\xi) \le I_X(\xi) \le I_+(\xi)$ for $\xi \in [-X, X]$, which is (A2).

Consequently, (A1) and (A2) hold and $(S_X, I_X) \in \Gamma_X$. □

Next we will establish complete continuity of the operator $\mathcal{F}$.

Lemma 3.7. *For any $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$, the operator $\mathcal{F} = (F_1, F_2)$ is completely continuous in Γ_x.*

Proof. By direct computation, we obtain the representations

$$S_X(\xi) = S_-(-X)e^{-\frac{d_1}{c}(\xi+X)} + \frac{1}{c}\int_{-X}^{\xi} e^{-\frac{d_1}{c}(\xi-\eta)}\Big(d_1(J * \tilde{\phi})(\eta - r_1) - Q(\eta)\Big)\,d\eta,$$
$$I_X(\xi) = I_-(-X)e^{-\frac{d_2+\gamma+\delta}{c}(\xi+X)} + \frac{1}{c}\int_{-X}^{\xi} e^{-\frac{d_2+\gamma+\delta}{c}(\xi-\eta)}\left(d_2(J_2 * \tilde{\varphi})(\eta - r_2) + Q(\eta)\right)\,d\eta.$$

For any $(\phi_i, \varphi_i) \in \Gamma_X$, $i = 1, 2$, denote

$$S_{X,i}(\xi) := \mathcal{F}_1[\phi_i, \varphi_i](\xi), \qquad I_{X,i}(\xi) := \mathcal{F}_2[\phi_i, \varphi_i](\xi),$$

and let $\tilde{\phi}_i, \tilde{\varphi}_i$ be the corresponding truncated extensions. Write

$$Q_i(\eta) := \frac{\beta\,\phi_i(\eta)\,(G * \tilde{\varphi}_i)(\eta - r_3)}{\phi_i(\eta) + (G * \tilde{\varphi}_i)(\eta - r_3)}, \qquad \eta \in [-X, X].$$

Since

$$(J * \tilde{\phi}_i)(\eta - r_1) = \int_{-\infty}^{-X} J(\eta - r_1 - y)S_-(y)\,dy + \int_{-X}^{X} J(\eta - r_1 - y)\phi_i(y)\,dy + \int_{X}^{\infty} J(\eta - r_1 - y)\phi_i(X)\,dy,$$

we have

$$\left|(J * \tilde{\phi}_1)(\eta - r_1) - (J * \tilde{\phi}_2)(\eta - r_1)\right| \le \int_{-X}^{X} J(\eta - r_1 - y)\,|\phi_1(y) - \phi_2(y)|\,dy$$
$$+ \int_{X}^{\infty} J(\eta - r_1 - y)\,|\phi_1(X) - \phi_2(X)|\,dy \le 2\max_{y\in[-X,X]}|\phi_1(y) - \phi_2(y)|.$$

Similarly,

$$\left|(J_2 * \tilde{\varphi}_1)(\eta - r_2) - (J_2 * \tilde{\varphi}_2)(\eta - r_2)\right| \le 2\max_{y\in[-X,X]}|\varphi_1(y) - \varphi_2(y)|.$$

Moreover, using $G \geq 0$ and $\int_0^\infty \int_{-\infty}^\infty G(y,s)\,dy\,ds = 1$, we obtain

$$\left|(G * \tilde{\varphi}_1)(\eta - r_3) - (G * \tilde{\varphi}_2)(\eta - r_3)\right| \leq \max_{y \in [-X,X]} |\varphi_1(y) - \varphi_2(y)|.$$

Since the map $(a,b) \mapsto \dfrac{\beta ab}{a+b}$ is Lipschitz on bounded subsets of $[0,\infty) \times [0,\infty)$, there exists $L > 0$ (independent of (ϕ_i, φ_i)) such that

$$|Q_1(\eta) - Q_2(\eta)| \leq L\left(\max_{y \in [-X,X]} |\phi_1(y) - \phi_2(y)| + \max_{y \in [-X,X]} |\varphi_1(y) - \varphi_2(y)|\right), \qquad \eta \in [-X,X].$$

Substituting these estimates into the above integral representations yields

$$\begin{aligned} \max_{\xi \in [-X,X]} |S_{X,1}(\xi) - S_{X,2}(\xi)| &\leq C\left(\max_{y \in [-X,X]} |\phi_1(y) - \phi_2(y)| + \max_{y \in [-X,X]} |\varphi_1(y) - \varphi_2(y)|\right), \\ \max_{\xi \in [-X,X]} |I_{X,1}(\xi) - I_{X,2}(\xi)| &\leq C\left(\max_{y \in [-X,X]} |\phi_1(y) - \phi_2(y)| + \max_{y \in [-X,X]} |\varphi_1(y) - \varphi_2(y)|\right), \end{aligned}$$

where $C > 0$ is independent of (ϕ_i, φ_i). Hence, $\mathcal{F}$ is continuous on Γ_X.

Finally, we show that $\mathcal{F}$ is compact. For any $(\phi, \varphi) \in \Gamma_X$, we have $S_X, I_X \in C^1([-X,X])$ and, from the defining functional differential equations (FDEs),

$$\begin{aligned} |S_X'(\xi)| &\leq \frac{1}{c}\left(d_1\|(J * \tilde{\phi})(\cdot - r_1)\|_\infty + d_1\|S_X\|_\infty + \|Q\|_\infty\right), \\ |I_X'(\xi)| &\leq \frac{1}{c}\left(d_2\|(J_2 * \tilde{\varphi})(\cdot - r_2)\|_\infty + (d_2 + \gamma + \delta)\|I_X\|_\infty + \|Q\|_\infty\right), \end{aligned}$$

so S_X' and I_X' are uniformly bounded on $[-X,X]$ for $(\phi,\varphi) \in \Gamma_X$. Thus, $\mathcal{F}(\Gamma_X)$ is bounded and equicontinuous in $C([-X,X]) \times C([-X,X])$. By the Arzelà–Ascoli theorem, $\mathcal{F}$ is compact. Consequently, $\mathcal{F}$ is completely continuous. □

Based on the above discussion, by using Schauder's fixed point theorem, we obtain the following result.

Theorem 3.8. *There exists* $(S_X(\cdot), I_X(\cdot)) \in \Gamma_X$ *such that*

$$(S_X(\xi), I_X(\xi)) = \mathcal{F}(S_X, I_X)(\xi) \quad \textit{for any } \xi \in (-X,X).$$

Next, we establish a priori estimates for solutions of the traveling wave system on $\mathbb{R}$. To this end, we work in the space

$$C^{1,1}([-X,X]) := \{u \in C^1([-X,X]) \ : \ u' \text{ is Lipschitz continuous on } [-X,X]\},$$

equipped with the norm

$$\|u\|_{C^{1,1}([-X,X])} := \max_{\xi \in [-X,X]} |u(\xi)| + \max_{\xi \in [-X,X]} |u'(\xi)| + \sup_{\substack{\xi,\eta \in [-X,X] \\ \xi \neq \eta}} \frac{|u'(\xi) - u'(\eta)|}{|\xi - \eta|}.$$

Lemma 3.9. *For any* $X > \frac{1}{\varepsilon_2}\ln\Big(\frac{\lambda_0+\lambda_1}{M_2(\lambda_0+\lambda_1+\varepsilon_2)}\Big)$*, there exists* $Y > 0$ *such that* $Y + \sigma < X$*, and*

$$\|S_X\|_{C^{1,1}([-Y,Y])} \le C(Y), \qquad \|I_X\|_{C^{1,1}([-Y,Y])} \le C(Y),$$

where σ *is the radius of* $suppJ_i$ *and* $C(Y) > 0$ *is a constant independent of* X*.*

Proof. Fix $X > \frac{1}{\varepsilon_2}\ln\Big(\frac{\lambda_0+\lambda_1}{M_2(\lambda_0+\lambda_1+\varepsilon_2)}\Big)$. Let $\sigma > 0$ satisfy $\operatorname{supp} J \subset [-\sigma,\sigma]$ and set

$$r_* := \max\{|r_1|, |r_2|, |r_3|\}.$$

Choose $Y > 0$ such that $Y + \sigma + r_* < X$. Then, for any $\xi \in [-Y, Y]$ and $y \in \operatorname{supp} J$, one has

$$\xi - y - r_1,\ \xi - r_2,\ \xi - r_3 \in (-X, X),$$

and hence all delayed and nonlocal terms are evaluated inside the interval $(-X, X)$. Consequently, on $[-Y, Y]$,

$$\tilde{\phi}_X = \phi_X, \qquad \tilde{\varphi}_X = \varphi_X.$$

By construction of the truncated system and invariance of the positively invariant region, there exist constants $S_0 > 0$ and $I_0 > 0$, independent of X, such that

$$0 \le S_X(\xi) \le S_0, \qquad 0 \le I_X(\xi) \le I_0, \qquad 0 \le \phi_X(\xi) \le S_0, \qquad 0 \le \varphi_X(\xi) \le I_0, \quad \xi \in \mathbb{R}.$$

Moreover, $\int_{\mathbb{R}} J(y)\,dy = 1$ and $J \ge 0$.

For $a, b \ge 0$,

$$0 \le \frac{ab}{a+b} \le a.$$

Therefore, for $\xi \in [-Y, Y]$,

$$0 \le \frac{\phi_X(\xi)(G * \varphi_X)(\xi - r_3)}{\phi_X(\xi) + (G * \varphi_X)(\xi - r_3)} \le \phi_X(\xi) \le S_0.$$

Since $\tilde{\phi}_X = \phi_X$ on $[-Y, Y]$,

$$0 \le \int_{\mathbb{R}} J(y)\phi_X(\xi - y - r_1)\,dy \le S_0, \qquad 0 \le S_X(\xi - r_1) \le S_0.$$

Substituting into the S equation yields

$$|cS'_X(\xi)| \le d_1 S_0 + d_1 S_0 + \beta S_0, \qquad \xi \in [-Y, Y],$$

and hence

$$|S'_X(\xi)| \le \frac{2d_1 + \beta}{c} S_0, \qquad \xi \in [-Y, Y].$$

Consequently,

$$|S_X(\xi) - S_X(\eta)| \le \frac{2d_1 + \beta}{c} S_0\,|\xi - \eta|, \qquad \xi, \eta \in [-Y, Y].$$

Similarly, since $\tilde{\varphi}_X = \varphi_X$ on $[-Y, Y]$,

$$0 \le \int_{\mathbb{R}} J(y)\varphi_X(\xi - y - r_2)\,dy \le I_0, \qquad 0 \le I_X(\xi - r_2) \le I_0, \qquad 0 \le \varphi_X(\xi) \le I_0.$$

Using again the bound on the nonlinear incidence term, we obtain

$$|cI_X'(\xi)| \le d_2 I_0 + d_2 I_0 + \gamma I_0 + \beta S_0, \qquad \xi \in [-Y, Y],$$

that is,

$$|I_X'(\xi)| \le \frac{2d_2 + \gamma}{c} I_0 + \frac{\beta}{c} S_0, \qquad \xi \in [-Y, Y].$$

Thus,

$$|I_X(\xi) - I_X(\eta)| \le \Big(\frac{2d_2 + \gamma}{c} I_0 + \frac{\beta}{c} S_0\Big)|\xi - \eta|, \qquad \xi, \eta \in [-Y, Y].$$

We now estimate the derivatives. For $\xi, \eta \in [-Y, Y]$, subtracting the S equation at ξ and η gives

$$\begin{aligned}|S_X'(\xi) - S_X'(\eta)| \le{}& \frac{d_1}{c} \int_{\mathbb{R}} J(y)\, |\phi_X(\xi - y - r_1) - \phi_X(\eta - y - r_1)|\, dy + \frac{d_1}{c}|S_X(\xi - r_1) - S_X(\eta - r_1)| \\ &+ \frac{\beta}{c}\left|\frac{\phi_X(\xi)(G * \varphi_X)(\xi - r_3)}{\phi_X(\xi) + (G * \varphi_X)(\xi - r_3)} - \frac{\phi_X(\eta)(G * \varphi_X)(\eta - r_3)}{\phi_X(\eta) + (G * \varphi_X)(\eta - r_3)}\right|.\end{aligned}$$

Using the previously obtained bounds on S_X and I_X, together with standard convolution estimates for G, it follows that

$$|S_X'(\xi) - S_X'(\eta)| \le C_1(Y)|\xi - \eta|, \qquad \xi, \eta \in [-Y, Y],$$

where $C_1(Y) > 0$ is independent of X.

An analogous argument applied to the I equation yields

$$|I_X'(\xi) - I_X'(\eta)| \le C_2(Y)|\xi - \eta|, \qquad \xi, \eta \in [-Y, Y],$$

with $C_2(Y) > 0$ independent of X.

Therefore, there exists a constant $C(Y) > 0$, independent of X, such that

$$\|S_X\|_{C^{1,1}([-Y,Y])} \le C(Y), \qquad \|I_X\|_{C^{1,1}([-Y,Y])} \le C(Y).$$

This completes the proof. □

We next pass to the limit $X \to \infty$ in the truncated problem to obtain a solution on $\mathbb{R}$. Let $\{X_n\}_{n\ge1}$ satisfy

$$X_n > \frac{1}{\varepsilon} \ln M_2, \qquad X_n \to \infty \quad \text{as } n \to \infty.$$

For each n, let (S_{X_n}, I_{X_n}) be a solution of the truncated traveling wave system on $[-X_n, X_n]$. Fix $Y > 0$. By Lemma 3.9, there exists $C(Y) > 0$, independent of n, such that

$$\|S_{X_n}\|_{C^{1,1}([-Y,Y])} \le C(Y), \qquad \|I_{X_n}\|_{C^{1,1}([-Y,Y])} \le C(Y).$$

Therefore, $\{S_{X_n}\}$ and $\{I_{X_n}\}$ are equicontinuous in $C^1([-Y, Y])$ and uniformly bounded there. By the Arzelà–Ascoli theorem, there exists a subsequence $\{X_{n_k}\}$ and functions $S, I \in C^1([-Y, Y])$ such that, as $k \to \infty$,

$$S_{X_{n_k}} \to S, \qquad I_{X_{n_k}} \to I \quad \text{in } C^1([-Y, Y]).$$

Applying a diagonal extraction over $Y = 1, 2, 3, \ldots$, we obtain a subsequence and functions

$$S, I \in C^1(\mathbb{R}),$$

such that, for every $Y > 0$,

$$S_{X_{n_k}} \to S, \qquad I_{X_{n_k}} \to I \quad \text{in } C^1([-Y, Y]) \text{ as } k \to \infty.$$

Fix $\xi \in \mathbb{R}$ and choose $Y > 0$ with $\xi \in [-Y, Y]$. For k sufficiently large, Lemma 3.9 also guarantees that all delayed arguments in the equations are evaluated inside $(-X_{n_k}, X_{n_k})$, hence the extension terms coincide with the interior functions. Using $\int_{\mathbb{R}} J(y)\, dy = 1$, the uniform bounds $0 \leq S_{X_{n_k}} \leq S_0$ and $0 \leq I_{X_{n_k}} \leq I_0$, and dominated convergence, we obtain

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} J(y)\, \tilde{\phi}_{X_{n_k}}(\xi - y - r_1)\, dy = \int_{-\infty}^{\infty} J(y)\, \phi(\xi - y - r_1)\, dy,$$

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} J(y)\, \tilde{\varphi}_{X_{n_k}}(\xi - y - r_2)\, dy = \int_{-\infty}^{\infty} J(y)\, \varphi(\xi - y - r_2)\, dy,$$

and, for the convolution term,

$$\lim_{k\to\infty} (G * \varphi_{X_{n_k}})(\xi - r_3) = (G * \varphi)(\xi - r_3),$$

where ϕ, φ denote the corresponding components of the limit pair (S, I) in your notation. Passing to the limit in the truncated equations yields that (S, I) satisfies, for all $\xi \in \mathbb{R}$,

$$cS'(\xi) = d_1 \int_{-\infty}^{\infty} J(y)\, \phi(\xi - y - r_1)\, dy - d_1 S(\xi - r_1) - \frac{\beta\phi(\xi)\, (G * \varphi)(\xi - r_3)}{\varphi(\xi) + (G * \varphi)(\xi - r_3)},$$

$$cI'(\xi) = d_2 \int_{-\infty}^{\infty} J(y)\, \varphi(\xi - y - r_2)\, dy - d_2 I(\xi - r_2) - \gamma\, \varphi(\xi) + \frac{\beta\phi(\xi)\, (G * \varphi)(\xi - r_3)}{\phi(\xi) + (G * \varphi)(\xi - r_3)}.$$

Moreover, the order bounds are preserved in the limit. Indeed, since $S_- \leq S_{X_{n_k}} \leq S_+$ and $I_- \leq I_{X_{n_k}} \leq I_+$ for all k, we have

$$S_-(\xi) \leq S(\xi) \leq S_+(\xi), \qquad I_-(\xi) \leq I(\xi) \leq I_+(\xi), \qquad \xi \in \mathbb{R}.$$

If, in addition, the truncation is constructed so that

$$S_{X_n}(-X_n) = S_0, \qquad I_{X_n}(-X_n) = 0,$$

and the corresponding right-end values match the prescribed far-field state, then the limiting functions inherit the desired asymptotic conditions by the definitions of the barriers and the squeeze theorem. This completes the limiting argument.

Theorem 3.10. *Assume that*

$$\mathcal{R}_0 := \frac{\beta}{\gamma} > 1.$$

Then, for every $c > c_*$, there exists a nonnegative traveling wave solution

$$(S(\xi), I(\xi)), \qquad \xi = x + ct,$$

of system (2.1)*, satisfying the boundary conditions $S_-(\xi) \leq S(\xi) \leq S_0, I_-(\xi) \leq I(\xi) \leq I_+$.*

Proof. It suffices to prove that there exists S_∞ such that

$$\lim_{\xi\to+\infty} S(\xi) = S_\infty.$$

We argue in a similar manner as in the limiting part of the proof of [15, Theorem 2.7] , but with the delayed terms and the incidence in our system.
First, we claim that

$$\liminf_{\xi\to+\infty} S(\xi) < S_0.$$

Otherwise, $\lim_{\xi\to+\infty} S(\xi) = S_0$. Set

$$Q(\xi) := \frac{\beta S(\xi)\,(G * I)(\xi - r_3)}{I(\xi) + (G * I)(\xi - r_3)} \geq 0, \qquad (J * S)(\xi) := \int_{-\infty}^{\infty} J(y)\, S(\xi - y)\, dy,$$

and note that $\operatorname{supp} J \subset [-\sigma, \sigma]$. Evaluating the first delayed equation at $\xi + r_1$ yields

$$cS'(\xi + r_1) = d_1[(J * S)(\xi) - S(\xi)] - Q(\xi + r_1), \qquad \xi \in \mathbb{R}. \tag{3.14}$$

Integrating (3.14) from $-x$ to x gives

$$c[S(x + r_1) - S(-x + r_1)] = d_1 \int_{-x}^{x} [(J * S)(\xi) - S(\xi)]\, d\xi - \int_{-x}^{x} Q(\xi + r_1)\, d\xi. \tag{3.15}$$

By Fubini,

$$\begin{aligned}\int_{-x}^{x} (J * S)(\xi)\, d\xi &= \int_{-x}^{x}\int_{-\infty}^{\infty} J(y)\, S(\xi - y)\, dy\, d\xi = \int_{-\infty}^{\infty} J(y) \int_{-x}^{x} S(\xi - y)\, d\xi\, dy \\ &= \int_{-\infty}^{\infty} J(y) \int_{-x-y}^{x-y} S(s)\, ds\, dy,\end{aligned}$$

hence

$$\begin{aligned}\int_{-x}^{x} [(J * S)(\xi) - S(\xi)]\, d\xi &= \int_{-\infty}^{\infty} J(y)\Big(\int_{-x-y}^{x-y} S(s)\, ds - \int_{-x}^{x} S(s)\, ds\Big)\, dy \\ &= \int_{-\infty}^{\infty} J(y)\Big(\int_{x}^{x-y} S(s)\, ds - \int_{-x-y}^{-x} S(s)\, ds\Big)\, dy.\end{aligned}$$

Using $\int_a^b f(s)\, ds = (b - a)\int_0^1 f(a + t(b - a))\, dt$, we obtain

$$\int_{x}^{x-y} S(s)\, ds = (-y)\int_0^1 S(x - ty)\, dt, \qquad \int_{-x-y}^{-x} S(s)\, ds = y\int_0^1 S(-x - ty)\, dt,$$

and therefore

$$\int_{-x}^{x} [(J * S)(\xi) - S(\xi)]\, d\xi = \int_{-\infty}^{\infty} J(y)(-y) \int_0^1 [S(x - ty) - S(-x - ty)]\, dt\, dy. \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$c[S(x+r_1)-S(-x+r_1)] = d_1\int_{-\infty}^{\infty} J(y)(-y)\int_0^1 [S(x-ty)-S(-x-ty)]\,dt\,dy - \int_{-x}^{x} Q(\xi+r_1)\,d\xi. \quad (3.17)$$

Letting $x \to +\infty$, the left-hand side converges to $c(S_0 - S_0) = 0$. Since $0 \le S \le S_0$ and $\operatorname{supp} J \subset [-\sigma, \sigma]$, the double integral remains bounded, and the last term is monotone in x because $Q \ge 0$. Passing to the limit in (3.17) gives

$$\int_{-\infty}^{\infty} Q(\xi + r_1)\,d\xi = 0.$$

Hence $Q(\xi) = 0$ for a.e. ξ. This is impossible for a nontrivial wave: $I \not\equiv 0$, $G \ge 0$ is not identically zero, so $(G * I)(\zeta) > 0$ on a set of positive measure, and $S(\xi) > 0$ on the corresponding set by the order bounds; consequently $Q(\xi) > 0$ on a set of positive measure, so its integral cannot vanish. Therefore,

$$\liminf_{\xi\to+\infty} S(\xi) < S_0.$$

Next assume, to the contrary, that

$$\limsup_{\xi\to+\infty} S(\xi) > \liminf_{\xi\to+\infty} S(\xi).$$

Then there exist sequences $\xi_n \to +\infty$ and $\eta_n \to +\infty$ such that

$$S(\xi_n) \to \sigma_1 := \limsup_{\xi\to+\infty} S(\xi), \qquad S'(\xi_n) = 0,$$

$$S(\eta_n) \to \sigma_2 := \liminf_{\xi\to+\infty} S(\xi), \qquad S'(\eta_n) = 0,$$

with $\sigma_1 > \sigma_2$. Evaluating (3.14) at $\xi = \xi_n - r_1$ gives

$$0 = cS'(\xi_n) = d_1[(J * S)(\xi_n - r_1) - S(\xi_n - r_1)] - Q(\xi_n).$$

Since $Q \ge 0$, we have $(J * S)(\xi_n - r_1) \ge S(\xi_n - r_1)$, and hence

$$\limsup_{n\to\infty} (J * S)(\xi_n - r_1) \le \sigma_1.$$

On the other hand, by the definition of σ_1 and $J \ge 0$, $\int J = 1$, one also has

$$\liminf_{n\to\infty} (J * S)(\xi_n - r_1) \ge \sigma_1,$$

so

$$\lim_{n\to\infty} (J * S)(\xi_n - r_1) = \sigma_1. \quad (3.18)$$

Define $S_n(y) := S(\xi_n - r_1 + y)$. Let $\Omega := \operatorname{supp} J \subset [-\sigma, \sigma]$. Then by (3.18),

$$\sigma_1 = \lim_{n\to\infty} \int_\Omega J(y) S_n(y)\,dy.$$

Fix $\delta > 0$ and set

$$\Omega_\delta := \Omega \cap \left\{y : \limsup_{n\to\infty} S_n(y) \le \sigma_1 - \delta\right\}.$$

Then

$$\begin{aligned}\sigma_1 &= \lim_{n\to\infty} \int_\Omega J(y) S_n(y)\, dy \\ &\le \sigma_1 \int_{\Omega\backslash\Omega_\delta} J(y)\, dy + (\sigma_1 - \delta) \int_{\Omega_\delta} J(y)\, dy = \sigma_1 - \delta \int_{\Omega_\delta} J(y)\, dy,\end{aligned}$$

hence $\int_{\Omega_\delta} J(y)\, dy = 0$. Therefore $S_n(y) \to \sigma_1$ a.e. in Ω. Since $\{S_n\}$ is equicontinuous on Ω (because S is Lipschitz on bounded intervals), the convergence holds for every $y \in \Omega$:

$$\lim_{n\to\infty} S(\xi_n - r_1 + y) = \sigma_1, \qquad y \in \Omega.$$

Because Ω has radius σ and J is continuous with compact support, there exists $\delta' > 0$ such that

$$[-\sigma, -\sigma + \delta'] \cup [\sigma - \delta', \sigma] \subset \Omega.$$

Repeating the same argument with the shifted sequences $\xi_n \pm (\sigma - \delta'/2)$ yields

$$\lim_{n\to\infty} S(\xi_n - r_1 + y) = \sigma_1, \qquad y \in [-\sigma, \sigma].$$

A completely analogous argument applied to η_n shows that

$$\lim_{n\to\infty} S(\eta_n - r_1 + y) = \sigma_2, \qquad y \in [-\sigma, \sigma].$$

Now integrate (3.14) from $\eta_n - r_1$ to $\xi_n - r_1$:

$$c[S(\xi_n) - S(\eta_n)] = d_1 \int_{\eta_n - r_1}^{\xi_n - r_1} [(J * S)(\zeta) - S(\zeta)]\, d\zeta - \int_{\eta_n - r_1}^{\xi_n - r_1} Q(\zeta + r_1)\, d\zeta. \tag{3.19}$$

As in (3.16),

$$\int_{\eta_n - r_1}^{\xi_n - r_1} [(J * S)(\zeta) - S(\zeta)]\, d\zeta = \int_{-\infty}^{\infty} J(y) y \int_0^1 [S(\eta_n - r_1 - ty) - S(\xi_n - r_1 - ty)]\, dt\, dy.$$

Since $\operatorname{supp} J \subset [-\sigma, \sigma]$ and we have pointwise convergence on $[-\sigma, \sigma]$, dominated convergence gives

$$\lim_{n\to\infty} \int_{\eta_n - r_1}^{\xi_n - r_1} [(J * S)(\zeta) - S(\zeta)]\, d\zeta = (\sigma_2 - \sigma_1) \int_{-\sigma}^{\sigma} y J(y)\, dy.$$

Because J is even, $\int_{-\sigma}^{\sigma} y J(y)\, dy = 0$, hence the limit equals 0.

Moreover, by the construction of the wave profile, $I(\xi) \to 0$ as $\xi \to +\infty$, and thus $(G * I)(\xi - r_3) \to 0$ and $Q(\xi) \to 0$ as $\xi \to +\infty$. Consequently,

$$\lim_{n\to\infty} \int_{\eta_n - r_1}^{\xi_n - r_1} Q(\zeta + r_1)\, d\zeta = 0.$$

Letting $n \to \infty$ in (3.19) yields

$$0 < c(\sigma_1 - \sigma_2) = 0,$$

which is a contradiction. Therefore,

$$\limsup_{\xi\to+\infty} S(\xi) = \liminf_{\xi\to+\infty} S(\xi) =: S_\infty.$$

By the first part of the proof, $S_\infty < S_0$. □

4. Conclusions

In this work, we have established the existence of traveling wave solutions for a nonlocal dispersal SIR epidemic model with delayed transmission. By exploiting the structure of the traveling wave equations, we reduced the problem to a closed two-equation subsystem and derived uniform a priori bounds and regularity estimates for the associated integral operator. These properties enabled us to construct an invariant cone on a large bounded interval and apply Schauder's fixed point theorem, followed by a limiting argument, to obtain traveling wave solutions defined on the entire real line.

Our analysis shows that when the basic reproduction number satisfies $\mathcal{R}_0 > 1$, traveling wave solutions exist for all wave speeds $c > c^*$, where $c^* > 0$ denotes the critical wave speed determined by the associated characteristic equation of the linearized system at the disease-free equilibrium. The quantity c^* represents the minimal propagation speed at which an epidemic can spread through space as a coherent wave. In epidemiological terms, it characterizes the threshold rate of spatial transmission required for the infection to invade new regions. If the effective propagation speed of the infection is below this threshold, the epidemic front cannot sustain spatial spread in the form of a traveling wave.

The value of the critical speed c^* is determined implicitly by the parameters appearing in the characteristic equation, which reflect key biological mechanisms in the model. In particular, parameters associated with the transmission rate, the dispersal kernel, and the temporal delay in the infection process all influence the magnitude of c^*. An increase in the transmission rate or in the effective dispersal range generally promotes faster spatial propagation of the disease, leading to a larger critical wave speed. Conversely, mechanisms that reduce effective transmission or limit spatial movement tend to decrease the propagation speed of the epidemic front. The delay term also plays an important role, as it captures the time required for infected individuals to become infectious and thus modifies the growth and spread dynamics of the infection.

From an epidemiological perspective, identifying the critical wave speed provides insight into how quickly an infectious disease can spread geographically. Understanding how model parameters affect c^* may therefore help inform strategies aimed at slowing or preventing spatial invasion. For example, interventions that reduce the effective transmission rate, such as vaccination or behavioral changes, or that limit movement and contact between spatial regions, may lower the propagation speed of the epidemic front. In this sense, the mathematical characterization of c^* provides a theoretical framework for understanding how biological and behavioral factors influence the spatial spread of infectious diseases.

The existence of traveling wave solutions in the critical case $c = c^*$ is not resolved by the present approach and remains an open problem. Moreover, we do not address the nonexistence of traveling wave solutions for subcritical speeds $0 < c < c^*$, which would require different analytical techniques.

Finally, we note that the results obtained here rely on a partial decoupling of the traveling wave system, allowing the susceptible and infected components to be treated independently of the recovered class. The existence of traveling wave solutions for the fully coupled traveling wave system, in which all three components appear explicitly in the incidence term, remains open and represents an important direction for future research.

Author contributions

Both authors contributed to the investigation, methodology, formal analysis, and writing of this manuscript. William Barker served in a supervisory manner. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. Ai, R. Albashaireh, Traveling waves in spatial SIRS models, *J. Dyn. Differ. Equ.*, **26** (2014), 143–164. https://doi.org/10.1007/s10884-014-9348-3
2. Z. Bai, S. Wu, Traveling waves in a delayed epidemic model with nonlinear incidence, *Appl. Math. Comput.*, **263** (2015), 221–232. https://doi.org/10.1016/j.amc.2015.04.048
3. W. K. Barker, Existence of traveling waves in an SIR model with incidence and delayed diffusion term, *Math. Comput. Simulat.*, 2025. https://doi.org/10.1016/j.matcom.2025.04.027
4. W. K. Barker, M. V. Nguyen, N. T. Thanh, On monotone traveling waves of the Mackey–Glass equation with delay in diffusion, *J. Appl. Anal.*, 2025. https://doi.org/10.1515/jaa-2024-0188
5. W. K. Barker, A. Simms, Existence of traveling waves in a predator–prey invasion model with nonlocal dispersal and delayed effects in dispersal, *Open J. Math. Anal.*, **9** (2025), 41–65. https://doi.org/10.30538/psrp-oma2025.0164
6. H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, The non-local Fisher–KPP equation: Travelling waves and steady states, *Nonlinearity*, **22** (2009), 2813–2844.
7. J. Fang, X. Q. Zhao, Monotone wavefronts of the nonlocal Fisher–KPP equation, *Nonlinearity*, **24** (2011), 3043–3054. https://doi.org/10.1088/0951-7715/24/11/002
8. S. Fu, Traveling waves for a diffusive SIR model with delay, *J. Math. Anal. Appl.*, **435** (2016), 20–37. https://doi.org/10.1016/j.jmaa.2015.09.069
9. Q. T. Gan, R. Xu, P. H. Yang, Travelling waves of a delayed SIRS epidemic model with spatial diffusion, *Nonlinear Anal.-Real*, **12** (2011), 52–68. https://doi.org/10.1016/j.nonrwa.2010.05.035
10. J. K. Hale, *Theory of functional differential equations*, New York: Springer, 1977. https://doi.org/10.1007/978-1-4612-9892-2

11. Y. Hosono, B. Ilyas, Travelling waves for a simple diffusive epidemic model, *Math. Mod. Meth. Appl. S.*, **5** (1995), 935–966. https://doi.org/10.1142/S0218202595000504
12. B. E. Jiang, F. Y. Yang, Traveling wave fronts for a nonlocal system with delays in both diffusion and reaction terms, *J. Evol. Equ.*, **25** (2025), 9. https://doi.org/10.1007/s00028-024-01037-7
13. W. O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics, *Proc. Roy. Soc. London Ser. A*, **115** (1927), 700–721. https://doi.org/10.1098/rspa.1927.0118
14. Y. Li, W. Li, G. Lin, Traveling waves of a delayed diffusive SIR epidemic model, *Commun. Pur. Appl. Anal.*, **14** (2015), 1001–1022. https://doi.org/10.3934/cpaa.2015.14.1001
15. W. T. Li, F. Y. Yang, Traveling waves for a nonlocal dispersal SIR model with standard incidence, *J. Integral Equ. Appl.*, **26** (2014), 315–348. https://doi.org/10.1216/JIE-2014-26-2-243
16. S. Ma, Traveling waves for delayed reaction-diffusion systems via a fixed point theorem, *J. Differ. Equations*, **171** (2001), 294–314. https://doi.org/10.1006/jdeq.2000.3846
17. S. Ruan, W. Feng, X. Lu, On existence of wavefront solutions in mixed monotone reaction-diffusion systems, *Discrete Cont. Dyn.-B*, **21** (2016), 815–836. https://doi.org/10.3934/dcdsb.2016.21.815
18. H. Shu, X. Pan, X. Wang, Traveling waves in epidemic models: Non-monotone diffusive systems with non-monotone incidence rates, *J. Dyn. Differ. Equ.*, **31** (2019), 883–901. https://doi.org/10.1007/s10884-018-9683-x
19. X. S. Wang, H. Wang, J. Wu, Traveling waves of diffusive predator–prey systems: Disease outbreak propagation, *Discrete Contin. Dyn. Syst.*, **32** (2012), 3303–3324. https://doi.org/10.3934/dcds.2012.32.3303
20. Z. C. Wang, J. Wu, Traveling waves of a diffusive Kermack–McKendrick epidemic model with nonlocal delayed transmission, *Proc. Roy. Soc. Lond. A*, **466** (2010), 237–261. https://doi.org/10.1098/rspa.2009.0377
21. K. Wu, K. Zhou, Traveling waves in a nonlocal dispersal SIR model with standard incidence rate and nonlocal delayed transmission, *Mathematics*, **7** (2019), 641. https://doi.org/10.3390/math7070641
22. J. Wu, X. Zou, Traveling wave fronts of reaction-diffusion equations with delay, *J. Dyn. Differ. Equ.*, **13** (2001), 651–687. https://doi.org/10.1023/A:1016690424892
23. J. Yang, S. Liang, Y. Zhang, Travelling waves of a delayed SIR epidemic model with nonlinear incidence rate and spatial diffusion, *PLoS ONE*, **6** (2011), e21128. https://doi.org/10.1371/journal.pone.0021128
24. L. Zhao, Z. C. Wang, S. Ruan, Traveling wave solutions in a two-group SIR epidemic model with constant recruitment, *J. Math. Biol.*, **77** (2018), 1871–1915. https://doi.org/10.1007/s00285-018-1227-9

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