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Research article

Triple systems with bilinear forms and the associated graph theory

Antonio J. Calderón*

Departament of Mathematics, University of Cádiz, Puerto Real, Spain

* **Correspondence:** Email: ajesus.calderon@uca.es.

Abstract: We establish a decomposition theorem for the category of triple systems with multiplicative bases equipped with bilinear forms. We demonstrate that any object in this category can be expressed as a direct sum of irreducible, orthogonal ideals. This structural decomposition is intrinsically linked to graph theory. By constructing an appropriate directed graph associated with the triple system and its bilinear form, this decomposition can be directly retrieved from the graph's structure, thus highlighting a potent connection between the algebraic structure and its graph representation. Additionally, we provide a necessary condition for the simplicity of this class of triple systems, through the path symmetry of their associated graphs.

Keywords: directed graph; triple systems; multiplicative basis; structure theory
Mathematics Subject Classification: 05C25, 05C20, 05C50, 15A03, 17A40

1. Introduction

The main objects of the paper are triple systems T, that are vector spaces equipped with a trilinear operation. These spaces will be endowed with a nondegenerate bilinear form as well as a multiplicative basis that is compatible with the triple product.

We wish to study the inner structure of these objects using the theory of graphs as a tool.

We will prove that any of the above T can be represented as a direct sum of mutually orthogonal ideals, where this decomposition is the finest one. Hence the study of T is reduced to the analysis of their irreducible components. To do that we will associate an adequate graph Γ to T and show that any of the connected components of Γ corresponds to just one of theaforementioned ideals. Thus, the paper is organized as follows.

After the present *Introduction* section, there is a *Preliminaries* section where the main concepts, both about triples systems and graphs, are presented. Then in Section 2, the associated graph of T is constructed and we show how graph theory allows us to prove that any triple system with a multiplicative basis and a bilinear form $(T, \mathcal{B}, \gamma)$ admits a fundamental decomposition into a direct sum of orthogonal, irreducible ideals, that is,

$$T = \bigoplus_{k \in K} I_k,$$

where

- each I_k is an ideal of A,
- distinct ideals are mutually orthogonal, I_j and I_k are orthogonal for $j \neq k$,
- each ideal I_k is irreducible, that is, I_k cannot be decomposed into a direct sum $I_k = P \oplus Q$, where P and Q are non-zero, orthogonal ideals of I_k.

Thus, we will show that the structural decomposition of T can be directly recovered from the topological features of this graph. This establishes a powerful correspondence between the theory of triple systems with multiplicative bases and with bilinear forms and graph theory. The fundamental link is established by proving that the decomposition of the triple system exactly corresponds to the decomposition of the graph into its connected components.

In Section 3, we study the relationship between two decompositions of T when we take two different multiplicative bases. Finally, Section 4 provides a necessary condition under which the decomposition of T is by means of the family of its simple ideals through the path symmetry of its associated graph.

This approach effectively translates a complex algebraic structural problem into a much simpler, well-understood problem in combinatorics (graph connectivity).

Additionally, we note that the inner structure of different mathematical objects with multiplicative bases has been studied using algebraic tools (for instance, see [1–5]), however, a useful topic in graph theory and in abstract algebra is characterizing the structure of an algebraic object by the properties satisfied for some graph associated with it (for instance, see [6–9]). Hence, in this paper, we extend the results of [7] to triple systems.

Finally, we note that our techniques have potential applications in future research concerning split or graded triple systems endowed with bilinear forms. Furthermore, these techniques could be extended to algebraic objects with several products (such as Lie-Rinehart or Bol algebras) that involve multiplicative bases and bilinear products to further explore their inner structures.

2. Preliminaries

This section presents the different concepts needed in our study. First, we will introduce those related to triple systems and those related to graph theory.

Definition 2.1. *A* **triple system** T *over a base field* $\mathbb{K}$ *is a vector space* T *equipped with a trilinear operation, referred to as the triple product, defined by the map:*

$$T \times T \times T \to T, \quad (x, y, z) \mapsto \langle x, y, z \rangle.$$

It is common to consider triple systems in which the triple product verifies some identity. Then we deal with Lie triple systems, Jordan triple systems, Leibniz triple systems and so on (for instance, see the classical paper [10]). The triple systems in the present paper are considered with neither restriction on the dimension nor the base field, and no identities are assumed for the triple product.

Definition 2.2. *A* ***subtriple*** $S \subset T$ *is a linear subspace such that* $\langle S, S, S\rangle \subset S$*. An* **ideal** $I \subset T$ *is a subspace such that* $\langle I, T, T\rangle + \langle T, I, T\rangle + \langle T, T, I\rangle \subset I$*.*

Definition 2.3. *An* **automorphism** *of a triple system* T

$$f : T \to T$$

is a linear isomorphism that satisfies

$$\langle f(x), f(y), f(z)\rangle = f(\langle x, y, z\rangle)$$

for all $x, y, z \in T$*.*

Definition 2.4. *A* **basis** $B = \{v_i\}_{i\in J}$ *of* T *is said to be* ***multiplicative*** *if for any* $i, j, k \in J$*, the triple product satisfies the following:*

$$\langle v_i, v_j, v_k\rangle = 0 \quad \textit{or} \quad \langle v_i, v_j, v_k\rangle \in \mathbb{K}^{\times} v_r \quad \textit{for some (unique) } r \in J,$$

(where $\mathbb{K}^{\times} = \mathbb{K} \setminus \{0\}$*).*

Many well-known triple systems admit multiplicative bases, including: matrix triple systems, group triple systems, and Heisenberg triple systems. Furthermore, classical examples such as the real triple systems associated to $\mathbb{R}$, $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions), and $\mathbb{O}$ (octonions) possess both a multiplicative basis $\mathcal{B}$ and a canonical non-degenerate bilinear form $\gamma : T \times T \to \mathbb{R}$. These objects $(T, \mathcal{B}, \gamma)$ are crucial in the theory of absolute valued triple systems (for instance, see [11–13]).

In this paper, we center on the category of triple systems that admit a multiplicative basis with bilinear forms. The objects in this category are $(T, \mathcal{B}, \gamma)$, where T is a $\mathbb{K}$-triple system, $\mathcal{B} = \{v_i\}_{i\in I}$ is a fixed multiplicative basis for T, and $\gamma : T \times T \to \mathbb{K}$ is a bilinear form.

Definition 2.5. *An* **ideal** *of a triple system with a multiplicative basis and bilinear form* $(T, \mathcal{B}, \gamma)$ *is an object* $(I, \mathcal{B}_I, \gamma_I)$ *such that* I *is an ideal of* T*, the basis* $\mathcal{B}_I$ *is a subset of* $\mathcal{B}$*, and* γ_I *is the restriction of* γ *to* $I \times I$*. For simplicity, we typically refer to the ideal by* I*.*

Definition 2.6. *A triple system with multiplicative basis and bilinear form* $(T, \mathcal{B}, \gamma)$ *is called* **simple** *if its only ideals are* $\{0\}$ *and* T*.*

Definition 2.7. *For an object* $(T, \mathcal{B}, \gamma)$*, two elements* $x, y \in T$ *are called* **orthogonal** *if* $\gamma(x, y) = \gamma(y, x) = 0$*. Similarly, two non-empty subsets* $B, C \subset T$ *are mutually orthogonal if* $\gamma(B, C) = \gamma(C, B) = 0$*, meaning* $\gamma(x, y) = \gamma(y, x) = 0$ *for all* $x \in B$ *and* $y \in C$*.*

Definition 2.8. *Given a triple system with a multiplicative basis and a bilinear form* $(T, \mathcal{B}, \gamma)$*, we will say that it is* **irreducible***, where* T *cannot be expressed as* $T = I \oplus K$ *with* I *and* K *nonzero orthogonal ideals of* T*.*

Definition 2.9. *We will say that two pairs* $(\mathcal{B}, \gamma)$ *and* $(\mathcal{B}', \gamma')$*, where* $\mathcal{B}$*,* $\mathcal{B}'$ *are multiplicative bases of* T*; and* γ*,* γ' *are bilinear forms on* T*, are* ***equivalent*** *if there exists an automorphism*

$$f : T \to T$$

such that

- $f(\mathcal{B}) = \mathcal{B}'$.
- $\gamma'(f(v_i), f(v_j)) = \gamma(v_i, v_j)$ *for any* $v_i, v_j \in \mathcal{B}$.

Since the main topic of this paper is to use graph theory to study the inner structure of T, we are going to introduce several notation and concepts of graph theory.

Definition 2.10. *A* **directed graph** *is an ordered pair* $\Gamma = (V, E)$*, where* V *is the set of vertices and* $E \subset V \times V$ *is the set of directed edges. An edge* $(v_1, v_2) \in E$ *denotes a connection that originates at* v_1 *and terminates at* v_2*.*

Definition 2.11. *An* **undirected path** *between two vertices* $v_i, v_j \in V$ *is a sequence of vertices*

$$\{v_{i_1}, \dots, v_{i_n}\}$$

with $v_{i_1} = v_i$ *and* $v_{i_n} = v_j$*, such that for every step r, the vertices* v_{i_r} *and* $v_{i_{r+1}}$ *are joined by an edge in some direction, (i.e., either* $(v_{i_r}, v_{i_{r+1}}) \in E$ *or* $(v_{i_{r+1}}, v_{i_r}) \in E$*.)*

Definition 2.12. *A subset of vertices* $C \subset V$ *is* **connected** *if, for every pair of vertices in C, there exists an undirected path between them (or if C is a singleton).*

A **connected component** *is a maximal connected subgraph.*

Definition 2.13. *We recall that two graphs* (V, E) *and* (V', E') *are* **isomorphic** *if there exists a bijection* $f : V \to V'$ *such that* $(v_i, v_j) \in E$ *if and only if* $(f(v_i), f(v_j)) \in E'$*.*

Definition 2.14. *A graph* (V, E) *is defined as* **symmetric** *if the presence of the edge* $(v_i, v_j) \in E$ *implies that the edge* (v_j, v_i) *is also in E.*

This motivates the introduction of a more relaxed graph property.

Definition 2.15. *Given a graph* (V, E) *and* $v_i, v_j \in V$*, an ordered family* $\{v_{i_1}, \dots, v_{i_n}\} \subset V$ *is called a* **directed path from** v_i **to** v_j *if* $v_{i_1} = v_i$*,* $v_{i_n} = v_j$ *and* $(v_{i_r}, v_{i_{r+1}}) \in E$ *for every* $1 \le r \le n - 1$*.*

Definition 2.16. *A graph* (V, E) *is designated as* **path symmetric** *if, for every pair of different vertices* $v_i, v_j \in V$ *such that there exists a directed path from* v_i *to* v_j*, there also exists a directed path from* v_j *to* v_i*.*

Example 2.1. *The first graph is path symmetric while the second one is not (see Figure 1).*

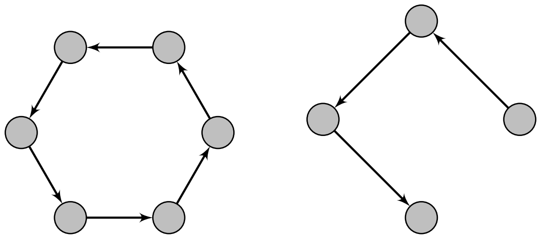

Figure 1. A path symmetric graph and a non-path symmetric graph.

3. Associated graph and structural decomposition

We begin by formally defining the graph structure used to analyze the triple systems with multiplicative bases with bilinear forms.

Let $(T, \mathcal{B}, \gamma)$ be a triple system with a multiplicative basis and a bilinear form. The **associated directed graph**, denoted $\Gamma(T, \mathcal{B}, \gamma) := (V, E)$, is constructed as follows:

Vertices (V): The set of vertices is the basis itself: $V = \mathcal{B}$.

Edges (E): The set of directed edges E consists of all ordered pairs $(v_i, v_r) \in V \times V$ that satisfy one of the following criteria:

- There exist elements $v_j, v_k \in \mathcal{B}$ such that $\langle v_{\sigma(i)}, v_{\sigma(j)}, v_{\sigma(k)} \rangle \in \mathbb{K}^{\times} v_r$ for some $\sigma \in S_3$, (where S_3 denotes the group of permutations of three elements), or
- The bilinear form evaluated at $\{v_i, v_r\}$ is non-zero, that is, either $\gamma(v_i, v_r) \neq 0$ or $\gamma(v_r, v_i) \neq 0$.

Example 3.1. *Consider the* 10*-dimensional real triple system T with the basis*

$$\mathcal{B} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$$

and non-zero products, among the elements of $\mathcal{B}$, given by

$$\begin{aligned}
&\langle v_1, v_2, v_3 \rangle = 7v_4,\\
&\langle v_5, v_5, v_4 \rangle = -3v_4,\\
&\langle v_4, v_5, v_1 \rangle = 9v_1,\\
&\langle v_1, v_2, v_2 \rangle = 6v_2,\\
&\langle v_2, v_2, v_5 \rangle = -v_5,\\
&\langle v_2, v_3, v_5 \rangle = 2v_3,\\
&\langle v_6, v_7, v_7 \rangle = -\langle v_7, v_6, v_7 \rangle = 8v_6\\
&\langle v_8, v_9, v_9 \rangle = -5\langle v_9, v_8, v_9 \rangle = 4v_8\\
&\langle v_8, v_9, v_{10} \rangle = -7\langle v_9, v_8, v_{10} \rangle = -\langle v_9, v_{10}, v_8 \rangle = \langle v_{10}, v_9, v_8 \rangle = 2v_9\\
&\langle v_{10}, v_8, v_8 \rangle = -6\langle v_8, v_{10}, v_8 \rangle = 4v_8\\
&\langle v_8, v_{10}, v_{10} \rangle = -\langle v_{10}, v_8, v_{10} \rangle = 2v_{10}\\
&\langle v_9, v_{10}, v_9 \rangle = \langle v_{10}, v_9, v_9 \rangle = -4v_{10}.
\end{aligned}$$

In T, we define the bilinear form

$$\gamma : T \times T \to \mathbb{R},$$

which is determined by the non-null products among the elements of the basis:

$$\begin{aligned}
&\gamma(v_2, v_4) = 3,\\
&\gamma(v_3, v_3) = -1,\\
&\gamma(v_6, v_8) = 2,\\
&\gamma(v_{10}, v_7) = -5,\\
&\gamma(v_6, v_6) = 1,
\end{aligned}$$

$$\gamma(v_{10}, v_9) = -3,$$
$$\gamma(v_8, v_8) = 4.$$

Then, $(T, \mathcal{B}, \gamma)$ *is a triple system with a multiplicative basis and a bilinear form. Its associated graph* $\Gamma(T, \mathcal{B}, \gamma)$ *is represented in Figure 2.*

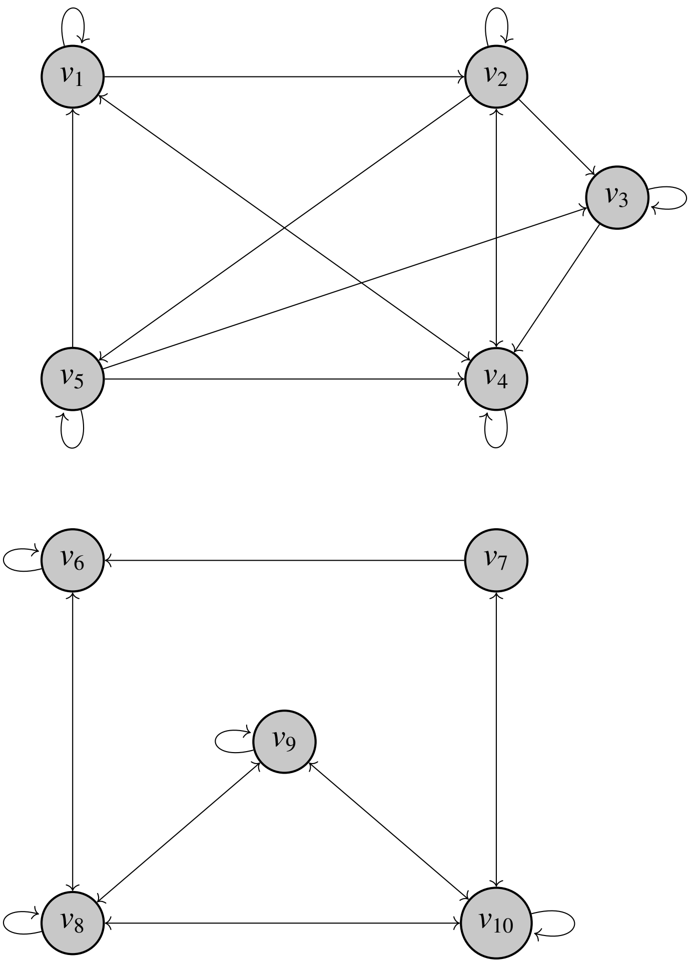

Figure 2. Graph associated to a triple system with a bilinear form.

Example 3.2. *Let the 13-dimensional real triple system T with linear basis*

$$\mathcal{B} = \{v_1, v_2, ..., v_{13}\}$$

and non-zero products among the elements of $\mathcal{B}$ *be given by the following:*

$$\langle v_1, v_3, v_1 \rangle = 5v_3,$$
$$\langle v_1, v_2, v_8 \rangle = 7v_2,$$
$$\langle v_{12}, v_3, v_8 \rangle = 15v_3,$$
$$\langle v_3, v_5, v_2 \rangle = 7v_{12},$$

$$\langle v_9, v_9, v_9\rangle = -4v_{13},$$

where the remaining products among elements of $\mathcal{B}$ are zero.

In T, we consider the bilinear form

$$\gamma : T \times T \to \mathbb{R},$$

which is defined by

$$\begin{aligned}\gamma(v_5, v_8) &= -5,\\ \gamma(v_8, v_8) &= 5,\\ \gamma(v_4, v_7) &= -4,\\ \gamma(v_6, v_{10}) &= 8,\\ \gamma(v_{10}, v_7) &= 6,\end{aligned}$$

where the remaining products, through γ, among elements of the bases are zero.

Then $(T, \mathcal{B}, \gamma)$ is a triple system with a multiplicative basis and a bilinear form.

Its associated graph $\Gamma(T, \mathcal{B}, \gamma)$ is represented in Figure 3.

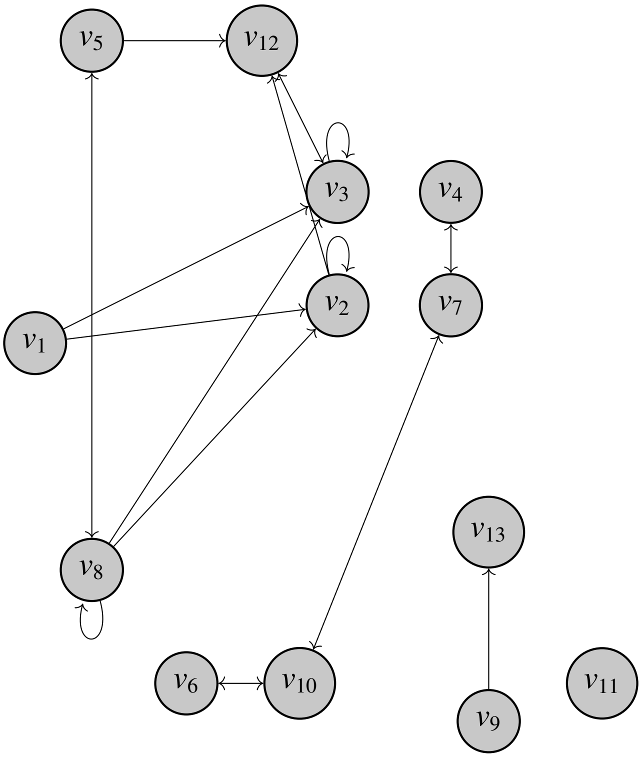

Figure 3. Graph associated to a triple system with a bilinear form.

We introduce an equivalence relation $\sim$ on the set of vertices V (the basis $\mathcal{B}$) of the associated graph $\Gamma(T, \mathcal{B}, \gamma)$. Two vertices $v_i, v_j \in V$ are considered equivalent, $v_i \sim v_j$, if and only if they are the same vertex ($v_i = v_j$) or if there exists an undirected path that connects v_i and v_j in $\Gamma(T, \mathcal{B}, \gamma)$. In this context, v_i and v_j are said to be **connected**.

The equivalence classes generated by this relation, denoted $[v_i] \in V/\sim$, precisely correspond to the connected components $C_{[v_i]}$ of the graph $\Gamma(T, \mathcal{B}, \gamma)$. This results in a decomposition of the entire graph into a disjoint union of its connected components:

$$\Gamma(T, \mathcal{B}, \gamma) = \dot{\bigcup_{[v_i] \in V/\sim}} C_{[v_i]}. \tag{3.1}$$

For each connected component $C_{[v_i]}$, we define the corresponding linear subspace $T_{C_{[v_i]}}$ as the span of the basis elements contained within the set of vertex elements of the component $[v_i]$:

$$T_{C_{[v_i]}} := \bigoplus_{v_j \in [v_i]} \mathbb{K} v_j. \tag{3.2}$$

This directly leads to our main structural result.

Theorem 3.1. *(Irreducible ideals decomposition via graph components). Let T be a triple system with a multiplicative basis* $\mathcal{B} = \{v_i\}_{i \in I}$ *and with a bilinear form* $\gamma : T \times T \to \mathbb{K}$. *The triple system T admits an orthogonal direct sum decomposition into irreducible ideals determined by the connected components of the graph:*

$$T = \bigoplus_{[v_i] \in V/\sim} T_{C_{[v_i]}}.$$

Proof. The fact that T is the direct sum of the subspaces $T_{C_{[v_i]}}$ is evident from the partition of the basis $\mathcal{B}$ induced by the equivalence relation $\sim$ (Eq (3.1)) and the definition of the subspaces (Eq (3.2)). By construction, each $T_{C_{[v_i]}}$ inherits the basis $[v_i]$ as a multiplicative basis.

To establish that $T_{C_{[v_i]}}$ is an ideal and that the decomposition is orthogonal, we first consider the triple product. Assume there exists a triple product that involves $v_j \in [v_i]$ (a basis element of $T_{C_{[v_i]}}$) and three basis elements $v_k, v_l, v_r \in \mathcal{B}$ such that, for some permutation $\sigma \in S_3(\{j, k, l\})$, the product is a non-zero scalar multiple of v_r:

$$\langle v_{\sigma(j)}, v_{\sigma(k)}, v_{\sigma(l)} \rangle \in \mathbb{K}^{\times} v_r.$$

By the definition of the graph $\Gamma(T, \mathcal{B}, \gamma)$, the non-zero product implies that v_r must be connected to v_j, v_k, and v_l. Since $v_j \in [v_i]$, and v_r is connected to v_j, all elements v_k, v_l, v_r must belong to the same connected component $[v_i]$. This confinement, $v_k, v_l, v_r \in T_{C_{[v_i]}}$, implies two critical consequences:

Ideal property: Since the product of any three basis elements (including one from $T_{C_{[v_i]}}$ and two from T) results in an element confined to $T_{C_{[v_i]}}$, the subspace $T_{C_{[v_i]}}$ is an ideal of T.

Null product between different ideals: Since any non-zero product must involve basis elements exclusively from a single component, the triple product of elements from different ideals must be zero.

Second, in case $\gamma(v_i, v_j) \neq 0$ for some $v_i, v_j \in \mathcal{B}$, we have that $v_i \sim v_j$ and so $T_{C_{[v_i]}} = T_{C_{[v_j]}}$. This guarantees that the direct sum is orthogonal.

Finally, let us show that any $T_{C_{[v_i]}}$ is an irreducible ideal. To do that, suppose the following:

$$T_{C_{[v_i]}} = P \oplus Q \tag{3.3}$$

with P and Q nonzero ideals of $T_{C_{[v_i]}}$ and

$$\gamma(P, Q) = \gamma(Q, P) = 0. \tag{3.4}$$

If we consider the (multiplicative) basis $[v_i]$ of $T_{C_{[v_i]}}$, then Eq (3.3) gives us the following:

$$[v_i] = \mathcal{B}_P \cup \mathcal{B}_Q$$

with $\mathcal{B}_P$ basis of P, $\mathcal{B}_Q$ basis of Q and

$$\mathcal{B}_P \cap \mathcal{B}_Q = \emptyset. \tag{3.5}$$

From here, since $C_{[v_i]}$ is connected and $\mathcal{B}_P \cup \mathcal{B}_Q$ are vertices of $C_{[v_i]}$, there exist $v_p \in \mathcal{B}_P$ and $v_q \in \mathcal{B}_Q$ such that either (v_p, v_q) or (v_q, v_p) is an edge of $C_{[v_i]}$. From here, we get the following:

(i) either there exist $v_r, v_s \in [v_i]$ such that $\langle v_p, v_r, v_s\rangle = \lambda v_q \neq 0$ or $\langle v_r, v_p, v_s\rangle = \lambda v_q \neq 0$ or $\langle v_r, v_s, v_p\rangle = \lambda v_q \neq 0$;
(ii) or there exist $v_t, v_k \in [v_i]$ such that $\langle v_q, v_t, v_k\rangle = \beta v_p \neq 0$, or $\langle v_t, v_q, v_k\rangle = \beta v_p \neq 0$, or $\langle v_t, v_k, v_q\rangle = \beta v_p \neq 0$;
(iii) or $\gamma(v_p, v_q) \neq 0$, or $\gamma(v_q, v_p) \neq 0$.

Now, we have that $v_q \in P$ in the first above possibility, which contradicts Eq (3.5). In the second above possibility, $v_p \in Q$, which also contradicts Eq (3.5). Finally, in the third above possibility $\gamma(P, Q) \neq 0$ or $\gamma(Q, P) \neq 0$, which contradicts Eq (3.4). We conclude that the decomposition given in Eq (3.3) never holds and so $T_{C_{[v_i]}}$ is irreducible. □

Example 3.3. *Consider the 10-dimensional triple system with a multiplicative basis and with a bilinear form as in Example 3.1. Its associated graph, which is depicted in Figure 2, has two connected components:*

$$C_{[v_1]} \text{ and } C_{[v_6]}.$$

Hence, Theorem 3.1 allows us to assert that T decomposes as a direct sum of irreducible orthogonal ideals in the following form (see Example 3.1):

$$T = T_{C_{[v_1]}} \oplus T_{C_{[v_6]}}, \tag{3.6}$$

where

$$T_{C_{[v_1]}} = \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus \mathbb{R}v_3 \oplus \mathbb{R}v_4 \oplus \mathbb{R}v_5,$$

and

$$T_{C_{[v_6]}} = \mathbb{R}v_6 \oplus \mathbb{R}v_7 \oplus \mathbb{R}v_8 \oplus \mathbb{R}v_9 \oplus \mathbb{R}v_{10}.$$

Example 3.4. *Consider the 13-dimensional triple system with a multiplicative basis and with a bilinear form as in 3.2. Its associated graph, which is depicted in Figure 3, has four connected components:*

$$C_{[v_1]}, C_{[v_4]}, C_{[v_9]}, C_{[v_{11}]}.$$

From here, Theorem 3.1 gives us that T decomposes as a direct sum of irreducible orthogonal ideals in the following form (see Example 3.2):

$$T = T_{C_{[v_1]}} \oplus T_{C_{[v_4]}} \oplus T_{C_{[v_9]}} \oplus T_{C_{[v_{11}]}}, \tag{3.7}$$

where

$$T_{C_{[v_1]}} = \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus \mathbb{R}v_3 \oplus \mathbb{R}v_5 \oplus \mathbb{R}v_{12},$$

$$T_{C_{[v_4]}} = \mathbb{R}v_4 \oplus \mathbb{R}v_6 \oplus \mathbb{R}v_7 \oplus \mathbb{R}v_{10},$$

$$T_{C_{[v_9]}} = \mathbb{R}v_9 \oplus \mathbb{R}v_{13},$$

$$T_{C_{[v_{11}]}} = \mathbb{R}v_{11}.$$

4. Relating graphs from different basis choices

If a fixed triple system T has different multiplicative bases, $\mathcal{B}$ and $\mathcal{B}'$, or different bilinear forms γ and γ', then the graphs $\Gamma(T, \mathcal{B}, \gamma)$ and $\Gamma(T, \mathcal{B}', \gamma')$ can be distinct, which leads to different decompositions. In this section, we study under which conditions the graphs associated to $(T, \mathcal{B}, \gamma)$ and $(T, \mathcal{B}', \gamma')$ are isomorphic.

Example 4.1. *Consider the real triple system*

$$(T, \mathcal{B}, \gamma)$$

with a multiplicative basis

$$\mathcal{B} = \{v_1, v_2, ..., v_{13}\}$$

and a bilinear form γ defined in Example 3.2.

If we take a new basis

$$\mathcal{B}' = \{u_1, u_2, ..., u_{13}\}$$

where $u_i = v_i$ for any $i \neq 11$, $u_{11} = v_9 + v_{11}$, and a new bilinear form γ' determined by the nonzero products on $\mathcal{B}'$:

$$\begin{aligned}
\gamma'(u_4, u_7) &= -2,\\
\gamma'(u_{10}, u_7) &= -5,\\
\gamma'(u_9, u_{13}) &= 2,\\
\gamma'(u_3, u_4) &= 1,\\
\gamma'(u_6, u_8) &= -1,
\end{aligned}$$

then we have that

$$(T, \mathcal{B}', \gamma')$$

is a triple system with a multiplicative basis and a bilinear form, which has as associated graph to the one in Figure 4.

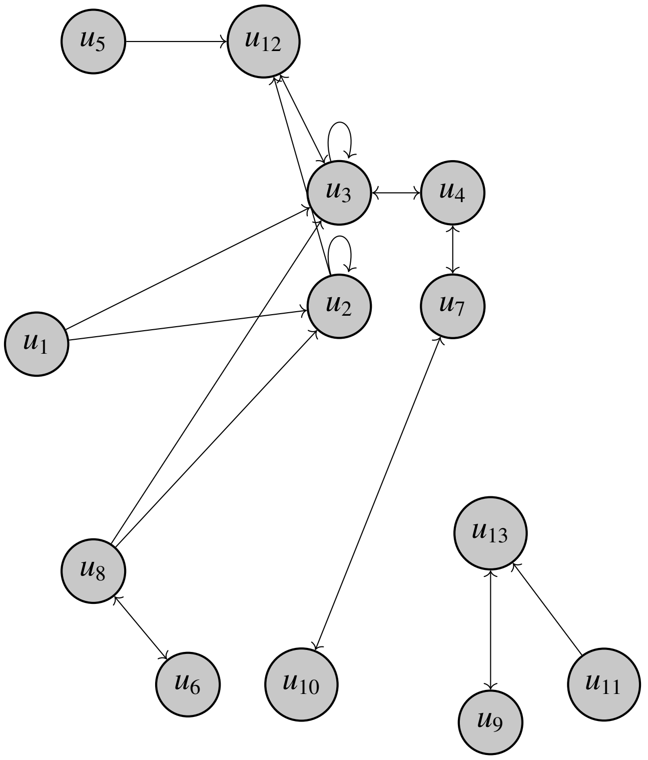

Figure 4. Graph associated to a triple system with a bilinear form.

From here, we have a different decomposition of T to the one determined in Example 3.2 (see Theorem 3.1), namely

$$T = T_{C_{[u_1]}} \oplus T_{C_{[u_9]}}$$

being

$$T_{C_{[v_1]}} := \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3 \oplus \mathbb{R}u_4 \oplus \mathbb{R}u_5 \oplus \mathbb{R}u_6 \oplus \mathbb{R}u_7 \oplus \mathbb{R}u_8 \oplus \mathbb{R}u_{10} \oplus \mathbb{R}u_{12}$$

and

$$T_{C_{[u_9]}} = \mathbb{R}u_9 \oplus \mathbb{R}u_{11} \oplus \mathbb{R}u_{13}.$$

Proposition 4.1 (Graph isomorphism via basis equivalence)**.** *If the pairs $(\mathcal{B}, \gamma)$ and $(\mathcal{B}', \gamma')$ are equivalent, then the associated graphs $\Gamma(T, \mathcal{B}, \gamma)$ and $\Gamma(T, \mathcal{B}', \gamma')$ are isomorphic.*

Proof. Let $f : T \to T$ be an automorphism such that $f(\mathcal{B}) = \mathcal{B}'$. The restriction of f to the vertices induces a bijection between the vertex sets $V = \mathcal{B}$ and $V' = \mathcal{B}'$. We need to show that $(v_i, v_r) \in E$ if and only if $(f(v_i), f(v_r)) \in E'$.

If $(v_i, v_r) \in E$, we have two cases to consider.

In the first case, there exist $v_j, v_k \in \mathcal{B}$ and $\sigma \in S_3$ such that

$$\langle v_{\sigma(i)}, v_{\sigma(j)}, v_{\sigma(k)} \rangle = \lambda v_r, \quad \lambda \in \mathbb{K}^{\times}.$$

Applying the automorphism f and using its defining property leads to the following:

$$\langle f(v_{\sigma(i)}), f(v_{\sigma(j)}), f(v_{\sigma(k)}) \rangle = f(\lambda v_r) = \lambda f(v_r).$$

Since $f(v_r) \in \mathcal{B}'$ and $\lambda \in \mathbb{K}^{\times}$, this implies $(f(v_i), f(v_r)) \in E'$.

In the second case, either $\gamma(v_i, v_r) \neq 0$ or $\gamma(v_r, v_i) \neq 0$. Hence $\gamma'(f(v_i), f(v_r)) = \gamma(v_i, v_r) \neq 0$ or $\gamma'(f(v_r), f(v_i))\gamma(v_r, v_i) \neq 0$. We get that $(f(v_i), f(v_r)) \in E'$ in any case.

The reverse implication is similar using f^{-1}. Hence, f is a graph isomorphism. □

5. On the simplicity of the ideals in the decomposition

Our aim in this section is to provide a necessary condition for the simplicity of the ideals in the decomposition given in Theorem 3.1, through the "path symmetry" that satisfies the associated graph $\Gamma(T, \mathcal{B}, \gamma)$.

Proposition 5.1. Let $(T, \mathcal{B}, \gamma)$ be a triple system with a multiplicative basis and a bilinear form. If

$$T = \bigoplus_{k \in K} \mathcal{I}_k$$

is the orthogonal direct sum of the family of its simple ideals then $\Gamma(T, \mathcal{B}, \gamma)$ is path symmetric.

Proof. Suppose that T decomposes as the orthogonal direct sum of the simple ideals $\mathcal{I}_k$. Let $(V, E) := \Gamma(T, \mathcal{B}, \gamma)$ be the associated graph to $(T, \mathcal{B}, \gamma)$ and take a pair of different $v_i, v_j \in V$ such that there exists a directed path from v_i to v_j. Let us write

$$\{v_{i_1}, v_{i_2}, \dots, v_{i_{n-1}}, v_{i_n}\} \subset V \tag{5.1}$$

as some directed path from v_i to v_j. Additionally, let us show that there is a directed path from v_j to v_i.

We have that for any $r \in \{1, ..., n-1\}$, the pair is as follows:

$$(v_{i_r}, v_{i_{r+1}}) \in E. \tag{5.2}$$

By the orthogonal direct sum character of the decomposition, we have that $v_{i_r}, v_{i_{r+1}} \in \mathcal{I}_k$ for some k and, by the simplicity of $\mathcal{I}_k$, we get that $v_{i_r} \in \mathcal{I}_k = \mathcal{I}(v_{i_{r+1}})$.

Taking into account that $\mathcal{B}$ is a multiplicative basis, we have that

$$v_{i_r} = \lambda\langle_\sigma\langle_\sigma \cdots \langle_\sigma\langle_\sigma v_{i_{r+1}}, v_1, v_2\rangle, v_3, v_4\rangle, \cdots\rangle, v_{2m-1}, v_{2m}\rangle$$

for $v_1, \dots, v_{2m} \in \mathcal{B}$ and $0 \neq \lambda \in \mathbb{K}$, being any

$$\langle_\sigma v_r, v_s, v_t\rangle \in \{\langle v_{\sigma(r)}, v_{\sigma(s)}, v_{\sigma(t)}\rangle : \sigma \in S_3\}.$$

From here, by writing

$$w_k := \mathbb{K}\langle_\sigma\langle_\sigma \cdots \langle_\sigma\langle_\sigma v_{i_{r+1}}, v_1, v_2\rangle, v_3, v_4\rangle, \cdots\rangle, v_{2k-1}, v_{2k}\rangle \cap \mathcal{B}$$

for $k \in \{1, \dots, m-1\}$ we get that

$$\{v_{i_{r+1}}, w_1, \dots, w_{m-1}, v_{i_r}\}$$

is a directed path from $v_{i_{r+1}}$ to v_{i_r} for any $r \in \{n-1, ..., 1\}$. Now, taking Eqs (5.1) and (5.2) into account, we conclude that there exists a directed path from v_j to v_i and so the graph $\Gamma(T, \mathcal{B}, \gamma)$ is path symmetric. □

Now, the next result is an immediate consequence of Theorem 3.1 and Proposition 5.1.

Corollary 5.1. *Let $(T, \mathcal{B}, \gamma)$ be a triple system with a multiplicative basis and a bilinear form. If T is simple then the associated graph $\Gamma(T, \mathcal{B}, \gamma)$ is path symmetric and has just one connected component.*

6. Conclusions

We have demonstrated a profound correspondence between graph theory and the theory of abstract triple systems. Specifically, our results show that the internal structure of an abstract algebraic object, (defined as a triple system equipped with a multiplicative basis and a bilinear form), can be fully reconstructed from its associated graph. These findings suggest that such graph-theoretic methodologies may be successfully extended to a broader class of algebraic structures.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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