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Research article

H_∞ boundary control design of spatial 2-D linear stochastic parabolic PDE systems

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Abstract: In practical applications involving spatiotemporal processes, the two-dimensional (2-D) spatial scenario is often more relevant. However, higher spatial dimensionality poses a major challenge for control design, drastically increasing its complexity. This work investigated the H_∞ boundary control problem for a class of 2-D linear stochastic parabolic partial differential equation (PDE) systems subject to multiplicative noise in the state. We used multidimensional Green's formula and alternative inequalities to handle complex terms in 2-D systems. A static output feedback (SOF) control strategy was introduced to achieve stabilization within a stochastic paradigm. By employing Lyapunov-based technique, a constructive method for SOF controller design was established, ensuring that the closed-loop system achieved exponential stability in the mean square sense with H_∞ performance. Numerical simulations of a stochastic thermal process demonstrated the effectiveness and applicability of the proposed control methodology.

Keywords: 2-D linear stochastic parabolic PDE systems; H_∞ boundary control; static output feedback; mean square exponential stability
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1. Introduction

In the real world, a great many processes can be described by distributed parameter systems (DPSs), such as neural science, fluid mechanics, heat conduction, etc. [1–3]. Due to the continuous distribution of these system states in both time and space, these systems are generally described by partial differential equations (PDEs). In control theory, the research on infinite-dimensional systems has led to the development of various analysis and synthesis methods that can preserve the distributed characteristics of PDEs. Among them, the application of Volterra integral transformation in the boundary control of linear parabolic PDEs is a pioneering work [4]. This achievement has promoted the formation of several specialized control strategies, such as active interference suppression strategies [5], output regulation techniques [6, 7], boundary stabilization methods [8], and adaptive boundary regulation [9]. With the deepening of model control methods, the observer design field has also achieved synchronous development. Reference [10] proposed a finite-dimensional observer design scheme for linear partial differential systems with time delays. At the same time, data-driven analysis methods based on data have gradually become an important approach for handling nonlinear problems. Research [11] constructed a spatiotemporal least squares support vector machine modeling framework, which can be used to describe the dynamics of complex nonlinear PDEs. In addition, reference [12] further integrated adaptive Karhunen-Loève decomposition and fuzzy system modeling, achieving tracking of time-varying spatiotemporal dynamics. The above theoretical advancements have been widely applied in engineering practice. For instance, study [13] designed a boundary control law for a flexible robotic arm system.

The dynamic evolution process of PDE systems is generally disturbed by external stochastic perturbations, which are mostly caused by environmental condition fluctuations, fluid wave force effects, and measurement errors. As a result, the research on stochastic PDEs (SPDEs) has gradually become a hot topic in the academic field, and the stability analysis of the corresponding SDPSs has also received extensive and in-depth discussions [14, 15]. Reference [16] constructed drive-response stochastic space-time discrete non-identical two-hub genetic regulatory networks. Compared with stochastic ordinary differential equation control systems, the research on SPDE systems faces more severe challenges due to the inherent infinite-dimensional nature of SPDEs [17, 18]. Among the various branches of PDEs, parabolic SPDEs have irreplaceable mathematical research value, as they can accurately describe a series of fundamental physical phenomena. In the past decade, considerable research efforts have been devoted to control design for parabolic SPDE systems, leading to numerous effective and practical control strategies [19–21].

In a bounded domain system, the boundary conditions need to be set in accordance with the actual physical background. For example, for the heat conduction equation, the homogeneous Neumann condition indicates an adiabatic boundary, while the homogeneous Dirichlet condition corresponds to a constant temperature boundary. When using the boundary control strategy, the actuator should be placed at the boundary position of the system area, which provides a feasible approach to achieving the specific performance of the SPDEs [22–24]. The backward propagation method, as a classic approach in boundary control research, was initially mainly applied to deterministic linear systems [25–27]. These methods usually require the full system state information [28, 29], but in practice, global measurement is often difficult to achieve. Therefore, observers can be used to estimate the state [30–32]. In the field of stochastic system control, there have been numerous research results based on observers [33–35].

In the presence of external disturbances, the H_∞ performance of a system is an important indicator for evaluating its disturbance rejection capability [36, 37]. Meanwhile, the H_∞ sampled-data control problem of PDEs has gradually attracted attention [26, 27]. For instance, [26] adopted a spatially discrete sampling strategy to study the H_∞ control problem of output synchronization for coupled time-delay stochastic neural networks. Another reference [27] analyzed the H_∞ control strategy for a class of semi-linear transport-reaction equations under sampled data conditions. Although certain progress has been made in related research, existing achievements are mostly concentrated on the exploration of one-dimensional linear systems, while the spatiotemporal dynamics in reality often involve complex features of multi-dimensionality and nonlinearity.

The above research results are currently applicable only to one-dimensional (1-D) spatial PDE systems. In fact, two-dimensional (2-D) PDE systems in space better describe the spatiotemporal characteristics of actual physical processes and thus have stronger physical basis. However, it is worth noting that as the system expands to 2-D space, the complexity of designing the mobile controller will increase sharply. Compared with 1-D systems where sensors and actuators can only be arranged along a 1-D line segment, 2-D systems allow them to be arbitrarily placed within a planar domain. This flexibility brings greater design freedom to static output feedback (SOF), but at the cost of making the matching conditions more complex. The solution to this problem is precisely the key focus of this study. The linear matrix inequality conditions we propose can be directly applied to the SOF design of the aforementioned 2-D systems, thereby demonstrating strong practical application prospects. This paper investigates a class of 2-D parabolic SPDE systems whose states are disturbed by multiplicative noise. For systems evolving in a two-dimensional spatial field, their behavioral mechanisms and control methods urgently require more comprehensive theoretical support and methodological construction. In light of this, we want to conduct in-depth exploration of control strategies for higher-dimensional dynamic systems with stochastic distributed parameters, in order to address the limitations of current theories when dealing with actual complex systems. The aim is to design an exponential stabilization scheme that guarantees H_∞ boundary performance. Our method is based on the SOF structure and utilizes infinite-dimensional system operators and Lyapunov stability theory to derive sufficient conditions described by linear matrix inequalities. Subsequently, a verification process for the H_∞ boundary controller design is established, ensuring that the closed-loop system has strict H_∞ performance in the mean-square sense. Through numerical simulations of the thermal process under stochastic perturbation, the effectiveness and feasibility of this control strategy are verified, indicating its applicability to the control of such DPSs.

Notations: In this paper, $\mathbb{R}$ and $\mathbb{R}^n$ indicate the set of one-dimensional and n-dimensional real numbers, respectively, and $(\cdot)^T$ is employed for the transpose operator for matrices or vectors. The notation $\Omega \leqslant (<)\ 0$ denotes that matrix Ω is a negative semidefinite (respectively, negative definite), for $\Omega = \Omega^T$. Given the constants $a > 0$ and $b > 0$, for a real-valued function $c(h, v) \in \mathbb{R}$, $(h, v) \in [0, a] \times [0, b] \subset \mathbb{R}^2$, one denotes $c(\cdot, \cdot) \triangleq c(h, v), (h, v) \in [0, a] \times [0, b]$. The collection of all square-integrable scalar functions $c(h, v) : [0, a] \times [0, b] \to \mathbb{R}$ forms a Hilbert space $\mathbb{Y} \triangleq \mathcal{L}^2([0, a] \times [0, b]; \mathbb{R})$ whose inner product is given by $\langle c_1(\cdot, \cdot), c_2(\cdot, \cdot)\rangle \triangleq \int_0^a \int_0^b c_1(h, v)c_2(h, v)dhdv$ and norm $\|c_1(\cdot, \cdot)\|_2 \triangleq \sqrt{\langle c_1(\cdot, \cdot), c_1(\cdot, \cdot)\rangle}$ with $c_1(\cdot, \cdot)$ and $c_2(\cdot, \cdot)$ being members in $\mathbb{Y}$. The Laplacian Δ is expressed as $\Delta c(h, v) \triangleq \frac{\partial^2 c(h,v)}{\partial h^2} + \frac{\partial^2 c(h,v)}{\partial v^2}$. The gradient operator ∇ is defined as $\nabla c(h, v) \triangleq [c_h(h, v), c_v(h, v)]^T$. Entries marked with $*$ in a matrix are understood to be equal to the symmetric counterpart across the diagonal.

2. Model and preliminary materials

The article considers a nolinear stochastic parabolic PDE system in two spatial dimensions as follows:

$$dc(h, v, t) = [\alpha \Delta c(h, v, t) + \varphi(c(h, v, t)) + \beta \varepsilon(h, v, t)]dt + \psi(c(h, v, t))dw(h, v, t), \tag{1}$$

subject to the Neumann boundary condition

$$\begin{cases} c_h(0, v, t) = c_v(h, 0, t) = 0, \\ c_h(a, v, t) = u_1(v, t), \\ c_v(h, b, t) = u_2(h, t), \end{cases} \tag{2}$$

coupled with the initial condition

$$c(h, v, 0) = c_0(h, v), \tag{3}$$

in which $\alpha > 0$ and β are the known constants; $c(h, v, t) \in \mathcal{L}^2(W; \mathbb{R})$ is the system state, where $\mathcal{L}^2(W; \mathbb{R})$ is a Hilbert space; $t \geq 0$ and $(h, v) \in W \triangleq [0, a] \times [0, b]$ respectively represent the time and the spatial position; $\varphi(c(\cdot, \cdot, t))$ and $\psi(c(\cdot, \cdot, t))$ are given as nonlinear local Lipschitz continuous functions satisfying $\varphi(0) = \psi(0) = 0$; $\varepsilon(\cdot, \cdot, t)$ is a random external disturbance with finite energy; $u_1(v, t)$ and $u_2(h, t)$ represent as the manipulated control boundary input; $c_0(h, v)$ is the deterministic initial system state; $w(h, v, t) \in \mathbb{R}$ represents 2-D standard Brownian motion.

Remark 1. *The system is subjected to a Gaussian random field $w(h, v, t)$ originating from a cylindrical Wiener process, with an average value of zero. Within the framework of the Hilbert space, the time process at a fixed spatial point (h, v) is continuous. The term $\psi(c(h, v, t))dw(h, v, t)$ represents a state-dependent continuous intrinsic fluctuation, indicating a linear coupling relationship between the internal fluctuations of the system and the state variable $c(h, v, t)$.*

Remark 2. *This study has transcended the dimensional limitations of the 1-D theory in reference [38], and has reconstructed this problem within a 2-D framework. Although the 1-D model can provide valuable theoretical insights, it is difficult to describe the complex multi-directional dynamics in real systems. The 2-D model constructed in this paper effectively addresses this deficiency and achieves explicit modeling of spatial gradient and directional changes.*

Then, to control the spatially 2-D linear stochastic PDE system (1)–(3), we address the following SOF control laws:

$$\begin{aligned} u_1(v, t) &= l_1 o_1(v, t), \\ u_2(h, t) &= l_2 o_2(h, t), \end{aligned} \tag{4}$$

in which the control gains l_1, l_2 are to be determined, and $o_1(v, t) \triangleq \int_0^a c(h, v, t)dh$ and $o_2(h, t) \triangleq \int_0^b c(h, v, t)dv$ denote the objective outputs of the nonlinear SPDE (1)–(3).

Remark 3. *It should be noted that the distributed state feedback controller (4) is an infinite-dimensional controller. Therefore, a finite-dimensional approximate form must be derived for this controller to facilitate online implementation. This task can be accomplished by using standard discretization techniques such as finite difference methods. Therefore, the number of actuators and sensors required to implement controller (4) must be sufficient. Recent significant progress in drive and sensing technologies, especially the advancement of micro-electromechanical systems technology, has made it possible to use a large number of actuators and sensors to implement such controllers. Distributed flow control for drag reduction, temperature control of catalytic surfaces, etc., are all examples of such applications.*

As is well known, H_∞ control is an excellent method that can mitigate the impact of external disturbances on the system. This work considers the following H_∞ performance with zero initial condition, i.e., $c_0 = 0$,

$$E\int_0^\infty (x\|c(\cdot,\cdot,t)\|_2^2 - y^2\|\varepsilon(\cdot,\cdot,t)\|_2^2)dt \le 0, \tag{5}$$

where $x > 0$ is a specified degree of interference attenuation and $y > 0$ is a given constant.

Definition 1. *The unforced disturbance-free stochastic PDE system (1)–(3) with equilibrium point $c(h, v, t) \equiv 0$ is said to be exponentially stable in mean square sense, if there exist some positive constants $\xi_1 > 0$ and $\xi_2 > 0$ such that the following holds:*

$$E[\|c(\cdot,\cdot,t)\|_2^2] \le \xi_1\|c_0(\cdot,\cdot)\|_2^2 \exp(-\xi_2 t). \tag{6}$$

Definition 2. *The unforced stochastic PDE system (1)–(3) is said to be exponentially stable in mean square sense with H_∞ performance, provided that: (1) the corresponding disturbance-free system is exponentially stable in mean square sense, i.e., (6) holds, and (2) the H_∞ performance (5) is satisfied.*

Assumption 1. *Suppose the functions $\varphi(\cdot)$ and $\psi(\cdot)$ satisfy the following Lipschitz condition:*

$$\|\varphi(c_1(\cdot,\cdot,t)) - \varphi(c_2(\cdot,\cdot,t))\|_2 \le \gamma\|c_1(\cdot,\cdot,t) - c_2(\cdot,\cdot,t)\|_2,$$
$$\|\psi(c_1(\cdot,\cdot,t)) - \psi(c_2(\cdot,\cdot,t))\|_2 \le \delta\|c_1(\cdot,\cdot,t) - c_2(\cdot,\cdot,t)\|_2,$$

for some nonnegative scalars $\gamma > 0$ and $\delta > 0$, and any $c_1(h, v, t), c_2(h, v, t) \in \mathcal{L}^2(W;\mathbb{R})$.

Lemma 1. *(Wirtinger's inequality* [39]*). For any function $c(h)$ satisfying $c(0) = 0$ or $c(a) = 0$, it holds*

$$\int_0^a c^2(h)dh \le \frac{4a^2}{\pi^2}\int_0^a c_h^2(h)dh. \tag{7}$$

This work thus aims to design a Lyapunov-based SOF controller in the form of (4) to ensure the exponential mean-square stability of the closed-loop SPDE system (1)–(3).

3. Boundary SOF control design

In this section, the H_∞ boundary SOF control design approach is developed. Let us construct the following candidate for Lyapunov functional:

$$V(t) = \frac{k}{2}\int_W c^2(h, v, t)dhdv. \tag{8}$$

Then, the infinitesimal operator of $V(t)$ is

$$\mathscr{L}[V(t)] = \int_W \Big[kc(h, v, t)\Big(\alpha\Delta c(h, v, t) + \varphi(c(h, v, t)) + \beta\varepsilon(h, v, t)\Big) + \frac{k}{2}\psi^2(c(h, v, t))\Big]dhdv. \tag{9}$$

Applying the Green formula, based on the boundary condition (2) and boundary SOF controller (4), it holds that

$$\int_W c(h, v, t)\frac{\partial^2 c(h, v, t)}{\partial h^2}dhdv = \oint_{\partial W} c(h, v, t)c_h(h, v, t)dv - \int_W c_h^2(h, v, t)dhdv$$

$$
\begin{aligned}
&= \int_0^b c(a,v,t)c_h(a,v,t)dv + \int_b^0 c(0,v,t)c_h(0,v,t)dv - \int_W c_h^2(h,v,t)dhdv \\
&= \int_W c(a,v,t)l_1 c(h,v,t)dhdv - \int_W c_h^2(h,v,t)dhdv. \qquad (10)
\end{aligned}
$$

Similarly, it can be obtained that

$$
\int_W c(h,v,t)\frac{\partial^2 c(h,v,t)}{\partial v^2}dhdv = \int_W c(h,b,t)l_2 c(h,v,t)dhdv - \int_W c_v^2(h,v,t)dhdv. \qquad (11)
$$

Combining (10) and (11), we have

$$
\begin{aligned}
&\int_W c(h,v,t)\Delta c(h,v,t)dhdv \\
&= \int_W c(h,v,t)[l_1 c(a,v,t) + l_2 c(h,b,t)]dhdv - \int_W \nabla c^T(h,v,t)\nabla c(h,v,t)dhdv \\
&\leq \int_W c(h,v,t)[l_1 c(a,v,t) + l_2 c(h,b,t)]dhdv - \frac{\pi^2}{4a^2}\int_W [c(h,v,t) - c(a,v,t)]^2 dhdv \\
&\quad - \frac{\pi^2}{4b^2}\int_W [c(h,v,t) - c(h,b,t)]^2 dhdv, \qquad (12)
\end{aligned}
$$

where the inequation has used Lemma 1.

From the Cauchy-Schwarz inequality and Assumption 1, one has

$$
\begin{aligned}
\int_W c(h,v,t)\varphi(c(h,v,t))dhdv &\leq \gamma \int_W c^2(h,v,t)dhdv, \\
\int_W \psi^2(c(h,v,t))dhdv &\leq \delta^2 \int_W c^2(h,v,t)dhdv. \qquad (13)
\end{aligned}
$$

Substituting (12) and (13) into (9), we have

$$
\begin{aligned}
\mathscr{L}[V(t)] \leq& k\alpha \int_W c(h,v,t)[l_1 c(a,v,t) + l_2 c(h,b,t)]dhdv - \frac{k\alpha\pi^2}{4a^2}\int_W [c(h,v,t) - c(a,v,t)]^2 dhdv \\
&- \frac{k\alpha\pi^2}{4b^2}\int_W [c(h,v,t) - c(h,b,t)]^2 dhdv + k\gamma \int_W c^2(h,v,t)dhdv \\
&+ k\beta \int_W c(h,v,t)\varepsilon(h,v,t)dhdv + \frac{k\delta^2}{2}\int_W c^2(h,v,t)dhdv \\
\triangleq& \int_W \tilde{\mathbf{c}}^T(h,v,t)\Phi\tilde{\mathbf{c}}(h,v,t)dhdv, \qquad (14)
\end{aligned}
$$

where

$$
\tilde{\mathbf{c}}(h,v,t) \triangleq \left[c(h,v,t)\ c(a,v,t)\ c(h,b,t)\ \varepsilon(h,v,t)\right]^T, \qquad (15)
$$

and

$$
\Phi \triangleq \begin{bmatrix} -\frac{k\alpha\pi^2}{4a^2} - \frac{k\alpha\pi^2}{4b^2} + k\gamma + \frac{k\delta^2}{2} & \frac{k\alpha l_1}{2} + \frac{k\alpha\pi^2}{4a^2} & \frac{k\alpha l_2}{2} + \frac{k\alpha\pi^2}{4b^2} & \frac{k\beta}{2} \\ * & -\frac{k\alpha\pi^2}{4a^2} & 0 & 0 \\ * & * & -\frac{k\alpha\pi^2}{4b^2} & 0 \\ * & * & * & 0 \end{bmatrix}, \qquad (16)
$$

in which entries marked with $*$ are understood to be equal to the symmetric counterpart across the diagonal.

Then, it holds

$$\mathscr{L}[V(t)] + \int_W (xc^2(h,v,t) - y^2\varepsilon^2(h,v,t))dhdv \le \int_W \tilde{\mathbf{c}}^T(h,v,t)\Gamma\tilde{\mathbf{c}}(h,v,t)dhdv, \tag{17}$$

where

$$\Gamma \triangleq \begin{bmatrix} -\frac{k\alpha\pi^2}{4a^2} - \frac{k\alpha\pi^2}{4b^2} + k\gamma + \frac{k\delta^2}{2} + x & \frac{k\alpha l_1}{2} + \frac{k\alpha\pi^2}{4a^2} & \frac{k\alpha l_2}{2} + \frac{k\alpha\pi^2}{4b^2} & \frac{k\beta}{2} \\ * & -\frac{k\alpha\pi^2}{4a^2} & 0 & 0 \\ * & * & -\frac{k\alpha\pi^2}{4b^2} & 0 \\ * & * & * & -y^2 \end{bmatrix}. \tag{18}$$

Therefore, if one has

$$\Gamma < 0, \tag{19}$$

then

$$\mathscr{L}[V(t)] + x\|c(\cdot,\cdot,t)\|_2^2 - y^2\|\varepsilon(\cdot,\cdot,t)\|_2^2 < 0. \tag{20}$$

Theorem 1. *The interest system is the spatial 2-D nonlinear SPDE given in (1)–(3) and assumes Assumption1 holds. For given constants $x > 0$ and $y > 0$, if the scalars s_1, s_2, and $k > 0$ can ensure*

$$\begin{bmatrix} -\frac{k\alpha\pi^2}{4a^2} - \frac{k\alpha\pi^2}{4b^2} + k\gamma + \frac{k\delta^2}{2} + x & \frac{\alpha s_1}{2} + \frac{k\alpha\pi^2}{4a^2} & \frac{\alpha s_2}{2} + \frac{k\alpha\pi^2}{4b^2} & \frac{k\beta}{2} \\ * & -\frac{k\alpha\pi^2}{4a^2} & 0 & 0 \\ * & * & -\frac{k\alpha\pi^2}{4b^2} & 0 \\ * & * & * & -y^2 \end{bmatrix} < 0, \tag{21}$$

then there exists an SOF control law (4) which yields the closed-loop PDE system (1)–(3) exponentially stable in the mean-square sense with H_∞ performance. Moreover, the control gains is obtained as

$$l_1 = s_1k^{-1}, l_2 = s_2k^{-1}. \tag{22}$$

Proof. (1) For the H_∞ performance, let's integrate (20) from 0 to ∞ and take mathematical expectation, then it holds that

$$E\int_0^\infty \mathscr{L}[V(t)]dt + E\int_0^\infty (x\|c(\cdot,\cdot,t)\|_2^2 - y^2\|\varepsilon(\cdot,\cdot,t)\|_2^2)dt < 0. \tag{23}$$

By Theorem 4.17 of [40], it holds

$$E\int_0^\infty \mathscr{L}[V(t)]dt = E[V(c(\cdot,\cdot,t))]. \tag{24}$$

Due to $E[V(c(\cdot,\cdot,t))] \ge 0$, combining (23) and (24), we have

$$E\int_0^\infty (x\|c(\cdot,\cdot,t)\|_2^2 - y^2\|\varepsilon(\cdot,\cdot,t)\|_2^2)dt < 0, \tag{25}$$

which completes the part of H_∞ performance proof.

(2) Now, let's prove the exponential stability. When $\varepsilon(h, v, t) = 0$, (14) is exchanged into

$$\mathscr{L}[V(t)] \leq \int_W \overline{\mathbf{c}}^T(h, v, t)\Psi\overline{\mathbf{c}}(h, v, t)dhdv, \tag{26}$$

where

$$\overline{\mathbf{c}}(h, v, t) \triangleq [c(h, v, t)\ c(a, v, t)\ c(h, b, t)]^T, \tag{27}$$

and

$$\Psi \triangleq \begin{bmatrix} -\frac{k\alpha\pi^2}{4a^2} - \frac{k\alpha\pi^2}{4b^2} + k\gamma + \frac{k\delta^2}{2} & \frac{k\alpha l_1}{2} + \frac{k\alpha\pi^2}{4a^2} & \frac{k\alpha l_2}{2} + \frac{k\alpha\pi^2}{4b^2} \\ * & -\frac{k\alpha\pi^2}{4a^2} & 0 \\ * & * & -\frac{k\alpha\pi^2}{4b^2} \end{bmatrix}. \tag{28}$$

From (21), one has $\Psi < 0$. Therefore, by choosing a sufficiently small constant $\lambda > 0$, it holds that

$$\Psi + 2\lambda I \leq 0. \tag{29}$$

Taking into account both (26) and (29), we have

$$\mathscr{L}[V(t)] \leq -2\lambda\|c(\cdot, \cdot, t)\|_2^2 = -\frac{4\lambda}{k}V(t), \tag{30}$$

where the equation is owing to (8).

Considering (9) and (30), we have

$$\int_W \Big[kc(h, v, t)\Big(\alpha\Delta c(h, v, t) + \varphi(c(h, v, t))\Big) + \frac{k}{2}\psi^2(c(h, v, t))\Big]dhdv \leq -\frac{4\lambda}{k}V(t). \tag{31}$$

Integrating (31) from 0 to t on both sides, and calculating the expectation, results in

$$E\int_0^t \int_W \Big[kc(h, v, \iota)\Big(\alpha\Delta c(h, v, \iota) + \varphi(c(h, v, \iota))\Big) + \frac{k}{2}\psi^2(c(h, v, \iota))\Big]dhdvd\iota \leq -\frac{4\lambda}{k}E\int_0^t V(\iota)d\iota. \tag{32}$$

By Theorem 4.17 of [40], one has

$$E[V(t)] = V(0) + E\int_0^t \int_W \Big[kc(h, v, \iota)\Big(\alpha\Delta c(h, v, \iota) + \varphi(c(h, v, \iota))\Big) + \frac{k}{2}\psi^2(c(h, v, \iota))\Big]dhdvd\iota. \tag{33}$$

Substituting (33) into (32), we get

$$E[V(t)] \leq V(0) - \frac{4\lambda}{k}E\int_0^t V(\iota)d\iota. \tag{34}$$

Taking the derivative of (34) with respect to t on both sides gives

$$\frac{dE[V(t)]}{dt} \leq -\frac{4\lambda}{k}E[V(t)]. \tag{35}$$

From (35), we can easily get

$$E[V(t)] \leq V(0)\exp\big(-\frac{4\lambda}{k}t\big). \tag{36}$$

Considering (8) and (36), we get

$$E\|c(\cdot,\cdot,t)\|_2^2 = \frac{2}{k}E[V(t)] \leq \frac{2}{k}V(0)\exp\big(-\frac{4\lambda}{k}t\big) = \frac{2}{k}\frac{k}{2}\|c_0(\cdot,\cdot)\|_2^2\exp\big(-\frac{4\lambda}{k}t\big) = \|c_0(\cdot,\cdot)\|_2^2\exp\big(-\frac{4\lambda}{k}t\big). \tag{37}$$

As Definition 2, the corresponding disturbance-free system is exponentially stable in mean square sense, which completes the proof. □

Similar to the proof of Theorem 1, we can get the following corollary about the linear SPDE system.

Corollary 1. *The interested linear parabolic SPDE system in two spatial dimensions is described as:*

$$dc(h,v,t) = [\alpha\Delta c(h,v,t) + \zeta c(h,v,t) + \beta\varepsilon(h,v,t)]dt + \eta c(h,v,t)dw(h,v,t), \tag{38}$$

subject to the Neumann boundary condition

$$\begin{cases} c_h(0,v,t) = c_v(h,0,t) = 0, \\ c_h(a,v,t) = u_1(v,t), \\ c_v(h,b,t) = u_2(h,t), \end{cases} \tag{39}$$

coupled with the initial condition

$$c(h,v,0) = c_0(h,v), \tag{40}$$

in which $\zeta > 0$ and η are the known constants. For given constants $x > 0$ and $y > 0$, if the scalars s_1, s_2, and $k > 0$ can ensure

$$\begin{bmatrix} -\frac{k\alpha\pi^2}{4a^2} - \frac{k\alpha\pi^2}{4b^2} + k\zeta + \frac{k\eta^2}{2} + x & \frac{\alpha s_1}{2} + \frac{k\alpha\pi^2}{4a^2} & \frac{\alpha s_2}{2} + \frac{k\alpha\pi^2}{4b^2} & \frac{k\beta}{2} \\ * & -\frac{k\alpha\pi^2}{4a^2} & 0 & 0 \\ * & * & -\frac{k\alpha\pi^2}{4b^2} & 0 \\ * & * & * & -y^2 \end{bmatrix} < 0, \tag{41}$$

then there exists an SOF control law (4) which yields the closed-loop PDE system (38)–(40) exponentially stable in the mean-square sense with H_∞ performance. Moreover, the control gains is obtained as

$$l_1 = s_1k^{-1}, l_2 = s_2k^{-1}. \tag{42}$$

Remark 4. *To ensure that the system design goals are achieved, the linear matrix inequalities (21) and (41) provide sufficient constraints. If there is a feasible solution to this inequality system, the SOF control law (4) that meets the design requirements can be directly derived from it; otherwise, other design strategies need to be considered to meet the system requirements.*

Remark 5. *In this paper, we have taken into account the Newman boundary conditions. In fact, if other boundary conditions, such as Dirichlet, are introduced, similar conclusions can also be drawn.*

Remark 6. *Extending the SPDEs to a two-dimensional space model is of great significance for accurately describing the planar dynamic behaviors of reaction-diffusion processes. However, the increase in spatial dimensions also significantly increases the complexity of numerical solution. Looking forward to future research, we plan to integrate the PDE control method with mobile robot technology to construct a control platform with dynamic perception and execution capabilities, in order to achieve mobile closed-loop regulation of systems with randomly distributed parameters. At the same time, we will further study the cooperative control architecture that can perform real-time path planning under the constraint of SPDEs. This research approach that integrates infinite-dimensional system theory with mobile robots is expected to provide new solutions for the control of complex spatiotemporal processes in industrial and environmental fields.*

4. Implementation and numerical tests

This section provides simulation results that corroborate the effectiveness of the presented approach. We focuse on the boundary control problem of thermal process, aiming to design a controller that meets the H_∞ performance index, where the system dynamics is described by the following SPDE.

$$\begin{aligned} dc(h,v,t) =&[\alpha\Delta c(h,v,t) + \rho_1(c(h,v,t) + sin(c(h,v,t))) + \beta\varepsilon(h,v,t)]dt \\ &+ \rho_2(c(h,v,t) + sin(c(h,v,t)))dw(h,v,t), \end{aligned} \tag{43}$$

subject to the Neumann boundary condition

$$\begin{cases} c_h(0,v,t) = c_v(h,0,t) = 0, \\ c_h(a,v,t) = u_1(v,t), \\ c_v(h,b,t) = u_2(h,t), \end{cases} \tag{44}$$

coupled with the initial condition

$$c(h,v,0) = 1.5 - 1.5\cos(2\pi h)\cos(2\pi v), \tag{45}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known parameters.

We choose the spatial domain as $W = [0,1] \times [0,1.2]$, and parameters as $\alpha = 0.3$, $\rho_1 = 0.18$, and ρ_2=0.52. Figure 1 shows the equilibrium profile distribution of the disturbance-free unforced system. It can be observed that the system state has not converged to the expected equilibrium point $c(h,v,t) = 0$, which indicates that the current disturbance-free open-loop structure lacks stability.

By the Lagrange mean value theorem, we can determine that the parameters in Assumption 1 can be set to $\gamma = 2\rho_1 = 0.36$ and $\delta = 2\rho_2 = 1.04$. Set the influence parameter as $\beta = 0.02$ and the parameters in (5) as $x = 0.1$ and $y = 0.2$, then based on the linear matrix inequality (LMI) toolbox in MATLAB, call the feasp solver to numerically solve the linear matrix inequality (21). This will yield a set of feasible solutions with $s_1 = -8.8206$, $s_2 = -6.1254$, and $k = 1.7874$. It can be further calculated that the control gains defined in (4) are $l_1 = -4.9348$ and $l_2 = -3.4269$. Substituting these control gains into (4), a boundary SOF controller is adopted. Under the condition that the external disturbance term $\varepsilon(h,v,t)$ is always zero, the closed-loop control is performed on the nonlinear distributed parameter system (43)–(45). The simulation results show the equilibrium profile distribution evolution within the system as shown in Figure 2. From the simulation results shown in Figure 2, it can be observed that, in the closed-loop system without external interference, the state variable can eventually converge to the target value. At the same time, Figure 3 presents the curve of the control action $u_1(v,t)$ and $u_2(h,t)$ during this process.

Taking into account the influence of external disturbances, the disturbance function is set as $\varepsilon(h,v,t) = \sin(t)\exp(-0.5t)$. In order to obtain the statistical characteristics of $E\int_0^{t_f} x\|c(\cdot,\cdot,\iota)\|_2^2 d\iota$ and $E\int_0^{t_f} \|\varepsilon(\cdot,\cdot,\iota)\|_2^2 d\iota$ with $t_f = 20$, we conducted 30 independent Monte Carlo simulations. All simulations used the same parameter configuration, and the following final results were the average of these 30 runs.

$$\sqrt{\frac{E\int_0^{30} x\|c(\cdot,\cdot,\iota)\|_2^2 d\iota}{E\int_0^{30} \|\varepsilon(\cdot,\cdot,\iota)\|_2^2 d\iota}} = 0.0752 < y = 0.2. \tag{46}$$

In consequence, the H_∞ control performance index defined by Eq (8) can also be satisfied.

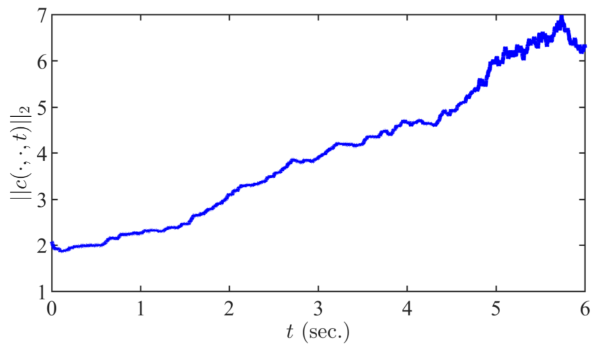

Figure 1. *The evolution process of* $\|c(\cdot,\cdot,t)\|_2$ *in open-loop condition.*

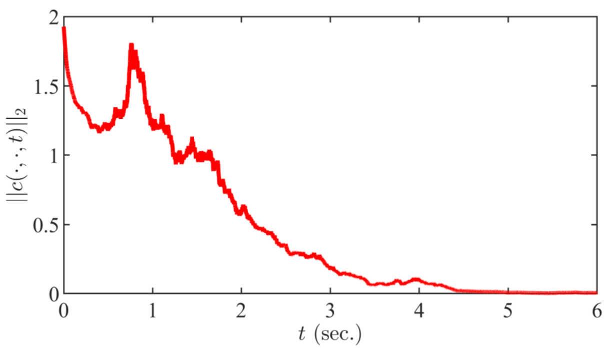

Figure 2. *The evolution process of* $\|c(\cdot,\cdot,t)\|_2$ *in closed-loop condition.*

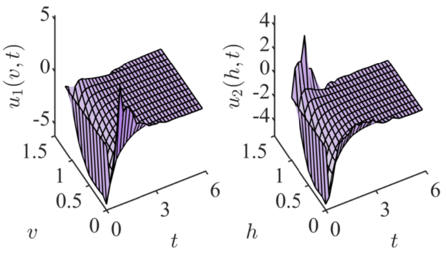

Figure 3. *Boundary SOF control $u_1(v,t)$ and $u_2(h,t)$ in closed-loop condition.*

5. Conclusions

This study established an H_∞ boundary control framework for a 2-D nonlinear parabolic SPDE system with state multiplicative noise. Based on the self-organizing feedback control strategy, combined with the Lyapunov method, this paper proposes a systematic design approach for constructing a SOF controller, which can ensure that the closed-loop system has exponential stability in the mean-square sense, as well as meets the H_∞ performance requirement. The numerical simulation results of the stochastic heat conduction equation show that this method has good control effects and practical feasibility, verifying its effectiveness as a robust control design tool for a class of complex DPSs. The SOF control strategy adopted in this paper has a simple structure and demonstrates significant advantages in practical deployment. However, the drawback of this method is that it is relatively conservative. To reduce the conservatism while improving the control quality, subsequent research should consider introducing more advanced control schemes, such as using mobile sensors or actuators.

Author contributions

Q. Wang and Z. Y. Lu wrote the main manuscript; Z. P. Wang conceived of the study, and participated in its design; X. W. Zhang helped to draft the manuscript. All authors reviewed the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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