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Research article

Exact solution for nonlinear deep-water waves

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Abstract: In this paper, we developed a generalized analytical framework for nonlinear deep-water waves by incorporating time-varying background flows ($s_1(t)$, $s_2(t)$), effectively extending the classical Gerstner wave theory to unsteady environments. Within the framework of nonlinear partial differential equations, exact solutions for particle trajectories, velocity fields, and vorticity were rigorously derived using a time-varying Lagrangian mapping, while volume conservation throughout the evolutionary process was ensured via Jacobian analysis. A self-consistent pressure function was constructed to satisfy the nonlinear dynamic boundary conditions at the free surface. The core contribution of this study lies in the derivation of a modulated dispersion relation, which reveals how the coupling mechanism between wave dynamics and background acceleration induces an effective gravity shift, thereby explicitly regulating the phase velocity. By integrating these time-varying background flow terms, this work provides a robust theoretical foundation for characterizing transient wave-current interactions and nonlinear evolutionary processes in complex, non-steady ocean systems.

Keywords: Gerstner wave; exact solution; unsteady environments; dispersion relation; Lagrangian coordinates
Mathematics Subject Classification: 53Q30, 76B15

1. Introduction

Waves are one of the most common and complex phenomena in ocean dynamics. Based on the classical Lagrangian description of water waves [1, 2], it is theoretically well-established that when surface waves propagate on deep water, the surface water particles trace roughly circular orbits with a diameter equal to the wave height (see Figure 1). As the distance from the free surface increases, these analytical particle paths gradually diminish. Waves in the equatorial region are influenced not only by the wind field and ocean depth but also by factors such as the Coriolis force, a result of the Earth's rotation [3]. As a nonlinear wave model, Gerstner wave describes the motion trajectory of water particles in deep water waves. Unlike the propagation of linear water waves in deep water, the Gerstner wave can more realistically simulate wave dynamics in scenarios with large wave amplitudes [4]. Gerstner wave, in particular, offers more precise predictions than linear wave theory when it comes to describing deep-water waves and large-scale fluctuations. Waves in the equatorial region are typically accompanied by strong winds, tropical cyclones, and other phenomena, which lead to more pronounced nonlinear characteristics of the waves. Consequently, the application of Gerstner wave is highly significant for the study of deep water wave propagation at the equator [5].

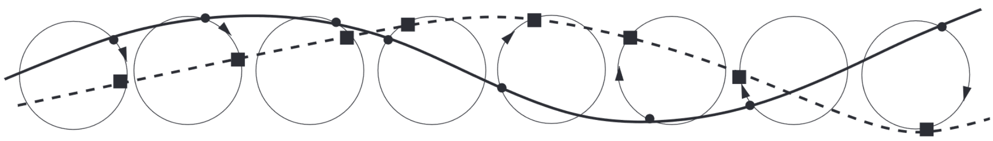

Figure 1. Theoretical schematic of the orbits of surface particles in a deep-water wave.

In recent years, research on Gerstner wave has been revitalized. Gerstner's solution has been extended to cases of nonconstant pressure on the free surface due to the action of the wind, and it has taken into account the effects of Earth's rotation and stratification [6]. Weber applied the Lagrange formula to study internal Gerstner and Stokes-type waves between horizontal planes, investigated the properties of weakly nonlinear waves at the interface between two fluids, and proposed that any wave without particle drift can be called a Gerstner-like wave [7, 8]. All families of Gerstner-like waves are vortical, which raises the question of the mechanism for vorticity generation. For some periodic traveling waves with vorticity, the qualitative features of the underlying flow have been rigorously established. Specifically, the existence of exact steady rotational waves has been mathematically proven [9, 10], and the inherent symmetry properties of their flow fields in various depth regimes have been thoroughly analyzed [11, 12]. For the incompressible Navier-Stokes equation with periodic boundary conditions, an auxiliary system can be constructed to approximate the original vorticity equation [13]. In the ideal fluid model, the vorticity is associated with shear flows, and Gerstner wave is generated in this context, with Lamb providing a reasonable explanation [1]. However, theories also suggest that viscous effects may lead to the generation of vortical waves near the free surface. For example, in the approximation of viscous fluid, standing waves on the water surface are potential waves in the linear approximation. However, in the quadratic approximation, they become vortical waves due to viscous effects [14]. Martin considered the full Coriolis force and centripetal terms in the three-dimensional geophysical water wave equations and examined the exact solutions for such flows with constant vorticity, which is consistent with Gerstner waves as an exact solution model for rotational flows [15]. To account for more complex flow environments, Kluczek systematically investigated nonlinear water wave models with vorticity, providing exact analytical solutions for particle paths in Gerstner-like waves within constant shear currents [16]. Furthermore, Henry and Lyons explored the dynamics of Lagrangian coherent structures and Pollard waves under underlying currents, further elucidating the influence of background shear on the geometric and stability characteristics of rotational fluid trajectories [17].

The oceans in the equatorial regions are generally deep, and in most cases, waves can be considered deep water waves. A proper description of deep water waves: Assume water is theoretically infinitely deep. The water body is the region in R^3 bounded by the free surface $z = h(t, x, y)$. Let the velocity field be $\boldsymbol{u} = (u, v, w)$. Moreover, we review the general problem of gravity wave propagation in deep water [18]. Due to water's homogeneity (constant density ρ) [19], we have the equation of mass conservation in the form

$$u_x + v_y + w_z = 0. \tag{1.1}$$

Under the assumption that water is inviscid, the equation of motion is the Euler's equation [4]

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}P_x, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}P_y, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}P_z - g. \tag{1.2}$$

The pressure is denoted by $P(t,x,y,z)$, g represents the gravitational acceleration constant, and $\frac{\mathrm{D}}{\mathrm{D}t}$ is the material time derivative, $\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$, which indicates the rate of change of the quantity f related to the same fluid particle as it moves. Moreover, it is the continuity equation (1.1) and Euler's equation (1.2) that are the exact equations of motion for the velocity field. Now, we introduce several boundary conditions, which will single out the water wave problem from all other possible solutions of Eqs (1.1) and (1.2). The boundary conditions for surface water waves are the dynamic and kinematic boundary conditions

$$P = P_{\text{atm}} \qquad on \qquad z = h(t,x,y), \tag{1.3}$$

$$w = h_t + uh_x + vh_y \qquad on \qquad z = h(t,x,y), \tag{1.4}$$

where P_{atm} is the constant atmospheric pressure, and $h(t,x,y)$ is the free surface. The dynamic boundary conditions decouple the motion of the air from that of the water. The kinematic boundary conditions means that the same particles always constitute the free water surface, that is, $\frac{\mathrm{D}}{\mathrm{D}t}(z-h) = 0$. The boundary condition at the bottom is

$$\boldsymbol{u} \to 0 \qquad as \qquad z \to -\infty, \tag{1.5}$$

which indicates that motion is not significant at very deep locations. Descriptions of deep water wave propagation are generally represented by Eqs (1.1)–(1.5), whose existence was pointed out by Gerstner in 1802 [20]. One of the distinguishing features is that the free surface is unknown and must be determined as part of the solution.

The science of nonlinear waves originated from Scott Russell's experiments in the 1830s-1840s, where he first observed the propagation of solitons on the surface of a shallow channel [21]. In 1895, Korteweg and de Vries mathematically described this phenomenon through an equation [22]. In fact, as early as 1804, Franz Gerstner proposed an exact solution for deep water waves-Gerstner wave, although this achievement was long overlooked [23]. Unlike the soliton solutions of the Korteweg-de Vries equation, Gerstner wave is an exact solution to the complete equations of hydrodynamics and is of greater mathematical significance to the theory of water wave analysis. While Constantin provided a rigorous mathematical justification for the Gerstner wave framework [24], the classical theory typically assumes wave propagation in a steady fluid, which limits its applicability in complex wave-current interactions. In realistic oceanic environments, particularly in equatorial regions, surface waves are frequently superimposed on large-scale, time-varying background currents driven by tidal forces or non-uniform density distributions.

In the framework of nonlinear dynamics, researchers have significantly expanded the investigation of evolution equations to capture complex wave behaviors in fluid mechanics. Specifically, studies on (3+1)-dimensional generalized shallow water wave and Kadomtsev-Petviashvili (KP) type equations have revealed novel evolutionary characteristics such as multi-wave interactions and the superposition of localized waves [25, 26]. By leveraging this framework, researchers have extended this to the investigation of extended and nonlocal Boussinesq equations for shallow water applications, further broadening the theoretical and practical scope of nonlinear wave dynamics [27, 28]. Furthermore, the pursuit of analytical exact solutions has been pushed to a more sophisticated frontier through the development of the inverse scattering transform (IST) and the Riemann-Hilbert approach, and was successfully applied to higher-order systems such as the fourth-order nonlinear Schrödinger equation and coupled Lakshmanan-Porsezian-Daniel (LPD) equations [29, 30]. These mathematical breakthroughs in handling non-vanishing backgrounds and intricate wave structures provide a vital theoretical parallel to our investigation of rotational water waves. In particular, the rigor in describing complex dynamics within non-trivial configurations offers essential insights into clarifying wave-current interactions and particle trajectories within unsteady marine environments.

Motivated by physical phenomena such as tidal surges and the transient acceleration of surface currents during wind-driven spin-up phases, we aim to bridge the gap between idealized models and natural conditions. Our primary objective of this research is to extend and generalize the classical Gerstner wave model by incorporating unsteady ambient currents. Specifically, building upon foundational Lagrangian theories, we propose an improved exact solution that incorporates time-varying fundamental velocities in the horizontal (x) and vertical (z) directions. By introducing the terms $s_1(t)$ and $s_2(t)$ as time-varying background current components to account for the bulk motion of the fluid, our model realistically captures wave-current interactions in non-uniform flow fields. Through this approach, and via a rigorous derivation of the velocity field, vorticity, and a time-consistent pressure function, we demonstrate the mathematical validity and physical rationality of this generalized solution, thereby providing a more refined framework for analyzing nonlinear deep-water wave dynamics. It should be noted that this model adopts a localized approximation where the wave frequency significantly exceeds the Coriolis parameter f. Consequently, f is not explicitly included in the governing equations to preserve the Lagrangian integrability of the exact solution. Instead, the physical influence of such geophysical forces is implicitly encapsulated within the time-varying background flows $(s_1(t), s_2(t))$, which represent the kinematic environment such as geostrophic currents or tidal advection-driven by larger-scale rotational effects. This approach ensures a self-consistent nonlinear framework that bridges the gap between localized exact solutions and broader geophysical contexts.

2. Exact solution

The Gerstner wave is a typical two-dimensional water wave, characterized by the circular or elliptical trajectories of its wave particles, rather than just vertical vibrations. For Gerstner wave, fluctuations in all planes parallel to a certain vertical fixed plane are identical. Consequently, it can typically be simplified to a two-dimensional analysis. We need to consider only the motion of water particles in the (x, z) plane, meaning that the velocity u, pressure P, and free surface h do not depend on the y variable. For convenience, we adopt the Lagrangian perspective and analyze by tracking the motion trajectories of individual water particles.

Let a and b be parameters that determine the position of a specific water particle before the wave passes through. The range of still water body is that $a \in \mathbb{R}$ and $b \leq b_0$ for some $b_0 < 0$ fixed to the lower half-plane. To account for the bulk motion of the fluid in unsteady environments, we incorporate the terms $s_1(t)$ and $s_2(t)$ as time-varying background current components. Unlike classical models that assume a steady or constant-velocity drift, we regard these components as univariate functions of time t to capture the transient acceleration of the ambient flow. Specifically, $s_1(t)$ denotes the time-dependent meridional velocity in the x-direction, while $s_2(t)$ represents the upward vertical velocity in the z-direction. Consequently, the position of a specific water particle at time t is expressed by the following exact solution:

$$\begin{aligned} x &= s_1(t)t + a - \frac{1}{k}\mathrm{e}^{kb}\sin k(a-ct), \\ z &= s_2(t)t + b + \frac{1}{k}\mathrm{e}^{kb}\cos k(a-ct), \end{aligned} \tag{2.1}$$

where $k > 0$ is the wavenumber and c is the phase speed. When the wave amplitude is $\frac{1}{k}$, the crest in the waveform is sharp (with an angle of zero), and such a limiting trochoid is called a cycloid. For convenience, we let $\theta = k(a - ct)$, and the above expression can be written as

$$\begin{aligned} x &= s_1(t)t + a - \frac{1}{k}\mathrm{e}^{kb}\sin\theta, \\ z &= s_2(t)t + b + \frac{1}{k}\mathrm{e}^{kb}\cos\theta, \end{aligned} \tag{2.2}$$

By calculation, we can obtain the velocity field

$$(u,w) = (\frac{\mathrm{D}x}{\mathrm{D}t}, \frac{\mathrm{D}z}{\mathrm{D}t}) = (s_1'(t)t + s_1(t) + c\mathrm{e}^{kb}\cos\theta, s_2'(t)t + s_2(t) + c\mathrm{e}^{kb}\sin\theta), \tag{2.3}$$

and the fluid acceleration

$$(\frac{\mathrm{D}u}{\mathrm{D}t}, \frac{\mathrm{D}w}{\mathrm{D}t}) = (s_1''(t)t + 2s_1'(t) + kc^2\mathrm{e}^{kb}\sin\theta, s_2''(t)t + 2s_2'(t) - kc^2\mathrm{e}^{kb}\cos\theta), \tag{2.4}$$

where $\frac{\mathrm{D}}{\mathrm{D}t}$ is the material (or Lagrangian) derivative. Moreover, Euler's equation (1.2) reduces to

$$\nabla_{(x,z)}P = -\rho(\frac{\mathrm{D}u}{\mathrm{D}t}, \frac{\mathrm{D}w}{\mathrm{D}t} + g),$$

which may be expressed by way of (2.3) and (2.4) as

$$\nabla_{(x,z)}P = -\rho\begin{pmatrix} s_1''(t)t + 2s_1'(t) + kc^2\mathrm{e}^{kb}\sin\theta \\ s_2''(t)t + 2s_2'(t) - kc^2\mathrm{e}^{kb}\cos\theta + g \end{pmatrix}. \tag{2.5}$$

To determine the gradient of the Lagrangian variables of P, we calculate the Jacobian matrix

$$J = \frac{\partial(x,z)}{\partial(a,b)} = \begin{pmatrix} 1 - \mathrm{e}^{kb}\cos\theta & -\mathrm{e}^{kb}\sin\theta \\ -\mathrm{e}^{kb}\sin\theta & 1 + \mathrm{e}^{kb}\cos\theta \end{pmatrix}, \tag{2.6}$$

and using $\nabla_{(a,b)}P = J^T \cdot \nabla_{(x,z)}P$, we can find

$$\nabla_{(a,b)}P = -\rho\begin{pmatrix} s_1''(t)t + 2s_1'(t) - (s_1''(t)t + 2s_1'(t))\mathrm{e}^{kb}\cos\theta + (kc^2 - s_2''(t)t - 2s_2'(t) - g)\mathrm{e}^{kb}\sin\theta \\ s_2''(t)t + 2s_2'(t) - (s_1''(t)t + 2s_1'(t))\mathrm{e}^{kb}\sin\theta + g - kc^2\mathrm{e}^{2kb} - (kc^2 - s_2''(t)t - 2s_2'(t) - g)\mathrm{e}^{kb}\cos\theta \end{pmatrix}. \tag{2.7}$$

According to (2.6), the determinant of the Jacobian matrix is $1 - e^{2kb}$, which is non-zero, and hence transformation (2.1) is well defined. Furthermore, since the determinant of the Jacobian is time-independent, Eq (2.1) satisfies $\nabla \cdot \boldsymbol{u} = 0$, which means the incompressibility of the fluid [31]. In particular, this time-independence ensures that the coordinate transformation remains a global diffeomorphism for all $t \geq 0$ as long as $b < 0$. This mathematically guarantees that the analytical solution remains physically self-consistent and free from particle trajectory crossing, even under the unbounded growth of the background flows $s_1(t)$ and $s_2(t)$. Furthermore, we can obtain that the inverse of the Jacobian matrix (2.6) is

$$J^{-1} = \frac{1}{1 - e^{2kb}} \begin{pmatrix} 1 + e^{kb}\cos\theta & e^{kb}\sin\theta \\ e^{kb}\sin\theta & 1 - e^{kb}\cos\theta \end{pmatrix},$$

because

$$\nabla_{(a,b)}\boldsymbol{u} = \begin{pmatrix} \frac{\partial u}{\partial a} & \frac{\partial w}{\partial a} \\ \frac{\partial u}{\partial b} & \frac{\partial w}{\partial b} \end{pmatrix} = kce^{kb} \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix},$$

according to $\nabla_{(x,z)}\boldsymbol{u} = (J^{-1})^T \cdot \nabla_{(a,b)}\boldsymbol{u}$, the velocity gradient tensor can be calculated,

$$\nabla_{(x,z)}\boldsymbol{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \frac{kce^{kb}}{1 - e^{2kb}} \begin{pmatrix} -\sin\theta & \cos\theta + e^{kb} \\ \cos\theta - e^{kb} & \sin\theta \end{pmatrix},$$

from which the vorticity component perpendicular to the plane of flow is

$$\omega = w_x - u_z = \frac{2kce^{2kb}}{1 - e^{2kb}}.$$

Next, we need to prove that Euler's equation (1.2) satisfies (2.3) and (2.4), which is equivalent to constructing a pressure function $P(a, b)$ such that (2.7) holds. Adopting the candidate

$$\begin{aligned} \tilde{P} = &-(s_1''(t)t + 2s_1'(t))\rho a - (s_2''(t)t + 2s_2'(t))\rho b + \rho\frac{s_1''(t)t + 2s_1'(t)}{k}e^{kb}\sin\theta \\ &- \rho g b + \rho\frac{kc^2}{2k}e^{2kb} + \rho\frac{kc^2 - s_2''(t)t - 2s_2'(t) - g}{k}e^{kb}\cos\theta + P_{\text{atm}}, \end{aligned} \tag{2.8}$$

and calculating its partial derivatives with respect to a, b, respectively, we have

$$\tilde{P}_a = -\rho(s_1''(t)t + 2s_1'(t) - (s_1''(t)t + 2s_1'(t))e^{kb}\cos\theta + (kc^2 - s_2''(t)t - 2s_2'(t) - g)e^{kb}\sin\theta),$$

$$\tilde{P}_b = -\rho(s_2''(t)t + 2s_2'(t) - (s_1''(t)t + 2s_1'(t))e^{kb}\sin\theta + g - kc^2e^{2kb} - (kc^2 - s_2''(t)t - 2s_2'(t) - g)e^{kb}\cos\theta).$$

To ensure the dynamic boundary condition, which requires the pressure at the free surface to remain time-varying, the following relation must be satisfied:

$$kc^2 - s_2''(t)t - 2s_2'(t) - g = 0. \tag{2.9}$$

The dispersion relation for the wave phase speed c under the influence of flow (2.1) is thus

$$c = \sqrt{\frac{s_2''(t)t + 2s_2'(t) + g}{k}}, \tag{2.10}$$

since we require $c > 0$. Given that $s_2(t) = m_2 t + n_2$ is a linear function, the acceleration term simplifies to $s_2''(t)t + 2s_2'(t) = 2m_2$. Consequently, the dispersion relation can be written as

$$c = \sqrt{\frac{2m_2 + g}{k}}, \tag{2.11}$$

where $g = 9.8m/s^{-2}$ is the gravitational acceleration and m_2 represents the constant vertical acceleration of the background flow. While $s_1(t) = m_1 t + n_1$ also implies a constant horizontal acceleration $2m_1$, its effect primarily manifests as a uniform pressure gradient that sustains the bulk advection of the fluid without directly modulating the wave's intrinsic restoring force. Consequently, the vertical acceleration m_2 becomes the dominant factor in redefining the wave dynamics.

According to Eq (2.11), the phase speed c is determined by the coupling of gravitational acceleration g and the background vertical acceleration m_2. In the absence of background flows (i.e., $m_2 = 0$), this relation naturally reduces to $c^2 = \frac{g}{k}$, which is the classical Gerstner wave dispersion relation. This consistency verifies that our model is a rigorous generalization of the classical theory, where the restoring force is adjusted by the effective gravity $g_{\text{eff}} = 2m_2 + g$. To ensure the physical integrity of the solution, two stability constraints are identified. First, a geometric constraint $b < 0$ is required to maintain a positive Jacobian, ensuring no particle trajectory crossing for all $t \geq 0$. Second, a dynamic stability constraint $2m_2 + g > 0$ must be satisfied to guarantee a real-valued phase speed. From a physical perspective, in the case of an accelerating upwelling ($m_2 > 0$), the transient vertical acceleration acts as an inertial force that effectively strengthens the restoring force. As shown in Figure 2, this leads to a positive shift in phase speed, which is particularly pronounced for the long-wave components ($k \to 0$). Such findings provide rigorous theoretical insights into wave dynamics within unsteady marine environments. Specifically, our model elucidates how background accelerations modulate the characteristic wave speed through an effective gravity shift, offering a more comprehensive framework than classical steady-state theories.

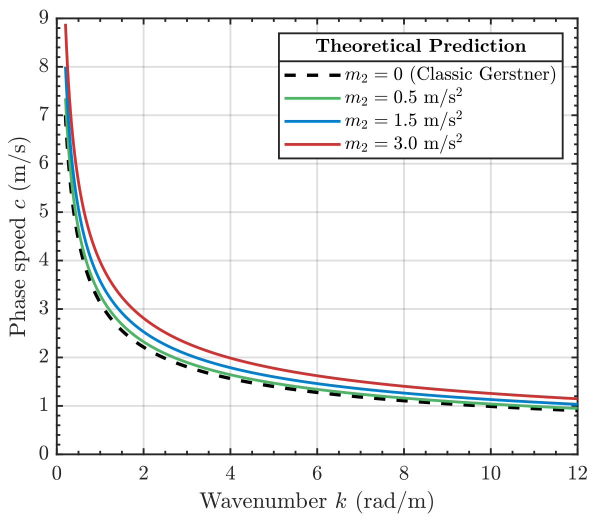

Figure 2. Phase speed c as a function of wavenumber k for different vertical acceleration coefficients m_2, plotted according to the analytical expression in Eq (2.11).

3. Gerstner wave

In this section, we explore some properties of the solution.

Theorem 3.1. *For any fixed $t \geq 0$, the mapping (2.1) establishes a smooth and invertible diffeomorphism from the Lagrangian reference region $\Omega_L = \{(a, b) : b \leq b_0, a \in \mathbb{R}\}$ to the physical fluid domain $\Omega_E(t) = \{(x, z) : z \leq h(t, x)\}$. Since $b_0 < 0$, the Jacobian of the mapping remains strictly positive, which ensures that the surface profile is smooth and free of self-intersections.*

Proof. To analyze the mapping for any $t > 0$, we consider that the general case can be deduced from the $t = 0$ state through a time-varying coordinate translation $(s_1(t), s_2(t))$. This conjugate transformation preserves the spatial derivatives and the mapping's topology. Thus, we need to only analyze the map

$$\begin{aligned} x &= a - \frac{1}{k} \mathrm{e}^{kb} \sin ka, \\ z &= b + \frac{1}{k} \mathrm{e}^{kb} \cos ka. \end{aligned} \tag{3.1}$$

The Jacobian of this transformation is given by

$$J = \frac{\partial(x, z)}{\partial(a, b)} = \begin{pmatrix} 1 - \mathrm{e}^{kb} \cos ka & -\mathrm{e}^{kb} \sin ka \\ \mathrm{e}^{kb} \sin ka & 1 + \mathrm{e}^{kb} \cos ka \end{pmatrix}. \tag{3.2}$$

The mapping is a diffeomorphism if $|J| > 0$, which is guaranteed when $b < 0$. Since the Lagrangian domain is restricted to $b \leq b_0 < 0$, we have $|J| \geq 1 - e^{2kb_0} > 0$, ensuring global invertibility and a smooth profile without cusps or singularities.

In Eq (3.1), whenever a increases by $\frac{2\pi}{k}$, x undergoes a linear shift of $\frac{2\pi}{k}$ while the value of z cycles through its period. Therefore, it is sufficient to restrict the analysis to $a \in [0, \frac{2\pi}{k}]$. Furthermore, since z is a strictly increasing function of b for all $b < 0$, the Lagrangian boundary $b = b_0$ maps uniquely to the evolving free surface $z = h(t, x)$.

Theorem 3.2. *Consider a fluid domain characterized by the time-varying background flow $s_1(t)$ and $s_2(t)$. When the still water surface $z = b_0$ ($b_0 \leq 0$) is mapped under (2.1), the equation for the profile of the free surface*

$$\begin{aligned} x &= s_1(t)t + a - \frac{1}{k} \mathrm{e}^{kb_0} \sin k(a - ct), \\ z &= s_2(t)t + b_0 + \frac{1}{k} \mathrm{e}^{kb_0} \cos k(a - ct). \end{aligned} \tag{3.3}$$

Thus, the boundary conditions (1.4) and (1.5) are satisfied.

Proof. The first part has been proven in the foregoing text. For the second statement, a particle on the free water surface will remain on the surface, which has been guaranteed by (1.4). Parameter $b_0 < 0$ is chosen to ensure a smooth, non-singular profile, as $b = 0$ represents the limit where the Gerstner wave develops a cusp. Furthermore, by differentiating Eq (2.1) with respect to time, the particle velocity is obtained

$$\boldsymbol{u}(x, z) = (s_1'(t)t + s_1(t) + c\mathrm{e}^{kb} \cos k(a - ct), s_2'(t)t + s_2(t) + c\mathrm{e}^{kb} \sin k(a - ct)).$$

As $b \to -\infty$, the wave-induced components ce^{kb} vanish, while the background velocities $s_1(t)$ and $s_2(t)$ persist to describe the non-uniform bulk motion.

Theorem 3.3. *The equation of continuity is satisfied by the flow defined by (2.1).*

Proof. Since the continuity equation reflects the property of mass conservation of fluids during the flow process, that is, the incompressibility of fluids, it is necessary to only demonstrate that the map

$$T : \begin{pmatrix} a - \frac{1}{k}e^{kb}\sin ka \\ b + \frac{1}{k}e^{kb}\cos ka \end{pmatrix} \mapsto \begin{pmatrix} s_1(t)t + a - \frac{1}{k}e^{kb}\sin k(a-ct) \\ s_2(t)t + b + \frac{1}{k}e^{kb}\cos k(a-ct) \end{pmatrix}$$

preserves the area invariant throughout the transformation from the initial state to any time $t > 0$. In other words, it is necessary to prove that the Jacobian determinant of this map remains equal to 1. We have defined two transformations, T_1 and T_2, respectively,

$$T_1 : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a - \frac{1}{k}e^{kb}\sin ka \\ b + \frac{1}{k}e^{kb}\cos ka \end{pmatrix},$$

$$T_2 : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} s_1(t)t + a - \frac{1}{k}e^{kb}\sin k(a-ct) \\ s_2(t)t + b + \frac{1}{k}e^{kb}\cos k(a-ct) \end{pmatrix}.$$

Through calculation, it is found that the Jacobian determinants of both transformations are equal to $(1 - e^{2kb})$ and independent of time. The map T we need to prove is $T_2 \circ T_1^{-1}$. According to the property of the Jacobian determinant of a composite map, the Jacobian determinant of $T_2 \circ T_1^{-1}$ is 1, which means that map T is area-preserving.

Theorem 3.4. *According to the flow (2.1), it can be determined that there exists a unique hydrodynamical pressure P, which satisfies the Euler equation (1.2) and the corresponding boundary conditions (1.3).*

Proof. By taking the second derivative of Eq (2.1) with respect to time, the acceleration of a particular fluid particle can be obtained,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = (s_1''(t)t + 2s_1'(t) + kc^2e^{kb}\sin k(a-ct), 0, s_2''(t)t + 2s_2'(t) - kc^2e^{kb}\cos k(a-ct)).$$

Thus, the equation of motion (1.2) can be written as

$$\begin{aligned} -[s_1''(t)t + 2s_1'(t) + kc^2e^{kb}\sin k(a-ct)] &= \tfrac{\partial}{\partial x}(\tfrac{P}{\rho} + gz), \\ -[s_2''(t)t + 2s_2'(t) - kc^2e^{kb}\cos k(a-ct)] &= \tfrac{\partial}{\partial z}(\tfrac{P}{\rho} + gz), \end{aligned}$$

without considering the y-component, we change the variables from (x, z) to (a, b). Considering (2.1) and (2.9), let $\theta = k(a - ct)$, and we get

$$\begin{aligned} \frac{\partial}{\partial a}(\frac{P}{\rho} + gz) &= -ge^{kb}\sin\theta + (s_1''(t)t + 2s_1'(t))e^{kb}\cos\theta - s_1''(t)t - 2s_1'(t), \\ \frac{\partial}{\partial b}(\frac{P}{\rho} + gz) &= g - kc^2 + kc^2e^{2kb} + (s_1''(t)t + 2s_1'(t))e^{kb}\sin\theta + ge^{kb}\cos\theta. \end{aligned} \tag{3.4}$$

Observing the above equation, we can construct a function F such that its partial derivatives with respect to a and b are the two terms on the right-hand side of Eq (3.4). Thus, by integration, we get

$$F = \frac{s_1''(t)t + 2s_1'(t)}{k}e^{kb}\sin\theta + \frac{g}{k}e^{kb}\cos\theta - (s_1''(t)t + 2s_1'(t))a - (kc^2 - g)b + \frac{c^2}{2}e^{2kb}.$$

Substituting $z = s_2(t)t + b + \frac{1}{k}e^{kb}\cos\theta$ and $s_1(t) = m_1 t + n_1$, we can obtain

$$P = C(t) + \rho\frac{2m_1}{k}e^{kb}\sin\theta - \rho kc^2 b - \rho g s_2(t)t - \rho 2m_1 a + \frac{c^2}{2}\rho e^{2kb}. \tag{3.5}$$

To ensure the pressure field satisfies the dynamic boundary condition $P = P_{atm}$ at the free surface $b = b_0$, the spatial dependence on a and phase θ must be eliminated. The term $2\rho m_1 a$ represents the pressure gradient induced by horizontal advection, while the $\sin\theta$ term arises from the dynamic coupling between the background flow and wave motion. Under the linear assumption, these terms, along with the bulk vertical displacement term $\rho g s_2(t)t$, represent spatially uniform contributions to the surface particles at any given instant t. To maintain a constant atmospheric pressure, these contributions are absorbed into the time-varying integration constant $C(t)$, which re-calibrates the pressure reference. By incorporating these simplifications and determining the constant at $b = b_0$, the final explicit expression for the hydrodynamic pressure is obtained,

$$P = P_{atm} - \rho g_{eff}(b - b_0) + \frac{c^2}{2}\rho(e^{2kb} - e^{2kb_0}).$$

This formula demonstrates that under time-varying background flows, the pressure field remains in a steady state relative to the depth b, ensuring physical self-consistency throughout the domain.

4. Conclusions

In this study, we establish a generalized framework for exact Gerstner wave solutions by incorporating time-varying background flow components $s_1(t)$ and $s_2(t)$, which successfully generalizes the classical model to unsteady oceanic environments. Moving beyond the traditional constraints of steady fluid assumptions, our proposed solution effectively encompasses the bulk motion of the fluid, providing a more realistic description of wave-current interactions where the background state evolves over time. By integrating time-varying displacements into the particle trajectories, the model characterizes wave propagation within an unsteady frame, enabling a rigorous description of fluid transport relative to an evolving background. The derivation of velocity and acceleration fields confirms that the wave-induced motion remains self-consistent despite the temporal shifts in the background flow. Furthermore, the evaluation of the Jacobian determinant confirms that the coordinate transformation remains a global diffeomorphism for all $t \geq 0$, ensuring volume conservation in this unsteady field. The construction of a time-consistent pressure function ensures that the free surface remains an isobaric surface even under the influence of time-varying background components. The resulting modified dispersion relation, $c = \sqrt{\frac{2s_2'(t)+g}{k}}$, accounts for the effective gravity shift induced by the unsteady vertical flow, justifying the physical existence of such exact rotational waves. The refined analytical framework established in this study provides a preliminary theoretical foundation for characterizing unsteady wave environments. While our work is limited to analytical derivations, the generalized model introduced serves as a basic reference for discussing local upwelling and complex wave-current interactions. We recognize that these unsteady effects are vital for evaluating hydrodynamic loads on offshore structures, compared to classical steady-state theories, the proposed model aims to offer a more rigorous physical starting point for further engineering analysis.

Author contributions

Shuwen Song: Writing–review & editing, Writing–original draft, Methodology, Investigation, Formal analysis, Conceptualization; Yuxin Wang: Writing–review & editing, Writing–original draft; Jian Song: Writing–review & editing, Writing–original draft, Validation, Supervision, Project administration, Funding acquisition, Conceptualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare no potential conflicts of interest in this paper.

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