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Research article

Some new (k, r)-Riemann-Liouville fractional Hermite-Hadamard-type inequalities for strongly modified $(\alpha, h - m) - p$-convex functions

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Abstract: In this paper, a strongly modified $(\alpha, h-m)-p$-convex function, which is a class of strongly convex functions, was defined, and the properties of this function were proved. Initially, the Hermite-Hadamard type inequality was proved for the presented function class. Then, the Hermite-Hadamard type inequality was obtained for the (k, r)-Riemann-Liouville fractional integrals. Also, some examples were included. The results were generalizations of various results in the literature.

Keywords: strongly modified $(\alpha, h - m) - p$-convexity; fractional integrals; Riemann-Liouville integrals; Hermite-Hadamard inequality; integral inequalities
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1. Introduction

In the applied sciences, the convexity theory has become a basis of motivation. This theory ensures an efficient study area. The notion of convexity relates to the characteristic of a function or a set, into the line segment linking any two points on the function or within the set, abiding completely upon the function or within the set. Due to their useful properties, they are very crucial in various areas, such as economics, geometry, engineering, mathematical optimization, and analysis. This peerless manner of convex functions leads to numerous important inequalities; this theory has also played a significant role in extending and generalizing inequality theory. Inequalities are useful in specifying the manner of a function on an interval. The investigation of these two concepts and their significant contributions has been extensively discussed in the literature. In [1], Ajmar et al. presented a new approach to estimating the fractional derivative of modified (h, m)-convex functions. In [2], Bayraktar and Valdes acquired some new integral inequalities of the Hermite-Hadamard type for (h, m)-convex modified functions. In [3], Breaz et al. acquired various Hermite-Hadamard and Fejér-type inequalities for modified h-convex functions. They proved several inequalities for the products of two modified h-convex functions. In [4], Nosheen et al. gave applications of modified (m, p, h)-convex functions as extensions of the Hermite-Hadamard, Jensen, and Fejér inequalities. In [5], Sarikaya et al. presented a new fractional integral approach that generalizes the Riemann-Liouville fractional integral. In [6], Yan et al. improved fractional integral inequalities for generalized strongly modified h-convex functions. Also, in [7], Yu et al. generated Hermite-Hadamard type inequalities for generalized geometrically strong modified h-convex functions. In [8], Wang et al. presented some properties of the Caputo-Fabrizio fractional integral operator for the modified h-convex function. In [9], Zhao et al. acquired Schur, Hermite-Hadamard, and Fejér type inequalities for generalized strong modified h-convex functions. In [10], Lakhdari and Saleh introduced the notion of multiplicative convexity using multiplicative Riemann-Liouville fractional integrals and established fractional Hermite-Hadamard inequalities for GG-convex functions. In [11], Li et al. introduced new versions of Hermite-Hadamard, midpoint, and trapezoid type inequalities that include fractional integral operators with exponential kernels.

In recent years, the below Hermite-Hadamard inequality, one of the most substantial classic inequalities, has attracted considerable attention. The Hermite-Hadamard inequality is a fundamental conclusion in convex analysis that ensures bounds for the integral of a convex function over a certain interval.

Let interval $I \subseteq \mathbb{R}$ and $\lambda, \mu \in I$ with $\lambda < \mu$. If $\omega : I \to \mathbb{R}$ is a convex function, then

$$\omega\left(\frac{\lambda+\mu}{2}\right) \leq \frac{1}{\mu-\lambda}\int_{\lambda}^{\mu} \omega(\kappa)d\kappa \leq \frac{\omega(\lambda)+\omega(\mu)}{2}. \tag{1.1}$$

This inequality is known as the Hermite-Hadamard inequality for convex functions.

2. Preliminaries

Definition 2.1. *[12] Let $\omega, h : J \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function $\omega : I \subset \mathbb{R} \to \mathbb{R}$ is named a modified h-convex function if*

$$\omega\left(\tau\kappa + (1-\tau)\iota\right) \leq h(\tau)\omega(\kappa) + (1-h(\tau))\omega(\iota), \tag{2.1}$$

for all $x, y \in J$ and $\tau \in [0, 1]$.

Definition 2.2. *[1] A modified (h, m)-convex function $\omega : I \to \mathbb{R}$ is a nonnegative function that satisfies the following inequality:*

$$\omega\left(\tau\kappa + m(1-\tau)\iota\right) \leq h(\tau)\omega(\kappa) + m(1-h(\tau))\omega(\iota), \tag{2.2}$$

and holds for all $\kappa, \iota \in I$, $\tau \in [0, 1]$, and $m \in [0, 1]$, where $h : J \subset \mathbb{R} \to \mathbb{R}$ is a nonnegative function.

Definition 2.3. *[13] Assume $h : J \subset \mathbb{R} \to \mathbb{R}$ to be a nonnegative and nonzero function. A function $\omega : I \to \mathbb{R}$, where I is p-convex set in $\mathbb{R}$, is named a modified-(p, h)-convex function, ω is nonnegative, and*

$$\omega\left(\left[\tau\kappa^p + (1-\tau)\iota^p\right]^{\frac{1}{p}}\right) \leq h(\tau)\omega(\kappa) + (1-h(\tau))\omega(\iota), \tag{2.3}$$

for all $\kappa, \iota \in I$, $\tau \in (0, 1)$, and $p > 0$.

Definition 2.4. *[14] A function $\omega : [0, b] \to \mathbb{R}$, with $b > 0$, is a strongly modified-(h, m)-convex function with modulus c if*

$$\omega\left(\tau\kappa + m(1-\tau)\iota\right) \leq h(\tau)\omega(\kappa) + m(1-h(\tau))\omega(\iota) - mc\tau(1-\tau)(\kappa-\iota)^2, \tag{2.4}$$

holds for all $\kappa, \iota \in [0, b]$, $c > 0$, $\tau \in [0, 1]$, and $m \in [0, 1]$.

Definition 2.5. *[15] Assume $h : J \subset \mathbb{R} \to \mathbb{R}$ to be a nonnegative and nonzero function and c be a real positive number. A function $\omega : I \to \mathbb{R}$, where I is a p-convex set in $\mathbb{R}$, is named a strongly modified-(p,h)-convex function with modulus c if*

$$\omega\left([\tau\kappa^p + (1-\tau)\iota^p]^{\frac{1}{p}}\right) \leq h(\tau)\omega(\kappa) + (1-h(\tau))\omega(\iota) - c\tau(1-\tau)(\iota^p - \kappa^p)^2, \tag{2.5}$$

for all $\kappa, \iota \in I$, $\tau \in (0,1)$, and $p > 0$.

Definition 2.6. *[16] Let $[\lambda,\mu] \subseteq [0,\infty]$, where $\lambda < \mu$, $\omega \in L_1[\lambda,\mu]$, and $k, r > 0$. The right and left-sided general (k,r)-Riemann-Liouville fractional integrals of order $\beta > 0$ are defined as*

$$_{\lambda^+}R^{\beta}_{\varphi(k,r)}\omega(\kappa) = \frac{1}{k\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right)}\int_{\lambda}^{\kappa}(\kappa-\tau)^{\frac{\beta}{\varphi(k,r)}-1}\omega(\tau)d\tau, \qquad \lambda < \kappa \leq \mu, \tag{2.6}$$

$$_{\mu^-}R^{\beta}_{\varphi(k,r)}\omega(\kappa) = \frac{1}{k\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right)}\int_{\kappa}^{\mu}(\tau-\kappa)^{\frac{\beta}{\varphi(k,r)}-1}\omega(\tau)d\tau, \qquad \lambda \leq \kappa < \mu, \tag{2.7}$$

where $\Gamma(k,r)$ is the (k,r)-Gamma function.

Remark 2.1. *(i) By taking $k = r$ in (2.6) and (2.7), we obtain k-Riemann-Liouville fractional integrals.*
(ii) By taking $k = r = 1$ in (2.6) and (2.7), we obtain classic Riemann-Liouville fractional integrals.

3. Strongly modified $(\alpha, h-m) - p$-convex function

In this section, we will define the strongly modified $(\alpha, h-m) - p$-convex function and prove some of its properties.

Definition 3.1. *Let $J \subseteq \mathbb{R}$ be an interval including $(0,1)$ and let $h : J \to \mathbb{R}$ be a nonnegative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $\omega : I \to \mathbb{R}$ is called the strongly modified $(\alpha, h-m) - p$-convex function, if*

$$\omega\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) \leq h(\tau^{\alpha})\,\omega(\kappa) + m\,(1-h(\tau^{\alpha}))\,\omega(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2 \tag{3.1}$$

holds $[\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}} \in I$ for $\tau \in [0,1]$, and $(\alpha, m) \in [0,1]^2$.

Remark 3.1. *(i) By setting $\alpha = 1$ and $p = 1$ in Definition 3.1, we obtain definition of strongly modified (SM) (h,m)-convex function (see [14]).*

(ii) By taking $\alpha = m = 1$ in Definition 3.1, we obtain definition of SM (p,h)-convex function (see [15]).

(iii) By taking $\alpha = m = 1$ and $c = 0$ in Definition 3.1, we obtain definition of modified (p,h)-convex function (see [13], Definition 1.9).

(iv) By taking $\alpha = m = 1$, $p = 1$, and $c = 0$ in Definition 3.1, we obtain definition of modified h-convex function (see [17]).

Proposition 3.1. *ω and ξ are SM-$(\alpha, h-m) - p$-convex functions, then their sum $\omega + \xi$ is also a SM-$(\alpha, h-m) - p$-convex function.*

Proof. For $\kappa, \iota \in I$, $p \geq 1$, $c > 0$, $\tau \in (0,1)$, we get

$$(\omega+\xi)\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) = \omega\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) + \xi\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right).$$

Since ω and ξ are SM-$(\alpha, h-m)-p$-convex,

$$\begin{aligned}(\omega+\xi)\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) &\leq h(\tau^\alpha)\omega(\kappa) + m(1-h(\tau^\alpha))\omega(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2 \\ &\quad + h(\tau^\alpha)\xi(\kappa) + m(1-h(\tau^\alpha))\xi(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2 \\ &= h(\tau^\alpha)(\omega+\xi)(\kappa) + m(1-h(\tau^\alpha))(\omega+\xi)(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2.\end{aligned}$$

Proposition 3.2. *Let ω be a SM-$(\alpha, h-m)-p$-convex function, then for scalar $n > 0$, $n\omega$ is also a SM-$(\alpha, h-m)-p$-convex function.*

Proof. For $\kappa, \iota \in I$, $p \geq 1$, $c > 0$, $\tau \in (0,1)$, we procure

$$\begin{aligned}n\omega\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) &\leq n\left(h(\tau^\alpha)\omega(\kappa) + m(1-h(\tau^\alpha))\omega(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2\right) \\ &= h(\tau^\alpha)n\omega(\kappa) + m(1-h(\tau^\alpha))n\omega(\iota) - mnc\tau(1-\tau)(\iota^p-\kappa^p)^2.\end{aligned}$$

Proposition 3.3. *Let h_1, h_2 be a nonnegative function on J and $h_2(\tau) \leq h_1(\tau)$. Under the assumption $m\omega(\iota) \leq \omega(\kappa)$, if ω is SM-$(\alpha, h_2-m)-p$-convex function, then, ω is also a SM-$(\alpha, h_1-m)-p$-convex function.*

Proof.

$$\begin{aligned}\omega\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) &\leq h_2(\tau^\alpha)\omega(\kappa) + m(1-h_2(\tau^\alpha))\omega(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2 \\ &\leq h_1(\tau^\alpha)\omega(\kappa) + m(1-h_1(\tau^\alpha))\omega(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2.\end{aligned}$$

Proposition 3.4. *Let $\omega_i : [0,b] \to \mathbb{R}$ be a SM-$(\alpha, h-m)-p$-convex function for $i \in \mathbb{N}$ and $\sum_{i=1}^d n_i = 1$; then also its linear combination, $\Delta(u) = \sum_{i=1}^d n_i\omega_i(u)$, is a strongly modified-$(\alpha, h-m)-p$-convex function.*

Proof. By taking $\kappa, \iota \in [0,b]$, with $b > 0$, and $[\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}} = u$,

$$\begin{aligned}\Delta\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) &\leq h(\tau^\alpha)\sum_{i=1}^d n_i\omega_i(\kappa) + m(1-h(\tau^\alpha))\sum_{i=1}^d n_i\omega_i(\iota) - \sum_{i=1}^d n_i mc\tau(1-\tau)(\iota^p-\kappa^p)^2 \\ &\leq h(\tau^\alpha)\Delta(\kappa) + m(1-h_1(\tau^\alpha))\Delta(\iota) - mc\tau(1-\tau)(\iota^p-\kappa^p)^2.\end{aligned}$$

4. The (k, r)-Riemann-Liouville fractional Hermite-Hadamard type inequalities

In this section, we primarily represent the below (k, r)-Riemann-Liouville fractional Hermite-Hadamard's inequality.

Theorem 4.1. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be nonnegative, nonzero, and integrable function and $\omega : [\lambda,\mu] \subseteq \mathbb{R} \to \mathbb{R}$ be a strongly modified $(\alpha, h - m) - p$-convex function with $\lambda < \mu$, then*

$$\begin{aligned}
&\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) + \frac{mc}{4}F_\tau \\
&\le h\left(\frac{1}{2^\alpha}\right)\frac{p}{m\mu^p - \lambda^p}\int_\lambda^{m^{\frac{1}{p}}\mu}\kappa^{p-1}\omega(\kappa)d\kappa + m^2\left(1 - h\left(\frac{1}{2^\alpha}\right)\right)\frac{p}{m\mu^p - \lambda^p}\int_{\frac{\lambda}{m^{\frac{1}{p}}}}^{\mu}\kappa^{p-1}\omega(\kappa)d\kappa \\
&\le h\left(\frac{1}{2^\alpha}\right)\int_0^1\left(h(\tau^\alpha)\omega(\lambda) + m(1 - h(\tau^\alpha))\omega(\mu) - mc\tau(1-\tau)(\mu^p - \lambda^p)^2\right)d\tau \\
&\quad + m\left(1 - h\left(\frac{1}{2^\alpha}\right)\right)\int_0^1\left(m(1 - h(\tau^\alpha))\omega\left(\frac{\lambda}{m^2}\right) + h(\tau^\alpha)\omega(\mu) - mc\tau(1-\tau)\left(\mu^p - \frac{\lambda^p}{m^2}\right)^2\right)d\tau,
\end{aligned} \tag{4.1}$$

where $F_\tau = \int_0^1\left(\tau(\mu^p - \lambda^p) + (1-\tau)\left(\frac{\lambda^p}{m} - m\mu^p\right)\right)^2 d\tau$.

Proof. For $\kappa, \iota \in [\lambda,\mu]$, $p \ge 1$, $c > 0$, $\tau \in [0,1]$, we procure

$$\omega\left([\tau\kappa^p + m(1-\tau)\iota^p]^{\frac{1}{p}}\right) \le h(\tau^\alpha)\omega(\kappa) + m(1 - h(\tau^\alpha))\omega(\iota) - mc\tau(1-\tau)(\iota^p - \kappa^p)^2. \tag{4.2}$$

If we put $\tau = \frac{1}{2}$ in (4.2), we get

$$\omega\left(\left[\frac{\kappa^p + m\iota^p}{2}\right]^{\frac{1}{p}}\right) \le h\left(\frac{1}{2^\alpha}\right)\omega(\kappa) + m\left(1 - h\left(\frac{1}{2^\alpha}\right)\right)\omega(\iota) - \frac{mc}{4}(\iota^p - \kappa^p)^2. \tag{4.3}$$

By choosing $\kappa^p = \tau\lambda^p + m(1-\tau)\mu^p$ and $\iota^p = (1-\tau)\frac{\lambda^p}{m} + \tau\mu^p$ in (4.3), we obtain

$$\begin{aligned}
\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) &\le h\left(\frac{1}{2^\alpha}\right)\omega\left([\tau\lambda^p + m(1-\tau)\mu^p]^{\frac{1}{p}}\right) + m\left(1 - h\left(\frac{1}{2^\alpha}\right)\right)\omega\left(\left[(1-\tau)\frac{\lambda^p}{m} + \tau\mu^p\right]^{\frac{1}{p}}\right) \\
&\quad - \frac{mc}{4}\left(\tau(\mu^p - \lambda^p) + (1-\tau)\left(\frac{\lambda^p}{m} - m\mu^p\right)\right)^2.
\end{aligned} \tag{4.4}$$

By integrating in (4.4) with respect to τ from 0 to 1, we get

$$\begin{aligned}
&\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) + \frac{mc}{4}F_\tau \\
&\le h\left(\frac{1}{2^\alpha}\right)\int_0^1\omega\left([\tau\lambda^p + m(1-\tau)\mu^p]^{\frac{1}{p}}\right)d\tau + m\left(1 - h\left(\frac{1}{2^\alpha}\right)\right)\int_0^1\omega\left(\left[(1-\tau)\frac{\lambda^p}{m} + \tau\mu^p\right]^{\frac{1}{p}}\right)d\tau.
\end{aligned} \tag{4.5}$$

Put $\kappa^p = \tau\lambda^p + m(1-\tau)\mu^p$ in the first integral of (4.5) and $\kappa^p = (1-\tau)\frac{\lambda^p}{m} + \tau\mu^p$ in the second integral of (4.5) to get

$$\begin{aligned}
&\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) + \frac{mc}{4}F_\tau \\
&\le h\left(\frac{1}{2^\alpha}\right)\frac{p}{m\mu^p - \lambda^p}\int_\lambda^{m^{\frac{1}{p}}\mu}\kappa^{p-1}\omega(\kappa)d\kappa + m^2\left(1 - h\left(\frac{1}{2^\alpha}\right)\right)\frac{p}{m\mu^p - \lambda^p}\int_{\frac{\lambda}{m^{\frac{1}{p}}}}^{\mu}\kappa^{p-1}\omega(\kappa)d\kappa.
\end{aligned} \tag{4.6}$$

By comparing the righthand side of (4.5) and (4.6),

$$h\left(\frac{1}{2^{\alpha}}\right)\frac{p}{m\mu^{p}-\lambda^{p}}\int_{\lambda}^{m^{\frac{1}{p}}\mu}\kappa^{p-1}\omega(\kappa)d\kappa+m^{2}\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\frac{p}{m\mu^{p}-\lambda^{p}}\int_{\frac{\lambda}{m^{\frac{1}{p}}}}^{\mu}\kappa^{p-1}\omega(\kappa)d\kappa$$
$$=h\left(\frac{1}{2^{\alpha}}\right)\int_{0}^{1}\omega\left(\left[\tau\lambda^{p}+m(1-\tau)\mu^{p}\right]^{\frac{1}{p}}\right)d\tau+m\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\int_{0}^{1}\omega\left(\left[(1-\tau)\frac{\lambda^{p}}{m}+\tau\mu^{p}\right]^{\frac{1}{p}}\right)d\tau.$$

Since ω is a strongly modified $(\alpha, h-m)-p$-convex function, we get

$$\begin{aligned}&h\left(\frac{1}{2^{\alpha}}\right)\frac{p}{m\mu^{p}-\lambda^{p}}\int_{\lambda}^{m^{\frac{1}{p}}\mu}\kappa^{p-1}\omega(\kappa)d\kappa+m^{2}\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\frac{p}{m\mu^{p}-\lambda^{p}}\int_{\frac{\lambda}{m^{\frac{1}{p}}}}^{\mu}\kappa^{p-1}\omega(\kappa)d\kappa\\&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{0}^{1}\left(h(\tau^{\alpha})\omega(\lambda)+m(1-h(\tau^{\alpha}))\omega(\mu)-mc\tau(1-\tau)(\mu^{p}-\lambda^{p})^{2}\right)d\tau\\&\quad+m\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\int_{0}^{1}\left((1-h(\tau^{\alpha}))\omega\left(\frac{\lambda}{m^{2}}\right)+h(\tau^{\alpha})\omega(\mu)-mc\tau(1-\tau)\left(\mu^{p}-\frac{\lambda^{p}}{m^{2}}\right)^{2}\right)d\tau.\end{aligned}\tag{4.7}$$

From (4.6) and (4.7), we obtain (4.1).

Remark 4.1. *(i) By taking $\alpha=m=1$ in (4.1), we obtain Theorem 2 of [15].*
(ii) By taking $\alpha=1$ and $p=1$ in (4.1), we obtain Theorem 2 of [14].
(iii) By taking $\alpha=p=1$, $h(\tau)=\tau$, and $m=1$ in (4.1), we procure Theorem 6 of [18].
(iv) By taking $\alpha=m=1$, $p=1$, and $c=0$ in (4.1), we obtain Theorem 3 of [12].
(v) By taking $\alpha=m=p=1$, $h(\tau)=\tau$, and $c=0$ in (4.1), we obtain classical the Hermite-Hadamard type inequality for convex functions.

Theorem 4.2. *Assume that $\omega:[\lambda,\mu]\subseteq\mathbb{R}\to\mathbb{R}$ is a strongly modified $(\alpha, h-m)-p$-convex function, then the following inequality holds:*

$$\begin{aligned}&\omega\left(\left[\frac{\lambda^{p}+m\mu^{p}}{2}\right]^{\frac{1}{p}}\right)+\frac{mc\beta}{4\varphi(k,r)}F_{\beta}\\&\leq\frac{pk^{\frac{\beta}{\varphi(k,r)}}\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right)}{(m\mu^{p}-\lambda^{p})^{\frac{\beta}{\varphi(k,r)}}}\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right){}_{\lambda+}R^{\beta}_{\varphi(k,r)}\omega(m\mu)+m^{\frac{\beta}{\varphi(k,r)}+1}h\left(\frac{1}{2^{\alpha}}\right){}_{\mu-}R^{\beta}_{\varphi(k,r)}\omega\left(\frac{\lambda}{m}\right)\right]\\&\leq\frac{\beta}{\varphi(k,r)}\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\omega(\lambda)-m\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\omega(\mu)-m^{2}h\left(\frac{1}{2^{\alpha}}\right)\omega\left(\frac{\lambda}{m^{2}}\right)+mh\left(\frac{1}{2^{\alpha}}\right)\omega(\mu)\right]\\&\quad\times\int_{0}^{1}\tau^{\frac{\beta}{\varphi(k,r)}-1}h(\tau^{\alpha})d\tau+m\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\omega(\mu)+mh\left(\frac{1}{2^{\alpha}}\right)\omega\left(\frac{\lambda}{m^{2}}\right)\right]\\&\quad-\frac{mc\beta\varphi(k,r)}{(\beta+\varphi(k,r))(\beta+2\varphi(k,r))}\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)(\mu^{p}-\lambda^{p})^{2}-mh\left(\frac{1}{2^{\alpha}}\right)\left(\mu^{p}-\frac{\lambda^{p}}{m^{2}}\right)^{2}\right],\end{aligned}\tag{4.8}$$

where $F_{\beta}=\int_{0}^{1}\tau^{\frac{\beta}{\varphi(k,r)}-1}\left(\tau(\mu^{p}-\lambda^{p})+(1-\tau)\left(\frac{\lambda^{p}}{m}-m\mu^{p}\right)\right)^{2}d\tau$.

Proof. Since ω is a strongly modified $(\alpha, h-m)-p$-convex function, then

$$\omega\left([m\tau\iota^p + (1-\tau)\kappa^p]^{\frac{1}{p}}\right) \leq (1-h(\tau^\alpha))\omega(\kappa) + mh(\tau^\alpha)\omega(\iota) - mc\tau(1-\tau)(\iota^p - \kappa^p)^2. \tag{4.9}$$

By choosing $\tau = \dfrac{1}{2}$, we have

$$\omega\left(\left[\frac{\kappa^p + m\iota^p}{2}\right]^{\frac{1}{p}}\right) \leq \left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\kappa) + mh\left(\frac{1}{2^\alpha}\right)\omega(\iota) - \frac{mc}{4}(\iota^p - \kappa^p)^2. \tag{4.10}$$

Assume that $\kappa^p = \tau\lambda^p + m(1-\tau)\mu^p$ and $\iota^p = (1-\tau)\frac{\lambda^p}{m} + \tau\mu^p$ in (4.10) to get

$$\begin{aligned}\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) \leq{}& \left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega\left([\tau\lambda^p + m(1-\tau)\mu^p]^{\frac{1}{p}}\right) + mh\left(\frac{1}{2^\alpha}\right)\omega\left(\left[(1-\tau)\frac{\lambda^p}{m} + \tau\mu^p\right]^{\frac{1}{p}}\right)\\ &- \frac{mc}{4}\left(\tau(\mu^p - \lambda^p) + (1-\tau)\left(\frac{\lambda^p}{m} - m\mu^p\right)\right)^2.\end{aligned} \tag{4.11}$$

Multiplying (4.11) with $\tau^{\frac{\beta}{\varphi(k,r)}-1}$, and then integrating with respect to τ from 0 to 1, we procure

$$\begin{aligned}&\frac{\varphi(k,r)}{\beta}\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) + \frac{mc}{4}F_\beta\\ &\leq \left(1-h\left(\frac{1}{2^\alpha}\right)\right)\int_0^1 \tau^{\frac{\beta}{\varphi(k,r)}-1}\omega\left([\tau\lambda^p + m(1-\tau)\mu^p]^{\frac{1}{p}}\right)d\tau\\ &\quad + mh\left(\frac{1}{2^\alpha}\right)\int_0^1 \tau^{\frac{\beta}{\varphi(k,r)}-1}\omega\left(\left[(1-\tau)\frac{\lambda^p}{m} + \tau\mu^p\right]^{\frac{1}{p}}\right)d\tau.\end{aligned} \tag{4.12}$$

Put $\kappa^p = \tau\lambda^p + m(1-\tau)\mu^p$ in the first integral of (4.12) and $\kappa^p = (1-\tau)\frac{\lambda^p}{m} + \tau\mu^p$ in the second integral of (4.12) to have

$$\begin{aligned}&\frac{\varphi(k,r)}{\beta}\omega\left(\left[\frac{\lambda^p + m\mu^p}{2}\right]^{\frac{1}{p}}\right) + \frac{mc}{4}F_\beta\\ &\leq \left(1-h\left(\frac{1}{2^\alpha}\right)\right)\frac{p}{(m\mu^p - \lambda^p)^{\frac{\beta}{\varphi(k,r)}}}\int_\lambda^{m^{\frac{1}{p}}\mu}(m\mu^p - \kappa^p)^{\frac{\beta}{\varphi(k,r)}-1}\kappa^{p-1}\omega(\kappa)d\kappa\\ &\quad + m^2h\left(\frac{1}{2^\alpha}\right)\frac{p}{(m\mu^p - \lambda^p)^{\frac{\beta}{\varphi(k,r)}}}\int_{\frac{\lambda}{m^{\frac{1}{p}}}}^{\mu}(m\kappa^p - \lambda^p)^{\frac{\beta}{\varphi(k,r)}-1}\kappa^{p-1}\omega(\kappa)d\kappa.\end{aligned} \tag{4.13}$$

Since

$$\int_\lambda^{m^{\frac{1}{p}}\mu}(m\mu^p - \kappa^p)^{\frac{\beta}{\varphi(k,r)}-1}\kappa^{p-1}\omega(\kappa)d\kappa = k\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right){}_{\lambda+}R^{\beta}_{\varphi(k,r)}\omega(\kappa),$$
$$\int_{\frac{\lambda}{m^{\frac{1}{p}}}}^{\mu}(m\kappa^p - \lambda^p)^{\frac{\beta}{\varphi(k,r)}-1}\kappa^{p-1}\omega(\kappa)d\kappa = k\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right){}_{\mu-}R^{\beta}_{\varphi(k,r)}\omega(\kappa).$$

Therefore, (4.13) becomes

$$\begin{aligned}&\omega\left(\left[\frac{\lambda^p+m\mu^p}{2}\right]^{\frac{1}{p}}\right)+\frac{mc\beta}{4\varphi(k,r)}F_\beta\\&\leq\frac{pk\frac{\beta}{\varphi(k,r)}\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right)}{(m\mu^p-\lambda^p)^{\frac{\beta}{\varphi(k,r)}}}\times\left[\left(1-h\left(\frac{1}{2^\alpha}\right)\right)_{\lambda+}R^{\beta}_{\varphi(k,r)}\omega(m\mu)+m^{\frac{\beta}{\varphi(k,r)}+1}h\left(\frac{1}{2^\alpha}\right)_{\mu-}R^{\beta}_{\varphi(k,r)}\omega\left(\frac{\lambda}{m}\right)\right].\end{aligned}\tag{4.14}$$

Also, ω is a strongly modified $(\alpha, h-m)-p$-convex function, then

$$\omega\left([\tau\lambda^p+m(1-\tau)\mu^p]^{\frac{1}{p}}\right)\leq h(\tau^\alpha)\omega(\lambda)+m(1-h(\tau^\alpha))\omega(\mu)-mc\tau(1-\tau)(\mu^p-\lambda^p)^2,\tag{4.15}$$

and

$$\omega\left(\left[(1-\tau)\frac{\lambda^p}{m}+\tau\mu^p\right]^{\frac{1}{p}}\right)\leq m(1-h(\tau^\alpha))\omega\left(\frac{\lambda}{m^2}\right)+h(\tau^\alpha)\omega(\mu)-mc\tau(1-\tau)\left(\mu^p-\frac{\lambda^p}{m^2}\right)^2.\tag{4.16}$$

Multiplying (4.15) with $\left(1-h\left(\frac{1}{2^\alpha}\right)\right)$ and multiplying (4.16) with $mh\left(\frac{1}{2^\alpha}\right)$, then adding (4.15) and (4.16), we get

$$\begin{aligned}&\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega\left([\tau\lambda^p+m(1-\tau)\mu^p]^{\frac{1}{p}}\right)+mh\left(\frac{1}{2^\alpha}\right)\omega\left(\left[(1-\tau)\frac{\lambda^p}{m}+\tau\mu^p\right]^{\frac{1}{p}}\right)\\&\leq\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\left[h(\tau^\alpha)\omega(\lambda)+m(1-h(\tau^\alpha))\omega(\mu)-mc\tau(1-\tau)(\mu^p-\lambda^p)^2\right]\\&\quad+h\left(\frac{1}{2^\alpha}\right)\left[m^2(1-h(\tau^\alpha))\omega\left(\frac{\lambda}{m^2}\right)+mh(\tau^\alpha)\omega(\mu)-m^2c\tau(1-\tau)\left(\mu^p-\frac{\lambda^p}{m^2}\right)^2\right]\\&=h(\tau^\alpha)\left[\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\lambda)-m\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\mu)-m^2h\left(\frac{1}{2^\alpha}\right)\omega\left(\frac{\lambda}{m^2}\right)+mh\left(\frac{1}{2^\alpha}\right)\omega(\mu)\right]\\&\quad+m\left[\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\mu)+mh\left(\frac{1}{2^\alpha}\right)\omega\left(\frac{\lambda}{m^2}\right)\right]\\&\quad-\left(1-h\left(\frac{1}{2^\alpha}\right)\right)mc\tau(1-\tau)(\mu^p-\lambda^p)^2-h\left(\frac{1}{2^\alpha}\right)m^2c\tau(1-\tau)\left(\mu^p-\frac{\lambda^p}{m^2}\right)^2.\end{aligned}\tag{4.17}$$

Multiplying (4.17) with $\tau^{\frac{\beta}{\varphi(k,r)}-1}$, and then integrating with respect to τ from 0 to 1, we have

$$\begin{aligned}&\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\int_0^1\tau^{\frac{\beta}{\varphi(k,r)}-1}\omega\left([\tau\lambda^p+m(1-\tau)\mu^p]^{\frac{1}{p}}\right)d\tau+mh\left(\frac{1}{2^\alpha}\right)\int_0^1\tau^{\frac{\beta}{\varphi(k,r)}-1}\omega\left(\left[(1-\tau)\frac{\lambda^p}{m}+\tau\mu^p\right]^{\frac{1}{p}}\right)d\tau\\&\leq\left[\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\lambda)-m\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\mu)-m^2h\left(\frac{1}{2^\alpha}\right)\omega\left(\frac{\lambda}{m^2}\right)+mh\left(\frac{1}{2^\alpha}\right)\omega(\mu)\right]\int_0^1\tau^{\frac{\beta}{\varphi(k,r)}-1}h(\tau^\alpha)d\tau\\&\quad+m\left[\left(1-h\left(\frac{1}{2^\alpha}\right)\right)\omega(\mu)+mh\left(\frac{1}{2^\alpha}\right)\omega\left(\frac{\lambda}{m^2}\right)\right]\int_0^1\tau^{\frac{\beta}{\varphi(k,r)}-1}d\tau\end{aligned}$$

$$-m\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)c(\mu^{p}-\lambda^{p})^{2}\int_{0}^{1}\tau(1-\tau)\tau^{\frac{\beta}{\varphi(k,r)}-1}d\tau$$
$$-m^{2}h\left(\frac{1}{2^{\alpha}}\right)c\left(\mu^{p}-\frac{\lambda^{p}}{m^{2}}\right)^{2}\int_{0}^{1}\tau(1-\tau)\tau^{\frac{\beta}{\varphi(k,r)}-1}d\tau. \tag{4.18}$$

Put $\kappa^p = \tau\lambda^p + m(1-\tau)\mu^p$ in the first integral of (4.18) and $\kappa^p = (1-\tau)\frac{\lambda^p}{m} + \tau\mu^p$ in the second integral of (4.18), and we obtain

$$\begin{aligned}
&\frac{pk\frac{\beta}{\varphi(k,r)}\Gamma_{(k,r)}\left(\frac{k\beta}{\varphi(k,r)}\right)}{(m\mu^{p}-\lambda^{p})^{\frac{\beta}{\varphi(k,r)}}}\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)_{\lambda+}R^{\beta}_{\varphi(k,r)}\omega(m\mu)+m^{\frac{\beta}{\varphi(k,r)}+1}h\left(\frac{1}{2^{\alpha}}\right)_{\mu-}R^{\beta}_{\varphi(k,r)}\omega\left(\frac{\lambda}{m}\right)\right]\\
&\le\frac{\beta}{\varphi(k,r)}\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\omega(\lambda)-m\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\omega(\mu)-m^{2}h\left(\frac{1}{2^{\alpha}}\right)\omega\left(\frac{\lambda}{m^{2}}\right)+mh\left(\frac{1}{2^{\alpha}}\right)\omega(\mu)\right]\\
&\times\left(\int_{0}^{1}\tau^{\frac{\beta}{\varphi(k,r)}-1}h(\tau^{\alpha})d\tau\right)+m\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)\omega(\mu)+mh\left(\frac{1}{2^{\alpha}}\right)\omega\left(\frac{\lambda}{m^{2}}\right)\right]\\
&-\frac{mc\beta\varphi(k,r)}{(\beta+\varphi(k,r))(\beta+2\varphi(k,r))}\left[\left(1-h\left(\frac{1}{2^{\alpha}}\right)\right)(\mu^{p}-\lambda^{p})^{2}-mh\left(\frac{1}{2^{\alpha}}\right)\left(\mu^{p}-\frac{\lambda^{p}}{m^{2}}\right)^{2}\right].
\end{aligned} \tag{4.19}$$

From (4.14) and (4.19), we obtain (4.8).

Remark 4.2. *(i) By taking $\alpha = m = 1$ and $\varphi(k,r) = I$ in* (4.8)*, we obtain Theorem 3 of [15].*
(ii) By taking $\alpha = 1$, $p = 1$, and $\varphi(k,r) = I$ in (4.8)*, we obtain Theorem 3 of [14].*
(iii) By taking $\alpha = p = 1$, $h(\tau) = \tau$, and $\varphi(k,r) = I$ in (4.8)*, we obtain Theorem 2.1 of [19].*

5. Application

Example 5.1. By taking $p = 2$, $\tau = \frac{1}{2}$, $c = \frac{1}{2}$, $m = 1$, $\alpha = 1$, $h(\tau) = \tau^5$, and $\omega(\kappa) = \kappa^4$ in Definition 3.1, we get

$$\begin{aligned}
&\left(\frac{\kappa^{2}+\iota^{2}}{2}\right)^{2}\le\frac{\kappa^{4}}{32}+\frac{31\iota^{4}}{32}-\frac{(\kappa^{2}-\iota^{2})^{2}}{8}\\
&\Rightarrow 6,25\le 14,40.
\end{aligned}$$

The presence of Hermite-Hadamard inequalities produced in Definition 3.1 is demonstrated in Figure 1.

Example 5.2. Assume that $p = 2$, $\tau = \frac{1}{4}$, $c = \frac{1}{2}$, $m = 1$, $\alpha = 1$, $h(\tau) = \tau^2$, then, the $\omega(\kappa) = \kappa^3$ is a convex function but is not a strongly modified $(\alpha, h-m) - p$-convex function.

$$\left(\frac{\kappa^{2}+3\iota^{2}}{4}\right)^{\frac{3}{2}}\nleq\frac{\kappa^{3}}{16}+\frac{15\iota^{3}}{16}-\frac{3(\kappa^{2}-\iota^{2})^{2}}{32}. \tag{5.1}$$

The presence of the above inequality is demonstrated in Figure 2.

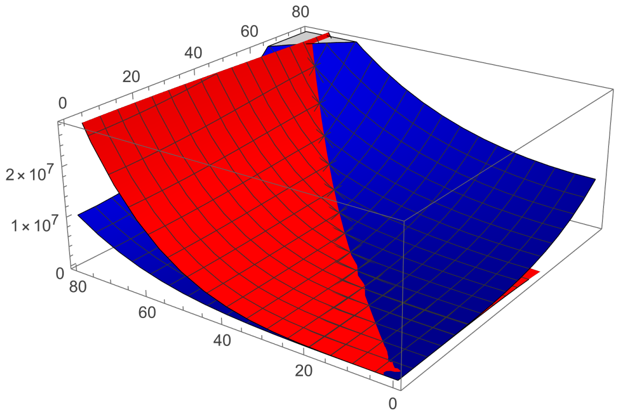

Figure 1. The graphical presentations of inequality (3.1).

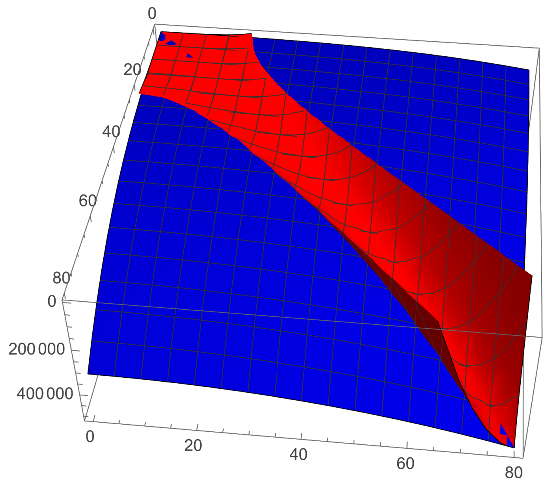

Figure 2. The graphical presentations of inequality (5.1).

Example 5.3. By taking $p = 2$, $\tau = \frac{1}{2}$, $c = \frac{1}{2}$, $m = 1$, $\alpha = \frac{1}{2}$, $h(\tau) = \tau^2$, and $\omega(\kappa) = \kappa^4$ in Theorem 4.1, we get

$$\left(\frac{a^2+b^2}{2}\right)^2 \leq \frac{b^6-a^6}{3(b^2-a^2)} \leq \frac{a^4+b^4}{2} - \frac{(b^2-a^2)^2}{8}$$
$$\Rightarrow 6,25 \leq 7 \leq 7,37.$$

The presence of Hermite-Hadamard inequalities produced in Theorem 4.1 is demonstrated in Figure 3.

Example 5.4. By taking $p = 2$, $\tau = \frac{3}{4}$, $c = \frac{1}{2}$, $m = 1$, $\beta = 1$, $\alpha = \frac{1}{2}$, $h(\tau) = \tau^2$, $\omega(\kappa) = \kappa^4$, and $\varphi(k, r) = I$ in Theorem 4.2, we get

$$\left(\frac{a^2+b^2}{2}\right)^2 + \frac{(b^2-a^2)^2}{32} \leq \frac{b^6-a^6}{3(b^2-a^2)} \leq \frac{a^4+b^4}{2}$$
$$\Rightarrow 6,53 \leq 7 \leq 8,50.$$

The presence of Hermite-Hadamard inequalities produced in Theorem 4.2 is demonstrated in Figure 4.

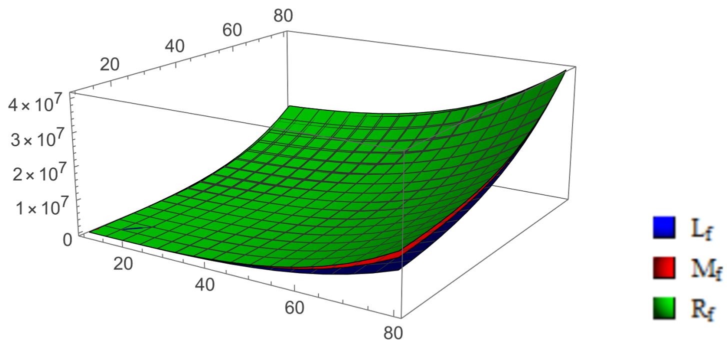

Figure 3. The graphical presentations of inequality (4.1).

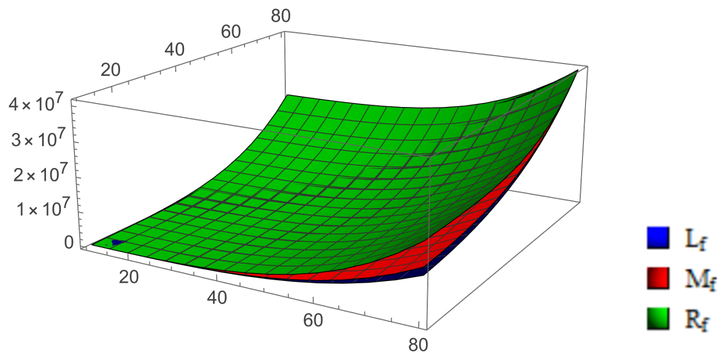

Figure 4. The graphical presentations of inequality (4.8).

6. Conclusions

In this paper, we introduced the concept of the strongly modified $(\alpha, h - m) - p$-convex function, a generalization of the strongly modified (h, m)-convex function [14] and strongly modified (p, h)-convex function [15] notions, and examined some of its properties. We also obtained Hermite-Hadamard type inequalities, including (k, r)-Riemann-Liouville fractional integrals, for this new concept. Some inequalities shown in [12–15, 19, 20] can be obtained from the Hermite-Hadamard inequalities presented in this paper. Some examples are also given to visualize the existence of this new class of convex functions and the inequalities obtained from this definition. Different inequalities, such as Fejer and Schur type, involving different fractional integrals, can be examined for this proposed class of convex functions. We believe that this newly defined the convex function class will be beneficial to inequality theory.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this paper.

Conflict of interest

The author declares that there is no conflict of interest in this paper.

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