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Research article

Study of modified Navier–Stokes equations in Fourier spaces

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Abstract: In this paper, we study the modified Navier–Stokes equations with a damping term, assuming that the initial data u^0 satisfies $\widehat{u^0} \in L^1(\mathbb{R}^3)$. We establish the existence and uniqueness of a local solution, together with a blow-up criterion in the case where the maximal time of existence is finite. The analysis relies on standard techniques from Fourier analysis.

Keywords: Navier–Stokes equations; global solution; local solution; Lei–Lin–Gevrey spaces
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1. Introduction and principal results

The mathematical analysis of fluid equations, particularly the three-dimensional Navier–Stokes system, remains a fundamental topic in nonlinear partial differential equations due to its complex interplay between nonlinearity and dissipation. In recent years, modified models incorporating damping terms have attracted increasing attention, as they provide additional mechanisms that may influence the regularity and long-time behavior of solutions. In particular, nonlinear damping terms of the form $\alpha|u|^2u$ arise naturally in various physical contexts and are known to enhance the dissipative effects of the system.

Motivated by these considerations, it is of interest to investigate such models in critical functional frameworks, where the balance between scaling and regularity plays a crucial role.

From a physical point of view, damping terms of the form $\alpha|u|^2u$ arise in various models of dissipative fluid flows and are known to enhance stability and energy dissipation mechanisms. Such terms appear in the modeling of porous media flows and in certain turbulence models, where nonlinear damping plays a significant role in controlling the behavior of solutions. This further justifies the mathematical analysis of the system considered in the present work.

In this paper, we consider the following modified Navier–Stokes equations:

$$(NSED)\begin{cases} \partial_t u - \nu\Delta u + u.\nabla u + \alpha|u|^2 u = -\nabla p \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0,x) = u^0(x) \text{ in } \mathbb{R}^3, \end{cases}$$

where $\nu > 0$ denotes the viscosity of the fluid, $u = (u_1, u_2, u_3)$ is the unknown velocity field, and p is the associated pressure at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. The term $\alpha|u|^2 u$, where $\alpha > 0$, represents the damping effect. To simplify the analysis, we assume throughout that $\nu = 1$. The function u^0 is a given initial velocity field. If u^0 is sufficiently regular, the divergence-free condition determines the pressure p. More precisely, under this regularity assumption, the Duhamel formula holds:

$$u = e^{t\Delta}u^0 - \mathcal{N}_1(u) - \alpha\mathcal{N}_2(u),$$

where

$$\begin{aligned} \mathcal{N}_1(u) &= \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\operatorname{div}(u \otimes u)\,d\tau = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\,(u.\nabla u)\,d\tau, \\ \mathcal{N}_2(u) &= \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\left(|u|^2 u\right)d\tau, \end{aligned}$$

and $\mathbb{P}$ denotes the Leray projector.
It is well known that, in [1], the authors introduced and studied the following system:

$$(P_\beta)\begin{cases} \partial_t u - \Delta u + u.\nabla u + \alpha|u|^{\beta-1} u = -\nabla p \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0,x) = u^0(x) \text{ in } \mathbb{R}^3. \end{cases}$$

In particular, they established the global existence of a weak solution for $\beta \geq 1$ with initial data $u^0 \in L^2(\mathbb{R}^3)$. Moreover, they proved the global existence of a strong solution for $\beta \geq 7/2$, as well as uniqueness for $7/2 \leq \beta \leq 5$ when $u^0 \in H^1(\mathbb{R}^3)$. The methods used in [1] are based on Galerkin approximation. In [2], the authors proved the existence of a global strong solution and established uniqueness for $\alpha \geq 1/2$ and $\beta = 3$. Moreover, several authors have investigated more general forms of evolution equations with damping than the problem $(NSED)$; see, for instance, [3–5].

Several works have investigated Navier–Stokes equations using Fourier-based approaches and critical functional frameworks. In particular, the spaces introduced by Lei and Lin [6] play a fundamental role in the analysis of such problems, and the monograph of Cannone [7] provides a comprehensive framework for Fourier analysis methods in fluid dynamics.

In the presence of damping terms, various results on regularity and uniqueness have been obtained; see, for instance, Zhou [8] and Zhang et al. [9]. More recently, the asymptotic behavior and long-time dynamics of damped Navier–Stokes equations have been studied in [10], highlighting the interplay between dissipation and nonlinear effects.

In these works, the analysis is mainly based on Friedrichs-type frequency approximations and standard analytical tools, sometimes combined with additional techniques.

Our objective in this work is to study the critical case $\beta = 3$ under a fairly regular initial condition:

$$(*) \qquad u^0 \in \mathcal{S}'(\mathbb{R}^3) \quad \text{and} \quad \widehat{u^0} \in L^1(\mathbb{R}^3).$$

It is easily seen that if u^0 satisfies Condition $(*)$, then u^0 is continuous, and

$$\lim_{|x|\to\infty} u^0(x) = 0.$$

More precisely, the system $(NSED)$ enjoys the following scaling property: If $u = u(t,x)$ is a solution of $(NSED)$ on $[0,T]$ with initial data $u(0,x) = u^0(x)$, then for any $\lambda > 0$, the rescaled function

$$u_\lambda(t,x) = \lambda\, u(\lambda^2 t, \lambda x)$$

is also a solution of $(NSED)$ on $[0, T/\lambda^2]$, with initial data

$$u_\lambda(0,x) = \lambda\, u^0(\lambda x).$$

The incompressible Navier–Stokes system with a damping term of the form $\alpha|u|^2u$ retains the same scaling invariance,

$$u(t,x) \mapsto \lambda\, u(\lambda^2 t, \lambda x), \qquad \lambda > 0,$$

and it therefore shares the same critical spaces as the classical incompressible Navier–Stokes equations (corresponding to $\alpha = 0$), namely $L^3(\mathbb{R}^3)$, $\dot{H}^{1/2}(\mathbb{R}^3)$, $\mathcal{X}^{-1}(\mathbb{R}^3)$, $\dot{C}^{-1}(\mathbb{R}^3)$, and others.

Moreover, blow-up criteria for fluid equations have been extensively studied in the literature. In particular, Chae [11] established important conditions related to the regularity of solutions to the three-dimensional Euler equations, and Li [12] derived critical regularity criteria for MHD-Boussinesq systems. These results are closely connected to the analysis of singularity formation and provide additional motivation for the blow-up criterion obtained in the present work.

The aim of this work is to study the problem $(NSED)$ in a critical Fourier framework, namely within the homogeneous family of functional spaces defined by

$$\mathcal{X}^\sigma(\mathbb{R}^3) = \{f \in S'(\mathbb{R}^3) :\ \widehat{f} \in L^1_{loc}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\xi|^\sigma |\widehat{f}(\xi)| d\xi < \infty\}.$$

It is clear that $\mathcal{X}^{-1}(\mathbb{R}^3)$ is the critical space for the problem $(NSED)$ within this family.

Although $\mathcal{X}^{-1}(\mathbb{R}^3)$ is the critical space associated with the scaling of the Navier–Stokes system, the analysis in this work is carried out in the space $\mathcal{X}^0(\mathbb{R}^3)$. This choice is motivated by the need to control the nonlinear damping term $\alpha|u|^2u$, which requires additional regularity at lower frequencies. Working in $\mathcal{X}^0(\mathbb{R}^3)$ ensures that the nonlinear estimates are well-defined and allows us to apply the fixed-point argument in a suitable functional framework.

The main difficulty lies in estimating the nonlinear term $\alpha|u|^2u$ in the space $\mathcal{X}^{-1}(\mathbb{R}^3)$, which cannot be handled easily using standard analytical techniques. Therefore, we require a higher regularity of the initial data at high frequencies. More precisely, we assume that $u^0 \in \mathcal{X}^0(\mathbb{R}^3)$ so that u^0 satisfies Condition $(*)$. It is worth recalling that this problem has already been studied by several authors (see, for example, [1, 13]). They showed that if the initial data belongs only to $L^2(\mathbb{R}^3)$, then there exists a unique global solution (for $\beta > 3$) in the space

$$C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)).$$

The construction of such solutions is based on the Friedrichs approximation method together with Cantor's diagonal extraction technique. Additional tools are then used to pass to the limit. The specific properties to problem $(NSED)$ associated with the space $L^2(\mathbb{R}^3)$ are

$$\int_{\mathbb{R}^3} u.\nabla u u = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} |u|^2 u u \geq 0,$$

which are not necessarily satisfied in space $\mathcal{X}^0(\mathbb{R}^3)$. Recall that the incompressible Navier-Stokes equations (corresponding to $\alpha = 0$) have been extensively studied in the critical space $\mathcal{X}^{-1}(\mathbb{R}^3)$; see, for instance, [14–16]. In this setting, local existence and uniqueness are known to hold, whereas the existence of a global solution remains an open problem.

In the present paper, we address the following issues: local existence, uniqueness, and blow-up criteria for maximal solutions, with the aim of characterizing their maximal time of existence. We now state our main results.

Theorem 1.1. *Let $u^0 \in \mathcal{X}^0(\mathbb{R}^3)$ be a divergence-free vector field. Then, there exists a time $T = T(u^0) > 0$ such that problem $(NSED)$ admits a unique solution*

$$u \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3)).$$

Our second result is the following.

Theorem 1.2. *Let $u^0 \in \mathcal{X}^0(\mathbb{R}^3)$ be a divergence-free vector field, and let*

$$u \in C([0, T^*), \mathcal{X}^0(\mathbb{R}^3)) \cap L^1_{loc}([0, T^*), \mathcal{X}^2(\mathbb{R}^3))$$

be the maximal solution of Problem $(NSED)$ given by Theorem 1.1. Then, for all $t \in [0, T^*)$, we have

$$\|u(t)\|_{\mathcal{X}^0} + \frac{1}{2}\int_0^t \|u(z)\|_{\mathcal{X}^2} dz \leq \|u^0\|_{\mathcal{X}^0} \exp\Big((\alpha + \frac{1}{2})\int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big). \tag{1.1}$$

If T^* is finite, then

$$\|u(t)\|_{\mathcal{X}^0} \geq \frac{(2\alpha + 1)^{-1/2}}{\sqrt{T^* - t}}, \ \forall t \in [0, T^*). \tag{1.2}$$

Remark 1.3. *(a) By using the inequality (1.1), we obtain that if u is bounded in $\mathcal{X}^0(\mathbb{R}^3)$ on $[0, T^*)$, and if $T^* < \infty$, then*

$$u \in L^1([0, T^*), \mathcal{X}^2(\mathbb{R}^3)).$$

(b) We have

$$\int_0^t \|u(z)\|_{\mathcal{X}^0}\|u(z)\|_{\mathcal{X}^1} dz$$
$$\leq \|u^0\|_{\mathcal{X}^0} \exp\Big((\alpha + \frac{1}{2})\int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big) + \frac{1}{2}\int_0^t \|u(z)\|^3_{\mathcal{X}^0} dz, \ \forall t \in [0, T^*).$$

Indeed: By using the fact that $\|u\|_{\mathcal{X}^1} \leq \|u\|^{1/2}_{\mathcal{X}^0}\|u\|^{1/2}_{\mathcal{X}^2}$ and the elementary inequality

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \ \forall a, b \geq 0,$$

we obtain

$$\|u(z)\|_{\mathcal{X}^0}\|u(z)\|_{\mathcal{X}^1} \leq \|u\|_{\mathcal{X}^0}^{3/2}\|u\|_{\mathcal{X}^2}^{1/2} \leq \frac{1}{2}\|u\|_{\mathcal{X}^0}^3 + \frac{1}{2}\|u\|_{\mathcal{X}^2}$$

and

$$\int_0^t \|u(z)\|_{\mathcal{X}^0}\|u(z)\|_{\mathcal{X}^1}dz \leq \frac{1}{2}\int_0^t \|u(z)\|_{\mathcal{X}^0}^3 dz + \frac{1}{2}\int_0^t \|u(z)\|_{\mathcal{X}^2}dz.$$

By using (1.1)*, we obtain the desired result.*

(c) As a consequence of Remark (b)*, if u is bounded in $\mathcal{X}^0(\mathbb{R}^3)$ on $[0, T^*)$, and if $T^* < \infty$, then*

$$(t \mapsto \|u(t)\|_{\mathcal{X}^0}\|u(t)\|_{\mathcal{X}^1}) \in L^1([0, T^*)).$$

The above results provide a rigorous analytical framework for the study of the Navier–Stokes system with nonlinear damping in critical Fourier spaces. The local well-posedness result ensures the existence and uniqueness of solutions under minimal regularity assumptions, which is consistent with the scaling properties of the system. Moreover, the blow-up criterion established in this work gives a precise characterization of the behavior of solutions near the maximal time of existence. In particular, it shows that the divergence of the $\mathcal{X}^0$-norm is unavoidable as the solution approaches a possible singularity. These results contribute to a better understanding of the interplay between nonlinearity, damping effects, and critical functional frameworks in fluid dynamics.

2. Preliminary results

Lemma 2.1. *$\mathcal{X}^0(\mathbb{R}^3)$ is an algebra. Precisely, if $f, g \in \mathcal{X}^0(\mathbb{R}^3)$, then $fg \in \mathcal{X}^0(\mathbb{R}^3)$, and*

$$\|fg\|_{\mathcal{X}^0} \leq \|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^0}.$$

Proof. Using $\widehat{(fg)} = \widehat{f} * \widehat{g}$ (in the sense of tempered distributions), we have

$$\int_{\mathbb{R}^3} |\widehat{fg}(\xi)|d\xi = \int_{\mathbb{R}^3} |(\widehat{f} * \widehat{g})(\xi)|d\xi.$$

By the triangle inequality and the Fubini–Tonelli theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |(\widehat{f} * \widehat{g})(\xi)|d\xi &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\widehat{f}(\eta)|.|\widehat{g}(\xi - \eta)|d\eta d\xi \\ &\leq \left(\int_{\mathbb{R}^3} |\widehat{f}(\eta)|d\eta\right)\left(\int_{\mathbb{R}^3} |\widehat{g}(\xi)|d\xi\right).\end{aligned}$$

Then, $fg \in \mathcal{X}^0(\mathbb{R}^3)$, and

$$\|fg\|_{\mathcal{X}^0} \leq \|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^0}.$$

Lemma 2.2. *Let $f \in \mathcal{X}^0(\mathbb{R}^3) \cap \mathcal{X}^2(\mathbb{R}^3)$. Then, $f \in \mathcal{X}^1(\mathbb{R}^3)$, and*

$$\|f\|_{\mathcal{X}^1} \leq \|f\|_{\mathcal{X}^0}^{1/2}\|f\|_{\mathcal{X}^2}^{1/2}.$$

Proof. We have

$$\int_{\mathbb{R}^3} |\xi||\widehat{f}(\xi)|d\xi = \int_{\mathbb{R}^3} \left(|\widehat{f}(\xi)|\right)^{\frac{1}{2}} \left(|\xi|^2|\widehat{f}(\xi)|\right)^{\frac{1}{2}} d\xi.$$

From the Cauchy–Schwarz inequality in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |\xi||\widehat{f}(\xi)|d\xi &\leq \left(\int_{\mathbb{R}^3} |\widehat{f}(\xi)|d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\xi|^2|\widehat{f}(\xi)|d\xi\right)^{\frac{1}{2}} \\ &\leq \|f\|_{\mathcal{X}^0}^{\frac{1}{2}}\|f\|_{\mathcal{X}^2}^{\frac{1}{2}}.\end{aligned}$$

Then, $f \in \mathcal{X}^1(\mathbb{R}^3)$, and the desired result is proved.

Lemma 2.3. *For all $\sigma \in \mathbb{R}$ and $a > 0$, we have*

$$\|e^{a\Delta} f\|_{\mathcal{X}^\sigma} \leq \|f\|_{\mathcal{X}^\sigma},\ \forall f \in \mathcal{X}^\sigma(\mathbb{R}^3).$$

Proof. Recall that the Fourier transform of the heat semigroup satisfies

$$\widehat{e^{a\Delta} f(\xi)} = e^{-a|\xi|^2}\widehat{f}(\xi).$$

Therefore,

$$\|\widehat{e^{a\Delta}} f\|_{\mathcal{X}^\sigma} = \int_{\mathbb{R}^3} |\xi|^\sigma e^{-a|\xi|^2}|\widehat{f}(\xi)|d\xi.$$

Because $0 < e^{-a|\xi|^2} \leq 1$ for all $\xi \in \mathbb{R}^3$, we obtain

$$\|\widehat{e^{a\Delta}} f\|_{\mathcal{X}^\sigma} \leq \int_{\mathbb{R}^3} |\xi|^\sigma|\widehat{f}(\xi)|d\xi = \|f\|_{\mathcal{X}^\sigma}.$$

Lemma 2.4. *Let $\sigma \in \mathbb{R}$; then,*

$$\|\mathbb{P}(f)\|_{\mathcal{X}^\sigma} \leq \|f\|_{\mathcal{X}^\sigma},\ \forall f \in \mathcal{X}^\sigma(\mathbb{R}^3),$$

where $\mathbb{P}$ denotes the Leray projector.

Proof. In Fourier variables, the Leray projector is given by

$$\mathbb{P}\widehat{(f)(\xi)} = \mathbb{P}(\xi)\widehat{f(\xi)} \quad \text{with} \quad \mathbb{P}(\xi) = I - \frac{\xi \otimes \xi}{|\xi|^2},\ \xi \neq 0.$$

The matrix $\mathbb{P}(\xi)$ is an orthogonal projection; hence its operator norm satisfies

$$\|\mathbb{P}(\xi)\|_{op} \leq 1.$$

Consequently,

$$|\mathbb{P}\widehat{(f)(\xi)}| \leq |\widehat{f(\xi)}| \text{ for a.e. } \xi \in \mathbb{R}^3,$$

and it follows that

$$\begin{aligned}\|\mathbb{P}\widehat{(f)(\xi)}\|_{\mathcal{X}^\sigma} &= \int_{\mathbb{R}^3} |\xi|^\sigma|\mathbb{P}\widehat{(f)(\xi)}|d\xi \\ &\leq \int_{\mathbb{R}^3} |\xi|^\sigma|\widehat{f(\xi)}|d\xi \\ &= \|f\|_{\mathcal{X}^\sigma}.\end{aligned}$$

Lemma 2.5. *Let $u, v \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))$; then,*

$$B(u, v) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\,(u.\nabla v)\,d\tau \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3)).$$

Precisely, we obtain the following estimates:

$$\|B(u, v)\|_{L^\infty_T(\mathcal{X}^0)} \leq \sqrt{T}\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^1_T(\mathcal{X}^2)}, \tag{2.1}$$

$$\|B(u, v)\|_{L^1_T(\mathcal{X}^2)} \leq \sqrt{T}\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^1_T(\mathcal{X}^2)}. \tag{2.2}$$

Proof. Let $u, v \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))$.

- We start by proving $B(u, v) \in C_T(\mathcal{X}^0(\mathbb{R}^3))$: From Lemmas 2.2–2.4, we have

$$\begin{aligned}
\|B(u, v)\|_{\mathcal{X}^0} &\leq \int_0^t \|u.\nabla v\|_{\mathcal{X}^0} d\tau \\
&\leq \int_0^t \|u\|_{\mathcal{X}^0}\|v\|_{\mathcal{X}^1} d\tau \\
&\leq \int_0^t \|v\|^{1/2}_{\mathcal{X}^0}\|u\|_{\mathcal{X}^0}\|v\|^{1/2}_{\mathcal{X}^2} d\tau \\
&\leq \|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^\infty_T(\mathcal{X}^0)} \int_0^t \|v\|^{1/2}_{\mathcal{X}^2} d\tau.
\end{aligned}$$

  Therefore,

$$\|B(u, v)\|_{L^\infty_T(\mathcal{X}^0)} \leq \sqrt{T}\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^1_T(\mathcal{X}^2)}.$$

- Now, prove that $B(u, v) \in L^1_T(\mathcal{X}^2(\mathbb{R}^3))$:

$$\begin{aligned}
\|B(u, v)\|_{L^1_T(\mathcal{X}^2)} &\leq \int_0^T \int_{\mathbb{R}^3} |\xi|^2 \int_0^t e^{-(t-\tau)|\xi|^2} |\mathcal{F}(u.\nabla v)(\tau, \xi)| d\tau d\xi dt \\
&\leq \int_{\mathbb{R}^3} |\xi|^2 \Big(\int_0^T \int_0^t e^{-(t-\tau)|\xi|^2} |\mathcal{F}(u.\nabla v)(\tau, \xi)| d\tau dt \Big) d\xi.
\end{aligned}$$

  As $\{(\tau, t) \in [0, T]^2 :\ 0 \leq \tau \leq t\} = \{(\tau, t) \in [0, T]^2 :\ \tau \leq t \leq T\}$, we obtain

$$\begin{aligned}
\|B(u, v)\|_{L^1_T(\mathcal{X}^2)} &\leq \int_{\mathbb{R}^3} |\xi|^2 \Big(\int_0^T [\int_\tau^T e^{-(t-\tau)|\xi|^2} dt] |\mathcal{F}(u.\nabla v)(\tau, \xi)| d\tau \Big) d\xi \\
&\leq \int_{\mathbb{R}^3} |\xi|^2 \Big(\int_0^T [\frac{1 - e^{-(T-\tau)|\xi|^2}}{|\xi|^2}] |\mathcal{F}(u.\nabla v)(\tau, \xi)| d\tau \Big) d\xi \\
&\leq \int_{\mathbb{R}^3} \Big(\int_0^T |\mathcal{F}(u.\nabla v)(\tau, \xi)| d\tau \Big) d\xi \\
&\leq \int_0^T \int_{\mathbb{R}^3} |\mathcal{F}(u.\nabla v)(\tau, \xi)| d\xi d\tau
\end{aligned}$$

$$\leq \int_0^T \|(u.\nabla v)(\tau)\|_{\mathcal{X}^0} d\tau.$$

Similarly, using Lemmas 2.2–2.4 and proceeding as in the previous estimate, we obtain

$$\|B(u,v)\|_{L^1_T(\mathcal{X}^2)} \leq \sqrt{T}\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^\infty_T(\mathcal{X}^0)}\|v\|^{1/2}_{L^1_T(\mathcal{X}^2)},$$

which completes the proof.

Lemma 2.6. *Let $u, v, w \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))$. Then,*

$$H(u,v,w) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\,([u.v]w)\,d\tau \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3)).$$

Precisely, we obtain the following estimates:

$$\|H(u,v,w)\|_{L^\infty_T(\mathcal{X}^0)} \leq T\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)}\|w\|_{L^\infty_T(\mathcal{X}^0)}, \tag{2.3}$$

$$\|H(u,v,w)\|_{L^1_T(\mathcal{X}^2)} \leq T\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)}\|w\|_{L^\infty_T(\mathcal{X}^0)}. \tag{2.4}$$

Proof. Let $u, v, w \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))$.

- We start by proving $H(u,v,w) \in C_T(\mathcal{X}^0(\mathbb{R}^3))$: Using Lemmas 2.3 and 2.4 and twice Lemma 2.1, we obtain

$$\begin{aligned}\|H(u,v,w)(t)\|_{\mathcal{X}^0} &\leq \int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}([u.v]w)(\tau)\|_{\mathcal{X}^0} d\tau\\ &\leq \int_0^t \|([u.v]w)(\tau)\|_{\mathcal{X}^0} d\tau\\ &\leq \int_0^t \|u(\tau)\|_{\mathcal{X}^0}\|v(\tau)\|_{\mathcal{X}^0}\|w(\tau)\|_{\mathcal{X}^0} d\tau.\end{aligned}$$

  Therefore,

$$\|H(u,v,w)(t)\|_{L^\infty_T(\mathcal{X}^0)} \leq T\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)}\|w\|_{L^\infty_T(\mathcal{X}^0)}.$$

- Second, we prove that $H(u,v,w) \in L^1_T(\mathcal{X}^2(\mathbb{R}^3))$: We have

$$\begin{aligned}\|H(u,v,w)\|_{L^1_T(\mathcal{X}^2)} &\leq \int_0^T \int_{\mathbb{R}^3} |\xi|^2 \int_0^t e^{-(t-\tau)|\xi|^2} |\mathcal{F}([u.v]w)(\tau,\xi)| d\tau d\xi dt\\ &\leq \int_{\mathbb{R}^3} |\xi|^2 \Big(\int_0^T \int_0^t e^{-(t-\tau)|\xi|^2} |\mathcal{F}([u.v]w)(\tau,\xi)| d\tau dt\Big) d\xi.\end{aligned}$$

  As $\{(\tau,t) \in [0,T]^2 :\ 0 \leq \tau \leq t\} = \{(\tau,t) \in [0,T]^2 :\ \tau \leq t \leq T\}$, we obtain

$$\begin{aligned}\|H(u,v,w)\|_{L^1_T(\mathcal{X}^2)} &\leq \int_{\mathbb{R}^3} |\xi|^2 \Big(\int_0^T [\int_\tau^T e^{-(t-\tau)|\xi|^2} dt] |\mathcal{F}([u.v]w)(\tau,\xi)| d\tau\Big) d\xi\\ &\leq \int_{\mathbb{R}^3} |\xi|^2 \Big(\int_0^T [\frac{1-e^{-(T-\tau)|\xi|^2}}{|\xi|^2}] |\mathcal{F}([u.v]w)(\tau,\xi)| d\tau\Big) d\xi\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^3}\Big(\int_0^T |\mathcal{F}([u.v]w)(\tau,\xi)|d\tau\Big)d\xi \\
&\leq \int_0^T\int_{\mathbb{R}^3} |\mathcal{F}([u.v]w)(\tau,\xi)|d\xi d\tau \\
&\leq \int_0^T \|([u.v]w)(\tau)\|_{\mathcal{X}^0}d\tau.
\end{aligned}
$$

Similarly, using twice Lemma 2.1, we obtain

$$
\|H(u,v,w)\|_{L^1_T(\mathcal{X}^2)} \leq T\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)}\|w\|_{L^\infty_T(\mathcal{X}^0)},
$$

which completes the proof.

3. Proof of Theorem 1.1

3.1. Local existence

Let R be a real number. For $T>0$, we define the Banach space

$$
E_T = C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))
$$

equipped with norm

$$
\|u\|_T = \|u\|_{L^\infty_T(\mathcal{X}^0)} + \|u\|_{L^1_T(\mathcal{X}^2)}.
$$

We denote the closed subset $F_{R,T}$ of E_T defined by

$$
F_{R,T} = \Big\{u \in E_T;\, \|u\|_{L^\infty_T(\mathcal{X}^0)} \leq 2\|u^0\|_{\mathcal{X}^0},\ \|u\|_{L^1_T(\mathcal{X}^2)} \leq R\Big\}.
$$

Now, consider the operator

$$
\begin{aligned}
\psi \quad : \quad & F_{R,T} \longrightarrow E_T \\
& u \longmapsto \psi(u) = e^{t\Delta}u^0 - \mathcal{N}_1(u)(t) - \alpha\mathcal{N}_2(u)(t).
\end{aligned}
$$

To apply the fixed-point theorem, it suffices to prove that the following estimates hold for T and R sufficiently small:

$$
\psi(F_{R,T}) \subset F_{R,T}, \tag{3.1}
$$

$$
\|\psi(u)-\psi(v)\|_T \leq \frac{1}{2}\|u-v\|_T,\ \forall u,v \in F_{R,T}. \tag{3.2}
$$

Proof of (3.1):

• To estimate $\psi(u)$ in $\mathcal{X}^0(\mathbb{R}^3)$, we have

$$
\|e^{t\Delta}u^0\|_{L^\infty_T(\mathcal{X}^0)} \leq \|u^0\|_{\mathcal{X}^0}.
$$

By the inequalities (2.1)–(2.3), we have

$$
\|\mathcal{N}_1(u)\|_{L^\infty_T(\mathcal{X}^0)} \leq \sqrt{T}\,\|u\|^{3/2}_{L^\infty_T(\mathcal{X}^0)}\|u\|^{1/2}_{L^1_T(\mathcal{X}^2)}
$$

$$\leq \quad 2\sqrt{2}\sqrt{T}R^{1/2}\,\|u^0\|^{3/2}_{L^\infty_T(\mathcal{X}^0)}$$

and

$$\|\mathcal{N}_2(u)\|_{L^\infty_T(\mathcal{X}^0)} \leq T\,\|u\|^3_{L^\infty_T(\mathcal{X}^0)} \leq 8T\,\|u^0\|^3_{\mathcal{X}^0}.$$

Then, there is a time $T_1 > 0$ such that

$$2\sqrt{2}\sqrt{T}R^{1/2}\,\|u^0\|^{3/2}_{\mathcal{X}^0} + 8\alpha T\,\|u^0\|^3_{\mathcal{X}^0} \leq \|u^0\|_{\mathcal{X}^0},\ \ \forall\, 0 < T \leq T_1.$$

By using this choice, we obtain

$$\|\psi(u)\|_{L^\infty_T(\mathcal{X}^0)} \leq 2\|u^0\|_{\mathcal{X}^0},\ \forall u \in F_{R,T}. \tag{3.3}$$

• To estimate $\psi(u)$ in $L^1_T(\mathcal{X}^2(\mathbb{R}^3))$, we have

$$\begin{aligned}\|e^{t\Delta}u^0\|_{L^1_T(\mathcal{X}^2)} &= \int_0^T \int_\xi |\xi|^2 e^{-t|\xi|^2}|\widehat{u^0(\xi)}|d\xi dt\\ &= \int_\xi \left(1 - e^{-T|\xi|^2}\right)|\widehat{u^0(\xi)}|d\xi.\end{aligned}$$

By the dominated convergence theorem, there is a time $T_2 > 0$ such that

$$\|e^{t\Delta}u^0\|_{L^1_T(\mathcal{X}^2)} < \frac{R}{3},\ \ \forall 0 < T \leq T_2.$$

By (2.2), we get

$$\begin{aligned}\|\mathcal{N}_1(u)\|_{L^1_T(\mathcal{X}^2)} &\leq \sqrt{T}\|u\|^{\frac{3}{2}}_{L^\infty_T(\mathcal{X}^0)}\|u\|^{\frac{1}{2}}_{L^1_T(\mathcal{X}^2)}\\ &\leq 2\sqrt{2}\sqrt{T}\|u^0\|^{\frac{3}{2}}_{\mathcal{X}^0}R^{1/2}.\end{aligned}$$

Inequality (2.4) holds:

$$\begin{aligned}\|\mathcal{N}_2(u)\|_{L^1_T(\mathcal{X}^2)} &\leq T\|u\|^3_{L^\infty_T(\mathcal{X}^0)}\\ &\leq 8T\|u^0\|^3_{\mathcal{X}^0}.\end{aligned}$$

Therefore, there is a time $T_3 > 0$ such that

$$2\sqrt{2}\sqrt{T}\|u^0\|^{\frac{3}{2}}_{\mathcal{X}^0}R^{1/2} + 8T\|u^0\|^3_{\mathcal{X}^0} \leq R,\ \ \forall 0 < T \leq T_3.$$

Then, if we choose $T > 0$ satisfying all the above conditions (i.e., $0 < T < \min(T_2, T_3)$), we deduce that

$$\|\psi(u)\|_{L^1_T(\mathcal{X}^2)} \leq R,\ \forall u \in F_{R,T}. \tag{3.4}$$

Consequently, (3.3) and (3.4) imply (3.1).
Proof of (3.2): By using the fact that

$$u.\nabla u - v.\nabla v = u.\nabla(u - v) + (u - v).\nabla v,$$

we get

$$\mathcal{N}_1(u) - \mathcal{N}_1(v) = B(u, u - v) + B(u - v, v).$$

By using (2.1), we obtain

$$\begin{aligned}
\|\mathcal{N}_1(u) - \mathcal{N}_1(v)\|_{L^\infty_T(\mathcal{X}^0)} &\leq \|B(u, u - v)\|_{L^\infty_T(\mathcal{X}^0)} + \|B(u - v, v)\|_{L^\infty_T(\mathcal{X}^0)} \\
&\leq \sqrt{T}\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|u - v\|^{\frac{1}{2}}_{L^\infty_T(\mathcal{X}^0)}\|u - v\|^{\frac{1}{2}}_{L^1_T(\mathcal{X}^2)} \\
&\quad + \sqrt{T}\,\|u - v\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{\frac{1}{2}}_{L^\infty_T(\mathcal{X}^0)}\|v\|^{\frac{1}{2}}_{L^1_T(\mathcal{X}^2)} \\
&\leq 2\sqrt{T}\,\|u^0\|_{\mathcal{X}^0}\|u - v\|_T + \sqrt{2}\sqrt{T}\|u^0\|^{1/2}_{\mathcal{X}^0}R^{1/2}\|u - v\|_T \\
&\leq \sqrt{2T}\|u^0\|^{1/2}_{\mathcal{X}^0}\left(\sqrt{2}\|u^0\|^{1/2}_{\mathcal{X}^0} + R^{1/2}\right)\|u - v\|_T. \qquad (3.5)
\end{aligned}$$

By using (2.2), we obtain

$$\begin{aligned}
\|\mathcal{N}_1(u) - \mathcal{N}_1(v)\|_{L^1_T(\mathcal{X}^2)} &\leq \|B(u, u - v)\|_{L^1_T(\mathcal{X}^2)} + \|B(u - v, v)\|_{L^1_T(\mathcal{X}^2)} \\
&\leq \sqrt{T}\,\|u\|_{L^\infty_T(\mathcal{X}^0)}\|u - v\|^{\frac{1}{2}}_{L^\infty_T(\mathcal{X}^0)}\|u - v\|^{\frac{1}{2}}_{L^1_T(\mathcal{X}^2)} \\
&\quad + \sqrt{T}\,\|u - v\|_{L^\infty_T(\mathcal{X}^0)}\|v\|^{\frac{1}{2}}_{L^\infty_T(\mathcal{X}^0)}\|v\|^{\frac{1}{2}}_{L^1_T(\mathcal{X}^2)} \\
&\leq 2\sqrt{T}\,\|u^0\|_{\mathcal{X}^0}\|u - v\|_T + \sqrt{2}\sqrt{T}\|u^0\|^{1/2}_{\mathcal{X}^0}R^{1/2}\|u - v\|_T \\
&\leq \sqrt{2T}\|u^0\|^{1/2}_{\mathcal{X}^0}\left(\sqrt{2}\|u^0\|^{1/2}_{\mathcal{X}^0} + R^{1/2}\right)\|u - v\|_T. \qquad (3.6)
\end{aligned}$$

Now, using the fact that

$$|u|^2u - |v|^2v = |u|^2(u - v) + [(u - v).(u + v)]v,$$

we get

$$\begin{aligned}
\mathcal{N}_2(u) - \mathcal{N}_2(v) &= \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\left[|u|^2(u - v)\right] + [(u - v).(u + v)]\,v d\tau \\
&= H(u, u, u - v) + H(u - v, u + v, v).
\end{aligned}$$

By using (2.3), we get

$$\begin{aligned}
\|\mathcal{N}_2(u) - \mathcal{N}_2(v)\|_{L^\infty_T(\mathcal{X}^0)} &\leq \|H(u, u, u - v)\|_{L^\infty_T(\mathcal{X}^0)} + \|H(u - v, u + v, v)\|_{L^\infty_T(\mathcal{X}^0)} \\
&\leq T\Big[\|u\|^2_{L^\infty_T(\mathcal{X}^0)} + \|u + v\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)}\Big]\|u - v\|_{L^\infty_T(\mathcal{X}^0)} \\
&\leq T\Big[\|u\|^2_{L^\infty_T(\mathcal{X}^0)} + \|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)} + \|v\|^2_{L^\infty_T(\mathcal{X}^0)}\Big]\|u - v\|_{L^\infty_T(\mathcal{X}^0)} \\
&\leq 2T\Big[\|u\|^2_{L^\infty_T(\mathcal{X}^0)} + \|v\|^2_{L^\infty_T(\mathcal{X}^0)}\Big]\|u - v\|_{L^\infty_T(\mathcal{X}^0)} \\
&\leq 2T\left(\|u^0\|^2_{\mathcal{X}^0} + R^2\right)\|u - v\|_T. \qquad (3.7)
\end{aligned}$$

By using (2.4), we get

$$\begin{aligned}
\|\mathcal{N}_2(u) - \mathcal{N}_2(v)\|_{L^1_T(\mathcal{X}^2)} &\leq \|H(u, u, u - v)\|_{L^1_T(\mathcal{X}^2)} + \|H(u - v, u + v, v)\|_{L^1_T(\mathcal{X}^2)} \\
&\leq T\Big[\|u\|^2_{L^\infty_T(\mathcal{X}^0)} + \|u + v\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)}\Big]\|u - v\|_{L^\infty_T(\mathcal{X}^0)} \\
&\leq T\Big[\|u\|^2_{L^\infty_T(\mathcal{X}^0)} + \|u\|_{L^\infty_T(\mathcal{X}^0)}\|v\|_{L^\infty_T(\mathcal{X}^0)} + \|v\|^2_{L^\infty_T(\mathcal{X}^0)}\Big]\|u - v\|_{L^\infty_T(L^2)}
\end{aligned}$$

$$\leq 2T\Big[\|u\|^2_{L^\infty_T(\mathcal{X}^0)} + \|v\|^2_{L^\infty_T(\mathcal{X}^0)}\Big]\|u-v\|_{L^\infty_T(\mathcal{X}^0)}$$
$$\leq 2T\left(\|u^0\|^2_{\mathcal{X}^0} + R^2\right)\|u-v\|_T. \qquad (3.8)$$

To conclude, it suffices to choose $T > 0$ satisfying

$$\sqrt{2T}\|u^0\|^{1/2}_{\mathcal{X}^0}\left(\sqrt{2}\|u^0\|^{1/2}_{\mathcal{X}^0} + R^{1/2}\right) \leq \frac{1}{4},$$
$$2T\left(\|u^0\|^2_{\mathcal{X}^0} + R^2\right) \leq \frac{1}{4}.$$

Therefore, (3.5)–(3.8) imply (3.2).
The fixed-point theorem gives the existence and uniqueness of the solution of $(NSED)$ in $C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))$. Therefore, we can deduce the existence and uniqueness of a local solution for critical Navier–Stokes equations with a damping system.

4. Proof of Theorem 1.2

4.1. *Proof of estimate* (1.1)

From the first equation of the system $(NSED)$, we obtain

$$\|u(t)\|_{\mathcal{X}^0} + \int_0^t \|u(z)\|_{\mathcal{X}^2}dz$$
$$\leq \|u^0\|_{\mathcal{X}^0} + \alpha\int_0^t \|u(z)\|^3_{\mathcal{X}^0}dz + \int_0^t \|u(z)\|_{\mathcal{X}^0}\|u(z)\|_{\mathcal{X}^1}dz. \qquad (4.1)$$

By using Lemma 2.2 and the elementary inequality

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2},\ \forall a, b \geq 0,$$

we get

$$\|u\|_{\mathcal{X}^0}\|u\|_{\mathcal{X}^1} \leq \|u\|^{3/2}_{\mathcal{X}^0}\|u\|^{1/2}_{\mathcal{X}^2} \leq \frac{1}{2}\|u\|^3_{\mathcal{X}^0} + \frac{1}{2}\|u\|_{\mathcal{X}^2}.$$

Using the last inequalities, (4.1) becomes

$$\|u(t)\|_{\mathcal{X}^0} + \frac{1}{2}\int_0^t \|u(z)\|_{\mathcal{X}^2}dz \leq \|u^0\|_{\mathcal{X}^0} + (\alpha + \frac{1}{2})\int_0^t \|u(z)\|^3_{\mathcal{X}^0}dz.$$

By Grönwall's lemma, we obtain the estimate (1.1).

4.2. *Proof of estimate* (1.2)*: Blow-up criteria*

Let $u \in C([0, T^*), \mathcal{X}^0(\mathbb{R}^3)) \cap L^1_{loc}([0, T^*), \mathcal{X}^2(\mathbb{R}^3))$ be the maximal solution of $(NSED)$ given by the last section, and suppose that T^* is finite.
The proof is divided into three steps:

First step. We prove that $\limsup_{t\to T^*}\|u(t)\|_{\mathcal{X}^0}=\infty$.
Suppose that u is bounded in $\mathcal{X}^0(\mathbb{R}^3)$, and set

$$M_*=\sup_{t\in[0,T^*)}\|u(t)\|_{\mathcal{X}^0}<\infty.$$

We prove that u satisfies the Cauchy property at T^*. For this, consider a sequence $(t_n)_{n\geq 1}$ in $[0,T^*)$ satisfying

$$\lim_{n\to\infty}t_n=T^*.$$

By using the Duhamel formula, we can write for $m,n\in\mathbb{N}$ (supposing that $t_m<t_n$):

$$\begin{array}{rcl} u(t_n) &=& e^{t_n\Delta}-\mathcal{N}_1(u)(t_n)-\mathcal{N}_2(u)(t_n),\\ u(t_m) &=& e^{t_m\Delta}-\mathcal{N}_1(u)(t_m)-\mathcal{N}_2(u)(t_m). \end{array}$$

Then,

$$u(t_n)-u(t_m)=\sum_{k=1}^{5}a_{n,m}^{(k)},$$

where

$$\begin{array}{rcl} a_{n,m}^{(1)} &=& e^{t_n\Delta}u^0-e^{t_m\Delta}u^0=(e^{t_n\Delta}-e^{t_m\Delta})u^0,\\ a_{n,m}^{(2)} &=& \displaystyle\int_0^{t_n}(e^{(t_m-z)\Delta}-e^{(t_n-z)\Delta})\mathbb{P}(|u|^2u)(z)dz,\\ a_{n,m}^{(3)} &=& \displaystyle\int_{t_n}^{t_m}e^{(t_m-z)\Delta}\mathbb{P}(|u|^2u)(z)dz,\\ a_{n,m}^{(4)} &=& \displaystyle\int_0^{t_n}(e^{(t_m-z)\Delta}-e^{(t_n-z)\Delta})\mathbb{P}(u.\nabla u)(z)dz,\\ a_{n,m}^{(5)} &=& \displaystyle\int_{t_n}^{t_m}e^{(t_m-z)\Delta}\mathbb{P}(u.\nabla u)(z)dz. \end{array}$$

- Study of $a_{n,m}^{(1)}$ in $\mathcal{X}^0(\mathbb{R}^3)$. We have

$$\begin{array}{rcl} \|a_{n,m}^{(1)}\|_{\mathcal{X}^0} &=& \displaystyle\int_{\mathbb{R}^3}|e^{-t_n|\xi|^2}-e^{-t_m|\xi|^2}|.|\widehat{u^0}(\xi)|d\xi\\ &=& \displaystyle\int_{\mathbb{R}^3}|e^{-(t_n-t_m)|\xi|^2}-1|.e^{-t_m|\xi|^2}|.|\widehat{u^0}(\xi)|d\xi\\ &=& \displaystyle\int_{\mathbb{R}^3}|e^{-(t_n-t_m)|\xi|^2}-1|.|\widehat{u^0}(\xi)|d\xi. \end{array}$$

  Then, by the dominated convergence theorem, we get $\lim_{m,n\to\infty}\|a_{n,m}^{(1)}\|_{\mathcal{X}^0}=0$.

- Study of $a_{n,m}^{(2)}$ in $\mathcal{X}^0(\mathbb{R}^3)$. We have

$$\begin{array}{rcl} \|a_{n,m}^{(2)}\|_{\mathcal{X}^0} &\leq& \displaystyle\int_{\mathbb{R}^3}\int_0^{t_n}|e^{-(t_m-z)|\xi|^2}-e^{-(t_n-z)|\xi|^2}|.|\mathcal{F}(|u|^2u)(z,\xi)|dzd\xi\\ &\leq& \displaystyle\int_{\mathbb{R}^3}\int_0^{t_n}|e^{-(t_m-t_n)|\xi|^2}-1|.e^{-(t_n-z)|\xi|^2}.|\mathcal{F}(|u|^2u)(z,\xi)|dzd\xi\\ &\leq& \displaystyle\int_{\mathbb{R}^3}\int_0^{t_n}|e^{-(t_m-t_n)|\xi|^2}-1|.|\mathcal{F}(|u|^2u)(z,\xi)|dzd\xi\\ &\leq& \displaystyle\int_{\mathbb{R}^3}\int_0^{T^*}|e^{-(t_m-t_n)|\xi|^2}-1|.|\mathcal{F}(|u|^2u)(z,\xi)|dzd\xi. \end{array}$$

By using the fact

$$|e^{-(t_m-t_n)|\xi|^2}-1|.|\mathcal{F}(|u|^2u)(z,\xi)|\leq|\mathcal{F}(|u|^2u)(z,\xi)|$$

and the fact $|u|^2u\in L^\infty([0,T^*),\mathcal{X}^0(\mathbb{R}^3))\subset L^1([0,T^*),\mathcal{X}^0(\mathbb{R}^3))$, implies $\lim\limits_{m,n\to\infty}\|a^{(2)}_{n,m}\|_{\mathcal{X}^0}=0$.

- Study of $a^{(3)}_{n,m}$ in $\mathcal{X}^0(\mathbb{R}^3)$. By Lemma 2.3, we have

$$\begin{aligned}
\|a^{(3)}_{n,m}\|_{\mathcal{X}^0} &\leq \int_{\mathbb{R}^3}\int_{t_n}^{t_m} e^{-(t_m-z)|\xi|^2}.|\mathcal{F}(|u|^2u)(z,\xi)|dzd\xi\\
&\leq \int_{t_n}^{t_m}\int_{\mathbb{R}^3}|\mathcal{F}(|u|^2u)(z,\xi)|d\xi dz\\
&\leq \int_{t_n}^{t_m}\|(|u|^2u)(z)\|_{\mathcal{X}^0}dz\\
&\leq \int_{t_n}^{t_m}\|u(z)\|^3_{\mathcal{X}^0}dz\\
&\leq M_*^3\,(t_m-t_n),
\end{aligned}$$

which implies $\lim\limits_{m,n\to\infty}\|a^{(3)}_{n,m}\|_{\mathcal{X}^0}=0$.

- Study of $a^{(4)}_{n,m}$ in $\mathcal{X}^0(\mathbb{R}^3)$. By using the fact that $\operatorname{div}u=0$, we can write

$$a^{(4)}_{n,m}=-\int_0^{t_n}(e^{(t_m-z)\Delta}-e^{(t_n-z)\Delta})\mathbb{P}\mathrm{div}\,(u\otimes u)(z)dz.$$

Then,

$$\begin{aligned}
\|a^{(4)}_{n,m}\|_{\mathcal{X}^0} &\leq \int_{\mathbb{R}^3}\int_0^{t_n}|e^{-(t_m-z)|\xi|^2}-e^{-(t_n-z)|\xi|^2}|.|\mathcal{F}(u.\nabla u)(z,\xi)|dzd\xi\\
&\leq \int_{\mathbb{R}^3}\int_0^{t_n}|e^{-(t_m-t_n)|\xi|^2}-1|.e^{-(t_n-z)|\xi|^2}|\mathcal{F}(u.\nabla u)(z,\xi)|dzd\xi\\
&\leq \int_{\mathbb{R}^3}\int_0^{t_n}|e^{-(t_m-t_n)|\xi|^2}-1|.|\mathcal{F}(u.\nabla u)(z,\xi)|dzd\xi\\
&\leq \int_{\mathbb{R}^3}\int_0^{T^*}|e^{-(t_m-t_n)|\xi|^2}-1|.|\mathcal{F}(u.\nabla u)(z,\xi)|dzd\xi.
\end{aligned}$$

By using Remark 1.3(c) and applying the dominated convergence theorem, we get $\lim\limits_{m,n\to\infty}\|a^{(4)}_{n,m}\|_{\mathcal{X}^0}=0$.

- Study of $a^{(5)}_{n,m}$ in $\mathcal{X}^0(\mathbb{R}^3)$. By Lemma 2.3, we have

$$\begin{aligned}
\|a^{(5)}_{n,m}\|_{\mathcal{X}^0} &\leq \int_{\mathbb{R}^3}\int_{t_n}^{t_m} e^{-(t_m-z)|\xi|^2}.|\mathcal{F}(u.\nabla u)(z,\xi)|dzd\xi\\
&\leq \int_{t_n}^{t_m}\int_{\mathbb{R}^3}|\mathcal{F}(u.\nabla u)(z,\xi)|d\xi dz\\
&\leq \int_{t_n}^{t_m}\|(u.\nabla u)(z)\|_{\mathcal{X}^0}dz\\
&\leq \int_{t_n}^{t_m}\|u(z)\|_{\mathcal{X}^0}\|\nabla u(z)\|_{\mathcal{X}^0}dz\\
&\leq \int_{t_n}^{t_m}\|u(z)\|_{\mathcal{X}^0}\|u(z)\|_{\mathcal{X}^1}dz\\
&\leq \int_{t_n}^{T^*}\|u(z)\|_{\mathcal{X}^0}\|u(z)\|_{\mathcal{X}^1}dz.
\end{aligned}$$

Again, by using Remark 1.3(c), we get $\lim_{m,n\to\infty} \|a^{(5)}_{n,m}\|_{\mathcal{X}^0} = 0$.

The five preceding results and the fact that $\mathcal{X}^0(\mathbb{R}^3)$ is a Banach space imply that

$$\exists u^* \in \mathcal{X}^0(\mathbb{R}^3) : \lim_{n\to\infty} \|u(t_n) - u^*\|_{\mathcal{X}^0} = 0.$$

The limit u^* is independent of the choice of the sequence $(t_n)_{n\geq 1}$.
Indeed, let $(s_n)_{n\geq 1}$ be another sequence in $[0, T^*)$ such that $\lim_{n\to\infty} s_n = T^*$. Following the same approach as before, we show that there exists $u^{**} \in \mathcal{X}^0(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|u(s_n) - u^{**}\|_{\mathcal{X}^0} = 0.$$

Consider the sequence $(r_n)_{n\geq 1}$ in $[0, T^*)$ defined by

$$r_{2n} = t_n, \quad r_{2n-1} = s_n.$$

Clearly, $\lim_{n\to\infty} r_n = T^*$. Following the same approach as before, we show that there exists $\varphi \in \mathcal{X}^0(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|u(r_n) - \varphi\|_{\mathcal{X}^0} = 0.$$

Using the definition of the sequence $(r_n)_{n\geq 1}$ and the previous results, we obtain

$$\begin{aligned} u^* &= \lim_{n\to\infty} u(t_n) = \lim_{n\to\infty} u(r_{2n}) = \varphi \quad \text{in } \mathcal{X}^0(\mathbb{R}^3), \\ u^{**} &= \lim_{n\to\infty} u(s_n) = \lim_{n\to\infty} u(r_{2n-1}) = \varphi \quad \text{in } \mathcal{X}^0(\mathbb{R}^3). \end{aligned}$$

Therefore, $u^* = u^{**}$, which shows that the limit is independent of the choice of the sequence $(t_n)_{n\geq 1}$.
Now, consider the following modified critical Navier–Stokes problem:

$$(S)\begin{cases} \partial_t v - \Delta v + v.\nabla v + \alpha|v|^2 v = -\nabla q \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\ v(0, x) = u^*(x) \text{ in } \mathbb{R}^3. \end{cases}$$

By Theorem 1.1, there is a positive time $T_0 = T_0(u^*) > 0$ and a unique solution $v \in C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3))$ of the problem (S). Then, the function w defined on $[0, T^* + T_0)$ by

$$w(t) = \begin{cases} u(t) \; if \; t \in [0, T^*), \\ v(t - T^*) \; if \; t \in [T^*, T^* + T_0), \end{cases}$$

is a solution of $(NSED)$, which contradicts the fact that u is a maximal solution of the problem $(NSED)$. Therefore, u is not bounded in $\mathcal{X}^0(\mathbb{R}^3)$, and $\limsup_{t\to T^*} \|u(t)\|_{\mathcal{X}^0} = \infty$.

Second step. We prove that $\displaystyle\int_0^{T^*} \|u(t)\|^2_{\mathcal{X}^0} dt = \infty$.
By (1.1), we get

$$\|u(t)\|_{\mathcal{X}^0} \leq \|u^0\|_{\mathcal{X}^0} \exp\Big((\alpha + \frac{1}{2}) \int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big), \; \forall t \in [0, T^*).$$

The fact that $\limsup_{t\to T^*} \|u(t)\|_{\mathcal{X}^0} = \infty$ implies that $\displaystyle\int_0^{T^*} \|u(z)\|^2_{\mathcal{X}^0} dz = \infty$.

Third step. We prove (1.2).
By (1.1), we get

$$\|u(t)\|_{\mathcal{X}^0} \leq \|u^0\|_{\mathcal{X}^0} \exp\Big((\alpha + \frac{1}{2}) \int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big),\ \forall t \in [0, T^*).$$

Passing to square

$$\|u(t)\|^2_{\mathcal{X}^0} \leq \|u^0\|^2_{\mathcal{X}^0} \exp\Big((2\alpha + 1) \int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big)$$

and

$$\|u(t)\|^2_{\mathcal{X}^0} \exp\Big(- (2\alpha + 1) \int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big) \leq \|u^0\|^2_{\mathcal{X}^0},$$

we multiply by $(2\alpha + 1)$ to obtain

$$(2\alpha + 1)\|u(t)\|^2_{\mathcal{X}^0} \exp\Big(- (2\alpha + 1) \int_0^t \|u(z)\|^2_{\mathcal{X}^0} dz\Big) \leq (2\alpha + 1)\|u^0\|^2_{\mathcal{X}^0},\ \forall t \in [0, T^*).$$

Integrate this inequality on $[0, T^*)$, taking into account $\displaystyle\int_0^{T^*} \|u(t)\|^2_{\mathcal{X}^0} dt = \infty$, to get

$$1 \leq (2\alpha + 1)T^* \|u^0\|^2_{\mathcal{X}^0}.$$

By the uniqueness of solution of $(NSED)$, we obtain the desired result.

5. Conclusions

In this paper, we studied the modified Navier–Stokes equations with a damping term in Fourier spaces. We established the local well-posedness of the system for divergence-free initial data in the space

$$C_T(\mathcal{X}^0(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^2(\mathbb{R}^3)).$$

Moreover, we derived a blow-up criterion for maximal solutions and obtained a lower bound on the $\mathcal{X}^0$-norm near the possible blow-up time.

Author contributions

Jamel Benameur: Supervision, conceptualization, validation, writing–review and editing; Lotfi Jlali: Methodology, formal analysis, investigation, writing–original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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