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Research article

Computation of edge metric dimension of zero divisor graph of matrices

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Abstract: This paper investigated the resolving parameters of the zero-divisor graph (ZDG) associated with the non-commutative ring of 3×3 upper triangular matrices over the field $\mathbb{Z}_2$. The structural complexity of non-commutative matrix rings, especially the difference between left and right zero-divisors, poses special difficulties for graph-theoretic characterization, although the metric dimensions of ZDGs for commutative rings have been well known. For this particular graph structure $G = ZDG[M_3(\mathbb{Z}_2)]$, we specifically calculated the metric dimension ($\dim_v(G)$) and the edge metric dimension ($\mathrm{edim}_e(G)$). We determined the minimal resolving sets and proved that $\dim_v(G) = [11]$ and $\mathrm{edim}_e(G) = [13]$ by combining combinatorial proofs with structural decomposition into equivalence classes. A basic framework for calculating the metric dimensions of generalized $n \times n$ matrix rings over finite fields is provided by these findings.

Keywords: metric dimension; ZDG; edge metric dimension; upper triangular matrix; resolving set
Mathematics Subject Classification: 05C12, 05C76, 05C90

1. Introduction

Graph theory is a branch of mathematics that has numerous applications in almost all the sciences such as, physics (electrical circuits, lattice structures, quantum mechanics, percolation theory, etc.); chemistry (topological indices, molecular structure, chemical informatics, reaction networks, etc.); biology (protein interaction networks, neural networks, etc.); computer science (graph algorithms, network design, artificial intelligence, interconnection networks, graph databases, social network analysis, etc.); and biotechnology (metabolic pathways, genomic analysis, drug design, etc.) [1]. Graph theory deals with two things, viz., vertices and edges, where vertices are the discrete objects and the edges are defined as the relationship among these discrete objects [2]. This definition of graph theory enhances its applicability and relevance to real-world problems, as many such problems involve discrete structures. It actually provides tools to analyze and model connections and relationships in systems made up of discrete elements. The ability of this subject to convert abstract, complex problems into manageable mathematical models makes it essential for simplifying many real-world challenges efficiently [3].

In graph theory, there are several active research topics; among them, resolvability parameters is the most important and interesting area. The property of identifying the vertices or edges uniquely in a graph G in some specific manner is collectively known as resolvability parameters. The most studied such parameter is the metric dimension of a graph G. The concepts of the minimum resolving set, resolving set and that of the metric dimension (MD) were introduced independently by Salter [4] and by Harary and Melter in 1976 [5]. The terms *"resolving set"* and *"metric dimension"* were coined by Harary and Melter, while Slater called these terms as *"locating set"* and *"location number"*, respectively. These concepts have wide-ranging applications in several disciplines. In the field of pattern recognition and image processing, Melter and Tomescu introduced the idea of metric bases in digital geometry [6]. In robotics and network navigation, Khuller, Raghavachari and Rosenfeld applied these concepts to landmark-based navigation in graphs [7]. Furthermore, Sebö and Tannier studied metric generators from the perspective of combinatorial optimization and graph theory [8]. These applications also extend to chemical sciences, network discovery and related areas.

We denote the collection of simple connected graphs by Φ. The collection Φ is said to have constant (fixed) MD if for any member, say $G_1 \in \Phi$, the MD is constant and equal to any other member sitting in Φ [9]. Chartrand et al. [10] proved that the MD of G is one iff $G = P_m$, where P_m is a path graph on $m \geq 2$ vertices. Furthermore, they proved that the cycle graph C_m has the MD two, for each positive integer $m \geq 3$. Sharma and Bhat [11], studied these concepts for an interesting family of planar graphs, known as heptagonal circular ladder H_m. They also introduced two planar graphs from H_m and also computed their MD. For these three graph families, they proved that their MD is three. Javaid et al. [12] investigated the minimum resolving set for Harary graphs $H_{4,m}$ and generalized Petersen graphs $P(m, 2)$ and proved that their MD is also constant. Therefore, we can say that for path graph P_n, cycle graph C_n, Harary graph, heptagonal circular ladder H_m, two planar graphs obtained from H_m, and generalized Petersen graph are the families of the graph with the constant MD.

In recent decades, the notion of the MD has been explored for several different families of the planar graphs [11,12]. Next, $G = G_1 + G_2$ represents a graph, known as the join of two graphs $G_1 = G_1(V_1, E_1)$ and $G_2 = G_2(V_2, E_2)$, and it is defined as a graph

$$G = G(V, E) = G_1 + G_2,$$

where $V = V_1 \cup V_2$ and

$$E = E_1 \cup E_2 \cup \{pq : p \in V_1 \text{ and } q \in V_2\}.$$

Using this definition of the join of two graphs, we have the wheel W_m defined as $W_m = C_m + K_1$ for $m \geq 3$, the fan graph F_m defined as $F_m = P_m + K_1$ for $m \geq 1$, and the Jahangir graph J_{2m} ($m \geq 2$) obtained from W_m by deleting m spokes alternately from it. Later, several researchers computed the MD for these graphs, viz., W_m, F_m, and J_{2m}. They proved that the MD for F_m with $m \geq 1$ is $\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{1, 2, 3, 6\}$ [13], for W_m with $m \geq 3$ is $\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{3, 6\}$ [14], and J_{2m} with $m \geq 4$ is $\lfloor \frac{2m}{3} \rfloor$ [15]. Now, one can clearly observed that the MD for F_m, J_{2m} and W_m depends upon the number of the vertices present in the respective graph, and these are the classes of the graphs with the non-constant MD. Zero-divisor graphs (ZDGs) form a significant link between algebra and graph theory by representing ring-theoretic properties in a combinatorial framework. Recent studies, including cozero-divisor graphs and their spectral analysis, have provided deeper insights into the structure of finite rings. These developments motivate further investigation of ZDGs of matrix rings, particularly with respect to metric and edge metric dimensions.

Algebraic graph theory is a significant and interesting area of mathematics that explores graphs using tools from abstract algebra, especially linear algebra, group theory, and ring theory. Algebraic graph theory enables us to understand the complex algebraic concepts through the lenses of graphs, and its importance lies in both theoretical insights and practical applications. One of such interesting concepts is ZDGs. This algebraic notion was first introduced by Beck [16], and the idea behind this concept is to construct an algebraic graph using the zero-divisors of a ring. Assume that the vertices of G correspond to the elements of the ring R, and two vertices x and y are adjacent if $x.y = 0$. We call such a graph G as a ZDG of R. Anderson and Livingston were the first to simplify Beck's ZDG [17]. Their goal was to give a more precise explanation of the zero-divisor structure of the ring R. The ZDG $G(R)$ is defined as follows: Two unique vertices a and b are adjacent to one another if and only if $ab = 0$. This is called as the collection of non-zero zero-divisors of R. Singh and Bhat presented a comprehensive survey on ZDGs of finite commutative rings, summarizing major developments and open problems in the area [18]. In another work, Singh and Bhat studied the adjacency matrix and Wiener index of the ZDGs $\Gamma(\mathbb{Z}_n)$ and analyzed their graph-theoretic properties [19]. They further examined graph invariants of the line graph associated with the ZDG of $\mathbb{Z}_n$ [20].

Redmond generalized the idea of a ZDG to include rings that are not necessarily commutative [21]. He established that if the left and right zero-divisor sets are identical, then $G(R)$ is connected for the non-commutative ring R. DeMeyer and colleagues [22] extended the notion of ZDGs of rings to semi-groups. The ZDGs for modules over commutative rings are then given by Behboodi [23]. Sharma and Bhat investigated the fault-tolerant metric dimension of zero-divisor graphs associated with commutative rings and established several structural results related to resolving sets [24]. More recently, Hanna, Alkandari, and Bhat explored the fault-tolerant metric dimension of ZDG arising from upper triangular matrices and discussed their applications [25]. These works provide significant background and motivation for the present study.

Recent advancements in combinatorial design have introduced novel methods for graph decompositions and their applications in secure communications. Algorithmic approaches have been developed to decompose circulant graphs for use in network topologies, while the construction of mutually orthogonal graph squares has provided a foundation for new graph-authentication codes and transversal designs. Furthermore, these techniques extend to the generation of partially balanced network designs, which enhance error-correction capabilities in graph-based coding schemes. While previous research has focused on the metric dimensions of ZDGs for commutative rings, this study addresses the structural hurdles of non-commutativity and high-dimensional vertex sets. We establish generalized bounds for $M_n(R)$ by utilizing matrix-theoretic properties such as nullity and row-space equivalence, which are not applicable in the commutative or trivial cases found in existing literature.

The study of ZDGs provides a powerful bridge between ring theory and graph theory by translating algebraic properties into combinatorial structures. While zero-divisor graphs of commutative rings have been extensively investigated, extending this concept to matrix rings, particularly upper triangular matrices, introduces a richer and more complex non-commutative setting. The motivation for studying zero-divisor graphs of matrices lies in extending the interplay between algebra and graph theory to non-commutative settings. Matrix rings, especially upper triangular ones, possess rich zero-divisor structures, and their graphical representation helps uncover algebraic properties through combinatorial analysis while enabling the study of graph invariants in a broader algebraic context. In this manuscript, we consider a graph, known as an undirected ZDG of an upper triangular matrix M_3 (space of 3×3 matrices), over $\mathbb{Z}_2$ and determine various resolvability parameters, i.e., minimum resolving set, minimum edge resolving set, metric dimension, and edge metric dimension for it.

Structure of the manuscript: Section 2 discusses the basic definition, previously obtained results, and theory regarding the notion of metric dimension and resolving set. In Sections 3 and 4, we investigate the minimum resolving set, minimum edge resolving set, metric dimension, and edge metric dimension of the ZDG arising from the upper triangular matrix ring over the field of order 2, denoted by $ZDG[M_3(\mathbb{Z}_2)]$. Finally, we conclude this manuscript with highlighting the key findings of the paper and shedding a light on possible future directions.

2. Materials and methods

In this section, we review the basic concepts required to establish our main results. In particular, we recall the definitions of the resolving set, minimum resolving set, independent set, and metric dimension, along with other related terminology.

2.1. Definitions

Suppose $G = G(V, E)$ is a graph with size $|E|$ and order $|V|$. The concept of distance is very important in graphs, and is defined between two vertices r_1 and r_2 as the length of the shortest path between them, and is denoted by $d(r_1, r_2)$. The degree of a vertex r in G, is the number of edges connecting it, and it is represented by d_r. The minimum and maximum degrees of G are, respectively, denoted by $\delta(G)$ and $\Delta(G)$ and are defined by

$$\delta(G) = \min\{d_r | r \in V(G)\}$$

and

$$\Delta(G) = \max\{d_r | r \in V(G)\}.$$

A subset $R = \{y_i \mid 1 \leq i \leq l\}$ of distinct vertices from $V(G)$ is said to be a resolving set in G if for any two vertices $p, q \in V(G)$, there exists a vertex $y_i \in R$ such that

$$d(p, y_i) \neq d(q, y_i).$$

Such a set R with minimum cardinality is called a minimum resolving set of G, and its cardinality is known as the **metric dimension** of G, denoted by $\dim_v(G)$.

For a minimum resolving set R, the representation of a vertex $k \in V(G)$ is defined as

$$r(k \mid R) = (d(y_1, k), d(y_2, k), \ldots, d(y_l, k)).$$

A subset $R_e = \{y_i \mid 1 \leq i \leq l\}$ of $V(G)$ is said to be an edge resolving set in G if for any two edges $p, q \in E(G)$, there exists a vertex $y_i \in R_e$ such that

$$d(p, y_i) \neq d(q, y_i),$$

where for an edge $p = ab$, the distance is defined by

$$d(p, y_i) = \min\{d(a, y_i), d(b, y_i)\}.$$

For example. Consider a planar graph G with vertices

$$V = \{v_1, \ldots, v_9\}$$

and edges

$$E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_2v_7, v_4v_8, v_6v_9\}.$$

Metric dimension. We claim that $dim_v(G) = 3$ with the resolving set $S = \{v_1, v_3, v_5\}$.

Distance vectors with respect to S:

v_1	(0,2,2)	v_2	(1,1,3)	v_3	(2,0,2)	v_4	(3,1,1)	v_9	(2,4,1)
v_5	(2,2,0)	v_6	(1,3,1)	v_7	(2,2,4)	v_8	(4,2,1)		

Since all vertex representations are distinct, it follows that $dim_v(G) = 3$.

Edge metric dimension. We claim that $edim_e(G) = 4$ with the edge-resolving set

$$S_e = \{v_1, v_2, v_3, v_5\}.$$

The distance representations of edges with respect to S_e are given below:

v_1v_2	(0,1,2,0)	v_2v_3	(1,0,2,0)	v_3v_4	(2,0,1,1)	v_4v_5	(2,1,0,2)	v_5v_6	(1,2,0,2)
v_6v_1	(0,2,0,1)	v_2v_7	(1,2,3,0)	v_4v_8	(3,1,0,2)	v_6v_9	(1,3,0,2)		

It is observed that all edge representations are distinct. Hence, $edim_e(G) = 4$.

Therefore, for this nine vertex planar graph, we obtain

$$\boxed{dim_v(G) = 3 \quad \text{and} \quad edim_e(G) = 4.}$$

This example demonstrates a graph where metric dimension and edge metric dimension differ suitably for illustrating this property in graph theory and network analysis.

2.2. Construction of undirected ZDG of an upper triangular matrix

We now construct the ZDG of the upper triangular matrix ring $M_3(R)$ over the finite field of order 2. Next, the general form of an upper triangular matrices in M_3, the space of 3×3 matrices is as follows:

$$A = \begin{pmatrix} p & q & r \\ 0 & s & t \\ 0 & 0 & u \end{pmatrix},$$

where p, q, r, s, t, and u are the elements of any field (finite or infinite). In such a case, p, q, r, s, t, and u can each independently choose any value from the considered field. However, we may calculate the finite number of possible upper triangular matrices in M_3 if we need to determine the number of matrices that can be constructed given a specific range of values of such integer modulo n. For example, every element can separately take two values if we limit p, q, r, s, t, and u to being in the set $\{0, 1\}$ (i.e., modulo 2). Therefore, the possible number of upper triangular matrices in this situation are listed below:

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_7 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_8 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_9 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{10} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{14} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{15} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{16} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{17} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{18} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{19} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{22} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{24} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{25} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_{26} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{27} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{28} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{29} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{30} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{31} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{32} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{33} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{34} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{35} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{36} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{37} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{38} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, vX_{39} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{40} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{41} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{42} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{43} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{44} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{45} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{46} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{47} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{48} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{49} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{50} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{51} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{52} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{53} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{54} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{55} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_{56} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X_{57} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{58} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{59} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{60} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{61} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ X_{62} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ X_{63} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the ring $M_3(\mathbb{Z}_2)$ contains 64 elements, the resulting ZDG is highly complex. To aid understanding, we present a labeled graphical representation (see Figure 1), where adjacency is verified using the condition $XY = YX = 0$. Furthermore, structural features such as pendant vertices and symmetric vertex pairs help describe the graph more clearly. From the list above, one can find that there are only 63 possible upper triangular matrices M_3, if we consider a finite field of order two, i.e., $\mathbb{Z}_2 = \{0, 1\}$. Now, interestingly one can see that, if we apply the definition of ZDGs [16] on these upper triangular matrices, we may get two types of ZDGs: (i) directed graphs, where $x.y = 0$ but $y.x \neq 0$, where $x, y \neq 0$, and (ii) undirected graphs, where $x.y = 0$ and also $y.x = 0$, where $x, y \neq 0$. In matrix rings, it is possible that $xy = 0$ while $yx \neq 0$. Such pairs are included in the directed ZDG, where a directed edge from x to y is defined by $xy = 0$. However, for the undirected ZDG, we only consider pairs satisfying $xy = yx = 0$.

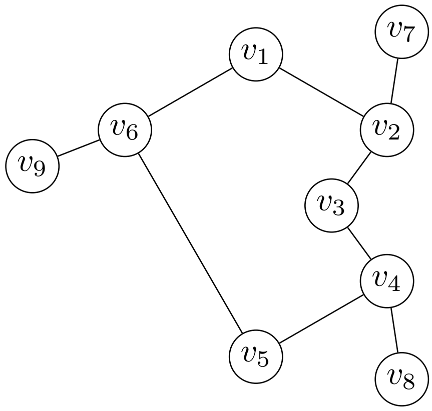

Figure 1. Planar graph G.

In this work, we consider the ZDG of the ring of 3×3 upper triangular matrices over $\mathbb{Z}_2$, denoted by $M_3(\mathbb{Z}_2)$. Since this ring is non-commutative, we distinguish between left and right zero-divisors. An element $X \neq 0$ is called a left (respectively, right) zero-divisor if there exists a non-zero matrix Y such that $XY = 0$ (respectively, $YX = 0$). The vertex set of the ZDG consists of all non-zero matrices in $M_3(\mathbb{Z}_2)$ that are either left or right zero-divisors. For the undirected (commutative) version of the graph, two distinct vertices X and Y are adjacent if and only if $XY = YX = 0$. For the directed (non-commutative) version, there is a directed edge from X to Y if $XY = 0$. In this work, we concentrate on an undirected graph, i.e., undirected ZDG of an upper triangular matrix M_3.

2.3. *Undirected ZDG of an upper triangular matrix $ZDG[M_3(\mathbb{Z}_2)]$*

We consider the undirected graph with vertex set $\{X_i \mid 1 \leq i \leq 27\}$. The possible number of upper triangular matrix in $M_3(\mathbb{Z}_2)$, that were described above are $X_1, X_2, \ldots, X_{63}$. We now construct the undirected ZDG of the upper triangular matrix ring, denoted by $ZDG[M_3(\mathbb{Z}_2)]$.

The graph $ZDG[M_3(\mathbb{Z}_2)]$ is generated in such way, that if there is an edge between X_1 and X_2, then $X_1 \cdot X_2 = 0$ and $X_2 \cdot X_1 = 0$. Therefore, whenever two vertices are joined by an edge, their product is zero, meaning that they correspond to zero-divisors. By linking all such zero-divisors, we obtain the ZDG. In the case of upper triangular matrices, the elements $X_{28}, X_{29}, \ldots, X_{63}$ do not form any pairs (X_i, X_j) satisfying $X_iX_j = X_jX_i = 0$. Hence, they are not adjacent to any vertex and are excluded from the graph. Consequently, the undirected ZDG of the upper triangular matrix, denoted by $ZDG[M_3(\mathbb{Z}_2)]$, is shown in Figure 2.

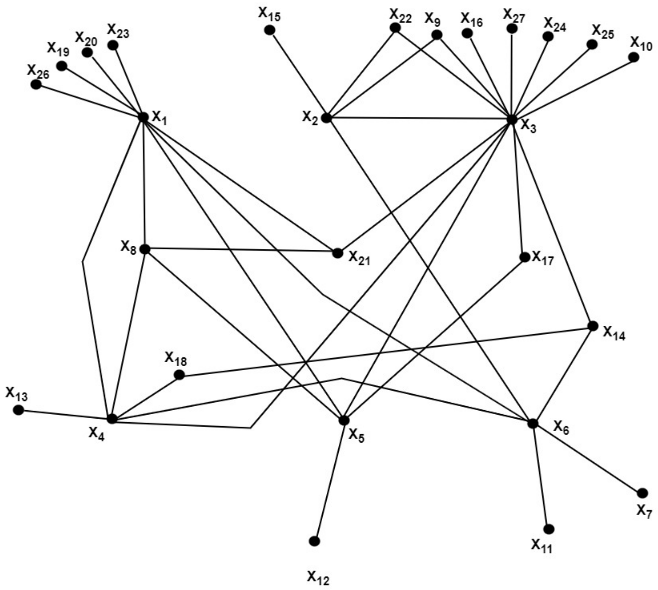

Figure 2. ZDG of an upper triangular matrix $ZDG[M_3(\mathbb{Z}_2)]$.

3. Results

The metric dimension and edge metric dimension of the zero-divisor graphs connected to $M_2(\mathbb{Z}_2)$ and $M_3(\mathbb{Z}_2)$ are investigated in this section. Using the relevant lemmas and results, we find the minimum resolving set, minimum edge resolving set, and associated metric dimensions for $ZDG[M_3(\mathbb{Z}_2)]$.

Theorem 1. Let $R = M_2(\mathbb{Z}_2)$ and consider its undirected ZDG defined by the condition $XY = YX = 0$. If there are no distinct nonzero matrices $X, Y \in R$ satisfying $XY = YX = 0$, then the graph contains no edges and is totally disconnected. Consequently, the metric dimension and edge metric dimension are not defined for this graph.

Proof. In the undirected ZDG associated with the ring $M_2(\mathbb{Z}_2)$, the vertex set consists of all nonzero zero-divisors, and two distinct vertices X and Y are adjacent if and only if $XY = YX = 0$. Since $M_2(\mathbb{Z}_2)$ is a non-commutative ring, this condition is highly restrictive. Suppose that there are no distinct nonzero matrices $X, Y \in M_2(\mathbb{Z}_2)$ such that $XY = YX = 0$. Then, by the definition of adjacency, no edges can exist in the graph.

Thus, the graph consists entirely of isolated vertices and is therefore totally disconnected. In such a graph, there are no paths between distinct vertices, and hence distances between vertices are not well-defined. These parameters are not significant in this situation because the ideas of metric dimension and edge metric dimension depend on the presence of finite shortest-path distances in a connected graph. □

Lemma 1. More than half of pendant vertices must belong to a resolving set R for $ZDG[M_3(\mathbb{Z}_2)]$.

Proof. Suppose R is a resolving set for $ZDG[M_3(\mathbb{Z}_2)]$. From Figure 1, it is clear that the total number of pendant vertices in $ZDG[M_3(\mathbb{Z}_2)]$ are equal to 14. We have to prove that more than seven pendant vertices of $ZDG[M_3(\mathbb{Z}_2)]$, belong to the set R. Suppose, on the contrary, that only six (or less than or equal to six) pendant vertices are lying in R. Then, from Figure 1, one can clearly find that from remaining pendant vertices, say p and q, we have $r(p|R) = r(q|R)$, a contradiction. Thus, it is clear that more than half of the pendant vertices of $ZDG[M_3(\mathbb{Z}_2)]$ must belong to the resolving set R. □

Theorem 2. For the ZDG of upper triangular matrix $ZDG[M_3(\mathbb{Z}_2)]$, we find that

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) = 11.$$

Proof. First, we show that the graph $ZDG[M_3(\mathbb{Z}_2)]$ admits a resolving set of cardinality 11. Suppose that $R \subseteq V(ZDG[M_3(\mathbb{Z}_2)])$, where $R = \{x_{19}, x_{20}, x_{23}, x_{22}, x_{27}, x_{24}, x_{25}, x_{10}, x_{11}, x_{18}, x_5\}$. Next, to verify that R is a resolving set for $ZDG[M_3(\mathbb{Z}_2)]$, we compute the vertex representations of all vertices in $V(ZDG[M_3(\mathbb{Z}_2)])$ with respect to R. For the vertices $\{x_t : 1 \le t \le 27\}$ of $ZDG[M_3(\mathbb{Z}_2)]$, vertex codes are given below in Table 1.

Table 1. Vertex codes for vertices of $ZDG[M_3(\mathbb{Z}_2)]$.

x_1	(1,1,1,3,3,3,3,3,2,2,1)	x_{15}	(4,4,4,2,3,3,3,3,3,4,3)
x_2	(3,3,3,1,2,2,2,2,2,3,2)	x_{16}	(4,4,4,2,2,2,2,2,4,3,2)
x_3	(3,3,3,1,1,1,1,1,3,2,1)	x_{17}	(3,3,3,2,2,2,2,2,4,3,1)
x_4	(2,2,2,2,2,2,2,2,2,1,2)	x_{18}	(3,3,3,3,3,3,3,3,3,0,3)
x_5	(2,2,2,2,2,2,2,2,3,3,0)	x_{19}	(0,2,2,4,4,4,4,4,3,3,2)
x_6	(2,2,2,2,3,3,3,3,1,2,2)	x_{20}	(2,0,2,4,4,4,4,4,3,3,2)
x_7	(3,3,3,3,4,4,4,4,2,3,3)	x_{21}	(2,2,2,2,2,2,2,2,3,3,2)
x_8	(2,2,2,3,3,3,3,3,3,2,1)	x_{22}	(4,4,4,0,2,2,2,2,3,3,2)
x_9	(4,4,4,2,2,2,2,2,4,3,2)	x_{23}	(2,2,0,4,4,4,4,4,3,3,2)
x_{10}	(4,4,4,2,2,2,2,0,4,3,2)	x_{24}	(4,4,4,2,2,0,2,2,4,3,2)
x_{11}	(3,3,3,3,4,4,4,4,0,3,3)	x_{25}	(4,4,4,2,2,2,0,2,4,3,2)
x_{12}	(3,3,3,3,3,3,3,3,4,4,1)	x_{26}	(2,2,2,4,4,4,4,4,3,3,2)
x_{13}	(3,3,3,3,3,3,3,3,3,2,3)	x_{27}	(4,4,4,2,0,2,2,2,4,3,2)
x_{14}	(3,3,3,2,2,2,2,2,2,1,2)		

From the vertex codes listed above for the graph $ZDG[M_3(\mathbb{Z}_2)]$, we find that for any two vertices in $ZDG[M_3(\mathbb{Z}_2)]$, say p and q, we have $r(p|R) \neq r(q|R)$. This proves that the set R is a resolving set for $ZDG[M_3(\mathbb{Z}_2)]$, and hence we conclude that

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) \le 11.$$

Next, to achieve the other inequality, i.e.,

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) \ge 11,$$

we prove that any set R with cardinality less than 11 can never be a resolving set for $ZDG[M_3(\mathbb{Z}_2)]$. To establish this, we construct a set R such that its minimum cardinality is at least 11. We now discuss the choice of vertices to be included in R so that it forms a resolving set for $ZDG[M_3(\mathbb{Z}_2)]$. It is clear that, among the four pendant vertices, namely, x_{19}, x_{20}, x_{23}, and x_{26}, at least three must belong to the set R. Without loss of generality, we choose x_{19}, x_{20}, and x_{23} to be included in R. Suppose that if we take only two vertices in R from these four vertices (say x_{19} and x_{20}), then the remaining two vertices

(x_{23} and x_{26}) have the same vertex code (i.e., $r(x_{23}|R) = r(x_{23}|R)$) with respect to the set R, which is a contradiction to the set R to become resolving set for $ZDG[M_3(\mathbb{Z}_2)]$.

A similar claim applies to the five pendant vertices, namely, x_{10}, x_{16}, x_{24}, x_{25}, and x_{27} on the vertex x_3. Out of these five vertices, at least four must belong to the set R. In particular, we include x_{10}, x_{24}, x_{25}, and x_{27} in R. Furthermore, among the two pendant vertices x_7 and x_{11} adjacent to the vertex x_6, at least one must belong to R. Without loss of generality, we include x_{11} in R. Next, from Figure 1, we observe that the vertices x_9 and x_{22} have identical distances to all other vertices of $ZDG[M_3(\mathbb{Z}_2)]$. Therefore, at least one of them must belong to R; we choose $x_{22} \in R$. Similarly, the pairs of vertices $\{x_{13}, x_{18}\}$ and $\{x_5, x_{21}\}$ have identical distance representations with respect to the current set R. Hence, one vertex from each pair must be included in R. We choose x_{18} and x_5 to be in R. Therefore, the set R consists of a minimum number of 11 vertices, i.e., $R = \{x_{19}, x_{20}, x_{23}, x_{22}, x_{27}, x_{24}, x_{25}, x_{10}, x_{11}, x_{18}, x_5\}$, and so, we find that

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) \geq 11.$$

We establish the metric dimension $\dim_v(ZDG[M_3(\mathbb{Z}_2)])$ by proving both the upper and lower bounds. First, we construct an explicit resolving set R and show that it uniquely distinguishes all vertices of $ZDG[M_3(\mathbb{Z}_2)]$, which gives

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) \geq 11.$$

Next, we prove that no resolving set of smaller cardinality exists by analyzing structural properties of the graph such as pendant vertices and pairs of vertices with identical distance representations. These arguments ensure that at least 11 vertices are required, yielding

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) \geq 11,$$

and we conclude that

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) \geq 11.$$

Hence,

$$dim_v(ZDG[M_3(\mathbb{Z}_2)]) = 11,$$

which proves the result. □

Theorem 3. For the ZDG of upper triangular matrix $ZDG[M_3(\mathbb{Z}_2)]$, the diameter of the graph is $\text{diam}(ZDG[M_3(\mathbb{Z}_2)]) = 4$.

Proof. The diameter of a graph G, denoted $\text{diam}(G)$, is defined as the maximum distance between any pair of vertices in $V(G)$. We proceed with the proof in two steps:

(1) Establishing the lower bound (diam(G) ≥ 4)

We consider the vertex codes $r(x_i|R)$ provided in the proof for the metric dimension, where $R = \{r_1, r_2, \ldots, r_{11}\}$. The j-th entry in the code $r(x_i|R)$ represents the shortest path distance $d(x_i, r_j)$.

1) From the vertex Codes (Table 1), we inspect the code for vertex x_9:

$$r(x_9) = (4, 4, 4, 2, 2, 2, 2, 2, 4, 3, 2).$$

2) The maximum value observed in this code is 4.

3) Thus, there exists at least one pair of vertices, $\{x_9, x_{19}\}$ such that the shortest distance between them is 4, i.e., $d(x_9, x_{19}) = 4$.
4) By the definition of diameter, the existence of a path of length 4 implies:

$$\text{diam}(ZDG[M_3(\mathbb{Z}_2)]) \geq 4.$$

(2) Establishing the upper bound (diam(G) ≤ 4)

1) The largest distance value observed in all 27 vertex codes (which cover distances from every vertex x_i to the 11-element resolving set R) is 4.
2) We assume that 4 is the maximum distance between any two vertices in the graph. This is a standard inference drawn from the calculated distances to a resolving set in this context.
3) Therefore, no distance greater than 4 exists between any pair of vertices in the graph:

$$\text{diam}(ZDG[M_3(\mathbb{Z}_2)]) \leq 4.$$

By combining the lower bound ($\text{diam}(ZDG[M_3(\mathbb{Z}_2)]) \geq 4$) and the upper bound ($\text{diam}(ZDG[M_3(\mathbb{Z}_2)]) \leq 4$), we conclude that the diameter of the graph is exactly 4.

$$\text{diam}(ZDG[M_3(\mathbb{Z}_2)]) = 4.$$

This completes the proof. □

Remark 1. Since the considered ZDG has diameter 4, the distances $d(b, c)$ used in constructing resolving sets are correspondingly bounded. In our analysis, distances are computed using standard shortest-path definitions in the graph. Despite this bounded diameter, distinct vertices (or edges) can still be uniquely identified through their distance representations with respect to a carefully chosen resolving set. Thus, the diameter does not affect the validity of the resolving process.

Theorem 4. For the ZDG of upper triangular matrix $ZDG[M_3(\mathbb{Z}_2)]$, we find that

$$edim_e(ZDG[M_3(\mathbb{Z}_2)]) \leq 13.$$

Proof. First, we show that the graph $ZDG[M_3(\mathbb{Z}_2)]$ has an edge resolving set of size 13. Suppose that

$$R_e \subseteq V(ZDG[M_3(\mathbb{Z}_2)]),$$

where $R = \{x_{19}, x_{20}, x_{23}, x_{22}, x_{27}, x_{24}, x_{25}, x_{10}, x_{11}, x_{18}, x_{17}, x_8, x_9\}$. Next, to verify that R_e is an edge resolving set for $ZDG[M_3(\mathbb{Z}_2)]$, we compute the edge representations of all edges in $E(ZDG[M_3(\mathbb{Z}_2)])$ with respect to R_e. For the edges of $ZDG[M_3(\mathbb{Z}_2)]$, edge codes are given below in Table 2.

Table 2. Edge codes for edges of $ZDG[M_3(\mathbb{Z}_2)]$.

x_1x_4	(1,1,1,2,2,2,2,2,2,1,2,1,2)	x_3x_{16}	(3,3,3,1,1,1,1,1,3,2,1,2,1)
x_1x_5	(1,1,1,2,2,2,2,2,2,2,1,1,2)	x_3x_{17}	(3,3,3,1,1,1,1,1,3,2,0,2,1)
x_1x_6	(1,1,1,2,3,3,3,3,1,2,3,1,2)	x_3x_{21}	(2,2,2,1,1,1,1,1,3,2,1,1,1)
x_1x_8	(1,1,1,3,3,3,3,3,2,2,2,0,3)	x_3x_{22}	(3,3,3,0,1,1,1,1,3,2,1,2,1)
x_1x_{19}	(0,1,1,3,3,3,3,3,2,2,2,1,3)	x_3x_{24}	(3,3,3,1,1,0,1,1,3,2,1,2,1)
x_1x_{20}	(1,0,1,3,3,3,3,3,2,2,2,1,3)	x_3x_{25}	(3,3,3,1,1,1,0,1,2,2,1,2,1)
x_1x_{21}	(1,1,1,2,2,2,2,2,2,2,2,1,2)	x_3x_{27}	(3,3,3,1,0,1,1,1,3,2,1,2,1)
x_1x_{23}	(1,1,0,3,3,3,3,3,2,2,2,1,3)	x_4x_6	(2,2,2,2,2,2,2,2,1,1,2,1,2)
x_1x_{26}	(1,1,1,3,3,3,3,3,2,2,2,1,3)	x_4x_8	(2,2,2,2,2,2,2,2,2,1,2,0,2)
x_2x_3	(3,3,3,1,1,1,1,1,2,2,1,2,1)	x_4x_{13}	(2,2,2,2,2,2,2,2,2,1,2,1,2)
x_2x_6	(2,2,2,1,2,2,2,2,1,2,2,2,1)	x_4x_{18}	(2,2,2,2,2,2,2,2,2,0,2,1,2)
x_2x_9	(3,3,3,1,2,2,2,2,2,3,2,3,0)	x_5x_8	(2,2,2,2,2,2,2,2,3,2,1,0,2)
x_2x_{15}	(3,3,3,1,2,2,2,2,2,3,2,3,1)	x_5x_{12}	(2,2,2,2,2,2,2,2,3,3,1,1,2)
x_2x_{22}	(3,3,3,0,2,2,2,2,2,3,2,3,1)	x_5x_{17}	(2,2,2,2,2,2,2,2,3,3,0,1,2)
x_3x_4	(2,2,2,1,1,1,1,1,2,1,1,1,1)	x_6x_7	(2,2,2,2,3,3,3,3,1,2,3,2,2)
x_3x_5	(2,2,2,1,1,1,1,1,3,2,1,2,1)	x_6x_{11}	(2,2,2,2,3,3,3,3,0,2,3,2,2)
x_3x_9	(3,3,3,1,1,1,1,1,3,2,1,2,0)	x_6x_{14}	(2,2,2,2,2,2,2,2,1,1,2,2,2)
x_3x_{10}	(3,3,3,1,1,1,1,0,3,2,2,2,1)	x_8x_{21}	(2,2,2,2,2,2,2,2,3,2,2,0,2)
x_3x_{14}	(3,3,3,1,1,1,1,1,2,1,1,2,1)	$x_{14}x_{18}$	(3,3,3,2,2,2,2,2,2,0,2,2,2)

From the edge codes listed above for the graph $ZDG[M_3(\mathbb{Z}_2)]$, we find that for any two edges in $ZDG[M_3(\mathbb{Z}_2)]$, say p and q, we have $r_e(p|R_e) \neq r_e(q|R_e)$. This proves that the set R_e is an edge resolving set for $ZDG[M_3(\mathbb{Z}_2)]$, and hence we conclude that

$$edim_e(ZDG[M_3(\mathbb{Z}_2)]) \leq 13.$$

Next, to achieve the other inequality, i.e.,

$$edim_e(ZDG[M_3(\mathbb{Z}_2)]) \geq 13,$$

we prove that any set R_e with cardinality less than 13 can never be an edge resolving set for $ZDG[M_3(\mathbb{Z}_2)]$. To establish this, we construct a set R_e such that its minimum cardinality is at least 13. We now discuss the selection of vertices to be included in R_e so that it forms an edge resolving set for $ZDG[M_3(\mathbb{Z}_2)]$. It is clear that, among the four pendant vertices, namely, x_{19}, x_{20}, x_{23}, and x_{26}, at least three must belong to the set R_e. Without loss of generality, we include x_{19}, x_{20}, and x_{23} in R_e. Suppose if we take only two vertices in R_e from these four vertices (say x_{19} and x_{20}), then the remaining two vertices (x_{23} and x_{26}) have the same edge code (i.e., $r_e(x_{23}|R_e) = r_e(x_{23}|R_e)$) with respect to the set R_e, which is a contradiction to the set R_e to become an edge resolving for $ZDG[M_3(\mathbb{Z}_2)]$.

A similar claim applies to the five pendant vertices, namely, x_{10}, x_{16}, x_{24}, x_{25}, and x_{27} on the vertex x_3. Out of these five vertices, four must belong to the set R_e, i.e., x_{10}, x_{24}, x_{25}, and x_{27} belong to R_e. Again, among the two pendant vertices x_7 and x_{11} adjacent to the vertex x_6, at least one must belong to the set R_e. Without loss of generality, we include x_{11} in R_e. Next, from Figure 1, we find that the distance of the three edges $\{x_9x_3, x_{16}x_3, x_{22}x_3\}$ from any other vertex of $ZDG[M_3(\mathbb{Z}_2)]$ is the same, i.e., the distance of these three edges is same from the collected set of vertices

$$R_e^* = \{x_{19}, x_{20}, x_{23}, x_{10}, x_{24}, x_{25}, x_{27}, x_{11}\}.$$

Therefore, R_e^* is not an edge resolving for $ZDG[M_3(\mathbb{Z}_2)]$. To make it resolving for those three edges, we need to add at least two vertices in it. Thus, we include x_9 and x_{22} in the set R_e^*, that is,

$$R_e^1 = R_e^* \cup \{x_9, x_{22}\}.$$

Similarly, we again find that the distance of the pair of two edges $\{x_{13}x_4, x_{18}x_4\}$, $\{x_{26}x_1, x_8x_1\}$, $\{x_{16}x_3, x_{17}x_3\}$, and $\{x_5x_8, x_{21}x_8\}$ from the collected set of vertices R_e^1 is the same. Therefore, to resolve these pair of edges, we need to add at least three more vertices in it. Thus, we include x_{18}, x_{17}, and x_8 in the set R_e^1. Therefore, the set $R_e = R_e^1 \cup \{x_{18}, x_{17}, x_8\}$ is the minimum set of vertices, capable of resolving all the edges present in $ZDG[M_3(\mathbb{Z}_2)]$, i.e.,

$$R_e = \{x_{19}, x_{20}, x_{23}, x_{22}, x_{27}, x_{24}, x_{25}, x_{10}, x_{11}, x_{18}, x_{17}, x_8, x_9\}$$

is a minimum edge resolving set for $ZDG[M_3(\mathbb{Z}_2)]$. Therefore, we have concluded that

$$edim_e(ZDG[M_3(\mathbb{Z}_2)]) \geq 13.$$

Hence,

$$edim_e(ZDG[M_3(\mathbb{Z}_2)]) = 13,$$

which proves the result. □

Remark 2. For the graph $G = ZDG[M_3(\mathbb{Z}_2)]$, we obtain $\dim_v(G) = 11$ and $edim_e(G) = 13$. Hence, the vertex metric dimension and edge metric dimension do not coincide. This difference arises because edges, being pairs of vertices, involve more complex distance relationships, and therefore require a larger resolving set for unique identification.

Remark 3. In this work, we focus on the case of 3×3 upper triangular matrices over $\mathbb{Z}_2$. Although an explicit general formula for the metric and edge metric dimensions of $n \times n$ upper triangular matrices is not derived, the observed structure suggests that these parameters depend on the number of zero-divisors, pendant vertices, and symmetric vertex pairs, which grow with n. This indicates a potential pattern for larger values of n. A systematic investigation of the scalability and formulation of general results for arbitrary n remains an interesting direction for future research.

4. Conclusions

The selection of a minimal number of vertices in a graph to uniquely identify all vertices or edges is an interesting problem. In this article, we considered the undirected ZDG of the upper triangular matrix ring $ZDG[M_3(\mathbb{Z}_2)]$ and investigated its minimum edge resolving set, minimum resolving set, metric dimension, and edge metric dimension. We proved that this graph has constant and bounded metric and edge metric dimensions. A deeper investigation of the automorphism group of this graph could provide further insight into its structural properties and is left for future work.

In future work, one may explore other variants of metric dimension, such as strong metric, edge metric, fractional metric, mixed metric, fault-tolerant, local metric, partition, and edge partition dimensions for the algebraic structure $ZDG[M_3(\mathbb{Z}_2)]$.

Author contributions

Sahil Sharma: wrote the first draft, figures; Omaima Al Shanqiti: figures, validated results; Vijay Kumar Bhat: wrote the first draft, validated results, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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