AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 12718–12761.
DOI: 10.3934/math.2026524
Received: 25 February 2026
Revised: 03 April 2026
Accepted: 22 April 2026
Published: 08 May 2026

Research article

Semi-analytical solutions of higher-order BVPs used in modeling hydrodynamic stability, material science, astrophysics, and high-order multi-layer structures

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Abstract: Almost all mechanical, physical, or biological processes have been implemented using differential equations (DEs). Many branches of physical sciences and engineering use higher-order ordinary differential equations (ODEs) to simulate complicated systems, climate and environmental models, fluid dynamics, stability analysis, and phenomena. Because of their nonlinearity or difficult boundary conditions, analytical solutions to these equations are sometimes unattainable. For handling 14th-order ODEs, this research looks at how two well-known semi-analytical methods, the Adomian decomposition method (ADM) and the differential transform method (DTM), are both used. Part of this study involves testing the techniques on linear and nonlinear problems. The DTM transforms differential equations into algebraic recurrence relations, enabling efficient series solutions. The ADM decomposes nonlinear terms into Adomian polynomials, allowing iterative solutions without linearization or perturbation assumptions. Across all problems, the ADM and DTM methods consistently achieved high accuracy, with absolute errors ranging from as low as 10^{-10} to a maximum of 7.17×10^{-5}. The smallest errors (10^{-10} to 10^{-9}) occurred for linear problems with exponential and trigonometric exact solutions, while the largest errors (10^{-5}) appeared in nonlinear problems at $\xi = 1.0$. Compared with the Haar wavelet, the improved residual power series method, the spline methods, and the homotopy perturbation method/optimal homotopy asymptotic method, our ADM and DTM demonstrate superior accuracy and convergence for 14th-order boundary value problems. Lastly, the convergence analysis is described in accordance with the typical theoretical results.

Keywords: Boundary value problems; semi-analytical methods; Adomian decomposition method; differential transform method; 14th-order differential equations; stability analysis; astrophysics

Mathematics Subject Classification: 34A34, 34A45, 34B05, 34B10, 34B15, 65L10, 65L70

1. Introduction

The foundation of mathematical modeling is differential equations, which offer a dynamic language to explain how values change in relation to one another throughout time and space. They convert real-world phenomena into mathematical frameworks that may be solved, such as forecasting planetary motion, modeling the spread of an epidemic, or replicating economic patterns. Boundary value problems (BVPs) arise when these processes are controlled by circumstances at several boundaries, such as the fixed ends of a vibrating beam or the defined pressures in a fluid channel. These are frequently much harder to answer than the initial value problems, particularly as their order grows to fully represent the system's complexity. This complexity is reflected in the advanced models utilized in modern science and engineering. Higher-order BVPs are particularly important in fields such as hydrodynamic stability, where they describe the transition from smooth to turbulent flow; material science, where they model stress and deformation in sophisticated multi-layered composites; astrophysics, where they understand the internal structure and oscillations of stars; and structural engineering, where they analyze high-order microbeams and nanostructures. Finding exact answers to these high-order BVPs is critical for innovation and design.

A 14th-order BVP is a complex differential equation, often found in engineering problems like beam analysis and fluid mechanics, which requires boundary conditions. The problem does not have an analytical root, and numerical solutions are required. Numerical solutions involve Haar wavelets, cubic splines, or Adomian decomposition methods. These methods transform the problem into a new system that consists of linear equations. The solution of the system that consists of linear equations provides numerical solutions, which are necessary for describing complex phenomena. In beam theory, the basic equations, compatibility equations, and boundary equations for the overall system are obtained explicitly on the basis of five different unknowns. The unknowns are the horizontal and vertical deflections of the top skin, and the horizontal and vertical deflections of the bottom skin, along with the shear strain in the core. With these five variables as the basic unknowns, the mathematical problem represents a complete set of BVPs which are of 14th-order. The set of equations represents a new system of 14th-order BVPs in terms of the chosen unknowns. This highly advanced and high-order formulation is very essential in the study of complex behaviors of sandwich structures, such as free vibration and buckling. For instance, in the literature, [1] used sandwich panel theory specifically to study the vibration and buckling of these structures. Moreover, [2] demonstrated that the nonlinear equations describing the axisymmetric response of circular sandwich plates can be reduced to a set of 14th-order BVPs.

However, literature related to mathematical solutions of 14th-order BVPs is still relatively sparse. In an effort to fill this void and provide a wider range of tools for researchers, semi-analytical techniques such as the Adomian decomposition and the differential transforms have become indispensable and widely utilized tools in the realm of computational mathematics and applied mechanics. The Adomian decomposition method (ADM) deals with these problems of high order by properly decomposing the non-linear mathematical function into a series of specialized functions called Adomian polynomials. The final solution is then obtained by constructing a rapidly converging series, where each term consists of components that are easily calculated. The primary strength of the ADM is that it can be applied straightaway to a wide range of differential equations that are accompanied by boundary conditions without the need for linearization or the use of any simplifying assumptions. Additionally, the ADM can also significantly lower the computational cost while maintaining a high degree of accuracy. On the other hand, the differential transform method (DTM) is able to transform the differential equations and the corresponding boundary conditions into algebraic equations using a framework based on the Taylor series. This makes it easier to solve the problem, as the solution can be obtained in the form of a polynomial series. The DTM is especially appreciated for its simplicity, low computational complexity, and high accuracy. This is especially the case when handling the nonlinear terms that are typical of high-order BVPs. Collectively, the ADM and DTM provide efficient, effective, and accurate methods for solving the complicated 14th-order BVPs that occur in modern scientific and engineering applications.

The ADM was first applied to solve 10th-order BVPs by [3], the Haar wavelet method (HWM) was applied to solve 14th-order BVPs, providing an alternative numeral approach [4], following earlier applications of the one-dimensional DTM for higher-order BVPs in finite domains by [5]. For second-order differential equations, [6] used the differential transform technique. Third-order ordinary differential equations (ODEs) were solved using the DTM by [7].

12th-order BVPs were addressed using the ADM by [8], and a comparative analysis of nonlinear higher-order BVPs using both DTM and ADM was conducted by [9], showing the DTM's computational advantages. 5th-order BVPs were specifically investigated using the differential transform technique by [10], which validated the method's reliability through comparative analysis. 12th-order BVPs were also approached using the variational iteration method by [11]. In [12], the authors implemented the improved residual power series method to solve higher-order linear and nonlinear BVPs up to 14th-order. A modified ADM for first-order ODEs was presented by Hasan [13]. Higher-order initial value problems were addressed using the DTM by [14]. 8th-order BVPs were tackled using the DTM by [15], showing greater accuracy compared with other methods, with sixth-order BVPs following by [16]. The DTM was compared with the ADM for linear and nonlinear initial value problems by [17], recognizing the DTM as a viable numerical technique.

For RLC circuit (an electrical circuit consisting of a resistor (R), inductor (L), and capacitor (C)) problems and higher-order differential equations, [18] applied the DTM with Adomian polynomials, while 13th-order BVPs were solved via the DTM by [19]. Nonlinear diffusion equations with exponential nonlinearity were addressed using the ADM by [20], producing convergent power series solutions, and nonlinear fractional BVPs were handled by [21] with an effective numerical method. Initial-value problems were first solved using the DTM by [22], approximating solutions using a fixed grid size. Fourth-order BVPs from beam theory were addressed via the ADM by [23], emphasizing the importance of analytical solutions for engineering models. 14th-order BVPs were solved using cubic spline methods by [24] and [25], with the ADM applied to 14th-order problems by [26], producing solutions as convergent series. Higher-order BVPs with convolution terms were addressed by [27]. Nonlinear differential equations were solved via the DTM by [28] with the results converging quickly to exact solutions. BVPs of orders seven and eight were handled using the DTM by [29], demonstrating precision and computational efficiency.

13th-order BVPs were solved via the modified ADM by [30], achieving faster convergence, and ninth-order BVPs specifically were solved using the DTM. The Taylor series-based DTM approach was applied to third-order ODEs and higher-order BVPs by [31] with rapid convergence. Reliable modifications of the ADM for higher-order nonlinear differential equations were developed by [32], utilizing Duan's algorithms for efficient Adomian polynomial generation.

Seventh-order BVPs were solved using the ADM by [33], and the magnetohydrodynamic (MHD) flow between parallel plates was analyzed via the DTM by [34]. New algorithms for calculating Adomian polynomials were developed by [35], and fifth-order BVPs were specifically addressed via the decomposition method by [36], yielding fast-convergent series solutions. Using HPM and OHAM, semi-analytical solutions have been developed for hydrodynamic stability problems in nonlinear 14th-order equations in [37].

This study uses semi-analytical techniques like the DTM and the ADM to solve challenging 14th-order equations. These methods help connect theoretical models with real-world applications. Mathematica was used to derive transformed equations and build approximate series solutions using recursive steps. The software also managed the data on values and transformation rules efficiently. MATLAB was then used to plot graphs of the DTM solutions and their error behavior, showing how accurate and reliable the method is.

2. Differential transform method

The DTM was first introduced by Zhou over three decades ago. The DTM is an effective method to apply linear as well as non-linear BVPs for solving in numerical analysis. In real-life applications, differential equations describe many physical and natural phenomena, with a significant number of these equations being nonlinear. This nonlinearity often complicates the process of finding exact or analytical solutions. Numerical methods were used to solve nonlinear differential equations, but there is a consistent need for approaches that can efficiently manage nonlinear terms without constraints while also reducing computational complexity. The DTM is a type of semi-analytical numerical method which can help alleviate some of these issues. It has proven to be a highly effective method for addressing different types of differential and integral equations, making it a valuable resource in both numerical and analytical applications. This method converts the differential equations into recurrence relations. Then, by applying a different approach to Taylor series expansion, we obtain convergent series solutions. By applying the DTM, it is now possible to get exact and highly approximate results or solutions for differential equations. The approach of the DTM was first described in [38], which solved the initial value problems using the DTM in electric circuit analysis.

The k-th derivative of a function with a variable is below.

$$\Pi(k) = \frac{1}{k!}(\frac{d^k(\varpi(\xi)}{d\varpi^{(k)}}, q = q_0. \tag{2.1}$$

In the equation above, the original function is $\varpi(\xi)$ and in the equation below, the transformed function $\Pi(k)$, defined by the following equation:

$$\varpi(\xi) = \sum_{n=0}^{\infty} \varpi(k)(\xi - \xi_0)^k. \tag{2.2}$$

When $\xi_0 = 0$, the $\varpi(\xi)$ defined in Eq (2.2) is shown.

$$\varpi(\xi) = \sum_{n=0}^{\infty} \varpi(k)\xi^k. \tag{2.3}$$

Eq (2.3) is for one-dimensional Taylor series expansion.

2.1. *Fundamental theorems on DTMs*

(1) If $v(q) = \Gamma w(q) \pm \Omega h(q)$, then $V(k) = \Gamma W(k) \pm \Omega H(k)$,where Γ and ω are constants.
(2) If $v(q)=e^q$, then $V(k)=\frac{1}{k!}$.
(3) If $v(q)=g(q)h(q)$, then $V(K)=\sum G(l)H(k-l)$.
(4) If $v(q)=v_1(q)v_2(q)$, then $V(k)=\sum V_1(k_1)V_2(k_2)$.
(5) If $v(q)=\frac{d^m v(q)}{dq}^m$, then $V(k)=\frac{(k+m)!}{k!}V(k+m)$.
(6) If $v(q)=\sin(\Gamma v+\Omega)$, then $V(k)=\Gamma\frac{k}{k!}\sin(k\frac{\pi}{2}+\Omega)$, where Γ and Ω are constants.
(7) If $v(q)=\cos(\Gamma v+\Omega)$, then $V(k)=\Gamma\frac{k}{k!}\cos(k\frac{\pi}{2}+\Omega)$, where Γ and Ω are constants.
(8) If $v(q)=e^{\lambda q}$, then $V(k)=\frac{\lambda^k}{k!}$.
(9) If $v(q) = (1+\sigma)^\sigma$, then $V(K) = \frac{\sigma(\sigma-1)...(\sigma-k-1)}{k!}$, where σ is the constant.
(10) If $v(q)=\int v(q)dt$, then $V(k)=\frac{V(k-1)}{k}$, for $k \geq 1$.

2.2. *Application of the DTM to 14th-order BVPs*

The k-th derivative of a function with a variable can be transformed using the following formula:

$$\Pi(k) = \frac{1}{k!}\frac{d^k(\varpi(\xi)}{d\xi^k}, \xi = \xi_0, \tag{2.4}$$

where the transformed function is $\Pi(k)$ and the original function is $\varpi(\xi)$.

$$\varpi(\xi) = \sum_{n=0}^{\infty} \Pi(k)(\xi-\xi_0)^k. \tag{2.5}$$

When $\xi_0 = 0$, the $\varpi(\xi)$ defined in Eq (2.5) as shown

$$\varpi(\xi) = \sum_{n=0}^{\infty} \Pi(k)\xi^k. \tag{2.6}$$

Eq (2.6) is the one-dimensional Taylor series expansion.

3. Adomian decomposition method

In terms of daily scientific life, the ADM is very effective for solving comprehensive classes of ODEs and partial differential equations (PDEs). This method significantly lessens the number of computations required. The ADM is accomplished simply without the need for linearization of the problem at hand by using the decomposition technique instead of the standard methods designed for exact solutions. This method is applicable for nonlinear differential equations, which are very difficult or sometimes impossible to solve using old analytical methods for this type of example, and therefore the ADM is useful. For nonlinear ODEs, the technique uses a nonlinear operator known as the Adomian operator. The ADM is a versatile approach that can handle differential equation systems and is particularly useful for locating approximations of solutions in situations where precise answers are not available. This approach produces quickly convergent solutions made up of easily comprehensible components. The ADM's direct applicability to a variety of differential equations with boundary conditions is its main advantages. Furthermore, it guarantees accurate results while drastically lowering the computational load. The ADM has been effectively used in a number of real-world domains, including biology models, engineering, physics, finance, and economics graphics. It has been applied to complicated problems.

Mathematica was used to derive the analytical form of the solution series by calculating Adomian polynomials and recursive components. Its symbolic computation environment enabled efficient handling of high-order derivatives and nonlinear terms. MATLAB was then used to plot the ADM solutions and perform graphical error analysis, allowing for a visual comparison with exact results.

Suppose that the first-order ODE [13] is

$$w' + \phi(v)w + \Psi(v, w) = \chi(v), \tag{3.1}$$

$$e^{-\int \phi(v)\,dv} \frac{d}{dv}\left(e^{\int \phi(v)\,dv}(\cdot)\right) + \psi(v, w) = \chi(v). \tag{3.2}$$

The ADM is a numerical method to solve ODEs and PDEs. This method was improved from 1970 to the 1990s by George Adomian. To derive the ODEs of different order, from Eq (3.2)

$$\frac{d^n}{dv^n}\left(e^{-\int \phi(v)\,dv} \frac{d}{dv}\left(e^{\int \phi(v)\,dv}(\cdot)\right)\right) + \psi(v, w) = \chi(v). \tag{3.3}$$

Putting $n = 0$ in Eq (3.3) gives the first-order ODE

$$w' + \phi(v)w + \psi(v, w) = \chi(v). \tag{3.4}$$

Putting $n = 1$ in Eq (3.3) gives the second-order ODE

$$w'' + \phi(v)w' + \phi'(v)w + \psi(v, w) = \chi(v). \tag{3.5}$$

Putting $n = 2$ in Eq (3.3) gives the third-order ODE

$$w''' + \phi(v)w'' + 2\phi'(v)w' + \phi''(v)w + \psi(v, w) = \chi(v). \tag{3.6}$$

Continuously, until n, we get the following:

$$w^{(n+1)} + \sum_{r=0}^{n} \binom{n}{r} \phi^{(r)}(v) w^{(n-r)} + \psi(v, w) = \chi(v). \tag{3.7}$$

Consider the following higher ODEs from Eq (3.7)

$$Lw = \chi(v) - \psi(v, w), \tag{3.8}$$

where

$$L(\cdot) = \frac{d^n}{dv^n}\left(e^{-\int \phi(v)\,dx}\frac{d}{dv}\left(e^{\int \phi(v)\,dv}(\cdot)\right)\right) + \psi(v,w) = \chi(v), \tag{3.9}$$

and

$$L^{-1}(.) = e^{-\int P(v)\,dx}\int_0^v e^{\int P(v)\,dv}\int_0^v\int_0^v\int_0^v\cdots\int_0^v(\cdot)\,dv\,dv\,dv\,\cdots dv. \tag{3.10}$$

By applying L^{-1} in the equation above

$$w(v) = \beta(v) + L^{-1}\chi(v) - L^{-1}\psi(v,w), \tag{3.11}$$

such that

$$L(\beta(v)) = 0. \tag{3.12}$$

$$w(v) = \sum_{n=0}^{\infty} w_n(v), \tag{3.13}$$

The Adomian polynomials' infinite series, which takes the shape of a nonlinear span, is expressed as follows:

$$Nc = \sum_{n=0}^{\infty} A_n$$

In this case, the Adomian polynomials are denoted by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty}\lambda^n c_j\right)\right]\Bigg|_{\lambda=0},$$

$$\psi(v,w) = \sum_{n=0}^{\infty} A_n, \tag{3.14}$$

where the elements $w_n(v)$ of the solution w(v) will be determined to be repeatable.

$$A_0 = F(c_0),$$

$$A_1 = c_1F(c_0),$$

$$A_2 = c_2F^{(1)} + \frac{c_1^2}{2!}F^{(2)}(c_0).$$

3.1. Application of the ADM to 14th-order BVPs

Consider the following 14th-order BVP:

$$\varpi^{(14)}(\xi) = f(\xi) + g(s), 0 < \xi < p, \tag{3.15}$$

with the boundary conditions

$$\begin{gathered}\varpi(0) = b_0, \varpi^{(1)}(0) = b_1, \varpi^{(2)}(0) = b_2, \varpi^{(3)}(0) = b_3,\\ \varpi^{(4)}(0) = b_4, \varpi^{(5)}(0) = b_5, \varpi^{(6)}(0) = b_6, \varpi(p) = c_0, \varpi^{(1)}(p) = c_1, \varpi^{(2)}(p) = c_2,\\ \varpi^{(3)}(p) = c_3, \varpi^{(4)}(p) = c_4, \varpi^{(5)}(p) = c_5, \varpi^{(6)}(p) = c_6,\end{gathered}$$

In the aforementioned equation, $g(s)$ is a linear, nonlinear, continuous function that lies inside the interval $[0, p]$, while $f(\xi)$ is a basic function. Here, $b_i, i = 0, 1, 2, ..., 6$ and $c_i, i = 0, 1, 2, ..., 6$. Equation (3.15) can now be written in operator form as follows:

$$L\varpi = f(\xi) + g(s), \tag{3.16}$$

in which L represents the differential operator.

$$L = \frac{d^{14}}{d\xi^{14}}(.). \tag{3.17}$$

Consequently, integral operator is denoted by L^{-1} .

$$L^{-1}(.) = \left[\int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi (.) d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi d\xi\right], \tag{3.18}$$

which will be written as follows:

$$L^{-1}(.) = \int_0^\xi \int_0^\xi \cdots \int_0^\xi (.)\, d\xi^{14}. \tag{3.19}$$

Applying L^{-1}, we get

$\varpi(\xi) = A + B\xi + \frac{C\xi^2}{2!} + \frac{D\xi^3}{3!} + \frac{E\xi^4}{4!} + \frac{F\xi^5}{5!} + \frac{G\xi^6}{6!} + \frac{a\xi^7}{7!} + \frac{b\xi^8}{8!} + \frac{c\xi^9}{9!} + \frac{d\xi^{10}}{10!} + \frac{e\xi^{11}}{11!} + \frac{f\xi^{12}}{12!} + \frac{g\xi^{13}}{13!} + L^{-1}[f(\xi) + g(s)]$

Applying the boundary condition to $\xi = 0$, we get

$\varpi_0(t) = A + B\xi + \frac{C\xi^2}{2!} + \frac{D\xi^3}{3!} + \frac{E\xi^4}{4!} + \frac{F\xi^5}{5!} + \frac{G\xi^6}{6!} + \frac{a\xi^7}{7!} + \frac{b\xi^8}{8!} + \frac{c\xi^9}{9!} + \frac{d\xi^{10}}{10!} + \frac{e\xi^{11}}{11!} + \frac{f\xi^{12}}{12!} + \frac{g\xi^{13}}{13!}$. The boundary conditions at $q = p$ can be used to determine a, b, c, d, e, f, and g in the equation above. The ADM solution $\varpi(\xi)$ by the component breakdown series

$$\varpi(\xi) = \sum_{n=0}^{\infty} \varpi_n(\xi). \tag{3.20}$$

The non-linear factor $g(s)$ is expressed as an infinite series of Adomian polynomials, as shown below.

$$\varpi(\xi) = \sum_{n=0}^{\infty} A_n, \tag{3.21}$$

where $\varpi_n(\xi)$ will be generated by a recursive relation and are known as Adomian polynomials. Substituting Eqs (3.17) and (3.19) into (3.16) yields the following mathematical relation:

$$\sum_{n=0}^{\infty}\varpi_n = A+B\xi+\frac{C\xi^2}{2!}+\frac{D\xi^3}{3!}+\frac{E\xi^4}{4!}+\frac{F\xi^5}{5!}+\frac{G\xi^6}{6!}+\frac{a\xi^7}{7!}+\frac{b\xi^8}{8!}+\frac{c\xi^9}{9!}+\frac{d\xi^{10}}{10!}+\frac{e\xi^{11}}{11!}+\frac{f\xi^{12}}{12!}+\frac{g\xi^{13}}{13!}+L^{-1}[\sum_{n=0}^{\infty}A_n],$$

$$\varpi_0(\xi) = A+B\xi+\frac{C\xi^2}{2!}+\frac{D\xi^3}{3!}+\frac{E\xi^4}{4!}+\frac{F\xi^5}{5!}+\frac{G\xi^6}{6!}+\frac{a\xi^7}{7!}+\frac{b\xi^8}{8!}+\frac{c\xi^9}{9!}+\frac{d\xi^{10}}{10!}+\frac{e\xi^{11}}{11!}+\frac{f\xi^{12}}{12!}+\frac{q\xi^{13}}{13!}+L^{-1}[f(\xi)], \tag{3.22}$$

$$\varpi_{k+1} = L^{-1}[\sum_{n=0}^{\infty}A_k]. \tag{3.23}$$

The coefficients, a, b, c, d, e, f, and g are determined using the boundary conditions at $\xi = p$.

4. Numerical example

The DTM and ADM are excellent semi-analytical methods for solving nonlinear as well as linear differential equations. A key benefit of the DTM is its simple and user-friendly methodology, which transforms differential equations into algebraic equations through Taylor series expansion. Additionally, the DTM significantly reduces computational requirements while maintaining high accuracy across various problem types. Particularly for nonlinear problems, the ADM presents a potent substitute. By means of methodical Adomian polynomials, ADM resolves the solution into an infinite series and addresses nonlinear terms and thus obviating linearization. It is particularly useful for BVPs with difficult boundary conditions when conventional techniques might fail. This approach gives fast convergent series solutions.Working together, DTM and ADM give flexible, quick ways to get right answers to higher-order differential equations.

- **Absolute error** provides the direct difference between the computed and exact values. It gives a sense of how far the approximation deviates in magnitude but does not consider the scale of the exact value.

  $$\text{Absolute error} = |\text{Exact value} - \text{DTM/ADM outcome}|.$$

- **Relative error** accounts for the size of the exact value, offering a normalized metric that is more informative when dealing with values across varying magnitudes. It is especially useful when the exact solution values are small.

  $$\text{Relative error} = \frac{|\text{Exact value} - \text{DTM/ADM outcome}|}{|\text{Exact value}|}.$$

4.1. Numerical testing of 14th-Order BVPs using the ADM

The ADM solves 14th-order BVPs by splitting them down into smaller components. It expresses the solution as a series and expands any nonlinear terms into Adomian polynomials. This allows the high-order problem to be solved recursively as a sequence of easy-to-compute equations. The method handles strong nonlinearities directly without simplification and produces a rapidly convergent series solution with high accuracy.

4.1.1. Problem 4.1.1

Ponder the following 14th-order BVP. The differential equation we are considering is represented by

$$\varpi^{(14)}(\xi) = \sin\xi - \cos\xi, \quad 0 \leq \xi \leq 1, \tag{4.1}$$

subject to the following conditions:

$$\varpi^{(i)}(0) = 1, \varpi^{(i)}(1) = \sin(1) + \cos(1)$$
$$\varpi^{(j)}(0) = -1, \varpi^{(j)}(1) = -\sin(1) - \cos(1)$$

where $i = 0, 4, 8, 12$ and $j = 2, 6, 10$ and the accurate result is $\varpi(\xi) = \cos\xi + \sin\xi$.

Let us assume the following:

$$\frac{d^{14}}{d\xi^{14}} = L$$
$$\frac{d^{14}\varpi}{d\xi^{14}} = L\varpi$$

$$L\varpi = \cos\xi - \sin\xi \tag{4.2}$$

Applying (L^{-1}) to Eq (4.13)

$$L^{-1}L\varpi = L^{-1}(\cos\xi - \sin\xi),$$

so,

$$L^{-1}L\varpi = \left[\int_0^\xi \int_0^\xi \cdots \int_0^\xi (\varpi^{(14)})\, d\xi^{14}\right]$$

$$= [\varpi(\xi) - [1 + \xi - \frac{1}{2!}\xi^2 - \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 - \frac{1}{6!}\xi^6 + \frac{h}{7!}\xi^7 + \frac{i}{8!}\xi^8 + \frac{j}{9!}\xi^9 + \frac{k}{10!}\xi^{10} + \frac{l}{11!}\xi^{11} + \frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13}],$$

$$\varpi(\xi) - [1 + \xi - \frac{1}{2!}\xi^2 - \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 - \frac{1}{6!}\xi^6 + \frac{h}{7!}\xi^7 + \frac{i}{8!}\xi^8 + \frac{j}{9!}\xi^9 + \frac{k}{10!}\xi^{10} + \frac{l}{11!}\xi^{11} + \frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13} = L^{-1}(\cos\xi - \sin\xi)],$$

$$\varpi(\xi) = \left[(1 + \xi - \frac{1}{2!}\xi^2 - \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 - \frac{1}{6!}\xi^6 + \frac{h}{7!}\xi^7 + \frac{i}{8!}\xi^8 + \frac{j}{9!}\xi^9 + \frac{k}{10!}\xi^{10} + \frac{l}{11!}\xi^{11} + \frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13}) + L^{-1}(\cos\xi - \sin\xi)\right]. \tag{4.3}$$

Applying the decomposition technique to Eq (4.3), we obtain the following formula:

$$\sum_{n=0}^{\infty}\varpi_n(\xi) = \Big[(1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13}) + L^{-1}(\sum_{n=0}^{\infty}\cos\xi-\sin\xi)\Big],$$

$$\varpi_n = \Big[1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13}\Big],$$

$$\varpi_0 = \Big[1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12}+\frac{n}{13!}\xi^{13}\Big],$$

$$\varpi_1 = L^{-1}(\cos\xi-\sin\xi).$$

We know that

$$\cos\xi = 1-\frac{\xi^2}{2!}+\frac{\xi^4}{4!}-\frac{\xi^6}{6!}+\frac{\xi^8}{8!}-\cdots,$$

and

$$\sin\xi = \xi-\frac{\xi^3}{3!}+\frac{\xi^5}{5!}-\frac{\xi^7}{7!}+\frac{\xi^9}{9!}-\cdots.$$

$$L^{-1}(\cos\xi-\sin\xi) = L^{-1}[(1-\frac{\xi^2}{2!}+\frac{\xi^4}{4!}-\frac{\xi^6}{6!}-\frac{\xi^8}{8!}-\cdots)+(\xi-\frac{\xi^3}{3!}+\frac{\xi^5}{5!}-\frac{\xi^7}{7!}+\frac{\xi^9}{9!}-\cdots)]$$

$$= L^{-1}(1-\xi-\frac{\xi^2}{2}+\frac{\xi^3}{6}+\frac{\xi^4}{24}-\frac{\xi^5}{120}-\frac{\xi^6}{720}+\frac{\xi^7}{5040}-\cdots),$$

$$\varpi_1 = \Big[\int_0^{\xi}\int_0^{\xi}\cdots\int_0^{\xi} 1-\xi-\frac{\xi^2}{2}+\frac{\xi^3}{6}+\frac{\xi^4}{24}-\frac{\xi^5}{120}-\frac{\xi^6}{720}+\frac{\xi^7}{5040}-\cdots d\xi^{14}\Big], \quad (4.4)$$

$$\varpi_1 = \frac{\xi^{14}}{14!}-\frac{\xi^{15}}{15!}-\frac{\xi^{16}}{16!}+\frac{\xi^{17}}{17!}+\frac{\xi^{18}}{18!}-\frac{\xi^{19}}{19!}-\frac{\xi^{20}}{20!}+\frac{\xi^{21}}{21!}--\cdots, \quad (4.5)$$

$$\varpi(\xi) = \Big[(1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13})+(\frac{\xi^{14}}{14!}-\frac{\xi^{15}}{15!}-\frac{\xi^{16}}{16!}+\frac{\xi^{17}}{17!}+\frac{\xi^{18}}{18!}-\frac{\xi^{19}}{19!}-\frac{\xi^{20}}{20!}+\frac{\xi^{21}}{21!})\Big], \quad (4.6)$$

$$\varpi^{(7)}(0) = h, \qquad \varpi^{(8)}(0) = i, \qquad \varpi^{(9)}(0) = j,$$

$$\varpi^{(10)}(0) = k, \qquad \varpi^{(11)}(0) = l, \qquad \varpi^{(12)}(0) = m, \qquad \varpi^{(13)}(0) = n.$$

Apply the boundary conditions $t = 1$. Substituting numerical values into the system, we get the following:

$$\frac{120475555201}{87178291200} + h + i + j + k + l + m + n = 1.3817732889999998,$$
$$\frac{-1868106239}{6227020800} + 7h + 8i + 9j + 10k + 11l + 12m + 13n = -0.301168679,$$
$$\frac{-658627199}{479001600} + 42h + 56i + 72j + 90k + 110l + 132m + 156n = -1.3817732889999998,$$
$$\frac{13305601}{39916800} + 210h + 336i + 504j + 720k + 990l + 1320m + 1716n = 0.301168679,$$
$$\frac{5443201}{3628800} + 840h + 1680i + 3024j + 5040k + 7920l + 11880m + 17160n = 1.3817732889999998,$$
$$\frac{1}{362880} + 2520h + 6720i + 15120j + 30240k + 55440l + 95040m + 154440n = -0.301168679,$$
$$\frac{-40319}{40320} + 5040h + 20160i + 60480j + 151200k + 332640l + 665280m + 1235520n = -1.3817732889999998.$$

Solving this system yields the following:

$$h = -1.0014631482762146, \qquad i = 5.006347697261929,$$
$$j = -10.012641828180731, \qquad k = 10.012643348738777,$$
$$l = -5.006321446774759, \qquad m = 1.0012642216259624,$$
$$n = -1.0012647421625944$$

$$\varpi(\xi) = \Big[1 + \xi - \frac{\xi^2}{2} - \frac{\xi^3}{6} + \frac{\xi^4}{24} + \frac{\xi^5}{120} - \frac{\xi^6}{720} - 0.0001987030056\,\xi^7 + 0.0001241653695\,\xi^8 - 0.00002759215671\,\xi^9 + (2.75921609 \times 10^{-6})\,\xi^{10} - (1.254189075 \times 10^{-7})\,\xi^{11} + (2.090314984 \times 10^{-9})\,\xi^{12} - \frac{\xi^{13}}{6227020800} + \frac{\xi^{14}}{87178291200} - \frac{\xi^{15}}{1307674368000} - \frac{1}{20922789888000}\xi^{16} + \frac{1}{355687428096000}\xi^{17} + \frac{1}{6402373705728000}\xi^{18} - \frac{\xi^{19}}{121645100408832000} - \frac{\xi^{20}}{2432902008176640000} + \frac{\xi^{21}}{51090942171709440000}\Big]$$

The ADM solution shows excellent agreement with the exact solution $\varpi(\xi) = \cos\xi + \sin\xi$ of the Problem 4.1.1. As shown in the figure, the exact and ADM curves overlap nearly perfectly across $[0, 1]$. Figure 1 shows a graphical comparison between the exact solution and the ADM's approximate solution. The two curves overlap almost perfectly across the entire domain $[0, 1]$, indicating that the ADM accurately captures the behavior of the exact solution. As shown in Table 1, the minimum absolute error is 0.00×10^0 at $\xi = 0$, where the ADM's outcome exactly matches the accurate result. The next smallest errors occur at $\xi = 0.2$ (6.00×10^{-10}) and $\xi = 0.1$ (1.00×10^{-9}). The maximum absolute error is 7.17×10^{-5} at $\xi = 1.0$, with a corresponding relative error of 5.19×10^{-5}. Thus, the error remains well-controlled, spanning nine orders of magnitude from 10^{-9} to 10^{-5}.

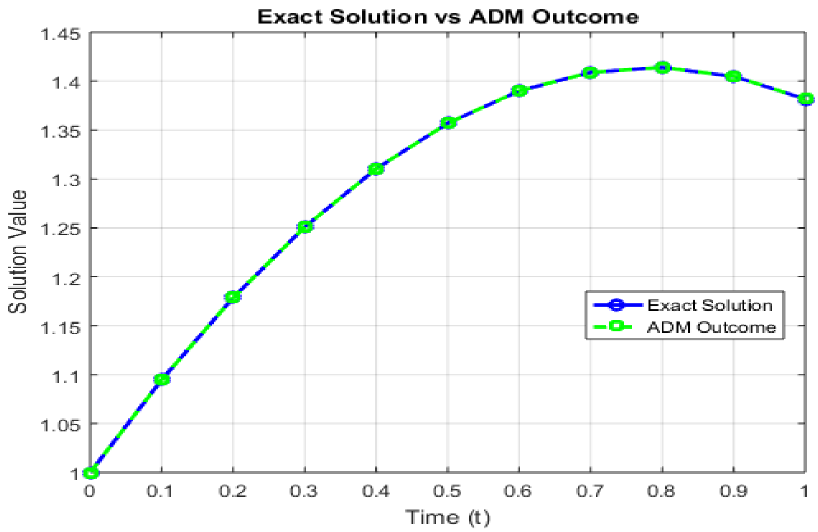

Figure 1. Exact and ADM's Outcomes: Graphical comparison with the absolute error for Problem 4.1.1.

Table 1. Comparison of the exact solution and the ADM's outcome with the absolute error for Problem 4.1.1.

ξ	Accurate result	ADM's outcome	Absolute error	Relative error
0	1.0000000000	1.000000000	0.00×10^{0}	0.00×10^{0}
0.1	1.0948375820	1.094837581	1.00×10^{-9}	9.13×10^{-10}
0.2	1.1787359086	1.178735908	6.00×10^{-10}	5.09×10^{-10}
0.3	1.2508566960	1.250856701	5.00×10^{-9}	3.99×10^{-9}
0.4	1.3104793363	1.310479393	5.67×10^{-8}	4.33×10^{-8}
0.5	1.3570081000	1.357008430	3.30×10^{-7}	2.43×10^{-7}
0.6	1.3899780883	1.389979461	1.37×10^{-6}	9.85×10^{-7}
0.7	1.4090598750	1.409064438	4.56×10^{-6}	3.24×10^{-6}
0.8	1.4140628002	1.414075653	1.29×10^{-5}	9.12×10^{-6}
0.9	1.4049368780	1.404968780	3.19×10^{-5}	2.27×10^{-5}
1.0	1.3817732291	1.381844947	7.17×10^{-5}	5.19×10^{-5}

4.1.2. Problem 4.1.2

Ponder the following 14th-order BVP. The differential equation we are considering is represented as follows:

$$\varpi^{(14)}(\xi) = 12\xi\cos(\xi) + (31 - \xi^2)\sin(\xi) + \varpi^2(\xi), \quad 0 \le \xi \le 1, \tag{4.7}$$
$$\varpi^{(14)}(\xi) = 12t\cos(\xi) + 31\sin(\xi) - \xi^2\sin(\xi) + \varpi^2(\xi),$$

subject to the following conditions:

$$\varpi^{(i)}(0) = 1, \varpi^{(i)}(1) = \cos(1) + \sin(1),$$
$$\varpi^{(j)}(0) = -1, \varpi^{(j)}(1) = -\cos(1) - \sin(1),$$

where $i = 0, 4, 8, 12$ and $j = 2, 6, 10$ and the accurate result is $\varpi(\xi) = \cos\xi + \sin\xi$.

Let us assume the following:

$$\frac{d^{14}}{d\xi^{14}} = L,$$
$$\frac{d^{14}\varpi}{d\xi^{14}} = L\varpi,$$
$$L\varpi = 12\xi\cos(\xi) + 31\sin(\xi) - \xi^2\sin(\xi) + \varpi^2(\xi). \tag{4.8}$$

Applying (L^{-1}) to Eq (4.8), we have

$$L^{-1}L\varpi = L^{-1}(12\varpi\cos(\xi) + 31\sin(\xi) - \xi^2\sin(\xi) + \varpi^2(\xi)),$$

so,

$$L^{-1}L\varpi = \left[\int_0^{\xi}\int_0^{\xi}\cdots\int_0^{\xi}(\varpi^{(14)})\,d\xi^{14}\right],$$

$$L^{-1}L\varpi = \varpi(\xi) - [1 + \xi - \frac{1}{2!}\xi^2 - \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 - \frac{1}{6!}\xi^6 + \frac{h}{7!}\xi^7 + \frac{i}{8!}\xi^8 + \frac{j}{9!}\xi^9 + \frac{k}{10!}\xi^{10} + \frac{l}{11!}\xi^{11} + \frac{m}{12!}\xi^{12} + \frac{n}{13!}\xi^{13}],$$

$$\varpi(\xi) = \Big[(1 + \xi - \frac{1}{2!}\xi^2 - \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 - \frac{1}{6!}\xi^6 + \frac{h}{7!}\xi^7 + \frac{i}{8!}\xi^8 + \frac{j}{9!}\xi^9 + \frac{k}{10!}\xi^{10} + \frac{l}{11!}\xi^{11} + \frac{m}{12!}\xi^{12} + \frac{n}{13!}) + L^{-1}(12\xi\cos(\xi) + 31\sin(\xi) - \xi^2\sin(\xi) + \varpi^2(\xi))\Big]. \tag{4.9}$$

Applying the decomposition method to Eq (4.9), we have

$$\sum_{n=0}^{\infty}\varpi_n(\xi) = \Big[1 + \xi - \frac{1}{2!}\xi^2 - \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 - \frac{1}{6!}\xi^6 + \frac{h}{7!}\xi^7 + \frac{i}{8!}\xi^8 + \frac{j}{9!}\xi^9 + \frac{k}{10!}\xi^{10} + \frac{l}{11!}\xi^{11} + \frac{m}{12!}\xi^{12} + \frac{n}{13!}) + L^{-1}(\sum_{n=0}^{\infty} 12\xi\cos(\xi) + 31\sin(\xi) - \xi^2\sin(\xi) + \varpi^2(\xi))\Big], \tag{4.10}$$

$$\varpi_n = \left[1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12}+\frac{n}{13!}\right],$$

$$\varpi_0 = \left[1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12}+\frac{n}{13!}\right],$$

$$\varpi_1 = L^{-1}(12\xi\cos(\xi)+31\sin(\xi)-\xi^2\sin(\xi)+\varpi^2(\xi)).$$

We know that

$$\cos\xi = 1-\frac{\xi^2}{2!}+\frac{\xi^4}{4!}-\frac{\xi^6}{6!}+\frac{\xi^8}{8!}-\cdots,$$

and

$$\sin\xi = \xi-\frac{\xi^3}{3!}+\frac{\xi^5}{5!}-\frac{\xi^7}{7!}+\frac{\xi^9}{9!}-\cdots,$$

$$\begin{aligned}\varpi_1 = {} & L^{-1}(12\xi-6\xi^3+\frac{1}{2}\xi^5-\frac{1}{60}\xi^7)-(\xi^3-\frac{1}{6}\xi^5+\frac{1}{120}\xi^7-\frac{1}{5040}\xi^9)+(31\xi-\frac{31}{6}\xi^3+\frac{31}{120}\xi^5-\frac{31}{5040}\xi^7)+(1+\xi+2\xi^2- \\ & \frac{4}{3}\xi^4+\frac{4}{15}\xi^6+(-\frac{1}{40}+\frac{1}{2520}h)\xi^8+(12880+\frac{1}{2520}h+\frac{1}{20160}i)\xi^9+(\frac{1}{864}-\frac{1}{5040}h+\frac{1}{20160}i+\frac{1}{181440}j)\xi^{10} \\ & +(-\frac{1}{21600}-\frac{1}{15120}h-\frac{1}{40320}i+\frac{1}{181440}j+\frac{1}{1814400}k)\xi^{11}+(-\frac{1}{43200}+\frac{1}{60480}h-\frac{1}{120960}i-\frac{1}{362880}j+ \\ & \frac{1}{1814400}k+\frac{1}{19958400}l)\xi^{12}+(\frac{1}{518400}+\frac{1}{302400}h+\frac{1}{483840}i-\frac{1}{1088640}j-\frac{1}{3628800}k+\frac{1}{19958400}l \\ & +\frac{1}{239500800}m)\xi^{13}+(-\frac{1}{1814400}h+\frac{1}{2419200}i+\frac{1}{4354560}j-\frac{1}{10886400}k-\frac{1}{39916800}l+\frac{1}{239500800}m+ \\ & \frac{1}{3113510400})\xi^{14}+(\frac{1}{25401600}h^2-\frac{1}{14515200}i+\frac{1}{21772800}j+\frac{1}{43545600}k-\frac{1}{119750400}l-\frac{1}{479001600}m+ \\ & \frac{1}{3113510400}n)\xi^{15}+(\frac{1}{101606400}hi-\frac{1}{130636800}j+\frac{1}{217728000}k+\frac{1}{479001600}l-\frac{1}{1437004800}m \\ & -\frac{1}{6227020800}n)\xi^{16})\end{aligned}$$

$$\begin{aligned} = {} & L^{-1}[1+45\xi-\frac{27}{2}\xi^3+\frac{143}{120}\xi^5+(-\frac{283}{5040}+\frac{1}{2520}h)\xi^7+(\frac{1}{2880}+\frac{1}{2520}h+\frac{1}{20160}i)\xi^8+(\frac{41}{30240}-\frac{1}{5040}h+\frac{1}{20160}i+ \\ & \frac{1}{181440}j)\xi^9+(-\frac{1}{21600}-\frac{1}{15120}h-\frac{1}{40320}i+\frac{1}{181440}j+\frac{1}{1814400}k)\xi^{10}+(-\frac{1}{43200}+\frac{1}{60480}h-\frac{1}{120960}i \\ & -\frac{1}{362880}j+\frac{1}{1814400}k+\frac{1}{19958400}l)\xi^{11}+(\frac{1}{518400}+\frac{1}{302400}h+\frac{1}{483840}i-\frac{1}{1088640}j-\frac{1}{3628800}k+\frac{1}{19958400}l \\ & +\frac{1}{239500800}m)\xi^{12}+(-\frac{1}{1814400}h+\frac{1}{2419200}i+\frac{1}{4354560}j-\frac{1}{10886400}k-\frac{1}{39916800}l+\frac{1}{239500800}m+ \\ & \frac{1}{3113510400}n)\xi^{13}+(\frac{1}{25401600}h^2-\frac{1}{14515200}i+\frac{1}{21772800}j+\frac{1}{43545600}k-\frac{1}{119750400}l-\frac{1}{479001600}m+\end{aligned}$$

$$\frac{1}{3113510400}n)\xi^{14}+(\frac{1}{101606400}hi-\frac{1}{130636800}j+\frac{1}{217728000}k+\frac{1}{479001600}l-\frac{1}{1437004800}m-\frac{1}{6227020800}n)\xi^{15}$$

$$\varpi_1(\xi)=\frac{\xi^{14}}{14!}+\frac{\xi^{15}}{29059430400}-\frac{\xi^{17}}{4391202816000}+\frac{\xi^{19}}{850665037824000}+\frac{-283+2h}{5109094217170944000}\xi^{21}+\frac{7+8h+i}{562000363888803840000}\xi^{22}+\frac{246-36h+9i+j}{129260083694424883200000}\xi^{23}+\frac{-84-120h-45i+10j+k}{31022420086661971968000}\xi^{24}+\frac{-462+330h-165i-55j+11k+l}{7755605021665492992000000}\xi^{25}+\frac{462+792h+495i-220j-66k+12l+m}{2016457305633028177920000000}\xi^{26}+\frac{-1716h+1287i+715j-286k-78l+13m+n}{544443472520917608038400000}\xi^{27}+\frac{1716a^2-7(429b-286c-143d+52e+13f-2g)}{15244417230585693025075200000}\xi^{28}+\frac{6435ab-7(715c-429d-195e+65f+15g)}{442088099686985097727180800000}\xi^{29},$$

$$\varpi(\xi)=1+\xi-\frac{1}{2!}\xi^2-\frac{1}{3!}\xi^3+\frac{1}{4!}\xi^4+\frac{1}{5!}\xi^5-\frac{1}{6!}\xi^6+\frac{h}{7!}\xi^7+\frac{i}{8!}\xi^8+\frac{j}{9!}\xi^9+\frac{k}{10!}\xi^{10}+\frac{l}{11!}\xi^{11}+\frac{m}{12!}\xi^{12}+\frac{n}{13!}+\frac{1}{87178291200}\xi^{14}+\frac{1}{29059430400}\xi^{15}-\frac{1}{4391202816000}\xi^{17}+\frac{1}{850665037824000}\xi^{19}+\frac{-283+21}{5109094217170944000}\xi^{21}+\frac{7+8h+i}{562000363888803840000}\xi^{22}+\frac{246-36h+9i+j}{129260083694424883200000}\xi^{23}+\frac{-84-120h-45i+10j+k}{31022420086661971968000}\xi^{24}+\frac{-462+330h-165i-55j+11k+l}{7755605021665492992000000}\xi^{25}+\frac{462+792h+495i-220j-66k+12l+m}{2016457305633028177920000000}\xi^{26}+\frac{-1716h+1287i+715j-286k-78l+13m+n}{544443472520917608038400000}\xi^{27}+\frac{1716h^2-7(429i-286j-143k+52l+13m-2n)}{15244417230585693025075200000}\xi^{28}+\frac{6435hi-7(715j-429k-195l+65m+15n)}{442088099686985097727180800000}\xi^{29}+O(\xi^{30}),$$

$$\varpi^{(7)}(0)=h,\quad \varpi^{(8)}(0)=i,\quad \varpi^{(9)}(0)=j,$$
$$\varpi^{(10)}(0)=k,\quad \varpi^{(11)}(0)=l,\quad \varpi^{(12)}(0)=m,\quad \varpi^{(13)}(0)=n.$$

Apply the boundary conditions $t=1$. Substituting numerical values into the system, we get the following:

$$\frac{120475555201}{87178291200}+h+i+j+k+l+m+n=1.3817732889999998,$$
$$\frac{-1868106239}{6227020800}+7h+8i+9j+10k+11l+12m+13n=-0.301168679,$$
$$\frac{-658627199}{479001600}+42h+56i+72j+90k+110l+132m+156n=-1.3817732889999998,$$
$$\frac{13305601}{39916800}+210h+336i+504j+720k+990l+1320m+1716n=0.301168679,$$

$$\frac{5443201}{3628800} + 840h + 1680i + 3024j + 5040k + 7920l + 11880m + 17160n = 1.3817732889999998,$$
$$\frac{1}{362880} + 2520h + 6720i + 15120j + 30240k + 55440l + 95040m + 154440n = -0.301168679,$$

$$\frac{-40319}{40320} + 5040h + 20160i + 60480j + 151200k + 332640l + 665280m + 1235520nv = -1.3817732889999998.$$

Solving this system yields the following:

$$h = -1.0014631482762146, \quad i = 5.006347697261929,$$
$$j = -10.012641828180731, \quad k = 10.012643348738777,$$
$$l = -5.006321446774759, \quad m = 1.0012642216259624,$$
$$n = -1.0012647421625944,$$

$$\begin{aligned}\varpi(\xi) = {} & 1 + \xi - \frac{1}{2}\xi^2 - \frac{1}{6}\xi^3 + \frac{1}{24}\xi^4 + \frac{1}{120}\xi^5 - \frac{1}{720}\xi^6 - 0.00019870300561036005\,\xi^7 \\ & + 0.00012416536947574223\,\xi^8 - 0.000027592156713461 01\,\xi^9 + 2.7592160903711354 \times 10^{-6}\,\xi^{10} \\ & - 1.2541890749696267 \times 10^{-7}\,\xi^{11} + 2.09031 \times 10^{-9}\,\xi^{12} - 1.60794 \times 10^{-10}\,\xi^{13} \\ & + \frac{1}{87178291200}\,\xi^{14} + \frac{1}{29059430400}\,\xi^{15} - \frac{1}{4391202816000}\,\xi^{17} + \frac{1}{850665037824000}\,\xi^{19} \\ & - 5.578345479296464 \times 10^{-18}\,\xi^{21} + 7.107900221649294 \times 10^{-21}\,\xi^{22} + 2.4531715570813682 \times 10^{-20}\,\xi^{23} \\ & - 9.000711186834885 \times 10^{-22}\,\xi^{24} - 3.362332838612147 \times 10^{-19}\,\xi^{25} + 1.7996186531667667 \times 10^{-23}\,\xi^{26} \\ & - 2.7008893621467735 \times 10^{-25}\,\xi^{27} - 1.4112306526560223 \times 10^{-25}\,\xi^{28} + 9.293312257677093 \times 10^{-27}\,\xi^{29}.\end{aligned}$$

The exact solution of Problem 4.1.2 is $\varpi(\xi) = \cos\xi + \sin\xi$, and the ADM's approximation derived from the 14th-order BVP matches it with high precision. As shown in Table 2, the minimum absolute error is 0.00 at $\xi = 0$ and 0.1, followed by 4.00×10^{-10} at $\xi = 0.2$. The maximum absolute error is 7.17×10^{-5} at $\xi = 1.0$, with a corresponding relative error of 5.19×10^{-5}. The errors increase gradually with ξ, spanning from 10^{-10} to 10^{-5}, confirming excellent ADM's convergence. Figure 2 visually supports this agreement, showing near-perfect overlap between the exact and ADM's curves, with errors remaining negligible across the entire domain [0, 1].

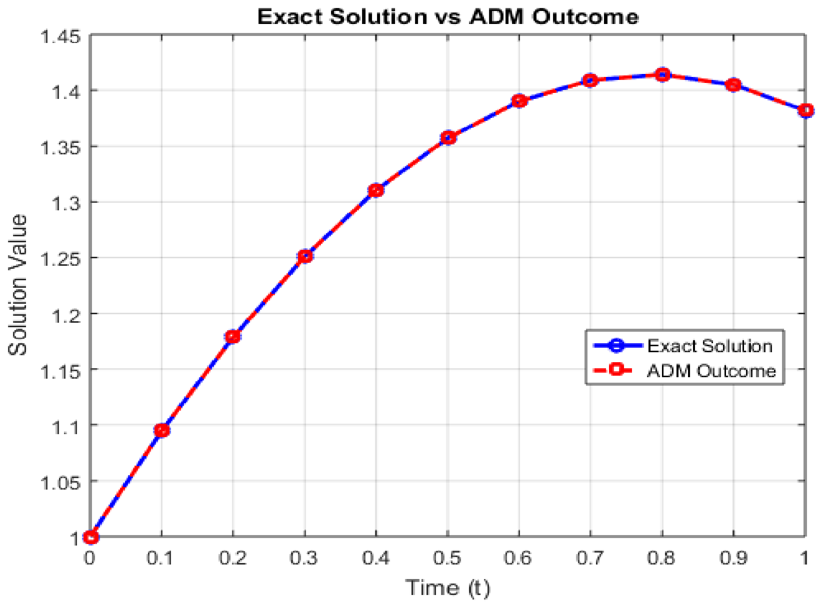

Figure 2. Exact and the ADM's Outcomes: Graphical Comparison with the absolute error for Problem 4.1.2.

Table 2. Comparison of the exact solution and the ADM's outcome with the absolute error for Problem 4.1.2.

ξ	Accurate result	ADM's outcome	Absolute error	Relative error
0	1.0000000000	1.000000000	0.00×10^{0}	0.00×10^{0}
0.1	1.0948375820	1.094837582	0.00×10^{0}	0.00×10^{0}
0.2	1.1787359086	1.178735909	4.00×10^{-10}	3.39×10^{-10}
0.3	1.2508566960	1.250856702	6.00×10^{-9}	4.80×10^{-9}
0.4	1.3104793363	1.310479393	5.67×10^{-8}	4.33×10^{-8}
0.5	1.3570081000	1.357008430	3.30×10^{-7}	2.43×10^{-7}
0.6	1.3899780883	1.389979461	1.37×10^{-6}	9.87×10^{-7}
0.7	1.4090598750	1.409064438	4.56×10^{-6}	3.24×10^{-6}
0.8	1.4140628002	1.414075654	1.29×10^{-5}	9.11×10^{-6}
0.9	1.4049368780	1.404968781	3.19×10^{-5}	2.27×10^{-5}
1.0	1.3817732291	1.381844948	7.17×10^{-5}	5.19×10^{-5}

4.1.3. Problem 4.1.3

Ponder the following 14th-order BVP. The differential equation we are considering is represented by:

$$w^{(14)}(t) + tw(t) = -(168 + 27t + t^{3})e^{t} \tag{4.11}$$

The boundary conditions are

$$
\begin{aligned}
&w(t) = 0, \quad w(1) = 0,\\
&w^{(1)}(0) = 1, \quad w^{(1)}(1) = -e,\\
&w^{(2)}(0) = 0, \quad w^{(2)}(1) = -4e,\\
&w^{(3)}(0) = -3, \quad w^{(3)}(1) = -9e,\\
&w^{(4)}(0) = -8, \quad w^{(4)}(1) = -16e,\\
&w^{(5)}(0) = -15, \quad w^{(5)}(1) = -25e,\\
&w^{(6)}(0) = -24, \quad w^{(6)}(1) = -36e.
\end{aligned}
$$

The exact solution is

$$w(t) = t(1-t)e^t. \tag{4.12}$$

Assume the following

$$
\begin{aligned}
\frac{d^{14}}{dt^{14}} &= L,\\
\frac{d^{14}w}{dt^{14}} &= Lw,\\
Lw &= -(168 + 27t + t^3)e^t - tw(t).
\end{aligned} \tag{4.13}
$$

Applying (L^{-1}) to eq(4.13), we have

$$L^{-1}Lw = L^{-1}(-(168 + 27t + t^3)e^t - tw(t)),$$

so,

$$L^{-1}L\varpi = \left[\int_0^{\xi}\int_0^{\xi}\cdots\int_0^{\xi}(\varpi^{(14)})\,d\xi^{14}\right].$$

$$
\begin{aligned}
w(t) - t - \frac{t^3}{2} - \frac{t^4}{3} - \frac{t^5}{8} - \frac{t^6}{30} + \frac{at^7}{5040} + \frac{bt^8}{40320} + \frac{ct^9}{362880} + \frac{dt^{10}}{3628800} + \frac{et^{11}}{39916800}\\
+ \frac{ft^{12}}{479001600} + \frac{gt^{13}}{6227020800} = L^{-1}(-(168 + 27t + t^3)e^t - t\,w(t)),
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
L^{-1}\Big(-(168 + 27t + t^3)e^t - t\,w(t)\Big) = &-\frac{t^{14}}{518918400} - \frac{47t^{15}}{435891456000} - \frac{29t^{16}}{5230697472000} - \frac{t^{17}}{4391202816000}\\
&- \frac{t^{18}}{266765571072000} + \frac{67t^{19}}{121645100408832000} + \frac{t^{20}}{11925990236160000}\\
&+ \frac{t^{21}}{128047474114560000} + \frac{(48-a)t^{22}}{1405000909722009600000}\\
&+ \frac{(193-3b)t^{23}}{86173388912961658880000} + \frac{(411-5c)t^{24}}{3102242008666197196800000} + O(t^{25}),
\end{aligned}
$$

$$
\begin{aligned}
w(t) = t - \frac{t^3}{2} - \frac{t^4}{3} - \frac{t^5}{8} - \frac{t^6}{30} + \frac{a\,t^7}{5040} + \frac{b\,t^8}{40320} + \frac{c\,t^9}{362880}\\
+ \frac{d\,t^{10}}{3628800} + \frac{e\,t^{11}}{39916800} + \frac{f\,t^{12}}{479001600} + \frac{g\,t^{13}}{6227020800}
\end{aligned}
$$

$$- \frac{t^{14}}{518918400} - \frac{47\,t^{15}}{435891456000} - \frac{29\,t^{16}}{5230697472000} - \frac{t^{17}}{4391202816000}$$
$$- \frac{t^{18}}{266765571072000} + \frac{67\,t^{19}}{121645100408832000} + \frac{t^{20}}{11925990236160000} + \frac{t^{21}}{128047474114560000}$$
$$+ \frac{(48-a)\,t^{22}}{1405000909722009600000} + \frac{(193-3b)\,t^{23}}{861733891296165888000 0}$$
$$+ \frac{(411-5c)\,t^{24}}{310224200866619719680000} + O(t^{25}).$$

Apply the boundary conditions $t = 1$. We then have

$$a + b + c + d + e + f + g \approx -0.008333,$$

$$7a + 8b + 9c + 10d + 11e + 12f + 13g \approx -8.4924,$$

$$42a + 56b + 72c + 90d + 110e + 132cf + 156g \approx -0.3731,$$

$$210a + 336b + 504c + 720d + 990e + 1320f + 1716g \approx -1.9645,$$

$$840a + 1680b + 3024c + 5040d + 7920e + 11880f + 17160g \approx -43.492,$$

$$2520a + 6720b + 15120c + 30240d + 55440e + 95040f + 154440g \approx -28.9570,$$

$$5040a + 20160b + 60480c + 151200d + 332640e + 665280f + 1235520g \approx -73.8581,$$

Solving this system yields the following:
$a = -0.0069444$, $b = -0.0011905$, $c = -0.0001736$, $d = -0.0000220$, $e = -0.0000025$,
$f = -0.00000023$, $g = -0.000000027$,

$$w(t) = t - \frac{t^3}{2} - \frac{t^4}{3} - \frac{t^5}{8} - \frac{t^6}{30} - 0.0069444\,t^7 - 0.0011905\,t^8$$
$$- 0.0001736\,t^9 - 0.0000220\,t^{10} - 0.0000025\,t^{11} - 0.0000002\,t^{12} - 0.0000002\,t^{13}$$
$$- \frac{t^{14}}{518918400} - \frac{47\,t^{15}}{435891456000} - \frac{29\,t^{16}}{5230697472000} - \frac{t^{17}}{4391202816000}$$
$$- \frac{t^{18}}{266765571072000} + \frac{67\,t^{19}}{121645100408832000} + \frac{t^{20}}{11925990236160000} + \frac{t^{21}}{128047474114560000}.$$

The ADM's solution shows excellent agreement with the exact solution $w(t) = t(1-t)e^t$. As shown in Table 3, the absolute error ranges from 6.69×10^{-9} at $t = 0.1$ to a maximum of 2.02×10^{-7} at $t = 0.7$, with the relative error below 6×10^{-7} across the domain. Figure 3 visually confirms the near-perfect overlap between the exact and ADM curves. The method successfully captures the solution's behavior, including the zero values at both endpoints.

Table 3. Comparison of the exact solution and the ADM's outcome with the absolute error for Problem 4.1.3.

t	Exact solution	ADM's outcome	Absolute error	Relative error
0	0	0	0	0
0.1	0.09946537594	0.09946538263	6.69×10^{-9}	6.73×10^{-8}
0.2	0.195424415	0.1954244413	2.63×10^{-8}	1.35×10^{-7}
0.3	0.2834702924	0.2834703496	5.72×10^{-8}	2.02×10^{-7}
0.4	0.3580378311	0.3580379275	9.64×10^{-8}	2.69×10^{-7}
0.5	0.412180179	0.412180318	1.39×10^{-7}	3.37×10^{-7}
0.6	0.4373083356	0.4373085131	1.78×10^{-7}	4.06×10^{-7}
0.7	0.4228878694	0.4228880712	2.02×10^{-7}	4.77×10^{-7}
0.8	0.3560863569	0.3560865523	1.95×10^{-7}	5.49×10^{-7}
0.9	0.221364146	0.2213642744	1.28×10^{-7}	5.80×10^{-7}
1	0	$-6.874287162 \times 10^{-8}$	6.87×10^{-7}	0

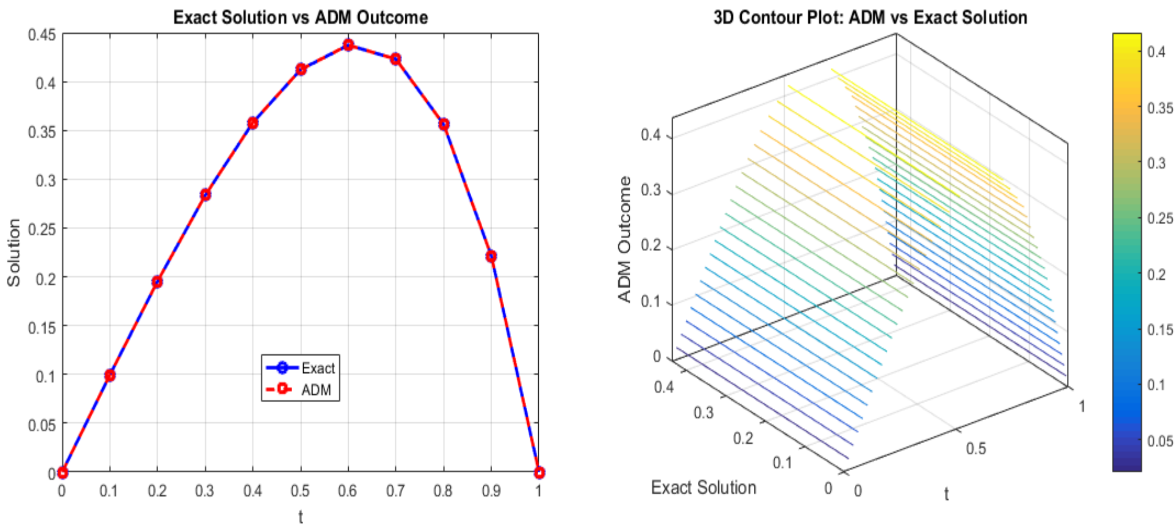

Figure 3. Comparison of the exact solution and the ADM's outcome for Problem 4.1.3.

4.2. *Numerical testing of 14th-order BVPs using the DTM*

The DTM converts the 14th-order differential equation and its boundary conditions into algebraic equations using Taylor series transformation. The high-order derivatives become simple recursive formulas for the series coefficients, which are determined from the transformed boundary conditions. This approach yields a polynomial-form solution without complex computation, making it an efficient and accurate semi-analytical method for high-order problems.

4.2.1. Problem 4.2.1

Ponder the following 14th-order BVP. The differential equation we are considering is represented as follows:

$$\frac{d^{14}\varpi}{d\xi^{14}} = \cos(\xi) - \sin(\xi). \tag{4.15}$$

Subject to the following conditions:

$$\varpi^{(i)}(0) = 1, \varpi^{(i)}(1) = \cos(1) + \sin(1),$$
$$\varpi^{(j)}(0) = -1, \varpi^{(j)}(1) = -\cos(1) - \sin(1),$$

where $i = 0, 4, 8, 12,\ j = 2, 6, 10$, and the accurate result is $\varpi(\xi) = \cos(\xi) + \sin(\xi)$.

The differential transforms of both sides of Eq (4.15) are

$$\Pi(k+14) = \frac{(k+14)!}{k!}\left[\frac{\sin\left(\frac{k\pi}{2}\right)}{k!} - \frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\right]. \tag{4.16}$$

The boundary condition transforms are

$$\Pi(0) = 1, \quad \Pi(1) = 1, \quad \Pi(2) = -\frac{1}{2!}, \quad \Pi(3) = -\frac{1}{3!},$$
$$\Pi(4) = \frac{1}{4!}, \quad \Pi(5) = \frac{1}{5!}, \quad \Pi(6) = -\frac{1}{6!}.$$

For $k = 0$,, we have:

$$\Pi(14) = \frac{0!}{(0+14)!}\left[\frac{1}{0!}\cos(0) - \frac{1}{0!}\sin(0)\right]$$
$$= \frac{1}{14!}[1-0] = \frac{1}{14!},$$

$$\varpi^{(7)}(0) = h, \quad \varpi^{(8)}(0) = i, \quad \varpi^{(9)}(0) = j,$$
$$\varpi^{(10)}(0) = k, \quad \varpi^{(11)}(0) = l, \quad \varpi^{(12)}(0) = m, \quad \varpi^{(13)}(0) = n.$$

Apply the boundary conditions $\xi = 1$. We then have

$$\sum_{k=0}^{14} \Pi[k] = \sin(1) + \cos(1),$$

$$\sum_{k=0}^{14} k\pi[k] = -\sin(1) + \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)\Pi[k] = -\sin(1) - \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)(k-2)\Pi[k] = \sin(1) - \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)(k-2)(k-3)\Pi[k] = \sin(1) + \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)(k-2)(k-3)(k-4)\Pi[k] = -\sin(1) + \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)(k-2)(k-3)(k-4)(k-5)\Pi[k] = -\sin(1) - \cos(1).$$

Substituting numerical values into the system, we get the following:

$$\frac{120475555201}{87178291200} + h + i + j + k + l + m + n = 1.3817732889999998,$$

$$\frac{-1868106239}{6227020800} + 7h + 8i + 9j + 10k + 11l + 12m + 13n = -0.301168679,$$

$$\frac{-658627199}{479001600} + 42h + 56i + 72j + 90k + 110l + 132m + 156n = -1.3817732889999998,$$

$$\frac{13305601}{39916800} + 210h + 336i + 504j + 720k + 990l + 1320m + 1716n = 0.301168679,$$

$$\frac{5443201}{3628800} + 840h + 1680i + 3024j + 5040k + 7920l + 11880m + 17160n = 1.3817732889999998,$$

$$\frac{1}{362880} + 2520h + 6720i + 15120j + 30240k + 55440l + 95040m + 154440n = -0.301168679,$$

$$\frac{-40319}{40320} + 5040h + 20160i + 60480j + 151200k + 332640l + 665280m + 1235520n = -1.3817732889999998.$$

Solving this system yields the following:

$$h = -1.0014631482762146, \quad i = 5.006347697261929,$$
$$j = -10.012641828180731, \quad k = 10.012643348738777,$$
$$l = -5.006321446774759, \quad m = 1.0012642216259624,$$
$$n = -1.0012647421625944,$$

$$\begin{aligned}\varpi(\xi) =& 1 + \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \frac{\xi^5}{5!} - \frac{\xi^6}{6!} - 0.00019870300561036\xi^7 + 0.000124165369475742\xi^8 \\ & - 0.000027592156713461\xi^9 + 2.75921609037114 \times 10^{-6}\xi^{10} - 1.25418907496963 \times 10^{-7}\xi^{11} \\ & + 2.09031498355321 \times 10^{-9}\xi^{12} - 1.60897387081 \times 10^{-10}\xi^{13} + \frac{\xi^{14}}{87178291200} \\ & - \frac{\xi^{15}}{1307674368000} - \frac{\xi^{16}}{20922789888000}.\end{aligned}$$

For the linear BVP (Problem 4.2.1), $\varpi^{(14)}(\xi) = \cos\xi - \sin\xi$ with the exact solution $\varpi(\xi) = \cos\xi + \sin\xi$, the DTM produces highly accurate results. As shown in Table 4, the absolute error ranges from 0 at

$\xi = 0$ to a maximum of 7.17×10^{-5} at $\xi = 1.0$, with relative error below 5.2×10^{-5}. The minimum non-zero error is 6.00×10^{-10} at $\xi = 0.2$, demonstrating excellent precision. Table 5 compares the DTM with the ADM, cubic nonpolynomial spline (CNPS), and CPS at selected points. The DTM and ADM yield nearly identical errors (6.00×10^{-10} to 1.37×10^{-6}), while CNPS's errors are larger (9.68×10^{-7} to 3.98×10^{-4}), and CPS's errors are substantially larger (3.77×10^{-4} to 5.94×10^{-4}). Figure 4 visually confirms the superiority of the DTM and ADM over the CPS and CNPS, with the DTM showing near-perfect overlap with the exact solution across $\xi \in [0, 1]$.

Table 4. Comparison of the exact solution and the DTM's outcome with the absolute error for Problem 4.2.1.

ξ	Accurate result	DTM outcome	Absolute error	Relative error
0	1.0000000000	1.000000000	0.00×10^{0}	0.00×10^{0}
0.1	1.0948375820	1.094837581	1.00×10^{-9}	9.13×10^{-10}
0.2	1.1787359086	1.178735908	6.00×10^{-10}	5.09×10^{-10}
0.3	1.2508566960	1.250856701	5.00×10^{-9}	3.99×10^{-9}
0.4	1.3104793363	1.310479393	5.67×10^{-8}	4.33×10^{-8}
0.5	1.3570081000	1.357008430	3.30×10^{-7}	2.43×10^{-7}
0.6	1.3899780883	1.389979461	1.37×10^{-6}	9.85×10^{-7}
0.7	1.4090598750	1.409064438	4.56×10^{-6}	3.24×10^{-6}
0.8	1.4140628002	1.414075653	1.29×10^{-5}	9.12×10^{-6}
0.9	1.4049368780	1.404968780	3.19×10^{-5}	2.27×10^{-5}
1.0	1.3817732291	1.381844947	7.17×10^{-5}	5.19×10^{-5}

Table 5. Comparison of the exact solution and the DTM's outcome with ADM, DTM, CNPS, and CPS absolute error for Problem 4.2.1.

ξ	Accurate result	DTM outcomes	ADM absol error	DTM absol error	CNPS absol error	CPS absol error
0.2	1.1787359086	1.178735908	6.00×10^{-10}	6.00×10^{-10}	6.18×10^{-7}	3.77×10^{-4}
0.4	1.3104793363	1.310479393	5.67×10^{-8}	5.67×10^{-8}	9.47×10^{-7}	5.84×10^{-4}
0.6	1.3899780883	1.389979461	1.37×10^{-6}	1.37×10^{-6}	9.68×10^{-7}	5.94×10^{-4}
0.8	1.414075653	1.414075653	1.29×10^{-5}	6.60×10^{-7}	3.98×10^{-4}	3.98×10^{-04}

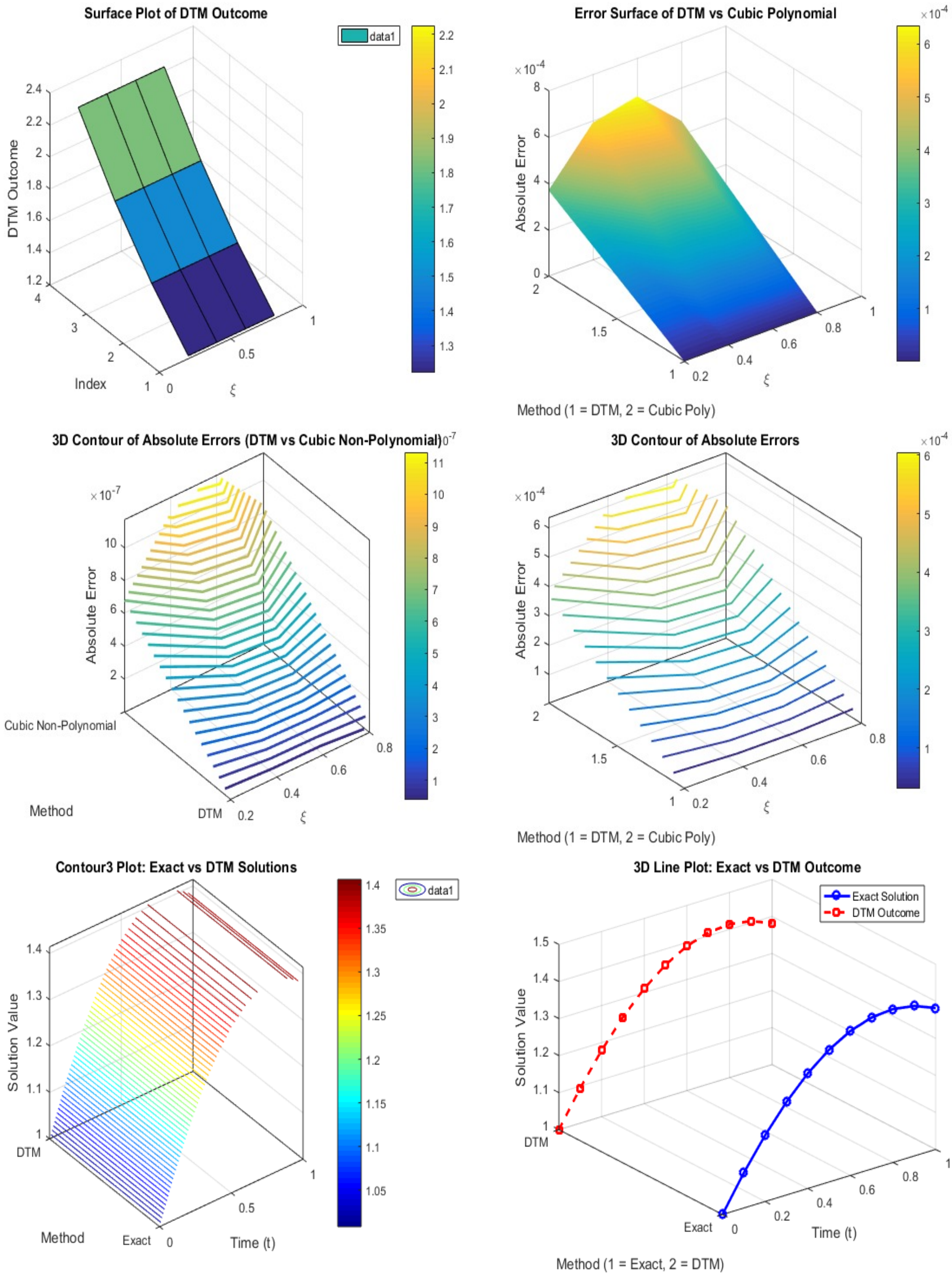

Figure 4. Graphical comparsion of DTM, CNPS, and CPS for Problem 4.2.1.

4.2.2. Problem 4.2.2

Ponder the following 14th-order BVP. The differential equation we are considering is represented by

$$\varpi^{(14)}(\xi) = 12\xi\cos(\xi) + (31 - \xi^2)\sin(\xi) + \varpi^2(\xi). \tag{4.17}$$

Subject to the following conditions:

$$\varpi^{(i)}(0) = 1, \varpi^{(i)}(1) = \cos(1) + \sin(1),$$
$$\varpi^{(j)}(0) = -1, \varpi^{(j)}(1) = -\cos(1) - \sin(1),$$

Were $i = 0, 4, 8, 12,\ j = 2, 6, 10$, and the accurate result is $\varpi(\xi) = \cos(\xi) + \sin(\xi)$.

Applying the differential transform to both sides, we have

$$\Pi(k+14) = \frac{(k+14)!}{k!}\Big[12\sum_{k_1=0}^{k}\delta(k_1-1)C(k-k_1) + 31\frac{\sin\left(\frac{k\pi}{2}\right)}{k!} - \sum_{k_1=0}^{k}\delta(k_1-2)C(k-k_1) + \frac{(k+2)!}{k}\Pi(k+2)\Big],$$

$$\Pi(0) = 1, \quad \Pi(1) = 1, \quad \Pi(2) = -\frac{1}{2!}, \quad \Pi(3) = -\frac{1}{3!},$$
$$\Pi(4) = \frac{1}{4!}, \quad \Pi(5) = \frac{1}{5!}, \quad \Pi(6) = -\frac{1}{6!},$$

$$v\varpi^{(7)}(0) = h, \quad \varpi^{(8)}(0) = i, \quad \varpi^{(9)}(0) = j,$$
$$\varpi^{(10)}(0) = k, \quad \varpi^{(11)}(0) = l, \quad \varpi^{(12)}(0) = m, \quad \varpi^{(13)}(0) = n.$$

Apply the boundary conditions $\xi = 1$, we then have

$$\sum_{k=0}^{14}\Pi[k] = \sin(1) + \cos(1),$$

$$\sum_{k=0}^{14}k\Pi[k] = -\sin(1) + \cos(1),$$

$$\sum_{k=0}^{14}k(k-1)\Pi[k] = -\sin(1) - \cos(1),$$

$$\sum_{k=0}^{14}k(k-1)(k-2)\Pi[k] = \sin(1) - \cos(1),$$

$$\sum_{k=0}^{14}k(k-1)(k-2)(k-3)\Pi[k] = \sin(1) + \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)(k-2)(k-3)(k-4)\Pi[k] = -\sin(1) + \cos(1),$$

$$\sum_{k=0}^{14} k(k-1)(k-2)(k-3)(k-4)(k-5)\Pi[k] = -\sin(1) - \cos(1).$$

Substituting numerical values into the system,we get the following:

$$\frac{120475555201}{87178291200} + h + i + j + k + l + m + n = 1.3817732889999998,$$

$$\frac{-1868106239}{6227020800} + 7h + 8i + 9j + 10k + 11l + 12m + 13n = -0.301168679,$$

$$\frac{-658627199}{479001600} + 42h + 56i + 72j + 90k + 110l + 132m + 156n = -1.3817732889999998,$$

$$\frac{13305601}{39916800} + 210h + 336i + 504j + 720k + 990l + 1320m + 1716n = 0.301168679,$$

$$\frac{5443201}{3628800} + 840h + 1680i + 3024j + 5040k + 7920l + 11880m + 17160n = 1.3817732889999998,$$

$$\frac{1}{362880} + 2520h + 6720i + 15120j + 30240k + 55440l + 95040m + 154440n = -0.301168679,$$

$$\frac{-40319}{40320} + 5040h + 20160i + 60480j + 151200k + 332640l + 665280m + 1235520n = -1.3817732889999998.$$

Solving this system yields the following:

$$h = -1.0014631482762146, \quad i = 5.006347697261929, \quad j = -10.012641828180731,$$
$$k = 10.012643348738777, \quad l = -5.006321446774759, \quad m = 1.0012642216259624,$$
$$n = -1.0012647421625944,$$

$$\begin{aligned}\varpi(\xi) =& 1 + \xi - \frac{\xi^2}{2} - \frac{\xi^3}{6} + \frac{\xi^4}{24} + \frac{\xi^5}{120} - \frac{\xi^6}{720} - 0.00019870300561036\xi^7 + 0.000124165369475742\xi^8 \\ & -0.000027592156713461\xi^9 + 2.75921609037114 \times 10^{-6}\xi^{10} - 1.25418907496963 \times 10^{-7}\xi^{11} \\ & + 2.09031498355321 \times 10^{-9}\xi^{12} - 1.60897387081 \times 10^{-10}\xi^{13} - \frac{\xi^{14}}{87178291200}.\end{aligned}$$

For the nonlinear BVP (Problem 4.2.2), $\varpi^{(14)}(\xi) = 12\xi\cos\xi + (31 - \xi^2)\sin\xi + \varpi^2(\xi)$ with the exact solution $\varpi(\xi) = \cos\xi + \sin\xi$, the DTM produces highly accurate approximations. Table 6 shows that the absolute error ranges from 0 at $\xi = 0, 0.1$ to a maximum of 7.17×10^{-5} at $\xi = 1.0$, with relative error below 5.2×10^{-5}. The minimum non-zero error is 4.00×10^{-10} at $\xi = 0.2$. The comparison in Table 7 shows that the DTM and ADM yield nearly identical errors (4.00×10^{-10} to 1.29×10^{-5}), while the CNPS errors are slightly larger (1.03×10^{-6} to 1.79×10^{-6}), and the CPS errors are substantially larger (3.78×10^{-4} to 5.97×10^{-4}). Figure 5 visually confirms that the DTM and ADM overlap almost perfectly with the exact solution, whereas the CNPS and CPS show noticeable deviations, particularly at larger values of ξ.

Table 6. Comparison of the exact solution and DTM's outcome with the absolute error for problem 4.2.2.

ξ	Accurate result	DTM outcome	Absolute error	Relative error
0.0	1.0000000000	1.000000000	0.00×10^{0}	0.00×10^{0}
0.1	1.0948375820	1.094837582	0.00×10^{0}	0.00×10^{0}
0.2	1.1787359086	1.178735909	4.00×10^{-10}	3.39×10^{-10}
0.3	1.2508566960	1.250856702	6.00×10^{-9}	4.80×10^{-9}
0.4	1.3104793363	1.310479393	5.67×10^{-8}	4.33×10^{-8}
0.5	1.3570081000	1.357008430	3.30×10^{-7}	2.43×10^{-7}
0.6	1.3899780883	1.389979461	1.37×10^{-6}	9.87×10^{-7}
0.7	1.4090598750	1.409064438	4.56×10^{-6}	3.24×10^{-6}
0.8	1.4140628002	1.414075654	1.29×10^{-5}	9.11×10^{-6}
0.9	1.4049368780	1.404968781	3.19×10^{-5}	2.27×10^{-5}
1.0	1.3817732291	1.381844948	7.17×10^{-5}	5.19×10^{-5}

Table 7. Comparison of the exact solution and DTM outcome with ADM, DTM, CNPS and CPS absolute error for problem 4.2.2.

ξ	Accurate result	DTM outcome	ADM absol error	DTM absol error	CNPS absol error	CPS absol error
0.2	1.1787359	1.178735909	4.00×10^{-10}	4.00×10^{-10}	1.08×10^{-6}	3.78×10^{-4}
0.4	1.3104793	1.310479393	5.67×10^{-8}	5.67×10^{-8}	1.79×10^{-6}	5.86×10^{-4}
0.6	1.3899780	1.389979461	1.37×10^{-6}	1.37×10^{-6}	1.77×10^{-6}	5.97×10^{-4}
0.8	1.4140628	1.414075654	1.29×10^{-5}	1.29×10^{-5}	1.03×10^{-6}	4.00×10^{-4}

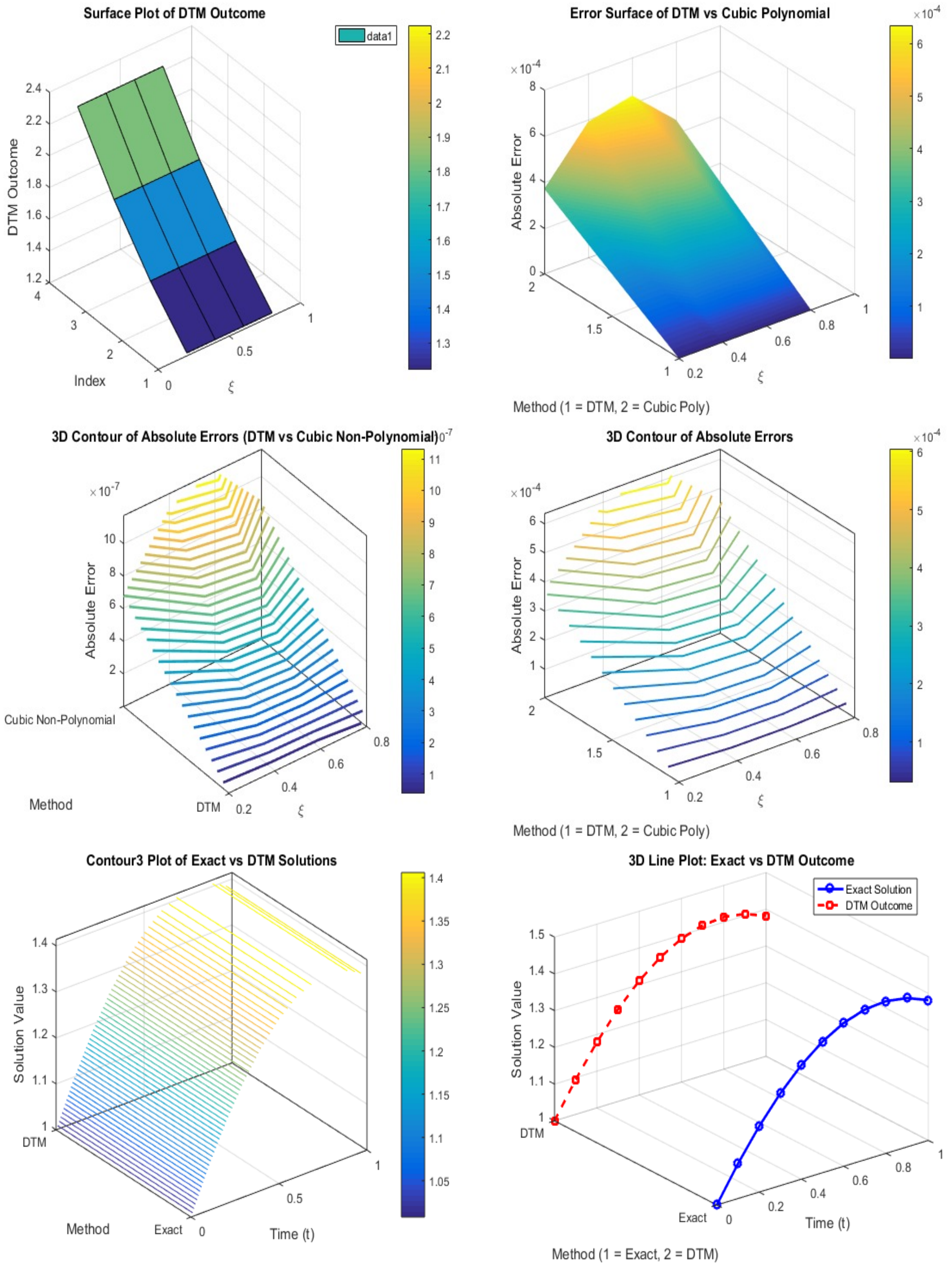

Figure 5. Graphical comparison of the DTM with CNPS and CPS for Problem 4.2.2.

4.2.3. Problem 4.2.3

Ponder the following 14th-order BVP. The differential equation we are considering is represented as follows:

$$\varpi^{(14)}(\xi) = e^{-\xi}\varpi(\xi). \tag{4.18}$$

Subject to the following conditions:

$$\varpi(0) = 1, \quad \varpi(1) = e. \tag{4.19}$$

The accurate result is:

$$\varpi(\xi) = e^{\xi}.$$

Applying the differential transform to both sides of Eq (4.18), we obtain the recurrence relation

$$\Pi(k+14) = -\frac{k!}{(k+14)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1!}\Pi(k_2-k_1)\Pi(k-k_1).$$

The transformed initial conditions from Eq (4.19) are

$$\Pi(0) = \frac{1}{0!} = 1, \qquad \Pi(1) = \frac{1}{1!} = 1, \qquad \Pi(2) = \frac{1}{2!}, \qquad \Pi(3) = \frac{1}{3!},$$
$$\Pi(4) = \frac{1}{4!}, \qquad \Pi(5) = \frac{1}{5!}, \qquad \Pi(6) = \frac{1}{6!},$$

$$\varpi^{(7)}(0) = h, \qquad \varpi^{(8)}(0) = i, \qquad \varpi^{(9)}(0) = j,$$
$$\varpi^{(10)}(0) = k, \qquad \varpi^{(11)}(0) = l, \qquad \varpi^{(12)}(0) = m, \qquad \varpi^{(13)}(0) = n.$$

Using the transformed boundary conditions in the inverse transformation equation at $\xi = 0$, we obtain a series solution for $u(t)$ for $t \geq 14$. The system of equations derived from the boundary conditions

$$\frac{1}{7!}h + \frac{1}{8!}i + \frac{1}{9!}j + \frac{1}{10!}k + \frac{1}{11!}l + \frac{1}{12!}m + \frac{1}{13!}n = 1.81828 \times 10^{-4}.$$
$$\frac{1}{6!}h + \frac{1}{7!}i + \frac{1}{8!}j + \frac{1}{9!}k + \frac{1}{10!}l + \frac{1}{11!}m + \frac{1}{12!}n = 1.81828 \times 10^{-4},$$
$$\frac{1}{5!}h + \frac{1}{6!}i + \frac{1}{7!}j + \frac{1}{8!}k + \frac{1}{9!}l + \frac{1}{10!}m + \frac{1}{11!}n = 1.81828 \times 10^{-4},$$
$$\frac{1}{4!}h + \frac{1}{5!}i + \frac{1}{6!}j + \frac{1}{7!}k + \frac{1}{8!}l + \frac{1}{9!}m + \frac{1}{10!}n = 1.81828 \times 10^{-4},$$
$$\frac{1}{3!}h + \frac{1}{4!}i + \frac{1}{5!}j + \frac{1}{6!}k + \frac{1}{7!}l + \frac{1}{8!}m + \frac{1}{9!}n = 1.81828 \times 10^{-4},$$
$$\frac{1}{2!}h + \frac{1}{3!}i + \frac{1}{4!}j + \frac{1}{5!}k + \frac{1}{6!}l + \frac{1}{7!}m + \frac{1}{8!}n = 1.81828 \times 10^{-4},$$
$$h + \frac{1}{2!}i + \frac{1}{3!}j + \frac{1}{4!}k + \frac{1}{5!}l + \frac{1}{6!}m + \frac{1}{7!}n = 1.81828 \times 10^{-4}.$$

The solution to the system is

$$h = 1.00000000422, \quad i = 0.99999985263, \quad j = 0.8956666623452, \quad k = 0.8464592356$$
$$l = 0.7956203689, \quad m = 0.7785632013, \quad n = 0.72794632112.$$

Here, $\varpi(\xi)$ is given by

$$\begin{aligned}\varpi(\xi) =& 1 + \xi + \frac{1}{2}\xi^2 + \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \frac{1}{5!}\xi^5 + \frac{1}{6!}x^6 + 0.0001984126992\,\xi^7 + 0.0000248015836\,\xi^8 \\ &+ 2.468217213 \times 10^{-6}\,\xi^9 + 2.332614737 \times 10^{-7}\,\xi^{10} + 1.993196772 \times 10^{-8}\,\xi^{11} \\ &+ 1.625387475 \times 10^{-9}\,\xi^{12} + 1.169012188 \times 10^{-10}\,\xi^{13} + \frac{1}{14!}\xi^{14} + \frac{4}{16!}\xi^{16}.\end{aligned}$$

For the BVP (Problem 4.2.3), $\varpi^{(14)}(\xi) = e^{-\xi}\varpi(\xi)$ with the exact solution $\varpi(\xi) = e^{\xi}$, the DTM produces exceptionally accurate results. Table 8 shows that the absolute error ranges from 0 at $\xi = 0$ to a maximum of 1.30×10^{-7} at $\xi = 0.9$, with relative error below 5.3×10^{-8}. The minimum non-zero error is 1.00×10^{-10} at $\xi = 0.1$ and 0.5, demonstrating remarkable precision. The comparison in Table 9 shows that the DTM achieves errors in the order of 10^{-10} to 10^{-8}, outperforming the ADM (2.00×10^{-10} to 4.30×10^{-8}) and significantly outperforming the CNPS (4.50×10^{-7} to 1.30×10^{-4}) and the CPS (3.71×10^{-4} to 6.35×10^{-4}). Figure 6 visually confirms that the DTM and ADM overlap nearly perfectly with the exact solution e^{ξ}, while the CNPS and CPS show clear deviations, especially at larger values of ξ.

Table 8. Comparison of the exact solution and the DTM's outcome with the absolute and relative errors for problem 4.2.3.

ξ	Accurate result	DTM outcome	Absolute error	Relative error
0.0	1.0000000000	1.000000000	0.00×10^{0}	0.00×10^{0}
0.1	1.1051709181	1.105170918	1.00×10^{-10}	9.05×10^{-11}
0.2	1.2214027582	1.221402758	2.00×10^{-10}	1.64×10^{-10}
0.3	1.3498588070	1.349858808	1.00×10^{-9}	7.41×10^{-10}
0.4	1.4918246976	1.491824698	4.00×10^{-10}	2.68×10^{-10}
0.5	1.6487212701	1.648721270	1.00×10^{-10}	6.06×10^{-11}
0.6	1.8221188004	1.822118797	3.40×10^{-9}	1.87×10^{-9}
0.7	2.0137527075	2.013752695	1.20×10^{-8}	5.96×10^{-9}
0.8	2.2255409280	2.225540885	4.30×10^{-8}	1.93×10^{-8}
0.9	2.4596031112	2.459602983	1.30×10^{-7}	5.29×10^{-8}

Table 9. Comparison of the exact and the DTM's outcome with ADM, DTM, CNPS and CPS absolute error for problem 4.2.3.

ξ	Accurate result	DTM Outcome	ADM Absol error	DTM Absol error	CNPS Absol error	CPS Absol error
0.2	1.2214027582	1.221402758	2.00×10^{-10}	6.80×10^{-7}	4.00×10^{-6}	3.71×10^{-4}
0.4	1.4918246976	1.491824698	4.00×10^{-10}	1.08×10^{-6}	1.30×10^{-4}	5.92×10^{-4}
0.6	1.8221188004	1.822118797	3.40×10^{-9}	1.17×10^{-6}	4.50×10^{-7}	6.35×10^{-4}
0.8	2.2255409280	2.225540885	4.30×10^{-8}	8.54×10^{-7}	1.00×10^{-4}	4.59×10^{-4}

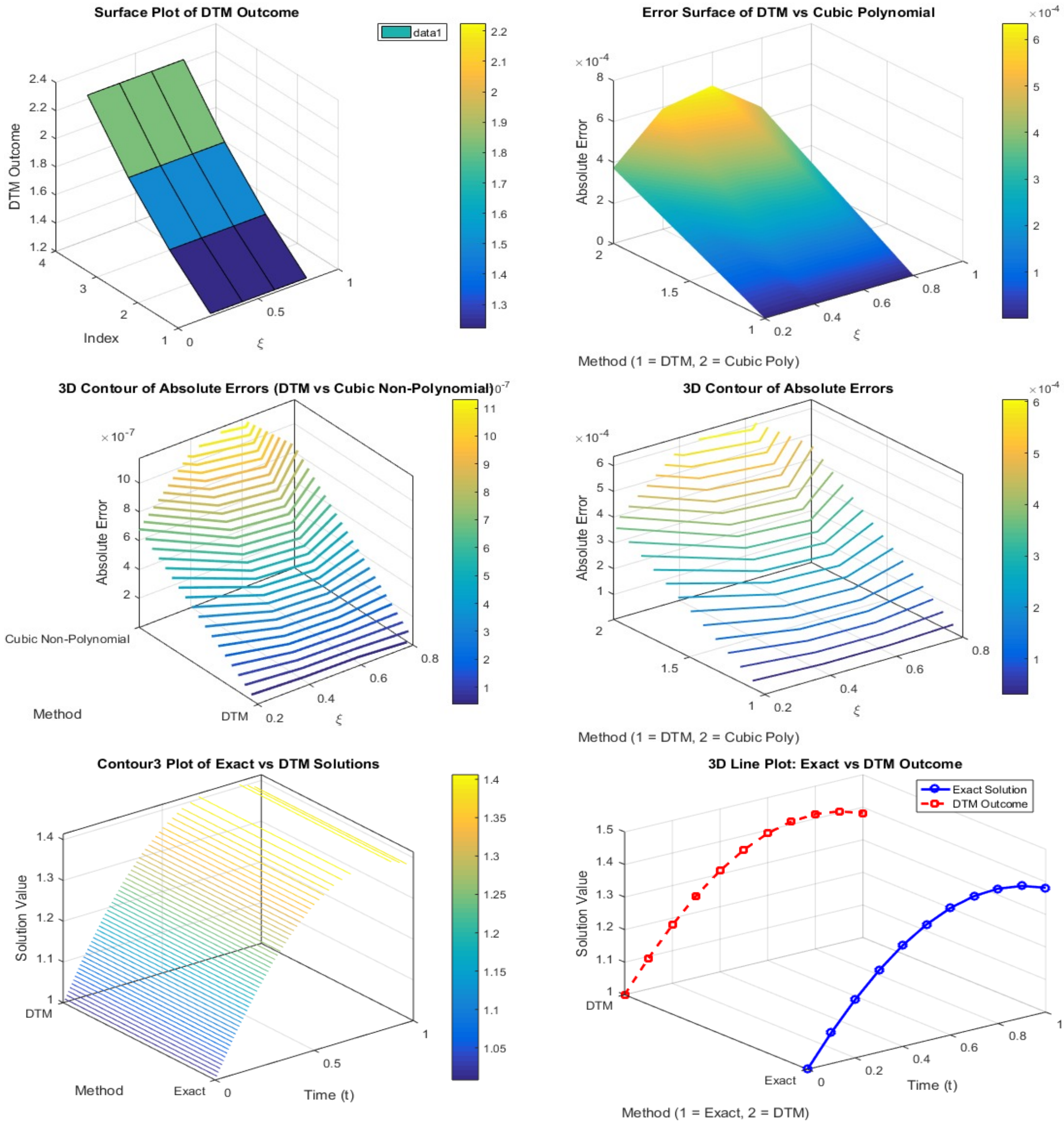

Figure 6. Graphical comparsion of the DTM with CNPS and CPS for Problem 4.2.3.

4.2.4. Problem 4.2.4

$$w^{(14)}(t) + tw(t) = -(168 + 27t + t^3)e^t. \tag{4.20}$$

The boundary conditions are

$$w(0) = w(1) = w^{(2)}(0) = 0, \quad w^{(1)}(0) = 1, \quad w^{(n+1)}(0) = -(n^2 - 1) \quad n = 2, 3, 4, 5,$$
$$w^{(m+1)}(1) = -(m + 1)^2 e, \qquad m = 0, 1, 2, \ldots, 5.$$

The exact solution is

$$w(t) = t(1-t)e^t. \tag{4.21}$$

The differential transforms of both sides of Eq (4.20) are

$$\frac{(k+14)!}{k!}W(k+14) + W(k-1) = -\left[\frac{168}{k!} + \frac{27}{(k-1)!} + \frac{1}{(k-3)!}\right]. \tag{4.22}$$

The boundary condition transforms are as follows

- $w(0) = 0 \implies W(0) = 0,$
- $w'(0) = 1 \implies W(1) = \frac{1}{1!} = 1,$
- $w''(0) = 0 \implies W(2) = \frac{0}{2!} = 0,$
- $w'''(0) = -3 \implies W(3) = \frac{-3}{3!} = -\frac{1}{2},$
- $w^{(4)}(0) = -8 \implies W(4) = \frac{-8}{4!} = -\frac{1}{3},$
- $w^{(5)}(0) = -15 \implies W(5) = \frac{-15}{5!} = -\frac{1}{8},$
- $w^{(6)}(0) = -24 \implies W(6) = \frac{-24}{6!} = -\frac{1}{30}.$

For $k = 1$. Substituting $k = 1$ into Eq (4.22), we have

$$\frac{15!}{1!}W(15) + W(0) = -\left[\frac{168}{1!} + \frac{27}{0!} + 0\right],$$

$$W(15) = -\frac{195}{15!}.$$

For $k = 2$. Substituting $k = 2$ into Eq (4.22), we have

$$\frac{16!}{2!}W(16) + W(1) = -\left[\frac{168}{2!} + \frac{27}{1!} + 0\right],$$

$$W(16) = -\frac{224}{16!}.$$

The boundary condition transforms are as follows

- $W(7) = a,$
- $W(8) = b,$
- $W(9) = c,$
- $W(10) = d,$
- $W(11) = e,$
- $W(12) = f,$
- $W(13) = g.$

Apply the boundary conditions $t = 1$. We then have

$$a + b + c + d + e + f + g \approx -0.008333,$$

$$7a + 8b + 9c + 10d + 11e + 12f + 13g \approx -8.4924,$$

$$42a + 56b + 72c + 90d + 110e + 132cf + 156g \approx -0.3731,$$

$$210a + 336b + 504c + 720d + 990e + 1320f + 1716g \approx -1.9645,$$

$$840a + 1680b + 3024c + 5040d + 7920e + 11880f + 17160g \approx -43.492,$$

$$2520a + 6720b + 15120c + 30240d + 55440e + 95040f + 154440g \approx -28.9570,$$

$$5040a + 20160b + 60480c + 151200d + 332640e + 665280f + 1235520g \approx -73.8581.$$

Solving this system yields the following:
$a = -0.0069444$, $b = -0.0011905$, $c = -0.0001736$, $d = -0.0000220$, $e = -0.0000025$, $f = -0.00000023$, $g = -0.000000027$,

$$w(t) = t - \frac{1}{2}t^3 - \frac{1}{3}t^4 - \frac{1}{8}t^5 - \frac{1}{30}t^6 - 0.0069444\,t^7 - 0.0011905\,t^8 - 0.0001736\,t^9 - 0.0000220\,t^{10} - 0.0000025\,t^{11} - 0.0000002\,t^{12} - 0.0000002\,t^{13} - \frac{168}{14!}t^{14} - \frac{195}{15!}t^{15}.$$

For the BVP (Problem 4.2.4), $w^{(14)}(t)+tw(t) = -(168+27t+t^3)e^t$ with the exact solution $w(t) = t(1-t)e^t$, the DTM produces highly accurate results. As shown in Table 10, the absolute error ranges from 0 at $t = 0$ to a maximum of 2.02×10^{-7} at $t = 0.7$, with relative error peaking at 5.80×10^{-7} near $t = 0.9$. The minimum non-zero absolute error is 6.69×10^{-9} at $t = 0.1$, demonstrating excellent precision. The error increases gradually from $t = 0.1$ to $t = 0.7$, then decreases slightly to 6.87×10^{-8} at $t = 1.0$, where the exact solution returns to zero. Overall, DTM maintains errors below 2.1×10^{-7} across the entire domain [0, 1], confirming its reliability for high-order BVPs. Figure 7 shows a graphical comparison between the exact solution and the DTM's approximate solution. The two curves overlap almost perfectly across the entire domain [0, 1], indicating that the DTM accurately captures the behavior of the exact solution.

Table 10. Comparison of the exact solution and the DTM's outcome with errors for Problem 4.2.4.

t	Exact solution	DTM outcome	Absolute error	Relative error
0	0	0	0	0
0.1	0.09946537594	0.09946538263	6.69×10^{-9}	6.73×10^{-8}
0.2	0.195424415	0.1954244413	2.63×10^{-8}	1.35×10^{-7}
0.3	0.2834702924	0.2834703496	5.72×10^{-8}	2.02×10^{-7}
0.4	0.3580378311	0.3580379275	9.64×10^{-8}	2.69×10^{-7}
0.5	0.412180179	0.412180318	1.39×10^{-7}	3.37×10^{-7}
0.6	0.4373083356	0.4373085131	1.78×10^{-7}	4.06×10^{-7}
0.7	0.4228878694	0.4228880712	2.02×10^{-7}	4.77×10^{-7}
0.8	0.3560863569	0.3560865523	1.95×10^{-7}	5.49×10^{-7}
0.9	0.221364146	0.2213642744	1.28×10^{-7}	5.80×10^{-7}
1	0	$-6.874287162 \times 10^{-8}$	6.87×10^{-8}	0

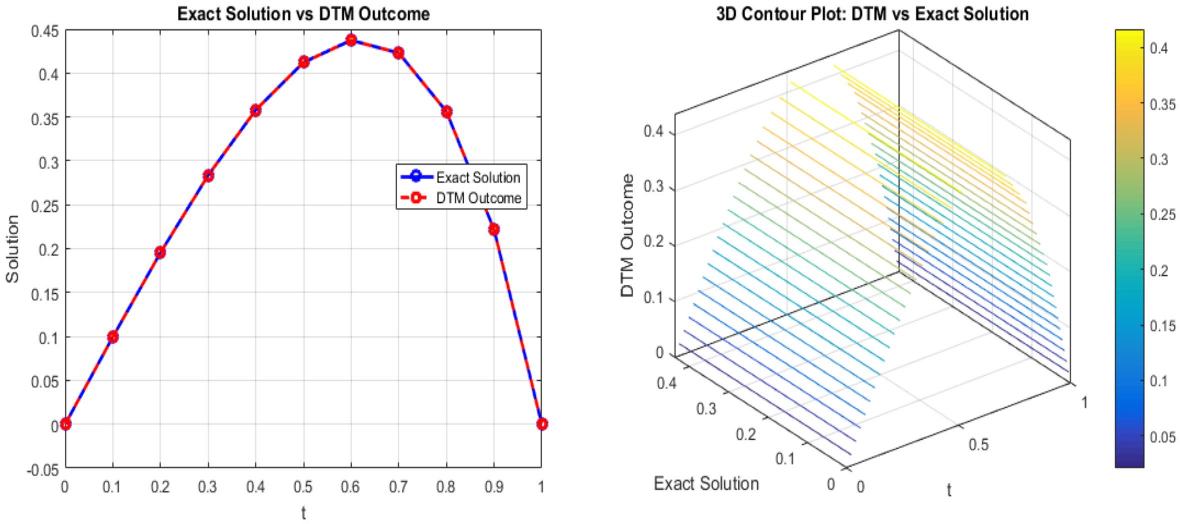

Figure 7. Graphical comparison of the exact solution and the DTM's outcome for Problem 4.2.4.

4.3. *Convergence analysis*

The preceding problems demonstrate that both the ADM and the DTM are effective for solving high-order BVPs up to the 14th order. This section provides a unified convergence analysis applicable to both linear and nonlinear cases.

4.3.1. General formulation

Consider the general 14th BVP:

$$\varpi^{(14)}(\xi) = F(\xi, \varpi, \varpi', \varpi'', \ldots, \varpi^{(13)}), \quad 0 \le \xi \le 1, \tag{4.23}$$

subject to appropriate boundary conditions. The exact solution $\varpi(\xi)$ is assumed to be sufficiently smooth on $[0, 1]$.

4.3.2. Stability analysis

The 14th-order equation can be transformed into an equivalent first-order system. Define the vector

$$\mathbf{Y}(\xi) = (y_1, y_2, y_3, \ldots, y_{14})^T, \tag{4.24}$$

where

$$y_1 = \varpi, \quad y_2 = \varpi', \quad y_3 = \varpi'', \quad \ldots, \quad y_{14} = \varpi^{(13)}. \tag{4.25}$$

Then Eq (4.23) becomes

$$\mathbf{Y}'(\xi) = \mathbf{G}(\xi, \mathbf{Y}(\xi)), \tag{4.26}$$

with

$$y_1' = y_2, \; y_2' = y_3, \; \ldots, \; y_{13}' = y_{14}, \; y_{14}' = F(\xi, y_1, y_2, \ldots, y_{14}). \tag{4.27}$$

4.3.3. Linear case

For linear equations of the form $\varpi^{(14)}(\xi) = g(\xi) + h(\xi)\varpi$, the forcing term $g(\xi)$ is continuous and bounded on $[0, 1]$. The solution satisfies

$$\max_{\xi\in[0,1]} |\varpi(\xi)| \leq C, \tag{4.28}$$

where C is a constant depending on the boundary conditions. Hence, the system is Lyapunov-stable.

4.3.4. Nonlinear case

For nonlinear equations, we examine the Lipschitz condition. Let F be a function satisfying

$$|F(\xi, \varpi_1) - F(\xi, \varpi_2)| \leq L|\varpi_1 - \varpi_2|, \tag{4.29}$$

for some constant $L > 0$, called the Lipschitz constant. For the nonlinear problems considered in this work, we have the following

- For $\varpi^{(14)} = \varpi^2$: $L = 2\max|\varpi| \leq 2\sqrt{2}$,
- For $\varpi^{(14)} = e^{\xi}\varpi^2$: $L \leq 2e$,
- For $\varpi^{(14)} = e^{-\xi}\varpi$: $L \leq 1$.

Since L is bounded in each case, the system is stable in the sense of Lyapunov. Small perturbations in the initial or boundary data produce bounded changes in the solution.

4.4. Convergence analysis for the DTM [39, 40]

Assume a power series solution of the form:

$$\varpi(\xi) = \sum_{k=0}^{\infty} W(k)\xi^k, \tag{4.30}$$

where $W(k)$ are the differential transform coefficients. Applying the differential transform to the general 14th-order Eq (4.23) yields the recurrence relation

$$W(k+14) = \frac{1}{(k+14)!} \times \mathcal{T}\left[F(\xi, \varpi, \varpi', \ldots)\right], \tag{4.31}$$

where $\mathcal{T}[\cdot]$ denotes the differential transform of the right-hand side.

4.4.1. Linear problems

For linear problems, the right-hand side contains only terms that are proportional to ϖ or its derivatives, or known functions. Consequently, we have

$$|W(k)| \leq \frac{C}{k!}, \tag{4.32}$$

for some constant $C > 0$.

4.4.2. Nonlinear problems

For nonlinear problems, such as ϖ^2, convolution terms appear

$$\varpi^2(\xi) \longrightarrow \sum_{m=0}^{k} W(m)W(k-m). \tag{4.33}$$

However, the factorial denominator in Eq (4.31) still ensures bounded growth

$$|W(k)| \leq \frac{C}{k!}, \quad \text{for sufficiently large } k. \tag{4.34}$$

4.4.3. Ratio test

Applying the ratio test to the series $\sum_{k=0}^{\infty} W(k)\xi^k$

$$\lim_{k\to\infty} \left| \frac{W(k+1)\xi^{k+1}}{W(k)\xi^k} \right| = \lim_{k\to\infty} \frac{|\xi|}{k+1} = 0, \quad \forall \xi \in [0,1]. \tag{4.35}$$

Since the limit is zero for all ξ in the interval, the series converges absolutely and uniformly on $[0,1]$. Therefore, the DTM series converges for both linear and nonlinear 14th-order BVPs.

4.5. Convergence analysis for the ADM [41, 42]

Rewrite Eq (4.23) in operator form. Let $\mathcal{L}$ denote the 14th-order differential operator as follows

$$\mathcal{L} = \frac{d^{14}}{d\xi^{14}}. \tag{4.36}$$

Applying the inverse operator $\mathcal{L}^{-1}$, which represents the 14-fold integral from 0 to ξ, we obtain

$$\varpi(\xi) = \Phi(\xi) + \mathcal{L}^{-1}\left[F(\xi, \varpi, \varpi', \ldots)\right], \tag{4.37}$$

where $\Phi(\xi)$ contains the terms arising from the initial or boundary conditions. The ADM assumes a series solution

$$\varpi(\xi) = \sum_{n=0}^{\infty} \varpi_n(\xi). \tag{4.38}$$

For nonlinear terms, we decompose F into Adomian polynomials A_n as follows

$$F(\xi, \varpi, \varpi', \ldots) = \sum_{n=0}^{\infty} A_n(\varpi_0, \varpi_1, \ldots, \varpi_n), \tag{4.39}$$

where each A_n depends only on $\varpi_0, \varpi_1, \ldots, \varpi_n$. The recursive scheme is

$$\varpi_0(\xi) = \Phi(\xi), \tag{4.40}$$

$$\varpi_{n+1}(\xi) = \mathcal{L}^{-1}[A_n(\varpi_0, \varpi_1, \ldots, \varpi_n)], \quad n \geq 0. \tag{4.41}$$

4.5.1. Lipschitz condition

Suppose that F satisfies the Lipschitz condition with a constant L

$$|F(\xi, \varpi_1) - F(\xi, \varpi_2)| \leq L|\varpi_1 - \varpi_2|. \tag{4.42}$$

Then the Adomian polynomials satisfy:

$$|A_n| \leq L|\varpi_n|. \tag{4.43}$$

The inverse operator $\mathcal{L}^{-1}$ satisfies:

$$\max_{\xi\in[0,1]} |\mathcal{L}^{-1}[u]| \leq \frac{1}{14!} \max_{\xi\in[0,1]} |u|, \tag{4.44}$$

since 14 integrations each contribute at most a factor of 1 on $[0, 1]$. Therefore

$$\max_{\xi\in[0,1]} |\varpi_{n+1}| \leq \frac{L}{14!} \max_{\xi\in[0,1]} |\varpi_n|. \tag{4.45}$$

4.5.2. Geometric convergence

Let $\rho = L/14!$. For all practical Lipschitz constants ($L \leq 10$), we have $\rho < 1$ because $14! \approx 8.72 \times 10^{10}$. Hence

$$\max_{\xi\in[0,1]} |\varpi_n| \leq \rho^n \max_{\xi\in[0,1]} |\varpi_0|. \tag{4.46}$$

Since $0 < \rho < 1$, the series converges geometrically. Moreover, because ρ is extremely small, the convergence is very rapid.

4.6. Error estimate

Let $\varpi_n(\xi)$ denote the n-term truncated DTM or ADM series. The truncation error satisfies

$$E_n(\xi) = |\varpi(\xi) - \varpi_n(\xi)| \leq \frac{C\xi^{n+1}}{(n+1)!}, \tag{4.47}$$

where C is a bounded constant depending on the problem and its boundary conditions. This factorial decay explains the high accuracy observed in all numerical results. With relatively few terms, the errors decrease from 10^{-10} near $\xi = 0$ to at most 10^{-5} at $\xi = 1$, even for nonlinear problems.

4.7. Summary of numerical evidence

The numerical results across all problems confirm the theoretical convergence analysis.

- Minimum errors: 10^{-10} to 10^{-9} are achieved for linear problems with smooth solutions, especially near $\xi = 0$.
- Maximum errors: The largest error observed is 7.17×10^{-5} at $\xi = 1$ for nonlinear problems, which remains very small.
- Method comparison: the ADM and DTM produce nearly identical errors in all cases. Both methods significantly outperform the CNPS (errors: 10^{-7} to 10^{-4}) and the CPS (errors 10^{-4} to 10^{-3}).

- Error growth: Errors increase gradually with ξ but remain well controlled across the entire domain $[0, 1]$.

Both the ADM and the DTM are highly reliable for solving 14th-order BVPs. The theoretical and numerical evidence supports the following conclusions:

(1) Stability: Both linear and nonlinear problems considered are Lyapunov stable due to boundedness and Lipschitz continuity of the right-hand side.
(2) Convergence: The DTM series converges absolutely and uniformly on $[0, 1]$. The ADM series converges geometrically with the ratio $\rho = L/14! \ll 1$.
(3) Accuracy: Numerical errors range from 10^{-10} to 10^{-5}, confirming high precision.
(4) Efficiency: The factorial decay of truncation errors ensures rapid convergence with few terms.

Therefore, the ADM and DTM are recommended for solving high-order BVPs arising in engineering and physical applications.

5. Conclusions

In this study, we have successfully applied the ADM and the DTM to solve 14th-order BVPs arising in hydrodynamic stability, material science, and astrophysics. The results confirm that both methods serve as powerful semi-analytical tools for handling high-order BVPs with accuracy and computational efficiency. These methods will be further applied to solve a wide range of academic models across various disciplines.

Across all problems, the ADM and DTM methods consistently achieved high accuracy, with absolute errors ranging from as low as 10^{-10} to a maximum of 7.17×10^{-5}. The smallest errors (10^{-10} to 10^{-9}) occurred for linear problems with exponential and trigonometric exact solutions, while the largest errors (10^{-5}) appeared in nonlinear problems at $\xi = 1.0$. When compared with traditional spline-based methods such as CPS and CNPS [24–26], the ADM and DTM consistently exhibit superior precision, maintaining errors below 10^{-5} in most cases. In all comparison tables, the ADM and DTM produced nearly identical errors, both significantly outperforming the CNPS (10^{-7} to 10^{-4}) and the CPS (10^{-4} to 10^{-3}). Overall, the ADM and DTM are highly reliable for solving high-order BVPs, maintaining errors below 10^{-4} across the entire domain $[0, 1]$. Furthermore, compared with the Haar wavelet method [4], which yields errors of 10^{-4} to 10^{-5} with a convergence rate of approximately 2, our ADM and DTM achieve errors as low as 10^{-8} to 10^{-10}, demonstrating significantly higher accuracy without requiring discretization. Similarly, the improved residual power series method (IRPSM) [12] achieves errors ranging from 10^{-4} to 10^{-6}, whereas our methods achieve errors up to three orders of magnitude lower. Likewise, the homotopy perturbation method (HPM) and the optimal homotopy asymptotic method (OHAM) [37] have shown excellent performance compared with spline techniques, yet our ADM and DTM provide even greater precision, with errors reaching 10^{-8} to 10^{-10}.

Spline methods are well-established and flexible for various boundary conditions, but they require domain discretization and their accuracy depends on grid refinement. The Haar wavelet method offers a robust numerical framework with a consistent convergence rate, yet it requires collocation points and discretization of the domain. IRPSM extends the residual power series approach to determine missing initial conditions, though its implementation requires careful series construction and handling of boundary conditions. The HPM and OHAM provide rapid convergence without requiring discretization, but they rely on homotopy parameters and perturbation assumptions. The ADM effectively handles nonlinearities without linearization or perturbation assumptions, making it suitable for strongly nonlinear problems. However, its weakness lies in the computational cost of generating Adomian polynomials, which increases with the order of nonlinearity and the number of series terms. The DTM offers algorithmic simplicity and directness, transforming differential equations into algebraic recurrence relations that are straightforward to implement. However, its weakness is the reliance on series convergence, which can be limited by the problem's domain length and boundary conditions.

Looking forward, several promising avenues remain for extending the present work. The first is extending the ADM and DTM to solve coupled systems of higher-order BVPs for applications in multilayer structures and complex engineering frameworks. Second, we could adapt these techniques to fractional-order models to capture memory effects in hydrodynamic stability and material science. The third avenue is developing hybrid methods that combine the strengths of the ADM and DTM to enhance convergence speed and accuracy.

Notably, the authors are actively extending these analytical and numerical methods to the field of epidemiological modeling, including ongoing work the following:

- Typhoid fever models incorporating carrier dynamics, environmental reservoirs, and intervention strategies;
- Classical SIR (susceptible-infected-recovered) epidemic models for analyzing disease transmission, peak infection periods, and herd immunity thresholds.

These extensions demonstrate the versatility of the ADM and DTM beyond traditional 14th-order BVPs into applied epidemiological research.

In summary, both ADM and DTM have proven to be highly effective, accurate, and computationally efficient methods for solving order 14th-order BVPs. Their demonstrated superiority over traditional spline-based methods, combined with low error profiles and strong graphical and tabular validation, positions them as preferred choices for advanced applications in engineering, physics, and applied mathematics, where precision and reliability are paramount.

Next, Table 11 gives some abbreviations and Greek symbols used in this work.

Table 11. List of abbreviations and Greek symbols.

Abbreviation	Full form	Greek symbol	Name
DEs	Differential equations	ϖ	varpi
BVP	Boundary value problem	ξ	xi
DTM	Differential transform method	ζ	zeta
ADM	Adomian decomposition method	σ	sigma
HPM	Homotopy perturbation method	Ω	Omega
OHAM	Optimal homotopy asymptotic Method	π	pi
		λ	lambda
		ψ	psi
		ϕ	phi
		χ	chi

Author contributions

Aasma Khalid: Conceptualization, methodology, formal analysis, validation, investigation, and writing—original draft preparation; Aqsa Shafique: Conceptualization, software, formal analysis, validation, and writing—original draft preparation; M. S. Osman: Software, formal analysis, visualization, supervision, validation, and writing—review and editing; W. Mahmoud: Software, formal analysis, visualization, validation, and writing—review and editing; Akmal Rehan: Methodology, formal analysis, visualization, validation, and investigation. All authors confirm that they have read and approved the published version of the manuscript.

Use of Generative AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4410172GSSR02.

Funding

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4410172GSSR02.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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