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Research article

Probabilistically-oriented analysis of a job execution in a semi-Markov environment

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Abstract: In this paper, a single machine's functioning process in a random environment is studied. The machine, which is subject to failures and renewals, executes stochastic jobs, with the lifetime of the machine and the renewal time being random variables with general distributions. Here, we present a novel, purely probabilistic approach to solve the stochastic system that describes the operation of the machine, without relying on Kolmogorov equations. The results are formulated as rigorous mathematical statements, thus offering a significant simplification in the reliability analysis of the considered stochastic system. These results, together with related developments in the literature, illustrate that for a broad family of semi–Markov reliability and queuing systems, purely probabilistic arguments based on renewal and regeneration properties could provide transparent and effective analytical tools. This approach can be applied, among many others, to semi–Markov models in which supplementary variables are also employed, including extensions of classical queuing systems (e.g., M/G/1 models with vacations, retrials, reneging, balking, or feedback) and repairable reliability systems with general lifetime and repair distributions. In conclusion, the proposed methodology provides an efficient probabilistically transparent framework for the transient analysis of semi–Markov systems; as a result, it may serve as a useful alternative or complement to classical analytical techniques with important practical implications in reliability and the queuing theory.

Keywords: failure; renewal; stochastic job execution; semi-Markov process; supplementary variables technique; non-classical boundary value problem; non-local boundary condition
Mathematics Subject Classification: 60K05, 60K15, 60K20, 60K25

1. Introduction

The semi-Markov process theory usually gives rise to mathematical models specified by boundary value problems (BVP) of a different nature than those traditionally found in physics. Such sophistication comes with the application of the supplementary variable technique (SVT), where the model contains partial differential equation (PDE) systems (Kolmogorov forward or backward equations) as a main part, together with boundary conditions of a non-local nature [17]; Gaver's parallel system is one example of this. The solutions of such BVPs are often exceedingly complicated and are not available in an explicit form. However, this paper demonstrates that an alternative, probabilistically oriented interpretation of the system dynamics provides a more intuitive and tractable path to a solution.

A new direction of methodology, initiated by [11] in queueing research, creates an original probabilistic approach that was subsequently applied to reliability models, (e.g., extensions of Gaver's system [12–14]). Continuing this direction, the objective of current research is to highlight the method's conceptual clarity and practical efficiency, thereby presenting it as a robust alternative to analyze systems governed by semi-Markov processes. This technique emphasizes the stochastic essence of such models, thus enabling a more natural understanding of their behaviour.

Experts in the field often describe semi-Markov processes as a "marriage" of the renewal theory and the Markov chain theory [2]. Until these systems are considered probabilistically, their underlying BVPs do not reveal much about their true nature. Since this research is motivated by the SVT, an excursion into its history and place is required. The SVT was originally introduced in 1955 [3] for queuing research, and D.P. Gaver, Jr. was the first to apply it in 1963 [6] for reliability modeling; since then, it has become famous in the study of semi-Markov systems, particularly in reliability and queueing applications.

The approach reduces the system dynamics to a two-dimensional Markov process, thus enabling researchers to derive the corresponding PDEs. This transition from stochastic modeling to PDEs naturally led to complex BVPs. Despite their difficulty, SVT-based methods have remained at the center of reliability system analyses, with Gaver's research standing out both in depth and influence. The present study not only stands on its own but also tries to reinterpret and simplify classical results in a probabilistic vein. Such methodological development has had attraction in the broader reliability research community, as reflected in works such as that of [4,10,20–24]. As repairable systems continue to be a focus in the reliability theory, Gaver's model remains a fundamental benchmark for both theoretical advancement and practical applications.

The stochastic system that describes the operation of a machine has been studied in [9, 16] using the SVT. As a result, non-classical BVPs of mathematical physics with non-local boundary conditions has been obtained. Naturally, the most significant part of the problem is a system of PDEs (PDE-Kolmogorov forward equations). In this study, we demonstrate that the PDEs are redundant, and the problem can be solved without them. Here, we present a novel, purely probabilistic approach for this. Related results can be found in [11, 13, 14].

In this paper, we discuss a single machine operation process which functions in a semi-Markov random environment. Namely, the machine is subject to random failures, the repair time is a random variable, and it executes a job of random length. The machine is repaired upon failure and is then returned to operation. After repairing, the machine is as good as new. The time to machine failure (its lifetime) is a random variable Y with an arbitrary continuous distribution function G; the corresponding hazard rate function is $\lambda(x)$.

The r.v. X represents the execution time during which the machine is in operation, with a continuous distribution function F and a hazard rate $\mu(x)$. Finally, the repair time is a random variable Z with an arbitrary continuous distribution function B; the corresponding hazard rate function is $\eta(x)$.

The notion of hazard rate function has wide applications in the reliability theory, in the queuing theory, and in related areas. The hazard rate, also called the hazard function, is precisely the same as the force mortality introduced by actuaries many years ago [19]. In statistics, its reciprocal for the normal distribution is known as "Mills ratio" [15]. In the extreme value theory, it is known as an "intensity function" [8] and plays an important role in determining the form of an extreme value distribution.

In contrast to the classical SVT, which typically leads to systems of PDEs with non-local boundary conditions, the present paper develops a purely probabilistic approach that yields explicit transient state–age representations without solving the associated PDE system. While probabilistic reasoning has appeared in earlier works, such as [11] and related follow-up papers, our contribution differs in three respects: (i) We provide a unified derivation of both $p_0(x,t)$ and $p_1(x,t)$ directly from regenerative arguments; (ii) we give a full verification theorem which demonstrates that the PDEs are redundant for this class of models; and (iii) we explicitly show how the renewal-theoretic structure leads to closed-form transient expressions. This positions our approach as a conceptually simpler and computationally more transparent alternative to SVT-based PDE derivations.

The paper progresses as follows: In Section 2, we formulate the problem and give all relevant definitions to the stochastic system and in Section 3, we rigorously prove the main results.

2. Formulation of the problem

In this section, we provide the precise specification of the system by providing a rigorous semi-Markov description, the Main Assumptions, and the Model Assumptions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We model the machine by the càdlàg process $\zeta(t) \in \{0,1\}$, $t \geq 0$, with state 0 = operational (job executed) and state 1 = under repair. Let $(T_n)_{n\geq 0}$ be the jump times of ζ, with $T_0 = 0$, and $J_n = \zeta(T_n)$ as the embedded chain. The sojourn times in state 0 and 1 are i.i.d. across visits and independent across states, with distribution functions F (lifetime in state 0) and B (repair duration in state 1), respectively. The job execution time has a distribution G and restarts from scratch after each repair (renewal).

For notational consistency, we emphasize that the process $S(t)$ used later in the manuscript is identical to the process $\zeta(t)$ introduced here. Both notations represent the same underlying càdlàg state process taking values in $\{0,1\}$, with 0 denoting the operational status and 1 denoting the repair status. From now on, to avoid ambiguity, we only use $S(t)$.

Main Assumptions A1–A4.
(A1) F, G, B are absolutely continuous on $[0,\infty)$ with densities f, g, b.
(A2) The hazard rates associated with X, Y, Z given in (2.1)–(2.3) exist and are locally bounded.
(A3) $\{X_k\}, \{Y_k\}, \{Z_k\}$ are mutually independent sequences, and independent of (J_0, T_0).
(A4) Upon each repair completion, the job restarts from scratch; hence, the elapsed execution age $X(t)$ equals the lifetime age $Y(t)$ while $\zeta(t) = 0$.

For mathematical completeness, we explicitly define the sequences $\{X_k\}_{k\geq 1}$, $\{Y_k\}_{k\geq 1}$, and $\{Z_k\}_{k\geq 1}$ as measurable mappings on $(\Omega, \mathcal{F}, \mathbb{P})$. Here, X_k denotes the duration of the k-th job execution attempt, Y_k denotes the k-th machine lifetime between two successive failures, and Z_k denotes the duration of the k-th repair period. Assumption A3 requires these three sequences to be independent of each other, of the initial state (J_0, T_0), and each sequence to be i.i.d. with distribution functions F, G, and B, respectively. This construction guarantees that the semi-Markov structure and regenerative arguments employed later in Section 3 are fully rigorous.

In regard to the lifetime Y of the above machine, we wish to compare the conditional probability that the machine will fail in the time interval $(x, x+h)$, given that it was operable for a length of time x, i.e., we wish to obtain

$$\mathbb{P}[x \leq Y < x+h \mid Y > x].$$

Using the definition of conditional probability and letting $h \to 0$, we have the hazard rate of the r.v. Y given as follows:

$$\lambda(x) = \frac{G'(x)}{1-G(x)} = \frac{g(x)}{1-G(x)}, \tag{2.1}$$

where $g(\cdot)$ is the pdf of Y.

Analogously, for the r.v. X, we obtain that its hazard rate is given as

$$\mu(x) = \frac{f(x)}{1-F(x)}, \tag{2.2}$$

and the hazard rate of the r.v. Z is given as

$$\eta(x) = \frac{b(x)}{1-B(x)}, \tag{2.3}$$

with $f(\cdot)$ and $b(\cdot)$ representing the pdfs of X and Z, respectively.

The consecutive lifetimes between failures are assumed to be independently distributed, as are the consecutive repair periods and the consecutive execution times.

Recall that by A3, the random variables Y, Z, X are also mutually independent.

We introduce the following random processes:

1. $S(t)$; $S(t) = 0$, if the machine is operable and executes the job at time (instant) t. $S(t) = 1$, if the machine is under repair at time t;
2. $X(t)$ is the elapsed time of a job execution at time t, from its beginning of operation;
3. $Y(t)$ is the elapsed lifetime at time t;
4. $Z(t)$ is the elapsed repair time at time instant t from the beginning of the repair.

Recall that the definition of $S(t)$ coincides with the earlier process $\zeta(t)$. From now on, the notation $S(t)$ is adopted to maintain consistency when formulating the boundary value problem and the associated state–age densities.

Here, $S(t)$ is the discrete state of the semi–Markov system, while $X(t)$, $Y(t)$, and $Z(t)$ denote the corresponding age (sojourn-time) processes. By A4, $X(t) = Y(t)$ whenever $S(t) = 0$.

Denote by the following:

$$p_0(x,t) = \lim_{h\to 0} \frac{1}{h} \mathbb{P}[S(t) = 0;\ x \leq X(t) = Y(t) < x+h], \tag{2.4}$$

$$p_1(x,t) = \lim_{h\to 0} \frac{1}{h} \mathbb{P}[S(t) = 1;\ x \le Z(t) < x+h]. \tag{2.5}$$

The limits above are understood as right derivatives with respect to the age variable x, which exist under A1 and A2.

Model assumptions

1. The lifetime begins at the time instant when the job execution starts, or at the time instant when the repair of the machine ends (is completed).
2. Additionally, it is assumed, that if the machine fails during the job execution, then the job execution starts from scratch upon repair (all over again).
3. Due to the previous, two model Assumptions, namely the random processes $X(t)$ and $Y(t)$, coincide (i.e., $X(t) = Y(t)$).
4. In other words, we can suppose, that the machine starts the job execution at time instant $t = 0$, which is, at the same time, the beginning of the machine lifetime and the beginning of the job execution.

Using the SVT [1], the following mathematical model is constructed [9]. Under A1–A4, the joint state–age densities satisfy the first-order transport equations with loss terms given by the hazard rates. The boundary terms encode the regeneration at jump times (repair completions and failures), thus leading to the non-local conditions below. The system of equations (2.6)–(2.11) constitutes a non-classical BVP of mathematical physics with non-local boundary conditions (2.8) and (2.9):

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) p_0(x,t) = -[\lambda(x) + \mu(x)]\, p_0(x,t), \tag{2.6}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) p_1(x,t) = -\eta(x)\, p_1(x,t), \tag{2.7}$$

$$p_0(0,t) = \delta(t) + \int_0^t p_1(x,t)\, \eta(x)\, dx, \tag{2.8}$$

$$p_1(0,t) = \int_0^t p_0(x,t)\, \lambda(x)\, dx, \tag{2.9}$$

$$p_0(x,0) = \delta(x), \tag{2.10}$$

$$p_1(x,0) \;=\; 0, \tag{2.11}$$

where $(x,t) \in [0,+\infty) \times [0,\infty)$, and $\delta(\cdot)$ is the Dirac delta function.

In the following section, we first provide the candidate solutions of the above system based on the probabilistic reasoning (Theorem 3.2), verify the PDE's associated with the system (Theorem 3.3), and finally verify the non-local boundaries with the use of the probabilistic based solutions. In other words, in the following section, we will show that the underlying probabilistic structure allows one to obtain explicit transient representations of the relevant state–age densities without relying on the direct solution of the associated PDEs.

3. Main results

Denote by the following:

$$A_0(x,t,h) = [S(t) = 0;\; x \le X(t) = Y(t) < x+h],$$

$$A_1(x,t,h) = [S(t) = 1;\; x \le Z(t) < x+h],$$

and

$$C(x,t) = [Z(t) > x].$$

Let $T(t)$ denote the a.s. unique start time of the repair interval that covers t whenever $S(t) = 1$ (i.e., $S(u) = 1$ for all $u \in [T(t), t]$ and $S(T(t)^-) = 0$).
Note that the process $S(t)$ is adapted to the natural filtration $\mathcal{F}_t = \sigma(S(u) : 0 \le u \le t)$, which is inherited from the càdlàg semi–Markov dynamics. For each n, the jump time

$$T_n = \inf\{t > T_{n-1} :\; S(t) \ne S(t^-)\}$$

is an $\mathcal{F}_t$–stopping time, since $\{T_n \le t\}$ is measurable with respect to $\mathcal{F}_t$. Therefore, the quantity $T(t) = \max\{T_n : T_n \le t\}$ is also an $\mathcal{F}_t$–stopping time. This justifies the use of the Strong Markov Property at time $T(t)$ in Lemma 3.1, which is a crucial step for the regeneration argument in Theorem 3.2 that leads to (3.1) and (3.2).

Lemma 3.1. *Fix $0 < x < t$. Let $T(t)$ denote the a.s. unique start time of the repair interval covering t whenever $S(t) = 1$ (i.e., $S(u) = 1$ for all $u \in [T(t), t]$ and $S(T(t)^-) = 0$). Then, under Assumptions A1–A4,*

$$\mathbb{P}(A_1(x,t,h)) = \mathbb{E}[\mathbf{1}_{\{T(t)\in[t-x,\,t-x+h)\}}]\; \mathbb{P}(x \le Z < x+h) + o(h);$$

therefore, taking $h \downarrow 0$,

$$p_1(x,t) = p_1(0, t-x)\,[1 - B(x)], \qquad 0 < x < t.$$

Moreover, if B is continuous, then $A_1(x,t,h) \subset C(x,t)$ a.s..

Proof. Observe the condition on $\mathcal{F}_{t-x}$, from the σ-algebra up to time $t-x$. On the event $\{T(t) \in [t-x, t-x+h)\}$ the repair begins within that short interval; thus, the repair age at time t equals x with the probability $\mathbb{P}(x \le Z < x+h)$. Using independence of the repair duration from the past (A3), we obtain the following:

$$\mathbb{P}(A_1(x,t,h) \mid \mathcal{F}_{t-x}) = \mathbf{1}_{\{T(t)\in[t-x,\,t-x+h)\}}\,(b(x)h + o(h)).$$

Taking expectations and using

$$\mathbb{P}(T(t) \in [t-x,\ t-x+h)) = p_1(0, t-x)h + o(h)$$

gives the claimed result. The continuity of B ensures $x \le Z(t) < x + h \ \Rightarrow\ Z(t) > x$ a.s., thus proving the final statement. □

The redundancy of the transport Eqs (2.6) and (2.7) is a direct consequence of the regenerative structure of the underlying semi–Markov model. The jump times $\{T_n\}$ form an embedded Markov chain, and the strong Markov property at these regeneration epochs implies that all transient age–state densities can be expressed in terms of the renewal survival probabilities associated with F, G, and B. In contrast with earlier works such as [11, 13], which relied on PDE systems as a starting point, the present work establishes that the probabilistic representations solely and completely determine the transient solution. To the best of our knowledge, this is the first formal proof showing that the PDE system is fully redundant once the renewal structure is made explicit for this class of semi–Markov reliability systems.

Using Lemma 3.1 and the alternative probabilistic approach to solution for semi–Markov systems [11], the following theorem holds.

Theorem 3.2. *For $0 < x < t$ (right derivatives in x understood), the functions $p_0(x,t)$ and $p_1(x,t)$ have the form*

$$p_0(x,t) \quad = \quad p_0(0, t-x)\,[1 - F(x)]\,[1 - G(x)] \tag{3.1}$$

and

$$p_1(x,t) \quad = \quad p_1(0, t-x)\,[1 - B(x)]. \tag{3.2}$$

Proof. We prove (3.2); the argument for (3.1) is analogous.

By definition of $p_1(x,t)$ as a right derivative,

$$p_1(x,t)\,h = \mathbb{P}(A_1(x,t,h)) + o(h).$$

Using Lemma 3.1,

$$\mathbb{P}(A_1(x,t,h)) = \mathbb{P}(T(t) \in [t-x,\ t-x+h))\,\mathbb{P}(x \le Z < x+h) + o(h).$$

The two factors admit the first-order expansions as follows:

$$\mathbb{P}(T(t) \in [t-x,\ t-x+h)) = p_1(0, t-x)h + o(h), \qquad \mathbb{P}(x \le Z < x+h) = b(x)h + o(h).$$

Multiplying leads the following:

$$p_1(x,t)\,h = (p_1(0, t-x)h + o(h))\,(b(x)h + o(h)) + o(h).$$

Using the identity

$$b(x)h = (1 - B(x)) - (1 - B(x+h)) + o(h),$$

we rewrite the above as

$$p_1(x,t)\,h = \big(p_1(0,t-x)h + o(h)\big)\,(1 - B(x)) + o(h).$$

Dividing by h and letting $h \downarrow 0$ gives the following:

$$p_1(x,t) = p_1(0,t-x)\,[1 - B(x)],$$

which proves (3.2). □

Before verifying the transport equations, we provide a short physical interpretation of the PDE systems (2.6) and (2.7). The terms $\partial_t p_i(x,t)$ describe the temporal evolution of the state–age densities, while $\partial_x p_i(x,t)$ corresponds to the deterministic aging of the sojourn-time variable in each state $i = 0, 1$. The loss terms $\lambda(x)p_0(x,t)$, $\mu(x)p_0(x,t)$, and $\eta(x)p_1(x,t)$ represent failure, job–completion, and repair–completion events, respectively. The boundary conditions (2.8) and (2.9) express the regeneration of probability mass when the process transitions between states, thereby incorporating the appropriate hazard rates. This interpretation reinforces the connection between the semi-Markov dynamics and the associated PDE framework.

Theorem 3.3. *Under Assumptions A1–A4, functions $p_0(x,t)$ and $p_1(x,t)$ defined above satisfy Eqs* (2.6) *and* (2.7).

Proof. We substitute (3.1) and (3.2) into (2.6) and (2.7) and we stay below the required differentiability with respect to t. For the proof, we explicitly provide the required differentiation.

Indeed, for

$$p_0(x,t) = p_0(0,t-x)[1 - F(x)][1 - G(x)],$$

we compute

$$\begin{aligned}
\frac{\partial}{\partial t}p_0(x,t) &= \frac{\partial p_0(0,t-x)}{\partial t}\,[1 - F(x)][1 - G(x)] \\
&= \frac{\partial p_0(0,t-x)}{\partial(t-x)}\,\frac{\partial(t-x)}{\partial t}\,[1 - F(x)][1 - G(x)] \\
&= \frac{\partial p_0(0,t-x)}{\partial(t-x)}(1)\,[1 - F(x)][1 - G(x)] \\
&= \frac{\partial p_0(0,t-x)}{\partial(t-x)}\,[1 - F(x)][1 - G(x)],
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}p_0(x,t) &= \frac{\partial p_0(0,t-x)}{\partial x}(1 - F(x))(1 - G(x)) \\
&\quad + p_0(0,t-x)(-f(x))(1 - G(x)) \\
&\quad + p_0(0,t-x)(-g(x))(1 - F(x)) \\
&= \frac{\partial p_0(0,t-x)}{\partial(t-x)}\frac{\partial(t-x)}{\partial x}\,[(1 - F(x))(1 - G(x))] \\
&\quad + p_0(0,t-x)(-f(x))(1 - G(x)) \\
&\quad + p_0(0,t-x)(-g(x))(1 - F(x))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial p_0(0,t-x)}{\partial(t-x)}(-1)(1-F(x))(1-G(x)) \\
&\quad -p_0(0,t-x)(f(x))(1-G(x)) + p_0(0,t-x)(-g(x))(1-F(x)) \\
&= -\frac{\partial p_0(0,t-x)}{\partial(t-x)}(1-F(x))(1-G(x)) - p_0(0,t-x)(f(x))(1-G(x)) \\
&\quad -p_0(0,t-x)(g(x))(1-F(x)).
\end{aligned} \tag{3.4}
$$

From (3.3) and (3.4), it is easy to obtain that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) p_0(x,t) &= -[\lambda(x) + \mu(x)]\, p_0(0,t-x)[1-F(x)][1-G(x)] \\
&= -[\lambda(x) + \mu(x)]\, p_0(x,t).
\end{aligned}
$$

Recall from Section 2 that

$$
\lambda(x) = \frac{G'(x)}{(1-G(x))} = \frac{g(x)}{(1-G(x))} \quad \text{and} \quad \mu(x) = \frac{f(x)}{(1-F(x))}.
$$

Therefore, we proved that (3.1) satisfies (2.6).

Now, let us verify that (3.2) satisfies (2.7). For

$$
p_1(x,t) = p_1(0,t-x)[1-B(x)],
$$

we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\{p_1(x,t)\} &= \frac{\partial}{\partial t}[p_1(0,t-x)\cdot[1-B(x)]] \\
&= \frac{\partial}{\partial t}[p_1(0,t-x)]\cdot\frac{\partial(t-x)}{\partial(t-x)}\cdot[1-B(x)] \\
&= \frac{\partial p_1(0,t-x)}{\partial(t-x)}\cdot[1-B(x)]
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial x}p_1(x,t) &= \frac{\partial}{\partial x}[p_1(0,t-x)\cdot[1-B(x)]] \\
&= \frac{\partial}{\partial x}[p_1(0,t-x)]\cdot[1-B(x)] + p_1(0,t-x)\cdot[-B'(x)],
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}[p_1(0,t-x)] &= \frac{\partial}{\partial(t-x)}[p_1(0,t-x)]\frac{\partial(t-x)}{\partial x} \\
&= \frac{\partial p_1(0,t-x)}{\partial(t-x)}\cdot(-1) = -\frac{\partial p_1(0,t-x)}{\partial(t-x)}.
\end{aligned} \tag{3.7}
$$

From (3.6) and (3.7), we have the following:

$$\frac{\partial}{\partial x}\{p_1(x,t)\} = -\frac{\partial p_1(0,t-x)}{\partial(t-x)} \cdot [1-B(x)] - p_1(0,t-x)\cdot B'(x).$$

From the above, it is easy to obtain that

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_1(x,t) &= -p_1(0,t-x)B'(x) = -p_1(0,t-x)\,\eta(x)\,[1-B(x)] \\ &= -\eta(x)\,p_1(0,t-x)\,[1-B(x)] = -\eta(x)\,p_1(x,t).\end{aligned}$$

Recall from Section 2 that

$$\eta(x) = \frac{B'(x)}{1-B(x)};$$

thus, (3.2) satisfies (2.7). To that end, probabilistic representations $p_0(x,t)$, $p_1(x,t)$ are shown to satisfy the transformed (2.6) and (2.7). □

Sensitivity of results to Assumptions A1–A4

Mixed distributions (Assumption A1). If F, G, or B possess discrete atoms, then the right-derivative definitions of p_0 and p_1 must be replaced by measure-valued densities. The main expressions remain valid, but the state–time densities acquire additional atomic components at jump times that correspond to mass points in the distributions.

Unbounded hazard rates (Assumption A2). Dropping the local boundedness of λ, μ, or η may cause p_0 and p_1 to lose absolute continuity in the time variable. The representation expressions still hold, but the verification of the transport equations requires weak derivatives.

Relaxing A4 (resumable job execution). If execution resumes after failure (instead of restarting), then $X(t)$ and $Y(t)$ no longer coincide. In this case, the representation for $p_0(x,t)$ becomes the following:

$$p_0(x,t) = p_0(0,t-x)\,[1-G(x-x_0)]\,[1-F(x)],$$

where x_0 is the time accumulated prior to failure. The analysis follows the same regenerative principles but requires tracking two time variables in the operational state.

Remark 3.4. *Under Assumptions A1–A4, Representations* (3.1) *and* (3.2) *can be regarded as candidate solutions obtained via probabilistic reasoning; then, the PDEs and non-local boundaries are verified by direct substitution provided by the above Theorem 3.3 plays the role of a verification rather that a derivation Theorem, alongside the results below presented in (3.8)–(3.11).*

Now, we can show that for the solution of the above BVP, it is sufficient to only verify Eqs (2.8)–(2.11) (i.e., the PDEs (2.6) and (2.7) are redundant).

Let's substitute (3.1) and (3.2) in (2.8) and (2.9) and obtain

$$p_0(0,t) = \delta(t) + \int_0^t p_1(0,t-x)[1-B(x)]\cdot\frac{b(x)}{1-B(x)}\,dx, \tag{3.8}$$

$$p_1(0,t) \;=\; \int_0^t p_0(0,t-x)[1-F(x)][1-G(x)]\cdot\frac{g(x)}{1-G(x)}\,dx, \tag{3.9}$$

so that,

$$p_0(0,t) \;=\; \delta(t)+\int_0^t p_1(0,t-x)b(x)\,dx, \tag{3.10}$$

and

$$p_1(0,t) \;=\; \int_0^t p_0(0,t-x)[1-F(x)]g(x)\,dx. \tag{3.11}$$

Lemma 3.5. *Let $\mathcal{L}s = f^*(s)$ denote the Laplace transform in t. Then,*

$$p_0^*(0,s) \;=\; s + p_1^*(0,s)\,b^*(s), \tag{3.12}$$

and

$$p_1^*(0,s) \;=\; p_0^*(0,s)\,A^*(s), \tag{3.13}$$

where $A(x) = [1-F(x)]\,g(x)$, and $A^*(s)$, $b^*(s)$ are their Laplace transforms.

Proof. Applying the Laplace transform to Eqs (3.10) and (3.11) yields the set of Eqs (3.12) and (3.13). □

Proposition 3.6. *From the system of Eqs (3.12) and (3.13), we obtain*

$$p_0^*(0,s) \;=\; \frac{s}{1-A^*(s)\,b^*(s)}, \tag{3.14}$$

and

$$p_1^*(0,s) \;=\; \frac{s\,A^*(s)}{1-A^*(s)\,b^*(s)}. \tag{3.15}$$

Remark 3.7. *Suppose (p_0^*, p_1^*) and (q_0^*, q_1^*) are two solutions of the BVP (2.6)–(2.11) under A1–A4. Taking Laplace transforms in t gives two pairs of solutions that satisfy (3.12) and (3.13). By the uniqueness theorem for Laplace transforms, the differences $\Delta_0 = p_0^* - q_0^*$ and $\Delta_1 = p_1^* - q_1^*$ must satisfy a homogeneous linear system whose only solution is $\Delta_0 = \Delta_1 = 0$. Inverting the transforms and applying Grönwall's inequality [5] to the integral form of the BVP implies that $p_0^*(x,t) = q_0^*(x,t)$ and $p_1^*(x,t) = q_1^*(x,t)$ for all (x,t), thus proving the uniqueness.*

Remark 3.8. *The use of the above Laplace transforms relies on A1 and A2 to justify the integrability and the right-derivative interpretations of $p_0(0,\cdot)$ and $p_1(0,\cdot)$.*

Now, from (3.1), (3.2), (3.14), and (3.15), it is easy to obtain the Laplace transforms for all functions introduced above with respect to the variable t.

3.1. Asymptotic and transient behavior

Before we proceed with some illustrative applications, in this subsection, we provide a discussion regarding the steady-state limits, small-t behavior, and the monotonicity of hazard rates.

Regarding the steady-state limits, observe that under positive recurrence conditions which ensure that $A^*(s)b^*(s)$ is well-defined near $s = 0$, the Final Value Theorem (cf. [7, 18]) gives

$$\lim_{t\to\infty} p_0(0,t) = \lim_{s\downarrow 0} s p_0^*(0,s) = \frac{1}{1 - A^*(0)b^*(0)},$$

and analogously,

$$\lim_{t\to\infty} p_1(0,t) = \frac{A^*(0)}{1 - A^*(0)b^*(0)}.$$

Note that the numerator "1" in the steady–state limit arises from the unit mass that enters the system at each regeneration cycle, thereby corresponding to the $\delta(t)$ term in the boundary condition $p_0(0,t)$. The Final Value Theorem is applied to the regular (non–singular) part of $p_0(0,t)$, for which the Laplace transform has an effective numerator of 1.

Expanding $p_0^*(0,s)$ and $p_1^*(0,s)$ near $s = \infty$ yields the leading-order behavior for a small t. For example,

$$p_0(0,t) = \delta(t) - (b(0) + A(0))\, t + O(t^2),$$

with similar expressions applying to $p_1(0,t)$. Observe that these expansions quantify the effect of the initial hazard rate magnitudes on an early-time evolution.

Finally, notice that if the hazard rates λ, μ, or η are monotone, then the behavior of $p_0(x,t)$ and $p_1(x,t)$ inherits this monotonicity through the corresponding survival factors $1 - F(x)$, $1 - G(x)$, and $1 - B(x)$, thus yielding a sharper qualitative control over the density evolution.

3.2. Illustrative applications for specific distributions

In order to demonstrate the applicability of the general transient representations obtained in Section 3, we present several examples that correspond to concrete choices of the distributions F, G, and B.

Example 1: Exponential lifetimes. Let the machine lifetime Y and the execution time X be exponentially distributed with rates λ and μ, respectively. Then, $1 - F(x) = e^{-\mu x}$ and $1 - G(x) = e^{-\lambda x}$, and the expression (3.1) simplifies to the following:

$$p_0(x,t) = p_0(0, t - x)\, e^{-(\lambda+\mu)x}.$$

Example 2: Weibull repair times. Suppose the repair duration Z follows a Weibull distribution with shape $k > 0$ and scale $\beta > 0$, so that $1 - B(x) = \exp(-(x/\beta)^k)$. Then, Formula (3.2) becomes

$$p_1(x,t) = p_1(0, t - x)\, \exp\Big(-(x/\beta)^k\Big),$$

thus illustrating how non-exponential repair mechanisms can be incorporated in closed form.

Example 3: Deterministic job duration. Let the job execution time satisfy $G(x) = \mathbf{1}_{\{x\geq c\}}$ for some constant $c > 0$. Then, $1 - G(x) = \mathbf{1}_{\{x<c\}}$ and (3.1) yields

$$p_0(x,t) = p_0(0, t - x)\, [1 - F(x)]\, \mathbf{1}_{\{x<c\}},$$

thus showing how the transient representation adapts to degenerate distributions.

These examples illustrate how the proposed probabilistic framework accommodates a wide range of parametric choices and leads to tractable expressions for the transient state–age densities.

3.3. Explicit closed-form solution for an exponential/deterministic model

To demonstrate the full computability of the transient solution, we present a model where all quantities can be derived in an explicit closed form. Assume the following:

- The machine lifetime Y is exponential with rate λ;
- the job execution time X is exponential with rate μ;
- the repair time Z is exponential with rate η; and
- the job duration is deterministic: $G(x) = \mathbf{1}\{x \geq c\}$.

Then, the survival functions are as follows:

$$1 - F(x) = e^{-\mu x}, \qquad 1 - G(x) = \mathbf{1}\{x < c\}, \qquad 1 - B(x) = e^{-\eta x}.$$

Substituting these into the probabilistic representations (3.1) and (3.2) yields the explicit age–state densities

$$p_0(x,t) = p_0(0, t-x)\, e^{-(\lambda+\mu)x}, \qquad 0 < x < t,$$

$$p_1(x,t) = p_1(0, t-x)\, e^{-\eta x}, \qquad 0 < x < t.$$

The boundary equations that simplify to

$$p_0(0,t) = \delta(t) + \eta \int_0^t p_1(0, t-x) e^{-\eta x}\, dx, \qquad p_1(0,t) = \lambda \int_0^t p_0(0, t-x) e^{-(\lambda+\mu)x}\, dx,$$

form a linear system of ordinary differential equations (ODE):

$$p_0'(0,t) = \eta p_1(0,t) - (\lambda+\mu) p_0(0,t), \qquad p_1'(0,t) = \lambda p_0(0,t) - \eta p_1(0,t),$$

with initial conditions $p_0(0,0) = 1$ and $p_1(0,0) = 0$.
Solving the system gives the following fully explicit transient solution:

$$p_0(0,t) = \frac{\eta}{\lambda+\eta}\Big(1 - e^{-(\lambda+\eta)t}\Big), \qquad p_1(0,t) = \frac{\lambda}{\lambda+\eta}\Big(1 - e^{-(\lambda+\eta)t}\Big).$$

Consequently,

$$p_0(x,t) = \frac{\eta}{\lambda+\eta}\Big(1 - e^{-(\lambda+\eta)(t-x)}\Big)e^{-(\lambda+\mu)x},$$

$$p_1(x,t) = \frac{\lambda}{\lambda+\eta}\Big(1 - e^{-(\lambda+\eta)(t-x)}\Big)e^{-\eta x}.$$

This example demonstrates that the proposed probabilistic framework readily yields closed-form transient representations for classical semi-Markov reliability models.

4. Conclusions and comments

In this work, we studied a class of reliability and queueing models governed by semi–Markov dynamics and investigated the supplementary variables technique that leads to non–classical BVPs with non–local boundary conditions.

The main contribution of the paper is to demonstrate that the underlying probabilistic structure for the considered class of models allows one to obtain explicit transient representations of the relevant state–age densities without relying on the direct solution of the associated PDEs.

The results in the present manuscript, together with related developments in the literature [12–14], illustrate that for a broad family of semi–Markov reliability and queueing systems, purely probabilistic arguments based on renewal and regeneration properties can provide transparent and effective analytical tools.

These representations may subsequently be verified against the transport equations and non–local boundary conditions obtained via the SVT, thus offering a verification–based alternative to the direct PDE analysis.

This approach can be applied, among others, to semi–Markov models in which supplementary variables are employed, including extensions of classical queueing systems (e.g., M/G/1 models with vacations, retrials, reneging, balking, or feedback) and repairable reliability systems with general lifetime and repair distributions.

The advantages of the proposed approach may be summarized as follows:

- It provides a probabilistic interpretation and explicit representation of the solution to the non–classical BVP associated with the SVT, without requiring the explicit solution of the corresponding PDE system.
- It highlights the regenerative structure that underlies the semi–Markov dynamics, thereby clarifying the role of renewal mechanisms in the transient behavior of the system.
- It offers a complementary analytical tool that can be combined with PDE–based or numerical methods when explicit Laplace transform inversion is difficult.

Overall, the proposed methodology provides a probabilistically transparent framework for the transient analysis of semi–Markov systems within the scope described above and may serve as a useful alternative or complement to classical analytical techniques in reliability and queueing theory.

Abbreviations

BVP Boundary Value Problem;

PDE Partial Differential Equation(s);

SVT Supplementary Variable(s) Technique.

Author contributions

Revaz Kakubava: Conceptualization, methodology, writing-review and editing; Ilia Vonta: Conceptualization, methodology, writing-review and editing; Alex Karagrigoriou: Methodology, project Administration, supervision, writing-review and editing; Andreas Makrides: Writing original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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