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Research article

Some properties of Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials and their fractional extension

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Abstract: This study presents a new, generalized family of special polynomials. These polynomials are designated as the Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials. Based on their generating function, the corresponding series expansions and summation formulas are obtained. Furthermore, determinant representation and several differential and integral representations of these polynomials are derived. The fractional extension of this polynomial family is explored, resulting in the formulation of several associated identities. Finally, the study utilizes Mathematica to deliver zero distributions and graphical representations.

Keywords: Hermite–Kampé de Fériet polynomials; Bell polynomials; Apostol-Bernoulli-type polynomials; Apostol-type polynomials; generating function; differential and integral representations; fractional derivatives
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1. Introduction

Special functions and polynomials are fundamental to numerous scientific disciplines. For example, the authors in [1] examined the applications of special functions in mathematical physics. Numerous applications of special functions in numerical analysis are documented in [2]. Some applications in engineering are discussed in [3]. Other studies encompass applications across various areas. For instance: probability theory [4, 5], combinatorial structures [6], approximation theory [7], umbral interpolation [8, 9], matrix calculus and related interpolation [10, 11], and fractional calculus [12]. In addition, these polynomials naturally generate a wide variety of useful identities and serve as a foundation for the construction of new families of polynomials, for example, Bell-based Bernoulli polynomials [13], a new family of Bernoulli-type polynomials [14], Gould-Hopper-Eulerian-type polynomials [15], and a new family of Fubini-type polynomials [16].

In recent years, many researchers have also developed generalized versions of well-known special polynomials such as Hermite, Bernoulli, and Bell polynomials, for example, Cesarano et al. [17] studied Apostol-type Hermite degenerated polynomials, Ramírez and Cesarano [18] applied the monomiality principle to introduce and study a new class of Apostol Hermite Bernoulli-type polynomials, Duran et al. [13] defined and examined the Bell-based Bernoulli polynomials with some applications, Duran and Acikgoz [19] established Bell-based Genocchi polynomials and obtained some related properties, Kim et al. [20] discussed and established some results on Bell and poly-Bell numbers and polynomials, Al-Jawfi et al. [21] defined and studied a generalized family of Bell polynomials with some geometric applications, Ramírez et al. [22] introduced and discussed the Bell-based Apostol-Bernoulli-type polynomials and their characteristics, Mohra et al. [23] presented some advancements in q-Hermite-Appell polynomials, Quintana et al. [24] introduced a degenerate version of hypergeometric Bernoulli polynomials, Ramírez et al. [25] investigated a new U-Bernoulli, U-Euler, and U-Genocchi polynomials and their matrices, and Urieles et al. [26] discussed the F-Frobenius-Euler polynomials and their matrix approach. Recent advancements in approximation theory and special functions indicate wider analytical and computational relationships for generalized polynomial systems; refer to the boundary interpolation framework in [27] and the q-statistical approximation context in [28].

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ represent sets of real numbers, complex numbers, and integer numbers, respectively. Additionally, $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Special polynomials in two variables serve as a potent mathematical instrument for analyzing solutions to partial differential equations in physics. This discovery is a fundamental component of mathematical physics, where special functions originated primarily as tools to address particular physical problems. Recall the Hermite–Kampé de Fériet polynomials $H_\rho(\alpha, \upsilon)$ [29] which are defined as:

$$e^{\alpha t+\upsilon t^2} = \sum_{\rho=0}^{\infty} H_\rho(\alpha, \upsilon)\frac{t^\rho}{\rho!} \tag{1.1}$$

and represented by

$$H_\rho(\alpha, \upsilon) = \rho! \sum_{\kappa=0}^{[\frac{\rho}{2}]} \frac{\alpha^{\rho-2\kappa}\, \upsilon^\kappa}{(\rho-2\kappa)!\kappa!}.$$

Bell polynomials represent a distinguished class of special polynomials in two variables. These polynomials, named for the mathematician Eric Temple Bell, provide a robust mathematical framework for analyzing number theory, combinatorics, and mathematical physics. Originating from the theory of exponential generating functions, they offer a systematic method to represent and manipulate certain polynomial sequences. Their capacity to encode complex combinatorial data renders them indispensable across a spectrum of disciplines, including number theory, mathematical physics, probability theory, and computer science. The bivariate Bell polynomials (BelP) $\mathrm{Bel}_\rho(\alpha, \upsilon)$ [13, 19] are defined by

$$e^{\alpha t+\upsilon(e^t-1)} = \sum_{\rho=0}^{\infty} \mathrm{Bel}_\rho(\alpha, \upsilon)\, \frac{t^\rho}{\rho!}. \tag{1.2}$$

Taking $\alpha = 0$ in generating function (1.2), we get

$$e^{\upsilon(e^t-1)} = \sum_{\rho=0}^{\infty} \mathrm{Bel}_\rho(\upsilon)\, \frac{t^\rho}{\rho!}, \tag{1.3}$$

where $\mathrm{Bel}_\rho(\upsilon)$ denotes the classical Bell polynomials [30]. Note that the BelP $\mathrm{Bel}_\rho(\upsilon)$ satisfy the following relation [31]:

$$\mathrm{Bel}_\rho(\upsilon) = \sum_{\kappa=0}^{\rho} \mathbb{S}_2(\rho, \kappa)\, \upsilon^\kappa,$$

where $\mathbb{S}_2(\rho, \kappa)$ denotes the second kind Stirling numbers [20], which are given by

$$\frac{(e^t-1)^\kappa}{\kappa!} = \sum_{\rho=\kappa}^{\infty} \mathbb{S}_2(\rho, \kappa)\, \frac{t^\rho}{\rho!}, \qquad \kappa \geq 0. \tag{1.4}$$

Recently, Al-Jawfi et al. [21] introduced Hermite–Kampé de Fériet-based Bell polynomials ${}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma)$ which are defined as follows:

$$e^{\alpha t+\upsilon t^2+\gamma(e^t-1)} = \sum_{\rho=0}^{\infty} {}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma)\frac{t^\rho}{\rho!} \tag{1.5}$$

and represented by the series

$${}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} H_{\rho-\kappa}(\alpha, \upsilon)\, \mathrm{Bel}_\kappa(\gamma).$$

Let $\sigma \in \mathbb{N}$; the generalized Bernoulli-type polynomials $R_\rho^{(\sigma)}(\alpha)$ of order σ are given by means of the following generating function [14]:

$$\left(\frac{t^2}{2e^t-2}\right)^\sigma e^{\alpha t} = \sum_{\rho=0}^{\infty} R_\rho^{(\sigma)}(\alpha)\, \frac{t^\rho}{\rho!}, \;\; |t| < 2\pi, \;\; 1^\sigma := 1. \tag{1.6}$$

Setting $\sigma = 1$ in (1.6), we get

$$\left(\frac{t^2}{2e^t-2}\right) e^{\alpha t} = \sum_{\rho=0}^{\infty} R_\rho(\alpha)\, \frac{t^\rho}{\rho!}, \tag{1.7}$$

where $R_\rho(\alpha)$ are classical Bernoulli-type polynomials.

Setting $\alpha = 0$ in (1.6), we get

$$\left(\frac{t^2}{2e^t-2}\right) = \sum_{\rho=0}^{\infty} R_\rho \frac{t^\rho}{\rho!},$$

where R_ρ are classical Bernoulli-type numbers.

The generalized Apostol–Bernoulli-type polynomials (GABTP) $R_\rho^{(\sigma)}(\alpha;\lambda)$ of order σ are given by means of the following generating function (see [22]):

$$\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{\alpha t} = \sum_{\rho=0}^{\infty} R_\rho^{(\sigma)}(\alpha;\lambda) \frac{t^\rho}{\rho!}, \qquad |t| < |\log\lambda|, \ \ 1^\sigma := 1. \tag{1.8}$$

Setting $\alpha = 0$ in (1.8), we get

$$\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma = \sum_{\rho=0}^{\infty} R_\rho^{(\sigma)}(\lambda) \frac{t^\rho}{\rho!}, \tag{1.9}$$

where $R_\rho^{(\sigma)}(\lambda)$ denote generalized Apostol–Bernoulli-type numbers of order σ.

The r-parametric Hermite-based Milne–Thomson type polynomials (rPHMTTP) $h(\rho, w, z; \vec{u}, r, a, b)$ [32] (see also [33, 34]) are defined as follows:

$$M_1(t, w, z, \vec{u}, r, a, b) = (b + f(t, a))^z\, G(t, w, \vec{u}, r) = \sum_{\rho=0}^{\infty} h(\rho, w, z; \vec{u}, r, a, b) \frac{t^\rho}{\rho!}, \tag{1.10}$$

where $a, b \in \mathbb{R}$, $z \in \mathbb{N}_0$, $r \in \mathbb{N}$, r-tuples $\vec{u} = (u_1, u_2, \ldots, u_r)$, and $w = \alpha + i\upsilon$. The function $f(t, a)$ denotes an analytic function or meromorphic function, and $G(t, w, \vec{u}, r)$ denotes the generating functions for r-parametric Hermite type polynomials $\mathcal{K}(\rho; w, \vec{u}, r)$, defined as follows:

$$G(t, w, \vec{u}, r) = \exp\left(wt + \sum_{j=1}^{r} u_j t^j\right) = \sum_{\rho=0}^{\infty} \mathcal{K}(\rho; w, \vec{u}, r) \frac{t^\rho}{\rho!}.$$

Setting $w = \alpha, \vec{u} = \vec{0}, b = 0, z \longrightarrow \sigma, a \longrightarrow \lambda$, and $f(t, \lambda) = \frac{t^2}{2\lambda e^t - 2}$ in (1.10), the rPHMTTP $h(n, w, z; \vec{u}, r, a, b)$ reduce to the GABTP $R_\rho^{(\sigma)}(\alpha;\lambda)$.

The generalized tangent polynomials $\mathcal{T}_{\rho,\epsilon}(\alpha)$ [35] are defined by

$$\frac{2}{e^{\epsilon t}+1} e^{\alpha t} = \sum_{\rho=0}^{\infty} \mathcal{T}_{\rho,\epsilon}(\alpha) \frac{t^\rho}{\rho!}, \qquad \epsilon \in \mathbb{R}^+. \tag{1.11}$$

Setting $\epsilon = 2$ in (1.11) yields the generating function of classical tangent polynomials $\mathcal{T}_\rho(\alpha)$ [36]:

$$\frac{2}{e^{2t}+1} e^{\alpha t} = \sum_{\rho=0}^{\infty} \mathcal{T}_\rho(\alpha) \frac{t^\rho}{\rho!}. \tag{1.12}$$

Setting $\epsilon = 1$ in (1.11) yields the generating function of Euler polynomials $E_\rho(\alpha)$ [37]:

$$\frac{2}{e^{t}+1} e^{\alpha t} = \sum_{\rho=0}^{\infty} E_\rho(\alpha) \frac{t^\rho}{\rho!}. \tag{1.13}$$

The determinant representation and matrix approach are essential tools for the analysis of special polynomials. The matrix and determinant methodology in umbral calculus originated from two nearly concurrent studies [38, 39], employing entirely distinct methodologies. One of these evolved autonomously and was codified in the literature [11]. This study extends the determinantal definition of Bernoulli polynomials, initially proposed by Costabile et al. [40] and further developed for Sheffer and Appell polynomials [38, 41], to a class of hybrid polynomials.

Motivated by the importance of Hermite–Kampé de Fériet polynomials, Bell polynomials, and generalized Apostol–Bernoulli-type polynomials, this study introduces an original hybrid unified family incorporating these special polynomials. Our unified family integrates features from three significant special polynomials, Hermite–Kampé de Fériet polynomials, Bell polynomials, and generalized Apostol–Bernoulli-type polynomials. This unified family strengthens the applicability of classical special polynomials by incorporating their advantages and providing increased versatility. This makes them more adaptable and strong in handling complicated, contemporary problems across multiple fields.

In comparison to other polynomial sequences, our introduced family can be regarded as a generalization of other existing polynomial sequences, such as Hermite–Kampé de Fériet polynomials, Bell polynomials, and generalized Apostol–Bernoulli-type polynomials, Bernoulli-type polynomials, Hermite–Kampé de Fériet–Bell polynomials, Hermite–Kampé de Fériet–Apostol–Bernoulli-type polynomials, Hermite–Kampé de Fériet–Bernoulli-type polynomials, Bell–Apostol–Bernoulli-type polynomials, and Bell–Bernoulli-type polynomials. Further, the fraction extension (fractional-type Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials) is also a generalization of many fractional special polynomials.

The structure of this paper is as follows: This study establishes a new generalized class of special polynomials, the Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials (see Definition 1); Section 2 introduces the series representations and significant properties of this new class; in Section 3, we investigate some related summation formulas and determinant representation; Section 4 establishes certain differential and integral representations; Section 5 presents the fractional extension for these polynomials; and finally, the study concludes with computational analyses and graphical illustrations.

2. The Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials

In this section, we introduce a new class of special polynomials known as Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials (AHKFBelBP) and investigate their associated properties and identities.

Definition 1. *The Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials* $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ *of order* σ *are defined as*

$$\left(\frac{t^2}{2\lambda e^t - 2}\right)^{\sigma} e^{\alpha t + \upsilon t^2 + \gamma(e^t - 1)} = \sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) \frac{t^\rho}{\rho!}, \quad (|t| < |\log \lambda|,\ \lambda \neq 1; |t| < 2\pi, \lambda = 1). \tag{2.1}$$

Setting $\alpha = \upsilon = 0$ and $\gamma = 1$ in generating function (2.1), we get Apostol–Hermite–Kampé de

Fériet–Bell–Bernoulli-type numbers $\psi_\rho^{(\sigma)}(\lambda)$ of order σ, which are given by

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(0,0,1;\lambda)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{(e^t-1)}=\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\lambda)\,\frac{t^\rho}{\rho!}.$$

Remark 1. *Setting* $\lambda=1$ *in (2.1), we get the Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials* $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma)$ *of order* σ*, which are given by*

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;1)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2e^t-2}\right)^\sigma e^{\alpha t+\upsilon t^2+\gamma(e^t-1)}=\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma)\,\frac{t^\rho}{\rho!}.$$

Remark 2. *Setting* $\upsilon=0$ *in (2.1), we get the Apostol–Bell–Bernoulli-type polynomials* ${}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)$ *of order* σ*, which are given by [22]*

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,0,\gamma;\lambda)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{\alpha t+\gamma(e^t-1)}=\sum_{\rho=0}^{\infty}{}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)\,\frac{t^\rho}{\rho!}. \tag{2.2}$$

Remark 3. *Setting* $\alpha=\upsilon=0$ *in (2.1), we get the classical Apostol–Bell–Bernoulli-type polynomials* ${}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\gamma;\lambda)$ *of order* σ*, which are given by*

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(0,0,\gamma;\lambda)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{\gamma(e^t-1)}=\sum_{\rho=0}^{\infty}{}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\gamma;\lambda)\,\frac{t^\rho}{\rho!}. \tag{2.3}$$

Remark 4. *Setting* $\gamma=0$ *in (2.1), we get the Apostol–Hermite–Kampé de Fériet–Bernoulli-type polynomials (AHKDFBTP)* ${}_HR_\rho(\alpha,\upsilon;\lambda)$ *of order* σ*, which are given by*

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,0;\lambda)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{\alpha t+\upsilon t^2}=\sum_{\rho=0}^{\infty}{}_HR_\rho^{(\sigma)}(\alpha,\upsilon;\lambda)\,\frac{t^\rho}{\rho!}. \tag{2.4}$$

Setting $w=\alpha,\vec{u}=(u_1,u_2)=(0,u_2)=(0,\upsilon),b=0,z\longrightarrow\sigma,a\longrightarrow\lambda$ and $f(t,\lambda)=\frac{t^2}{2\lambda e^t-2}$ in (1.10), the rPHMTTP $h(n,w,z;\vec{u},r,a,b)$ reduce to the AHKDFBTP ${}_HR_\rho^{(\sigma)}(\alpha,\upsilon;\lambda)$.

Remark 5. *Setting* $\gamma=0$ *and* $\sigma=1$ *in (2.1), we get the Apostol–Hermite–Kampé de Fériet–Bernoulli-type polynomials* ${}_HR_\rho(\alpha,\upsilon;\lambda)$*, which are given by [18]*

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(1)}(\alpha,\upsilon,0;\lambda)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)e^{\alpha t+\upsilon t^2}=\sum_{\rho=0}^{\infty}{}_HR_\rho(\alpha,\upsilon;\lambda)\,\frac{t^\rho}{\rho!}.$$

Remark 6. *Setting* $\upsilon=0$ *and* $\gamma=0$ *in (2.1), we get the generalized Apostol–Bernoulli-type polynomials* $R_\rho^{(\sigma)}(\alpha;\lambda)$ *of order* σ*, which are given by (see [22])*

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,0,0;\lambda)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{\alpha t}=\sum_{\rho=0}^{\infty}R_\rho^{(\sigma)}(\alpha;\lambda)\,\frac{t^\rho}{\rho!}. \tag{2.5}$$

Remark 7. *Setting* $\upsilon = \gamma = 0$ *and* $\lambda = 1$ *in (2.1), we get the generalized Bernoulli-type polynomials* $R_\rho^{(\sigma)}(\alpha)$ *of order* σ*, which are given by [14]*

$$\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, 0, 0; 1)\frac{t^\rho}{\rho!} = \left(\frac{t^2}{2e^t - 2}\right)^{\sigma} e^{\alpha t} = \sum_{\rho=0}^{\infty} R_\rho^{(\sigma)}(\alpha)\, \frac{t^\rho}{\rho!}.$$

Theorem 1. *The following series representation for the AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ *of order* σ *holds true:*

$$\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_H\mathrm{Bel}_{\rho-\kappa}(\alpha, \upsilon, \gamma)\, R_\kappa^{(\sigma)}(\lambda). \tag{2.6}$$

Proof. Using (1.5) and (1.9) in (2.1), we have

$$\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} = \Big(\sum_{\rho=0}^{\infty} {}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma)\, \frac{t^\rho}{\rho!}\Big)\Big(\sum_{\rho=0}^{\infty} R_\rho^{(\sigma)}(\lambda)\, \frac{t^\rho}{\rho!}\Big),$$

which, on using the Cauchy product rule and series rearrangement, gives

$$\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} = \sum_{\rho=0}^{\infty} \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_H\mathrm{Bel}_{\rho-\kappa}(\alpha, \upsilon, \gamma)\, R_\kappa^{(\sigma)}(\lambda) \frac{t^\rho}{\rho!}. \tag{2.7}$$

From (2.7), we get the asserted result (2.6). □

Theorem 2. *The following series representations for the AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ *of order* σ *hold true:*

$$\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_{\mathrm{Bel}}R_{\rho-\kappa}^{(\sigma)}(\gamma; \lambda)\, H_\kappa(\alpha, \upsilon), \tag{2.8}$$

$$\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = \rho! \sum_{\kappa=0}^{[\frac{\rho}{2}]} \frac{{}_{\mathrm{Bel}}R_{\rho-2\kappa}^{(\sigma)}(\alpha, \gamma; \lambda)\, \upsilon^\kappa}{(\rho - 2\kappa)!\, \kappa!}. \tag{2.9}$$

Proof. Using (1.1) and (2.3) in (2.1), then applying the Cauchy product rule, we have

$$\begin{aligned}\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} &= \Big(\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\gamma; \lambda)\, \frac{t^\rho}{\rho!}\Big)\Big(\sum_{\rho=0}^{\infty} H_\rho(\alpha, \upsilon)\, \frac{t^\rho}{\rho!}\Big)\\ &= \sum_{\rho=0}^{\infty} \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_{\mathrm{Bel}}R_{\rho-\kappa}^{(\sigma)}(\gamma; \lambda)\, H_\kappa(\alpha, \upsilon) \frac{t^\rho}{\rho!}. \end{aligned}\tag{2.10}$$

From (2.10), we get the asserted result (2.8). Next, in view of (2.1) and (2.2), we have

$$\begin{aligned}\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} &= \Big(\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha, \gamma; \lambda)\, \frac{t^\rho}{\rho!}\Big)\Big(\sum_{\rho=0}^{\infty} \upsilon^\rho\, \frac{t^{2\rho}}{\rho!}\Big)\\ &= \sum_{\rho=0}^{\infty} \sum_{\kappa=0}^{[\frac{\rho}{2}]} \frac{\rho!\, {}_{\mathrm{Bel}}R_{\rho-2\kappa}^{(\sigma)}(\alpha, \gamma; \lambda)\, \upsilon^\kappa}{(\rho - 2\kappa)!\, \kappa!} \frac{t^\rho}{\rho!}. \end{aligned}\tag{2.11}$$

From (2.11), we get the asserted result (2.9). □

Theorem 3. *The following series representation for the AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$ *of order* σ *holds true:*

$$\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)=\sum_{\eta=0}^{\rho}\sum_{\kappa=0}^{\eta}\binom{\rho}{\eta}\gamma^\kappa\,{}_HR_{\rho-\eta}^{(\sigma)}(\alpha,\upsilon;\lambda)\,\mathbb{S}_2(\eta,\kappa). \tag{2.12}$$

Proof. Using (1.4) and (2.4) in (2.1), we have

$$\begin{aligned}\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,\frac{t^\rho}{\rho!}&=\Big(\frac{t^2}{2\lambda e^t-2}\Big)^\sigma\,e^{\alpha t+\upsilon t^2+\gamma(e^t-1)}\\&=\Big(\sum_{\rho=0}^{\infty}{}_HR_\rho^{(\sigma)}(\alpha,\upsilon;\lambda)\,\frac{t^\rho}{\rho!}\Big)\Big(\sum_{\eta=0}^{\infty}\sum_{\kappa=0}^{\eta}\gamma^\kappa\,\mathbb{S}_2(\eta,\kappa)\,\frac{t^\eta}{\eta!}\Big),\end{aligned}$$

which, on using the Cauchy product rule and series rearrangement, gives

$$\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,\frac{t^\rho}{\rho!}=\sum_{\rho=0}^{\infty}\sum_{\eta=0}^{\rho}\sum_{\kappa=0}^{\eta}\binom{\rho}{\eta}\gamma^\kappa\,{}_HR_{\rho-\eta}^{(\sigma)}(\alpha,\upsilon;\lambda)\,\mathbb{S}_2(\eta,\kappa)\,\frac{t^\rho}{\rho!}. \tag{2.13}$$

From (2.13), we get the asserted result (2.12). □

3. Summation formulas and determinant representation

This section introduces useful identities for the Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$ of order σ, focusing on the implicit summation formulas and determinant representation established by the following theorems.

Theorem 4. *For* $\rho\in\mathbb{N}_0$ *and* $\sigma,\delta\in\mathbb{N}$, *we have*

$$\psi_\rho^{(\sigma+\delta)}(\alpha+w,\upsilon,\gamma;\lambda)=\sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}\psi_{\rho-\kappa}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,R_\kappa^{(\delta)}(w;\lambda). \tag{3.1}$$

Proof. Utilizing (2.1), we have

$$\begin{aligned}\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma+\delta)}(\alpha+w,\upsilon,\gamma;\lambda)\,\frac{t^\rho}{\rho!}&=\Big(\frac{t^2}{2\lambda e^t-2}\Big)^{(\sigma+\delta)}\,e^{(\alpha+w)t+\upsilon t^2+\gamma(e^t-1)}\\&=\Big(\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,\frac{t^\rho}{\rho!}\Big)\Big(\sum_{\rho=0}^{\infty}R_\rho^{(\delta)}(w;\lambda)\,\frac{t^\rho}{\rho!}\Big)\\&=\sum_{\rho=0}^{\infty}\sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}\psi_{\rho-\kappa}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,R_\kappa^{(\delta)}(w;\lambda)\,\frac{t^\rho}{\rho!}.\end{aligned} \tag{3.2}$$

From (3.2), we arrive at the asserted result (3.1). □

Remark 8. *Setting* $\gamma=0$ *in (3.1), we have*

$${}_HR_\rho^{(\sigma+\delta)}(\alpha+w,\upsilon;\lambda)=\sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}{}_HR_{\rho-\kappa}^{(\sigma)}(\alpha,\upsilon;\lambda)\,R_\kappa^{(\delta)}(w;\lambda).$$

Theorem 5. *For $\rho \in \mathbb{N}_0$ and $\sigma \in \mathbb{N}$, we have*

$$\psi_\rho^{(\sigma)}(\alpha + x, \upsilon, \gamma + z; \lambda) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_{\mathrm{Bel}}R_{\rho-\kappa}^{(\sigma)}(\alpha, \gamma; \lambda)\ {}_H\mathrm{Bel}_\kappa(x, \upsilon, z). \tag{3.3}$$

Proof. Utilizing (2.1), we have

$$\begin{aligned}\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha + x, \upsilon, \gamma + z; \lambda)\, \frac{t^\rho}{\rho!} &= \left(\frac{t^2}{2\lambda e^t - 2}\right)^{\sigma} e^{(\alpha+x)t+\upsilon t^2+(\gamma+z)(e^t-1)} \\ &= \left(\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha, \gamma; \lambda)\, \frac{t^\rho}{\rho!}\right)\left(\sum_{\rho=0}^{\infty} {}_H\mathrm{Bel}_\rho(x, \upsilon, z)\, \frac{t^\rho}{\rho!}\right) \\ &= \sum_{\rho=0}^{\infty}\sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_{\mathrm{Bel}}R_{\rho-\kappa}^{(\sigma)}(\alpha, \gamma; \lambda)\ {}_H\mathrm{Bel}_\kappa(x, \upsilon, z)\, \frac{t^\rho}{\rho!}. \end{aligned} \tag{3.4}$$

From (3.4), we arrive at the asserted result (3.3). □

Remark 9. *Setting $\gamma = 0$ in (3.3), we have*

$$\psi_\rho^{(\sigma)}(\alpha + x, \upsilon, \gamma; \lambda) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} R_\rho^{(\sigma)}(\alpha; \lambda)\ {}_H\mathrm{Bel}_\kappa(x, \upsilon, \gamma).$$

Theorem 6. *For $\rho \in \mathbb{N}_0$, $\epsilon \in \mathbb{R}^+$, and $\sigma \in \mathbb{N}$, we have*

$$\psi_\rho^{(\sigma)}(\alpha + w, \upsilon, \gamma; \lambda) = \frac{1}{2}\sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} \mathcal{T}_{\rho-\kappa,\epsilon}(w)\Big(\psi_\kappa^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda)\ + \psi_\kappa^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\Big). \tag{3.5}$$

Proof. Utilizing (2.1), we have

$$(e^{\epsilon\, t} + 1)\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha + w, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} = e^{wt}\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} + e^{wt}\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!}. \tag{3.6}$$

From (1.11) and (3.6), we have

$$\begin{aligned}\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha + w, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} &= \frac{e^{wt}}{e^{\epsilon\, t} + 1}\left(\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!} + \sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!}\right) \\ &= \frac{1}{2}\left(\sum_{\rho=0}^{\infty} \mathcal{T}_{\rho,\epsilon}(w)\frac{t^\rho}{\rho!}\right)\left(\sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda)\ + \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, \frac{t^\rho}{\rho!}\right) \\ &= \frac{1}{2}\sum_{\rho=0}^{\infty}\sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} \mathcal{T}_{\rho-\kappa,\epsilon}(w)\Big(\psi_\kappa^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda)\ + \psi_\kappa^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\Big)\, \frac{t^\rho}{\rho!}. \end{aligned} \tag{3.7}$$

From (3.7), we arrive at the asserted result (3.5). □

Remark 10. *Setting* $\epsilon = 2$ *in (3.5), we get*

$$\psi_\rho^{(\sigma)}(\alpha + w, \upsilon, \gamma; \lambda) = \frac{1}{2} \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} \mathcal{T}_{\rho-\kappa}(w) \Big(\psi_\kappa^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda) \ + \psi_\kappa^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) \Big).$$

Remark 11. *Setting* $\epsilon = 1$ *in (3.5), we get*

$$\psi_\rho^{(\sigma)}(\alpha + w, \upsilon, \gamma; \lambda) = \frac{1}{2} \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} E_{\rho-\kappa}(w) \Big(\psi_\kappa^{(\sigma)}(\alpha + \epsilon, \upsilon, \gamma; \lambda) \ + \psi_\kappa^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) \Big).$$

Theorem 7. *For* $\rho \in \mathbb{N}_0$, *we have*

$$\psi_\rho^{(1)}(\alpha + 1, \upsilon, \gamma; \lambda) = \frac{1}{2\lambda} \left\{ (\rho^2 - \rho)_H\mathrm{Bel}_{\rho-2}(\alpha, \upsilon, \gamma) + 2\, \psi_\rho^{(1)}(\alpha, \upsilon, \gamma; \lambda) \right\}. \tag{3.8}$$

Proof. Utilizing (1.5) and (2.1) for $\sigma = 1$, we have

$$\begin{aligned} \sum_{\rho=0}^{\infty} {}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma) \frac{t^\rho}{\rho!} &= \frac{2\lambda e^t - 2}{t^2} \sum_{\rho=0}^{\infty} \psi_\rho^{(1)}(\alpha, \upsilon, \gamma; \lambda) \frac{t^\rho}{\rho!} \\ &= \frac{2}{t^2} \left\{ \lambda \sum_{\rho=0}^{\infty} \psi_\rho^{(1)}(\alpha + 1, \upsilon, \gamma; \lambda) \frac{t^\rho}{\rho!} - \sum_{\rho=0}^{\infty} \psi_\rho^{(1)}(\alpha, \upsilon, \gamma; \lambda) \frac{t^\rho}{\rho!} \right\}. \end{aligned} \tag{3.9}$$

From (3.9), we have

$$\sum_{\rho=0}^{\infty} (\rho^2 - \rho)_H\mathrm{Bel}_{\rho-2}(\alpha, \upsilon, \gamma) \frac{t^\rho}{\rho!} = 2 \sum_{\rho=0}^{\infty} \left\{ \lambda \psi_\rho^{(1)}(\alpha + 1, \upsilon, \gamma; \lambda) \frac{t^\rho}{\rho!} - \psi_\rho^{(1)}(\alpha, \upsilon, \gamma; \lambda) \right\} \frac{t^\rho}{\rho!}. \tag{3.10}$$

From (3.10), we arrive at the asserted result (3.8). □

Remark 12. *Setting* $\gamma = 0$ *in (3.8), we have*

$${}_HR_\rho^{(1)}(\alpha + 1, \upsilon; \lambda) = \frac{1}{2\lambda} \left\{ (\rho^2 - \rho) H_{\rho-2}(\alpha, \upsilon) + 2\, {}_HR_\rho^{(1)}(\alpha, \upsilon; \lambda) \right\}.$$

Next, in view of the principles defined in [38], we present the determinant definition of the AHKFBelBP $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ of order σ. The determinant form captures the recurrence structure of the AHKFBelBP $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ in a unified manner and provides a compact algebraic characterization of the AHKFBelBP $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$.

Definition 2. *The AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ *of order* σ *and degree* ρ *are defined by*

$$\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = \frac{(-1)^\rho}{(\mu_0)^{\rho+1}} \begin{vmatrix} 1 & {}_H\mathrm{Bel}_1(\alpha, \upsilon, \gamma) & {}_H\mathrm{Bel}_2(\alpha, \upsilon, \gamma) & \cdots & {}_H\mathrm{Bel}_{\rho-1}(\alpha, \upsilon, \gamma) & {}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma) \\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{\rho-1} & \mu_\rho \\ 0 & \mu_0 & \binom{2}{1}\mu_1 & \cdots & \binom{\rho-1}{1}\mu_{\rho-2} & \binom{\rho}{1}\mu_{\rho-1} \\ 0 & 0 & \mu_0 & \cdots & \binom{\rho-1}{2}\mu_{\rho-3} & \binom{\rho}{2}\mu_{\rho-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_0 & \binom{\rho}{\rho-1}\mu_1 \end{vmatrix},$$

where $\mu_0, \mu_1, \mu_2, \ldots, \mu_\rho$ *are the coefficients of Maclaurin series of the function* $\left(\frac{t^2}{2\lambda e^t - 2}\right)^{-\sigma}$ *and* ${}_H\mathrm{Bel}_\rho(\alpha, \upsilon, \gamma)$ *are the Hermite–Kampé de Fériet-based Bell polynomials (1.5).*

4. Differential and integral formulas

In this section, utilizing the factorization method, we derive some differential and integral expressions for Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ of order σ.

Theorem 8. *For $\rho \in \mathbb{N}_0$ and $s \in \mathbb{N}$, we have*

$$\frac{\partial^s}{\partial \alpha^s}\left\{\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\} = \begin{cases} \frac{\rho!\ \psi_{\rho-s}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)}{(\rho-s)!}, & \rho \geq s; \\ 0, & 0 \leq \rho < s. \end{cases} \tag{4.1}$$

Proof. Upon differentiating (2.1) s-times partially with respect to α, gives

$$\sum_{\rho=0}^{\infty} \frac{\partial^s}{\partial \alpha^s}\left\{\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\} \frac{t^\rho}{\rho!} = t^s \sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) \frac{t^\rho}{\rho!} = \sum_{\rho=0}^{\infty} \psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) \frac{t^{\rho+s}}{\rho!},$$

which, after simplifying and then equating the coefficients of $\frac{t^\rho}{\rho!}$ on both sides, produces the stated result (4.1). □

Similarly, the following result can be proved.

Theorem 9. *For $\rho \in \mathbb{N}_0$ and $\sigma \in \mathbb{N}$, we have*

$$\frac{\partial}{\partial \upsilon}\left\{\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\} = \begin{cases} \frac{\rho!\ \psi_{\rho-2}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)}{(\rho-2)!}, & \rho \geq 2; \\ 0, & 0 \leq \rho < 2. \end{cases} \tag{4.2}$$

Theorem 10. *The AHKFBelBP $\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ of order σ satisfy the following operational representation:*

$$\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = \exp\left(\upsilon\ \frac{\partial^2}{\partial \alpha^2}\right)\left\{{}_{\text{Bel}}R_\rho^{(\sigma)}(\alpha, \gamma; \lambda)\right\}. \tag{4.3}$$

Proof. From (4.1) and (4.2), we have

$$\frac{\partial}{\partial \upsilon}\left\{\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\} = \frac{\partial^2}{\partial \alpha^2}\left\{\psi_\rho^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\}. \tag{4.4}$$

Also, from equations (2.2), it follows that

$$\psi_\rho^{(\sigma)}(\alpha, 0, \gamma; \lambda) = {}_{\text{Bel}}R_\rho^{(\sigma)}(\alpha, \gamma; \lambda). \tag{4.5}$$

Solving Eq (4.4) subject to initial condition (4.5), we arrive at assertion (4.3). □

Remark 13. *Setting $\gamma = 0$ in (4.3), we have*

$${}_{H}R_\rho^{(\sigma)}(\alpha, \upsilon; \lambda) = \exp\left(\upsilon\ \frac{\partial^2}{\partial \alpha^2}\right)\left\{R_\rho^{(\sigma)}(\alpha; \lambda)\right\}.$$

Theorem 11. *For $\rho \in \mathbb{N}_0$, we have*

$$\frac{\partial}{\partial\gamma}\left\{\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\} = \sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}\left\{\psi_{\rho-\kappa}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) - {}_{H}\mathrm{Bel}_{\rho-\kappa}(\alpha, \upsilon, \gamma)\, R_{\kappa}^{(\sigma)}(\lambda)\right\}. \tag{4.6}$$

Proof. We start with generating function (2.1). Differentiating it with respect to γ, followed by simplification using equation (2.6), results in

$$\begin{aligned}\sum_{\rho=0}^{\infty}\frac{\partial}{\partial\gamma}\left\{\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\}\frac{t^{\rho}}{\rho!} &= (e^{t}-1)\left\{\left(\frac{t^{2}}{2\lambda e^{t}-2}\right)^{\sigma} e^{\alpha t+\upsilon t^{2}+\gamma(e^{t}-1)}\right\}\\ &= \sum_{\rho=0}^{\infty}\frac{t^{\rho}}{\rho!}\left\{\sum_{\rho=0}^{\infty}\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\frac{t^{\rho}}{\rho!}\right\} - \left\{\sum_{\rho=0}^{\infty}\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\frac{t^{\rho}}{\rho!}\right\}\\ &= \sum_{\rho=0}^{\infty}\sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}\left\{\psi_{\rho-\kappa}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) - {}_{H}\mathrm{Bel}_{\rho-\kappa}(\alpha, \upsilon, \gamma)\, R_{\kappa}^{(\sigma)}(\lambda)\right\}\frac{t^{\rho}}{\rho!}. \end{aligned}\tag{4.7}$$

From (4.7), we get the asserted result (4.6). □

Similarly, the following result can be proved.

Theorem 12. *For $\rho \in \mathbb{N}_0$ and $\sigma \in \mathbb{N}$, we have*

$$\frac{\partial}{\partial\gamma}\left\{\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right\} = \sum_{\kappa=1}^{\rho}\binom{\rho}{\kappa}\psi_{\rho-\kappa}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda).$$

Theorem 13. *The following formula holds true:*

$$\psi_{\rho}^{(\sigma)}(x, \upsilon, \gamma; \lambda) = \psi_{\rho}^{(\sigma)}(a, \upsilon, \gamma; \lambda) + \rho\int_{a}^{x}\psi_{\rho-1}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, d\alpha. \tag{4.8}$$

Proof. Taking the integration of both sides of (2.1) with respect to α from a to x, we have

$$\begin{aligned}\sum_{\rho=0}^{\infty}\int_{a}^{x}\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, d\alpha\,\frac{t^{\rho}}{\rho!} &= \frac{1}{t}\left[\left(\frac{t^{2}}{2\lambda e^{t}-2}\right)^{\sigma} e^{\alpha t+\upsilon t^{2}+\gamma(e^{t}-1)}\right]_{a}^{x}\\ &= \frac{1}{t}\left\{\sum_{\rho=0}^{\infty}\psi_{\rho}^{(\sigma)}(x, \upsilon, \gamma; \lambda)\frac{t^{\rho}}{\rho!} - \sum_{\rho=0}^{\infty}\psi_{\rho}^{(\sigma)}(a, \upsilon, \gamma; \lambda)\frac{t^{\rho}}{\rho!}\right\},\end{aligned}$$

which yields the asserted result in (4.8). □

Theorem 14. *The following formulas hold true:*

$$\psi_{\rho}^{(\sigma)}(\alpha, x, \gamma; \lambda) = \psi_{\rho}^{(\sigma)}(\alpha, a, \gamma; \lambda) + (\rho^{2}-\rho)\int_{a}^{x}\psi_{\rho-2}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\, d\upsilon, \tag{4.9}$$

$$\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, x; \lambda) = \psi_{\rho}^{(\sigma)}(\alpha, \upsilon, a; \lambda) + \int_{a}^{x}\left(\psi_{\rho}^{(\sigma)}(\alpha+1, \upsilon, \gamma; \lambda) - \psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)\right)d\gamma. \tag{4.10}$$

Proof. Taking the integration of both sides of (2.1) with respect to υ from a to x, we have

$$\begin{aligned}\sum_{\rho=0}^{\infty}\int_a^x \psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,d\upsilon\,\frac{t^\rho}{\rho!}&=\frac{1}{t^2}\left[\left(\frac{t^2}{2\lambda e^t-2}\right)^{\sigma} e^{\alpha t+\upsilon t^2+\gamma(e^t-1)}\right]_a^x\\&=\frac{1}{t^2}\left\{\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,x,\gamma;\lambda)\,\frac{t^\rho}{\rho!}-\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,a,\gamma;\lambda)\,\frac{t^\rho}{\rho!}\right\},\end{aligned}$$

which yields the asserted result in (4.9). Next, taking the integration of both sides of (2.1) with respect to γ from a to x, we have

$$\begin{aligned}\sum_{\rho=0}^{\infty}\int_a^x \psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)\,d\gamma\,\frac{t^\rho}{\rho!}&=\frac{1}{e^t-1}\left[\left(\frac{t^2}{2\lambda e^t-2}\right)^{\sigma} e^{\alpha t+\upsilon t^2+\gamma(e^t-1)}\right]_a^x\\&=\frac{1}{e^t-1}\left\{\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,x;\lambda)\,\frac{t^\rho}{\rho!}-\sum_{\rho=0}^{\infty}\psi_\rho^{(\sigma)}(\alpha,\upsilon,a;\lambda)\,\frac{t^\rho}{\rho!}\right\},\end{aligned}$$

which yields the asserted result in (4.10). □

Example 1. *Setting $\lambda = 1$ and $\sigma = 2$ in (2.1), we get the Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials (2-oHKFBelBP) $\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)$ of order 2 which are given by the following generating function:*

$$\left(\frac{t^2}{2e^t-2}\right)^{2} e^{\alpha t+\upsilon t^2+\gamma(e^t-1)}=\sum_{\rho=0}^{\infty}\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)\,\frac{t^\rho}{\rho!}.$$

The first six 2-oHKFBelBP $\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)$ of order 2 are listed as:

$$\begin{aligned}&\psi_0^{(2)}(\alpha,\upsilon,\gamma)=0,\quad \psi_1^{(2)}(\alpha,\upsilon,\gamma)=0,\quad \psi_2^{(2)}(\alpha,\upsilon,\gamma)=\frac{1}{2},\quad \psi_3^{(2)}(\alpha,\upsilon,\gamma)=\frac{3}{2}(\alpha+\gamma-1),\\&\psi_4^{(2)}(\alpha,\upsilon,\gamma)=3\alpha^2+6\alpha\gamma-6\alpha+6\upsilon+3\gamma^2-3\gamma+\frac{5}{2},\\&\psi_5^{(2)}(\alpha,\upsilon,\gamma)=5\alpha^3+15\alpha^2\gamma-15\alpha^2+30\alpha\upsilon+15\alpha\gamma^2-15\alpha\gamma+\frac{25\alpha}{2}+30\upsilon\gamma-30\upsilon+5\gamma^3+\frac{5\gamma}{2}-\frac{5}{2}.\end{aligned}$$

In view of (2.6), (2.8), (2.9), and (2.12), the 2-oHKFBelBP $\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)$ of order 2 satisfy the following series representations:

$$\begin{aligned}\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)&=\sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}{}_H\mathrm{Bel}_{\rho-\kappa}(\alpha,\upsilon,\gamma)\,R_\kappa^{(2)},\\\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)&=\sum_{\kappa=0}^{\rho}\binom{\rho}{\kappa}{}_{\mathrm{Bel}}R_{\rho-\kappa}^{(2)}(\gamma)\,H_\kappa(\alpha,\upsilon),\\\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)&=\rho!\sum_{\kappa=0}^{[\frac{\rho}{2}]}\frac{{}_{\mathrm{Bel}}R_{\rho-2\kappa}^{(2)}(\alpha,\gamma)\,\upsilon^\kappa}{(\rho-2\kappa)!\,\kappa!},\\\psi_\rho^{(2)}(\alpha,\upsilon,\gamma)&=\sum_{\eta=0}^{\rho}\sum_{\kappa=0}^{\eta}\binom{\rho}{\eta}\gamma^\kappa\,{}_HR_{\rho-\eta}^{(2)}(\alpha,\upsilon)\,\mathbb{S}_2(\eta,\kappa).\end{aligned}$$

In view of (3.1) and (3.3), we present the following summation formulas for the 2-oHKFBelBP $\psi_{\rho}^{(2)}(\alpha, \upsilon, \gamma)$ *of order* 2*:*

$$\psi_{\rho}^{(2+\delta)}(\alpha + w, \upsilon, \gamma) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} \psi_{\rho-\kappa}^{(2)}(\alpha, \upsilon, \gamma)\, R_{\kappa}^{(\delta)}(w),$$

$$\psi_{\rho}^{(2)}(\alpha + x, \upsilon, \gamma + z) = \sum_{\kappa=0}^{\rho} \binom{\rho}{\kappa} {}_{\mathrm{Bel}}R_{\rho-\kappa}^{(2)}(\alpha, \gamma)\, {}_{H}\mathrm{Bel}_{\kappa}(x, \upsilon, z).$$

Definition 3. *The 2-oHKFBelBP* $\psi_{\rho}^{(2)}(\alpha, \upsilon, \gamma)$ *of order* 2 *and degree* ρ *are defined by*

$$\psi_{\rho}^{(2)}(\alpha, \upsilon, \gamma) = \frac{(-1)^{\rho}}{(\mu_0)^{\rho+1}} \begin{vmatrix} 1 & {}_{H}\mathrm{Bel}_1(\alpha, \upsilon, \gamma) & {}_{H}\mathrm{Bel}_2(\alpha, \upsilon, \gamma) & \cdots & {}_{H}\mathrm{Bel}_{\rho-1}(\alpha, \upsilon, \gamma) & {}_{H}\mathrm{Bel}_{\rho}(\alpha, \upsilon, \gamma) \\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{\rho-1} & \mu_{\rho} \\ 0 & \mu_0 & \binom{2}{1}\mu_1 & \cdots & \binom{\rho-1}{1}\mu_{\rho-2} & \binom{\rho}{1}\mu_{\rho-1} \\ 0 & 0 & \mu_0 & \cdots & \binom{\rho-1}{2}\mu_{\rho-3} & \binom{\rho}{2}\mu_{\rho-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_0 & \binom{\rho}{\rho-1}\mu_1 \end{vmatrix},$$

where $\mu_0, \mu_1, \mu_2, \ldots, \mu_{\rho}$ *are the coefficients of Maclaurin series of the function* $\left(\frac{t^2}{2e^t-2}\right)^{-2}$ *and* ${}_{H}\mathrm{Bel}_{\rho}(\alpha, \upsilon, \gamma)$ *are the Hermite–Kampé de Fériet-based Bell polynomials (1.5).*

Similarly, we can obtain the other results for the 2-oHKFBelBP $\psi_{\rho}^{(2)}(\alpha, \upsilon, \gamma)$ of order 2 from the corresponding results that are established in previous sections.

5. Fractional extension

Based on the obtained operational representation in Section 4 and in view of the Euler integral, we establish the fractional-type Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials.

We recall the Euler integral, a pivotal fractional operator in the study of special polynomials as characterized by Srivastava and Manocha [42]. Serving as a conceptual cornerstone, this integral has enabled researchers to forge new directions in polynomial theory. Notably, Dattoli et al. [43, 44] have made a substantial contribution to the subject by expanding the theoretical framework and the practical applications of these functions through the use of Euler integral representations. The Euler integral [42, p. 218] is given as

$$\zeta^{-\varepsilon} = \frac{1}{\Gamma(\varepsilon)} \int_0^{\infty} e^{-\zeta\, \omega}\, \omega^{\varepsilon-1}\, \mathrm{d}\omega, \qquad \min\{\mathrm{Re}(\zeta), \mathrm{Re}(\varepsilon)\} > 0, \tag{5.1}$$

which leads to the following [44]:

$$\left(\zeta - \frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^{-\varepsilon} f(\alpha) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\infty} e^{-\zeta\, \omega}\, \omega^{\varepsilon-1} \exp\left(\omega\, \frac{\mathrm{d}}{\mathrm{d}\alpha}\right) f(\alpha)\, \mathrm{d}\omega,$$

and, consequently, we can obtain

$$\left[\zeta - \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\right]^{-\varepsilon} f(\alpha) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\infty} e^{-\zeta\, \omega}\, \omega^{\varepsilon-1} \exp\left(\omega\, \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\right) f(\alpha)\, \mathrm{d}\omega.$$

These integral representations lay the groundwork for operational techniques used to construct and study special functions, including generalized polynomial classes.

Theorem 15. *The fractional-type Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials* $\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ *are defined by*

$$\left(\zeta-\upsilon\,\frac{\partial^2}{\partial\alpha^2}\right)^{-\varepsilon}{}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)=\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta). \tag{5.2}$$

Proof. Replacing ζ by $(\zeta-\upsilon\,\frac{\partial^2}{\partial\alpha^2})$ in relation to (5.1), then applying the resultant result to ${}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)$, we obtain the following transformation:

$$\left(\zeta-\upsilon\,\frac{\partial^2}{\partial\alpha^2}\right)^{-\varepsilon}{}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty e^{-\zeta\,\omega}\;\omega^{\varepsilon-1}\exp\left(\omega\upsilon\,\frac{\partial^2}{\partial\alpha^2}\right){}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)\;\mathrm{d}\omega\,,$$

which, on using operational relation (4.3), becomes

$$\left(\zeta-\upsilon\,\frac{\partial^2}{\partial\alpha^2}\right)^{-\varepsilon}{}_{\mathrm{Bel}}R_\rho^{(\sigma)}(\alpha,\gamma;\lambda)=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty e^{-\zeta\,\omega}\;\omega^{\varepsilon-1}\psi_\rho^{(\sigma)}(\alpha,\omega\upsilon,\gamma;\lambda)\;\mathrm{d}\omega\,. \tag{5.3}$$

The integral transform on the right-hand side of (5.3) yields a new class of special polynomials, known as the fractional-type Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials (FAHKFBelBP). By denoting these polynomials as $\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)$, we obtain

$$\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty e^{-\zeta\,\omega}\;\omega^{\varepsilon-1}\psi_\rho^{(\sigma)}(\alpha,\omega\upsilon,\gamma;\lambda)\;\mathrm{d}\omega\,. \tag{5.4}$$

Equations (5.3) and (5.4) imply assertion (5.2). □

Remark 14. *Setting* $\gamma=0$ *in (5.2), we have*

$$\left(\zeta-\upsilon\,\frac{\partial^2}{\partial\alpha^2}\right)^{-\varepsilon}R_\rho^{(\sigma)}(\alpha;\lambda)={}_HR_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon;\lambda,\zeta).$$

Now, we construct the generating function for the fractional-type Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials by proving the following result:

Theorem 16. *The following generating function for the FAHKFBelBP* $\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ *holds true:*

$$\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma\frac{e^{\alpha t+\gamma(e^t-1)}}{(\zeta-\upsilon t^2)^\varepsilon}=\sum_{\rho=0}^\infty\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)\frac{t^\rho}{\rho!}. \tag{5.5}$$

Proof. Multiplying both sides of (5.4) by $\frac{t^\rho}{\rho!}$ then summing over ρ, we obtain

$$\sum_{\rho=0}^\infty\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)\frac{t^\rho}{\rho!}=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty e^{-\zeta\omega}\;\omega^{\varepsilon-1}\left(\sum_{\rho=0}^\infty\psi_\rho^{(\sigma)}(\alpha,\omega\upsilon,\gamma;\lambda)\frac{t^\rho}{\rho!}\right)\mathrm{d}\omega\,,$$

which, in view of the Eq (2.1),

$$\sum_{\rho=0}^\infty\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)\frac{t^\rho}{\rho!}=\left(\frac{t^2}{2\lambda e^t-2}\right)^\sigma e^{\alpha t+\gamma(e^t-1)}\frac{1}{\Gamma(\varepsilon)}\int_0^\infty e^{-(\zeta-\upsilon t^2)\omega}\;\omega^{\varepsilon-1}\;\mathrm{d}\omega\,. \tag{5.6}$$

Finally, applying relation (5.1) to the right-hand side of (5.6) yields assertion (5.5). □

Remark 15. *Setting* $\gamma = 0$ *in (5.5), we get the generating function for the fractional-type Apostol–Hermite–Kampé de Fériet–Bernoulli-type polynomials (FAHKFBP)* ${}_{H}R^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon; \lambda, \zeta)$ *as*

$$\left(\frac{t^2}{2\lambda e^t - 2}\right)^{\sigma} \frac{e^{\alpha t}}{(\zeta - \upsilon t^2)^{\varepsilon}} = \sum_{\rho=0}^{\infty} {}_{H}R^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon; \lambda, \zeta)\frac{t^{\rho}}{\rho!}.$$

For any complex number ε, the Pochhammer symbol $(\varepsilon)_{\kappa}$ is defined by (see [37,42])

$$(\varepsilon)_{\kappa} = \begin{cases} \varepsilon(\varepsilon+1)\cdots(\varepsilon+\kappa-1), & \kappa \in \mathbb{N}; \\ 1, & \kappa = 0. \end{cases}$$

Next, we establish the series representation for the fractional-type Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials by proving the following results:

Theorem 17. *For the FAHKFBelBP* $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta)$*, we have*

$$\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta) = \frac{\rho!}{\zeta^{\varepsilon}} \sum_{\kappa=0}^{[\frac{\rho}{2}]} \frac{(\varepsilon)_{\kappa}\ {}_{\mathrm{Bel}}R^{(\sigma)}_{\rho-2\kappa}(\alpha, \gamma; \lambda)\ \upsilon^{\kappa}}{\zeta^{\kappa}\ (\rho-2\kappa)!\ \kappa!}. \tag{5.7}$$

Proof. In view of series definition (2.9), Eq (5.4) becomes

$$\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta) = \frac{\rho!}{\Gamma(\varepsilon)} \sum_{\kappa=0}^{[\frac{\rho}{2}]} \frac{{}_{\mathrm{Bel}}R^{(\sigma)}_{\rho-2\kappa}(\alpha, \gamma; \lambda)\ \upsilon^{\kappa}}{(\rho-2\kappa)!\ \kappa!} \int_0^{\infty} e^{-\zeta\omega}\ \omega^{\varepsilon+\kappa-1}\ \mathrm{d}\omega\,. \tag{5.8}$$

By applying (5.1) to the right-hand side of (5.8), we obtain assertion (5.7). □

Based on the monomiality principle [45, 46], we establish the operators for multiplication and differentiation and the differential equation for the FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta)$.

Theorem 18. *The FAHKFBelBP* $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta)$ *satisfy the following multiplicative and derivative operators:*

$$\hat{M}_{\psi} = \alpha + \gamma e^{D_{\alpha}} + \frac{2\sigma}{D_{\alpha}} + \frac{2\varepsilon\upsilon\, D_{\alpha}}{\zeta - \upsilon D_{\alpha}^2} - \frac{\sigma\lambda e^{D_{\alpha}}}{\lambda e^{D_{\alpha}} - 1} \tag{5.9}$$

and

$$\hat{P}_{\psi} = D_{\alpha}, \tag{5.10}$$

respectively.

Proof. Observe that

$$D_{\alpha}\left\{\left(\frac{t^2}{2\lambda e^t - 2}\right)^{\sigma} \frac{e^{\alpha t + \gamma(e^t-1)}}{(\zeta - \upsilon t^2)^{\varepsilon}}\right\} = t\ \left\{\left(\frac{t^2}{2\lambda e^t - 2}\right)^{\sigma} \frac{e^{\alpha t + \gamma(e^t-1)}}{(\zeta - \upsilon t^2)^{\varepsilon}}\right\}. \tag{5.11}$$

Partially differentiating (5.5) with respect to t yields

$$\left\{\alpha + \gamma e^t + \frac{2\sigma}{t} + \frac{2\varepsilon\upsilon\, t}{\zeta - \upsilon t^2} - \frac{\sigma\lambda e^t}{\lambda e^t - 1}\right\}\left\{\sum_{\rho=0}^{\infty} \psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta)\frac{t^{\rho}}{\rho!}\right\} = \sum_{\rho=0}^{\infty} \psi^{(\sigma)}_{\rho+1,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta)\frac{t^{\rho}}{\rho!}.$$

Using the identity (5.11) and comparing coefficients of each power of t, the following is obtained:

$$\left\{\alpha+\gamma e^{D_\alpha}+\frac{2\sigma}{D_\alpha}+\frac{2\varepsilon\upsilon\, D_\alpha}{\zeta-\upsilon D_\alpha^2}-\frac{\sigma\lambda e^{D_\alpha}}{\lambda e^{D_\alpha}-1}\right\}\left\{\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)\right\}=\psi_{\rho+1,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta). \tag{5.12}$$

From (5.12) and (5.11), we get asserted results (5.9) and (5.10), respectively. □

Remark 16. *Setting $\gamma = 0$ in (5.9) and (5.10), we get the following multiplicative and derivative operators for the FAHKFBP ${}_HR_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon;\lambda,\zeta)$ as*

$$\hat{M}_{HR}=\alpha+\frac{2\sigma}{D_\alpha}+\frac{2\varepsilon\upsilon\, D_\alpha}{\zeta-\upsilon D_\alpha^2}-\frac{\sigma\lambda e^{D_\alpha}}{\lambda e^{D_\alpha}-1}$$

and

$$\hat{P}_{HR}=D_\alpha,$$

respectively.

Theorem 19. *The FAHKFBelBP $\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ satisfy the following differential equation:*

$$\left(\alpha D_\alpha+\gamma e^{D_\alpha}D_\alpha+2\sigma+\frac{2\varepsilon\upsilon\, D_\alpha^2}{\zeta-\upsilon D_\alpha^2}-\frac{\sigma\lambda e^{D_\alpha}}{\lambda e^{D_\alpha}-1}D_\alpha-\rho\right)\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)=0. \tag{5.13}$$

Proof. In view of the monomiality principle [45, 46], utilizing operators (5.9) and (5.10) yields the asserted result (5.13). □

Remark 17. *Setting $\gamma = 0$ in (5.13), we get the following differential equation for the FAHKFBP ${}_HR_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon;\lambda,\zeta)$ as*

$$\left(\alpha D_\alpha+2\sigma+\frac{2\varepsilon\upsilon\, D_\alpha^2}{\zeta-\upsilon D_\alpha^2}-\frac{\sigma\lambda e^{D_\alpha}}{\lambda e^{D_\alpha}-1}D_\alpha-\rho\right){}_HR_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon;\lambda,\zeta)=0.$$

6. Computational and graphical frameworks

This section presents the advantages of applying computational analysis and investigates the zero distributions by displaying graphical illustrations of the AHKFBelBP $\psi_{\rho}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$ and FAHKFBelBP $\psi_{\rho,\varepsilon}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ for specific values and indices.

6.1. Computational and graphical representations of AHKFBelBP $\psi_{\rho}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$

The first seven terms of the AHKFBelBP $\psi_{\rho}^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$ for $\sigma = 1$ and $\lambda = 1$ are mentioned in Table 1.

Table 1. The first seven AHKFBelBP $\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$.

ρ	$\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$
0	0
1	$\frac{1}{2}$
2	$\alpha + \gamma - \frac{1}{2}$
3	$\frac{1}{4}\left(6\alpha^2 + 6\alpha(2\gamma - 1) + 12\upsilon + 6\gamma^2 + 1\right)$
4	$2\alpha^3 + 6\alpha^2\gamma - 3\alpha^2 + 12\alpha\upsilon + 6\alpha\gamma^2 + \alpha + 12\upsilon\gamma - 6\upsilon + 2\gamma^3 + 3\gamma^2$
5	$\frac{5\alpha^4}{2} + 10\alpha^3\gamma - 5\alpha^3 + 30\alpha^2\upsilon + 15\alpha^2\gamma^2 + \frac{5\alpha^2}{2} + 60\alpha\upsilon\gamma - 30\alpha\upsilon + 10\alpha\gamma^3 + 15\alpha\gamma^2 + 30\upsilon^2 + 30\upsilon\gamma^2$ $+5\upsilon + \frac{5\gamma^4}{2} + 10\gamma^3 + 5\gamma^2 - \frac{1}{12}$
6	$3\alpha^5 + 15\alpha^4\gamma - \frac{15\alpha^4}{2} + 60\alpha^3\upsilon + 30\alpha^3\gamma^2 + 5\alpha^3 + 180\alpha^2\upsilon\gamma - 90\alpha^2\upsilon + 30\alpha^2\gamma^3 + 45\alpha^2\gamma^2 + 180\alpha\upsilon^2$ $+180\alpha\upsilon\gamma^2 + 30\alpha\upsilon + 15\alpha\gamma^4 + 60\alpha\gamma^3 + 30\alpha\gamma^2 - \frac{\alpha}{2} + 180\upsilon^2\gamma - 90\upsilon^2 + 60\upsilon\gamma^3 + 90\upsilon\gamma^2 + 3\gamma^5 + \frac{45\gamma^4}{2}$ $+35\gamma^3 + \frac{15\gamma^2}{2}$

To show the shapes of the AHKFBelBP $\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda)$ for $\rho = 1; 2; 3; 4; 5; 6$, $-100 \leq \alpha \leq 100$, $\sigma = 1$, $\lambda = 1$, $\upsilon = \frac{1}{4}$, and $\gamma = \frac{1}{8}$, Figure 1 is given.

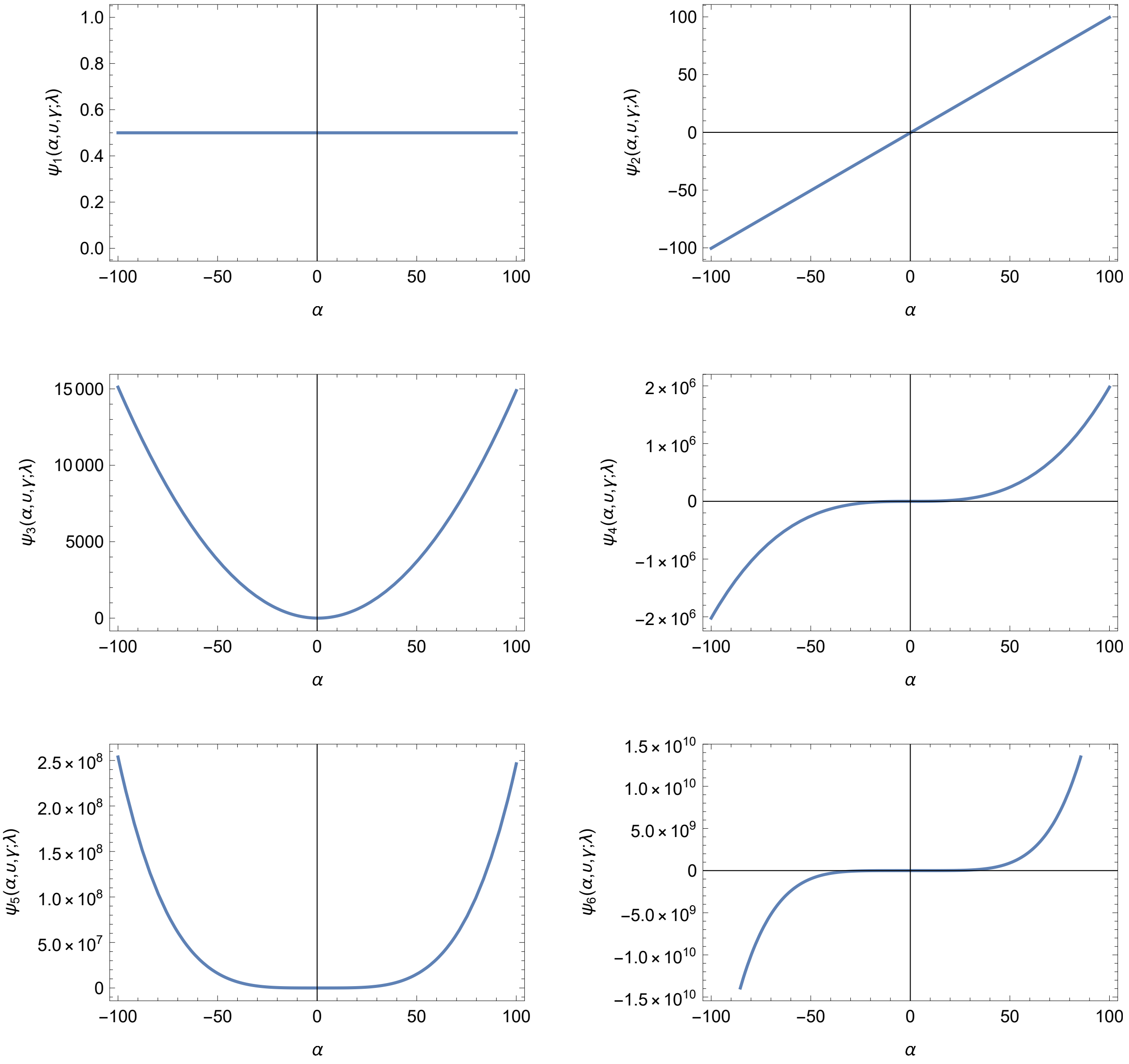


Figure 1. Graphs of $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$ for $-100 \le \alpha \le 100, \sigma = \lambda = 1, \upsilon = \frac{1}{4}, \gamma = \frac{1}{8}$, and for $\rho = 1$ (top-left); $\rho = 2$ (top-right); $\rho = 3$ (middle-left); $\rho = 4$ (middle-right); $\rho = 5$ (bottom-left); $\rho = 6$ (bottom-right).

Certain interesting zeros of the AHKFBelBP $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda) = 0$ for $\sigma = \lambda = 1$ and $\rho = 70$ are shown in Figure 2.

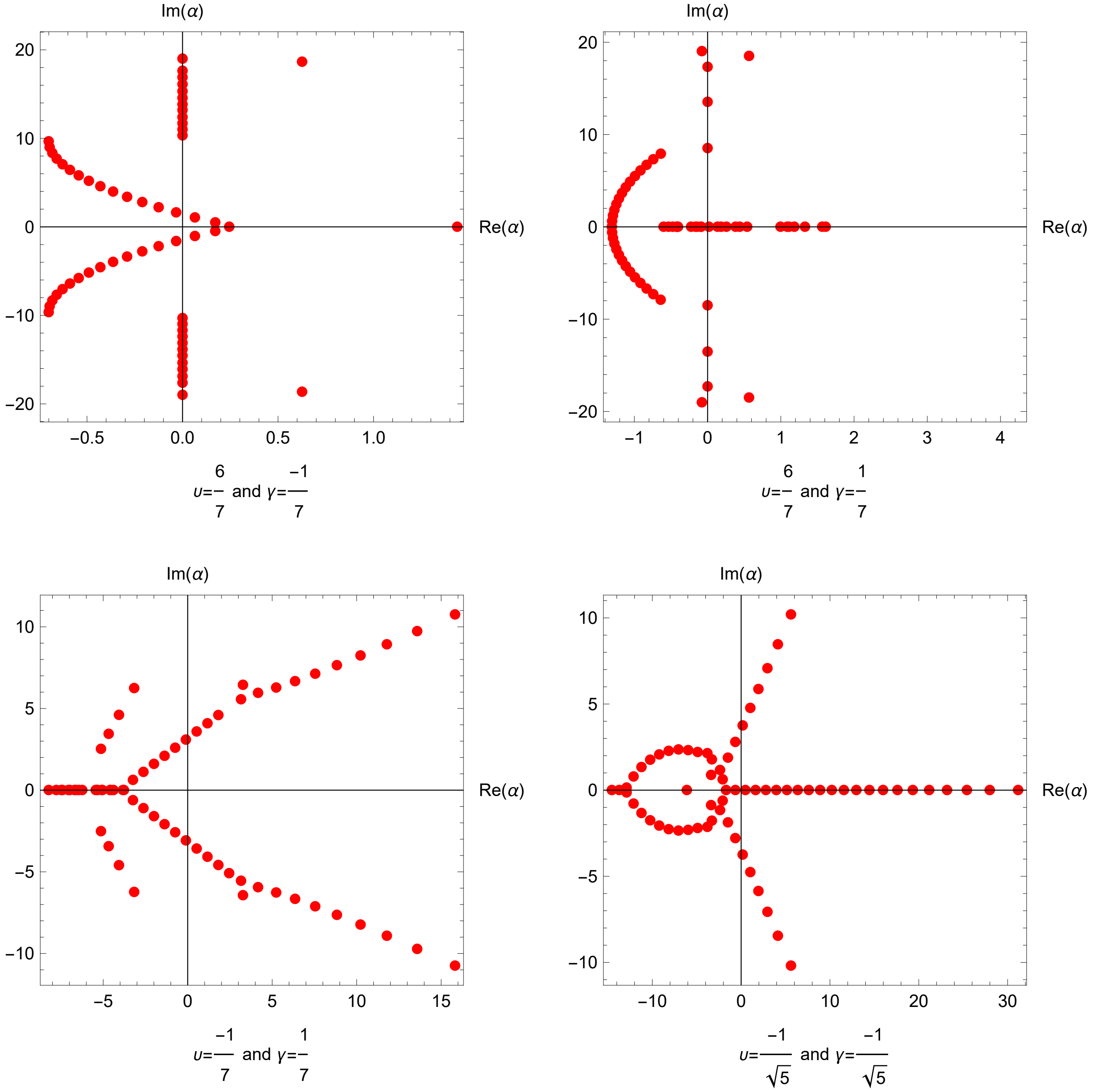
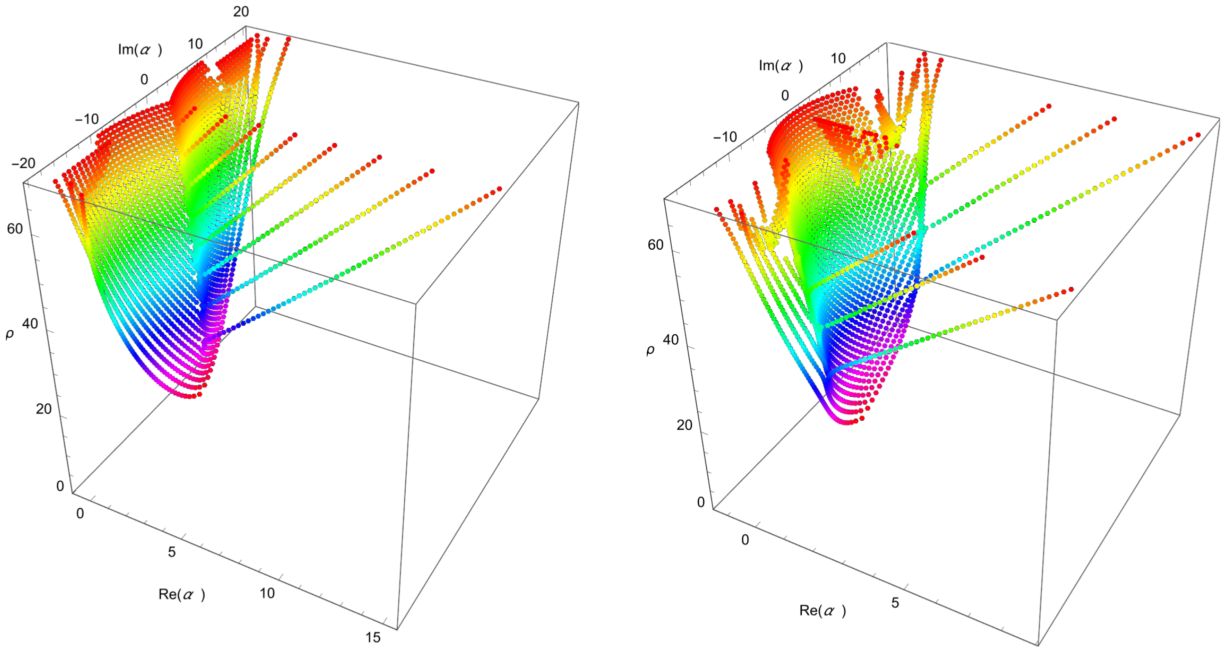


Figure 2. Zeros of $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)=0$ for $\sigma=\lambda=1,\rho=70$, and for $\upsilon=\frac{6}{7},\gamma=\frac{-1}{7}$ (top-left); $\upsilon=\frac{6}{7},\gamma=\frac{1}{7}$ (top-right); $\upsilon=\frac{-1}{7},\gamma=\frac{1}{7}$ (bottom-left); $\upsilon=\gamma=\frac{-1}{\sqrt{5}}$ (bottom-right).

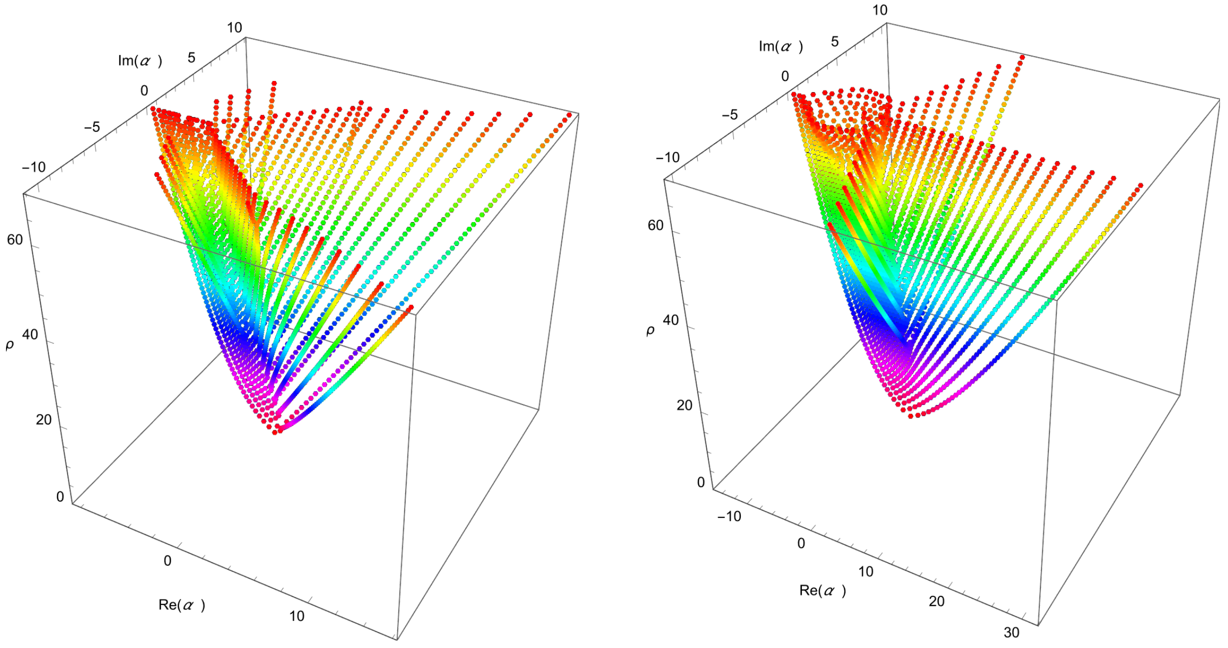
Remark 18. *We observe that the zeros of the AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)=0$ *for* $\sigma=\lambda=1$ *and* $\rho=70$ *have the following properties:*

1. *The AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)$ *of degree* ρ *possess exactly* $\rho-1$ *zeros.*
2. *Different zero distributions and a wide variety of graphical configurations are produced whenever the variables, parameters, or indices are brought into change.*
3. *The zeros (complex zeros) of AHKFBelBP* $\psi_\rho^{(\sigma)}(\alpha,\upsilon,\gamma;\lambda)=0$ *demonstrate symmetry with respect to the real axis.*

The stacking structures of approximation zeros of the AHKFBelBP $\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = 0$ for $\sigma = \lambda = 1$ and $1 \leq \rho \leq 70$ show 3D structures, which are given in Figure 3.



(a) $\upsilon = \frac{6}{7}$ and $\gamma = \frac{-1}{7}$ **(b)** $\upsilon = \frac{6}{7}$ and $\gamma = \frac{1}{7}$



(c) $\upsilon = \frac{-1}{7}$ and $\gamma = \frac{1}{7}$ **(d)** $\upsilon = \gamma = \frac{-1}{\sqrt{5}}$

Figure 3. Stacking structure zeros of $\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = 0$. This figure presents the 3D plot of the zeros of the $\psi_{\rho}^{(\sigma)}(\alpha, \upsilon, \gamma; \lambda) = 0$ for $\sigma = \lambda = 1, 1 \leq \rho \leq 70$, and for $\upsilon = \frac{6}{7}, \gamma = \frac{-1}{7}$ (top-left); $\upsilon = \frac{6}{7}, \gamma = \frac{1}{7}$ (top-right); $\upsilon = \frac{-1}{7}, \gamma = \frac{1}{7}$ (bottom-left); $\upsilon = \gamma = \frac{-1}{\sqrt{5}}$ (bottom-right).

6.2. Computational and graphical representations of FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta)$

The first seven terms of the FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ for $\sigma=\lambda=1,\varepsilon=2$, and $\zeta=3$ are listed in Table 2.

Table 2. The first seven FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta)$.

$\boldsymbol{\rho}$	$\boldsymbol{\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta)}$
0	0
1	$\frac{1}{18}$
2	$\frac{1}{18}(2\alpha+2\gamma-1)$
3	$\frac{1}{36}\left(6\alpha^2+6\alpha(2\gamma-1)+8\upsilon+6\gamma^2+1\right)$
4	$\frac{2\alpha^3}{9}+\frac{2\alpha^2\gamma}{3}-\frac{\alpha^2}{3}+\frac{8\alpha\upsilon}{9}+\frac{2\alpha\gamma^2}{3}+\frac{\alpha}{9}+\frac{8\upsilon\gamma}{9}-\frac{4\upsilon}{9}+\frac{2\gamma^3}{9}+\frac{\gamma^2}{3}$
5	$\frac{5\alpha^4}{18}+\frac{10\alpha^3\gamma}{9}-\frac{5\alpha^3}{9}+\frac{20\alpha^2\upsilon}{9}+\frac{5\alpha^2\gamma^2}{3}+\frac{5\alpha^2}{18}+\frac{40\alpha\upsilon\gamma}{9}-\frac{20\alpha\upsilon}{9}+\frac{10\alpha\gamma^3}{9}+\frac{5\alpha\gamma^2}{3}+\frac{20\upsilon^2}{9}+\frac{20\upsilon\gamma^2}{9}+\frac{10\upsilon}{27}+\frac{5\gamma^4}{18}$ $+\frac{10\gamma^3}{9}+\frac{5\gamma^2}{9}-\frac{1}{108}$
6	$\frac{\alpha^5}{3}+\frac{5\alpha^4\gamma}{3}-\frac{5\alpha^4}{6}+\frac{40\alpha^3\upsilon}{9}+\frac{10\alpha^3\gamma^2}{3}+\frac{5\alpha^3}{9}+\frac{40}{3}\alpha^2\upsilon\gamma-\frac{20\alpha^2\upsilon}{3}+\frac{10\alpha^2\gamma^3}{3}+5\alpha^2\gamma^2+\frac{40\alpha\upsilon^2}{3}+\frac{40}{3}\alpha\upsilon\gamma^2$ $+\frac{20\alpha\upsilon}{9}+\frac{5\alpha\gamma^4}{3}+\frac{20\alpha\gamma^3}{3}+\frac{10\alpha\gamma^2}{3}-\frac{\alpha}{18}+\frac{40\upsilon^2\gamma}{3}-\frac{20\upsilon^2}{3}+\frac{40\upsilon\gamma^3}{9}+\frac{20\upsilon\gamma^2}{3}+\frac{\gamma^5}{3}+\frac{5\gamma^4}{2}+\frac{35\gamma^3}{9}+\frac{5\gamma^2}{6}$

To show the shapes of the FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ for $\rho=1;2;3;4;5;6,-100\leq\alpha\leq 100,\sigma=\lambda=1,\varepsilon=2,\zeta=3,\upsilon=\frac{1}{4}$, and $\gamma=\frac{1}{8}$, Figure 4 is given.

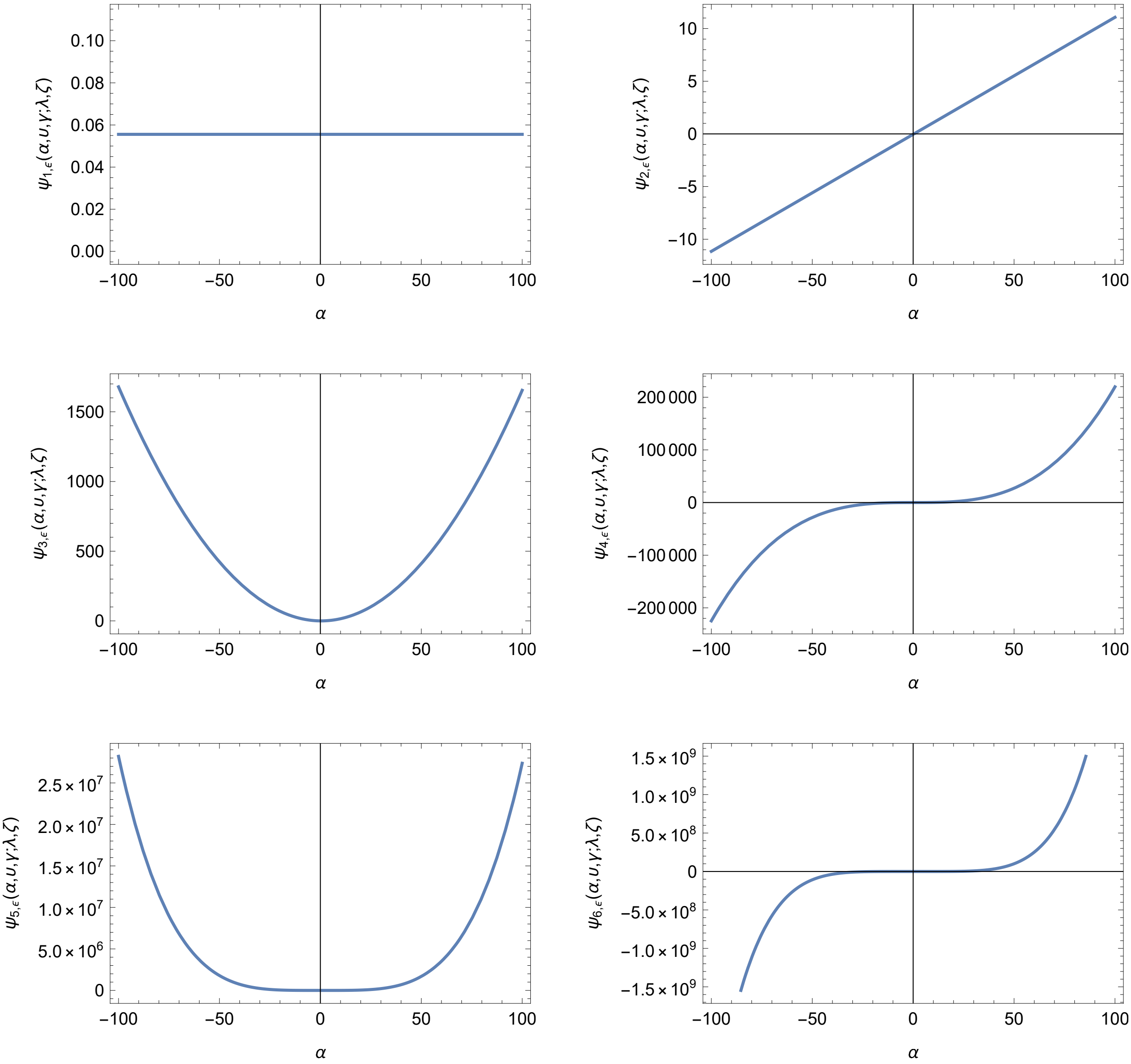


Figure 4. Graphs of $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta)$ for $-100 \leq \alpha \leq 100, \sigma = \lambda = 1, \varepsilon = 2, \zeta = 3, \upsilon = \frac{1}{4}$, $\gamma = \frac{1}{8}$, and for $\rho = 1$ (top-left); $\rho = 2$ (top-right); $\rho = 3$ (middle-left); $\rho = 4$ (middle-right); $\rho = 5$ (bottom-left); $\rho = 6$ (bottom-right).

Certain interesting zeros of the FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta) = 0$, for $\sigma = \lambda = 1, \varepsilon = 2, \zeta = 3$, and $\rho = 70$ are shown in Figure 5.

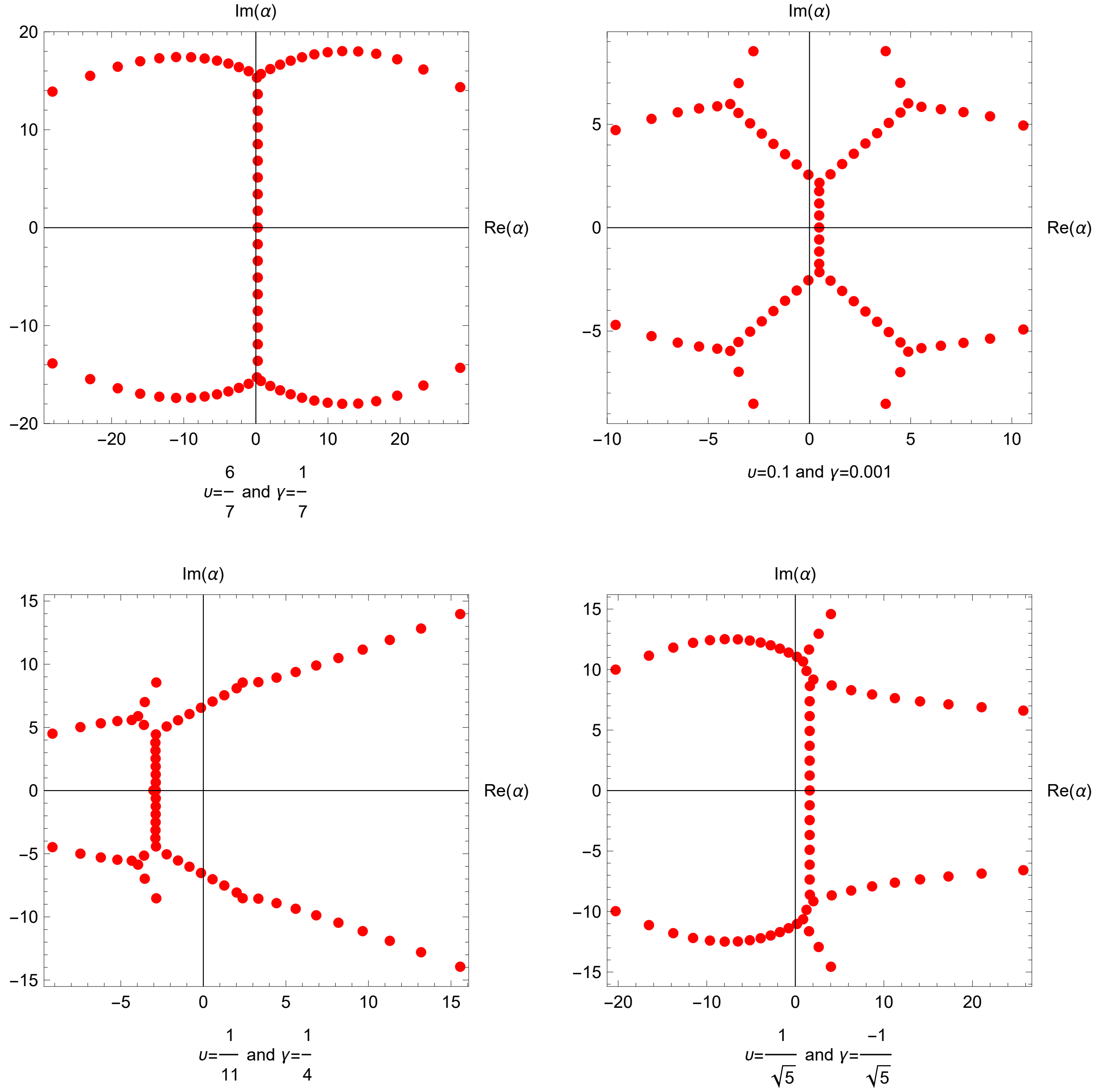
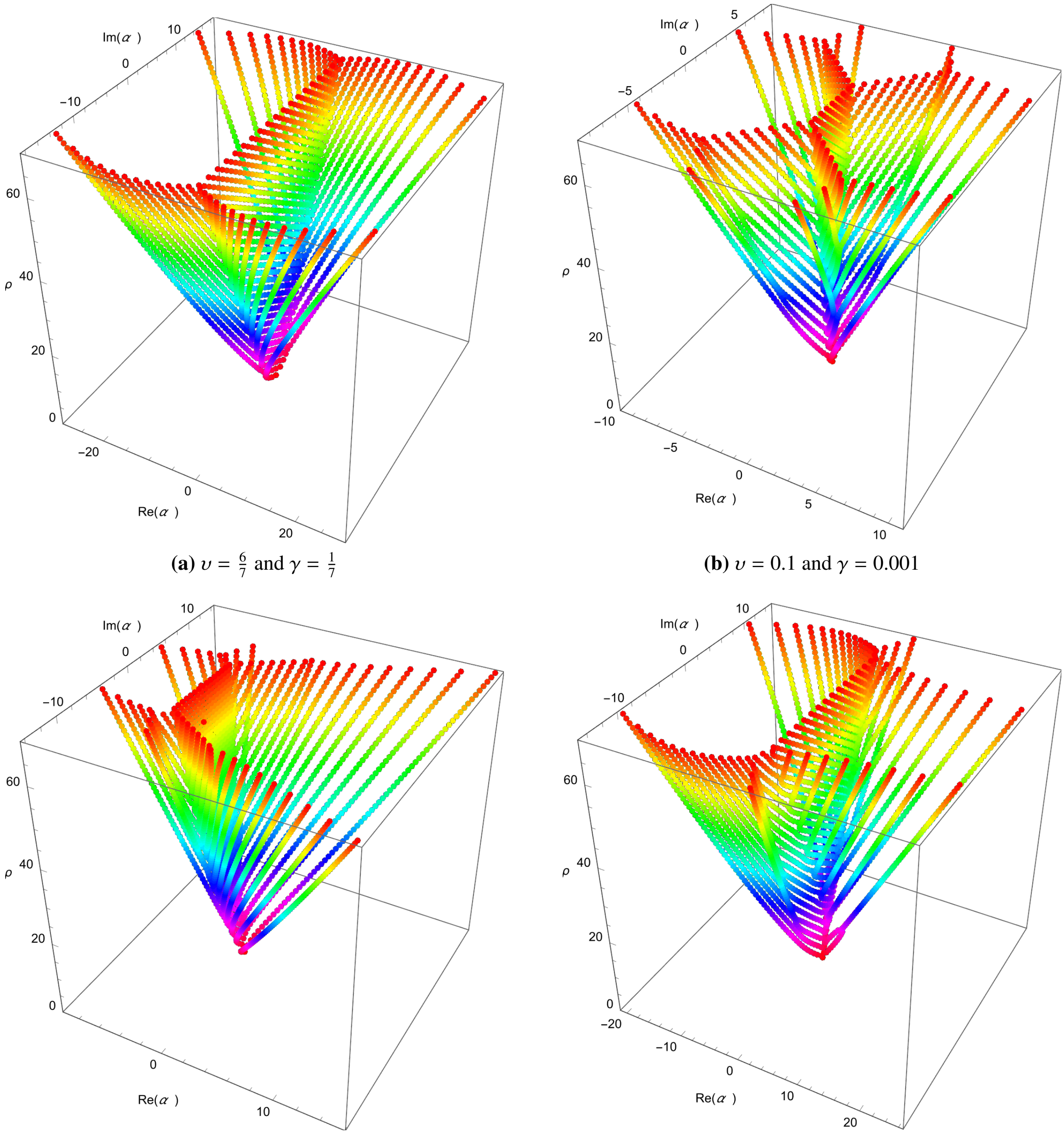


Figure 5. Zeros of $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta) = 0$ for $\sigma = \lambda = 1, \varepsilon = 2, \zeta = 3, \rho = 70$, and for $\upsilon = \frac{6}{7}, \gamma = \frac{1}{7}$ (top-left); $\upsilon = 0.1, \gamma = 0.001$ (top-right); $\upsilon = \frac{1}{11}, \gamma = \frac{1}{4}$ (bottom-left); $\upsilon = \frac{1}{\sqrt{5}}, \gamma = \frac{-1}{\sqrt{5}}$ (bottom-right).

The stacking structures of approximation zeros of the FAHKFBelBP $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha,\upsilon,\gamma;\lambda,\zeta) = 0$ for $\sigma = \lambda = 1, \varepsilon = 2, \zeta = 3$, and $1 \leq \rho \leq 70$ present 3D structures, which are given in Figure 6.



(a) $\upsilon = \frac{6}{7}$ and $\gamma = \frac{1}{7}$

(b) $\upsilon = 0.1$ and $\gamma = 0.001$

(c) $\upsilon = \frac{1}{11}$ and $\gamma = \frac{1}{4}$

(d) $\upsilon = \frac{1}{\sqrt{5}}$ and $\gamma = \frac{-1}{\sqrt{5}}$

Figure 6. Stacking structure zeros of $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta) = 0$. This figure presents the 3D plot of the zeros of the $\psi^{(\sigma)}_{\rho,\varepsilon}(\alpha, \upsilon, \gamma; \lambda, \zeta) = 0$ for $\sigma = \lambda = 1, \varepsilon = 2, \zeta = 3, 1 \leq \rho \leq 70$, and for $\upsilon = \frac{6}{7}, \gamma = \frac{1}{7}$ (top-left); $\upsilon = 0.1, \gamma = 0.001$ (top-right); $\upsilon = \frac{1}{11}, \gamma = \frac{1}{4}$ (bottom-left); $\upsilon = \frac{1}{\sqrt{5}}, \gamma = \frac{-1}{\sqrt{5}}$ (bottom-right).

7. Conclusions

The study of hybrid forms of special polynomials has garnered significant interest within the mathematical community. In this paper, we successfully established and analyzed a generalized family designated as Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials. Utilizing the generating function, we derived the fundamental properties of this family, including the associated series representations, summation formulas, determinant representation, and differential and integral representations. Furthermore, we provided a comprehensive theoretical framework by establishing a fractional extension of Apostol–Hermite–Kampé de Fériet–Bell–Bernoulli-type polynomials.

In addition to the theoretical investigation, we conducted computational investigations using Mathematica to illustrate the zero distributions and graphical behavior of these polynomials, providing a better understanding of their features.

In summary, this study not only introduces new generalized special polynomials, but it also unifies several polynomial families using a single approach. These results significantly enhance the extensive domain of special functions. Future research will concentrate on analyzing the degenerate versions of these specific polynomials and exploring their possible applicability across many domains.

Author contributions

Mesfer H. Alqahtani: Conceptualization, methodology, investigation; Abdulghani Muhyi: Conceptualization, formal analysis, investigation, writing-oiginal draft preparation; Amel Touati: Methodology, investigation, data curation; Muntasir Suhail: Validation, resources, visualization; L. M. Abdalgadir: Investigation, software, data curation; Khaled Aldwoah: Writing-review & editing, supervision, project administration; Amer Alsulami: Validation, resources, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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