AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 12414–12432.
DOI: 10.3934/math.2026510
Received: 06 February 2026
Revised: 08 April 2026
Accepted: 13 April 2026
Published: 06 May 2026

Research article

Achieving third-order accuracy via recursive time filtering: a seamless extension of the filtered backward Euler method

Ahmet Güzel*

Department of Accounting and Tax Applications, Batman University, Batıraman Campus, 72100, Türkiye

* **Correspondence:** Email: ahmet.guzel@batman.edu.tr.

Abstract: This paper introduces a novel recursive time-filtering framework designed to elevate the second-order filtered backward Euler (FBE) method to third-order accuracy while preserving its inherent computational simplicity. While FBE remains a staple in computational science due to its robust stability, its second-order convergence often necessitates prohibitively small time steps for high-fidelity simulations. We resolve this by proposing a non-intrusive extension that functions as a modular post-processing step, requiring no additional implicit solves or structural modifications to existing numerical solvers. By leveraging principles from discrete differential geometry, we generalize the framework to variable time-step regimes through a rigorous definition of discrete curvature based on quadratic interpolants. The theoretical foundation involves recasting the FBE scheme into its one-leg equivalent and applying the framework of linear multistep methods (LMM). Through a derivation of the local truncation error (LTE), we identify a unique, step-dependent filtering parameter $\beta(\tau_n, \tau_{n-1})$ that ensures a consistent transition to third-order convergence across non-uniform grids. Furthermore, we prove that the filtering operation satisfies a discrete maximum principle for curvature, demonstrating that the updated curvature is a convex combination of previous values. This ensures the scheme is strictly dissipative, effectively dampening high-frequency numerical artifacts without introducing new local extrema. Stability analysis via the boundary locus method confirms that the resulting recursive scheme is $A(\alpha)$-stable, providing a vast stability region for stiff differential systems. Numerical validations on a suite of oscillatory and quasiperiodic benchmark problems demonstrate that the method recovers the theoretical order of accuracy and significantly mitigates the excessive numerical dissipation characteristic of standard FBE schemes, offering a powerful and low-effort upgrade path for legacy codes in fluid dynamics and structural mechanics.

Keywords: numerical methods; time filter; linear multistep method; stability; error analysis
Mathematics Subject Classification: 65L04, 65L05, 65L07

1. Introduction

The numerical solution of stiff ordinary differential equations (ODEs) remains a cornerstone of computational physics and engineering [1,2]. Among the various temporal discretization schemes [3], the backward Euler (BE) method is perhaps the most pervasive, owing to its unconditional A-stability and the conceptual simplicity of its first-order implicit formulation. In large-scale industrial legacy codes, the BE method is often the default solver due to its robustness. However, its primary drawback is its first-order temporal accuracy, which necessitates exceptionally fine time steps [4] to satisfy the stringent error tolerances required for high-fidelity physics. To address this limitation without abandoning the robust BE framework, recent research has pivoted toward the use of time filters [5–7] as accuracy-enhancing post-processors. Building upon these foundations, the work in [8] introduced the second-order filtered BE (FBE) method, a pivotal development that serves as the basis for the higher-order extensions discussed herein. This approach utilizes a single-step, three- point time filter applied immediately after a standard BE prediction. By selecting a filter parameter that specifically targets the first-order truncation error, they demonstrated that a simple linear combination of the current prediction and the previous solution could cancel the terms, thereby recovering second-order accuracy without requiring an additional implicit solve. This work shifted the perspective on time filters from mere stabilization tools to active accuracy-enhancement mechanisms. In this paper, we extend this paradigm by proposing a recursive four- point time filter that elevates the FBE method output to strictly third-order accuracy. A significant challenge in modern computational frameworks is the transition from uniform to adaptive time-stepping, which is essential for the efficient simulation of multi-scale phenomena. We address this by extending our filtering framework to variable stepsize regimes [9–11] through a rigorous definition of discrete curvature [12] based on quadratic Lagrange interpolants. By framing the filtering step as a curvature-smoothing operator, we establish a discrete maximum principle, which proves that the updated curvature is a convex combination of previous values. This geometric foundation ensures that the filter is strictly dissipative, effectively dampening high-frequency numerical artifacts while precluding the formation of unphysical local extrema. The primary innovation of this FBE with four-point filter (FBEF) is its modularity; the core implicit solver remains untouched, acting as a predictor, while a sequence of this filter operates as a post-processing correction. The mathematical novelty lies in the derivation of a specific four- point linear combination that eliminates both the second-order error terms of the FBE step. Through a meticulous local truncation error (LTE) analysis of the resulting variable-coefficient linear multistep method (LMM), we derive a unique, step-dependent filtering parameter $\beta(\tau_n, \tau_{n-1})$. This analytical result ensures that third-order accuracy is maintained dynamically throughout the simulation, regardless of the stepsize ratios. The resulting algorithm is non-intrusive, requiring only the storage of three previous time levels and the addition of one algebraic line of code, making it an ideal candidate for upgrading existing computational frameworks.

The remainder of this paper is organized as follows: Section 2 establishes the mathematical preliminaries by summarizing the second-order FBE scheme. Its core properties, including A-stability and one-leg characterization, provide the necessary bridge between traditional first-order solvers and our proposed third-order extension. Section 3 details the theoretical derivation of the four- point filtering framework, beginning with the constant stepsize case and extending to adaptive regimes through the formal definition of discrete curvature and a discrete maximum principle. This section further maps the filtered evolution onto a variable-coefficient LMM to analytically derive the step-dependent parameter $\beta(\tau_n, \tau_{n-1})$ required for consistent third-order accuracy. Within this framework, we establish the 0-stability and boundary locus of the resulting scheme, confirming its robustness for the integration of stiff systems. Section 4 provides extensive numerical validations across a suite of oscillatory, simple pendulum, and quasiperiodic benchmark problems to verify the theoretical convergence rates and dissipative properties. Finally, Section 5 offers concluding remarks on the implications of this framework for enhancing large-scale legacy computational codes.

2. Materials and methods

The BE method is a staple of legacy code due to its robust stability, but it is often hampered by first-order accuracy. Upgrading these systems presents a significant challenge, as it typically requires a complete and costly refactoring of the existing architecture. We present a solution that enhances the FBE method by incorporating a four- point filter term [13] . This simple extension boosts accuracy to third order, provides an immediate error estimator, and mimics the behavior of the third-order backward differentiation formula (BDF3), offering a higher-order upgrade with minimal implementation effort.

To begin, consider the initial value problem

$$u'(t) = f\left(t, u(t)\right) \text{ for } t > 0 \text{ and } u(0) = u_0.$$

Let h denote the constant time stepsize, defining the discrete time points $t_{n+1} = t_n + h$, and u_n represents the numerical approximation to the exact solution $u(t_n)$ at time t_n. The proposed scheme is constructed by first advancing the solution via the standard fully implicit BE method, followed immediately by a linear time-filtering step parameterized by ν,

$$\frac{u_{n+1} - u_n}{h} = f(t_{n+1}, u_{n+1}), \tag{BE}$$

$$u_{n+1} \Leftarrow u_{n+1} - \frac{\nu}{2}\Big(u_{n+1} - 2u_n + u_{n-1}\Big). \tag{F}$$

The filter in step (F) is the unique three- point filter that forms a consistent approximation when combined with (BE). For $\nu = 2/3$, the combination is both A-stable and second-order accurate. With a constant time step, this parameter choice results in a second-order accurate, one-leg [14–16] two-step method. This resulting scheme is strongly A-stable and given by

$$\frac{3}{2}u_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf\Big(t_{n+1}, \frac{3}{2}u_{n+1} - u_n + \frac{1}{2}u_{n-1}\Big). \tag{FBE}$$

We introduce a higher-order [17, 18] four- point linear time filter that operates recursively on the computed time steps. Through this mechanism, we demonstrate that the dominant error terms of the second-order approximation are cancelled out, yielding a strictly third-order accurate scheme without the need for additional implicit that solves

$$\frac{3}{2}v_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf\Big(t_{n+1}, \frac{3}{2}v_{n+1} - u_n + \frac{1}{2}u_{n-1}\Big), \tag{FBE}$$

$$u_{n+1} = v_{n+1} - \frac{\beta}{2}\Big(v_{n+1} - 3u_n + 3u_{n-1} - u_{n-2}\Big), \tag{F}$$

where β is the filter parameter and v and u are the unfiltered and once- filtered values, respectively. We provide a comprehensive theoretical framework for the proposed method. We begin with a systematic derivation of the algorithmic structure, followed by a rigorous asymptotic error analysis to verify the third-order accuracy, and conclude with a detailed examination of the scheme's linear stability properties in the following subsection.

3. Results

3.1. Derivation of method for constant stepsize

We use a higher-order time filter to improve accuracy of FBE by adding a four- point filter. The properties of FBE are given in [8]. Therefore, the combination of a four- point linear time filter [13] with the FBE method improves accuracy to the third-order method:

$$\frac{3}{2}v_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf\Big(t_{n+1}, \frac{3}{2}v_{n+1} - u_n + \frac{1}{2}u_{n-1}\Big), \tag{FBE}$$

$$u_{n+1} = v_{n+1} + av_{n+1} + bu_n + cu_{n-1} + du_{n-2}, \tag{F}$$

where $a, b, c,$ and d are constants and v and u represent the unfiltered and once-filtered values, respectively. To analyze this system, we apply it to Dahlquist test equation [19]

$$u'(t) = \lambda u \text{ for } t > 0 \text{ and } u(0) = 1. \tag{3.1}$$

Then we transform it into an LMM. Eliminate v_{n+1} in (FBE), and we obtain

$$v_{n+1} = \frac{1}{1-h\lambda}u_n - \frac{1}{3}\Big(\frac{1}{1-h\lambda}u_n - 2u_n + u_{n-1}\Big).$$

Then substitute it in (F) step, and

$$u_{n+1} = (1+a)\left(\frac{1}{1-h\lambda}u_n - \frac{1}{3}\left(\frac{1}{1-h\lambda}u_n - 2u_n + u_{n-1}\right)\right) + bu_n + cu_{n-1} + du_{n-2}.$$

Rearrange the terms, and we get the following LMM:

$$\begin{aligned}&\frac{3}{1+a}u_{n+1} - \frac{3b}{1+a}u_n - \frac{3c}{1+a}u_{n-1} - \frac{3d}{1+a}u_{n-2} - 4u_n + u_{n-1}\\ &= h\lambda\Big(\frac{3}{1+a}u_{n+1} - \frac{3b}{1+a}u_n - \frac{3c}{1+a}u_{n-1} - \frac{3d}{1+a}u_{n-2} - 2u_n + u_{n-1}\Big).\end{aligned}$$

Multiply all by $(1+a)$ and rewrite it, and we get

$$\begin{aligned}&3u_{n+1} - 3bu_n - 3cu_{n-1} - 3du_{n-2} - 4(1+a)u_n + (1+a)u_{n-1}\\ &= h\lambda\Big(3u_{n+1} - 3bu_n - 3cu_{n-1} - 3du_{n-2} - 2(1+a)u_n + (1+a)u_{n-1}\Big).\end{aligned} \tag{3.2}$$

The LTE of (3.2) is

$$
\begin{aligned}
h\tau_n = &\Big(3 - 3b - 3c - 3d - 4 - 4a + 1 + a\Big)u_n \\
&+ \Big(3 + 3c + 6d - 1 - a - 3 + 3b + 3c + 3d + 2 + 2a - 1 - a\Big)hu_n' \\
&+ \Big(3 - 3c - 12d + 1 + a - 6 - 6c - 12d + 2 + 2a\Big)\frac{h^2}{2}u_n'' \\
&+ \Big(3 + 3c + 24d - 1 - a - 9 + 9c + 36d - 3 - 3a\Big)\frac{h^3}{6}u_n''' + O(h^4).
\end{aligned}
$$

Thus, the method achieves second-order accuracy if and only if

$$
\begin{cases}
3 - 3b - 3c - 3d - 4 - 4a + 1 + a = 0 \implies a + b + c + d = 0, \\
3c + 6d + 3b + 3c + 3d = 0 \implies b + 2c + 3d = 0, \\
-3c - 12d + a - 6c - 12d + 2a = 0 \implies a - 3c - 8d = 0 \implies a = 3c + 8d.
\end{cases}
$$

Solving these equations yields

$$
c = -3d, \quad b = 3d, \quad \text{and} \quad a = -d.
$$

Thus, the FBE scheme with a four- point filter, using the parameter $d = \frac{\beta}{2}$, is given by

$$
\frac{3}{2}v_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf\Big(t_{n+1}, \frac{3}{2}v_{n+1} - u_n + \frac{1}{2}u_{n-1}\Big), \tag{FBE}
$$

$$
u_{n+1} = v_{n+1} - \frac{\beta}{2}\Big(v_{n+1} - 3u_n + 3u_{n-1} - u_{n-2}\Big), \tag{F}
$$

which is generally second-order accurate for any β given as

$$
u_{n+1} - \frac{8+5\beta}{6}u_n + \frac{1+4\beta}{3}u_{n-1} - \frac{\beta}{2}u_{n-2} = h\lambda\Big(u_{n+1} - \frac{4+7\beta}{6}u_n + \frac{1+4\beta}{3}u_{n-1} - \frac{\beta}{2}u_{n-2}\Big) \tag{3.3}
$$

and third-order accurate when $\beta = \frac{5}{7}$. The equivalent third-order accurate one-leg multistep method is

$$
u_{n+1} - \frac{27}{14}u_n + \frac{18}{14}u_{n-1} - \frac{5}{14}u_{n-2} = h\lambda\Big(u_{n+1} - \frac{21}{14}u_n + \frac{18}{14}u_{n-1} - \frac{5}{14}u_{n-2}\Big). \tag{3.4}
$$

3.2. Local truncation error (LTE) analysis

We analyze the LTE by invoking the localization assumption [20, 21], i.e., $u(t_{n+1}) = u_{n+1}, u(t_n) = u_n, \ldots, u(t_1) = u_1$. By expanding the terms around t_n using a Taylor series, the error is defined as

$$
\begin{aligned}
LTE = &u(t_{n+1}) - \frac{8+5\beta}{6}u(t_n) + \frac{1+4\beta}{3}u(t_{n-1}) - \frac{\beta}{2}u(t_{n-2}) \\
&- h\left(u'(t_{n+1}) - \frac{4+7\beta}{6}u'(t_n) + \frac{1+4\beta}{3}u'(t_{n-1}) - \frac{\beta}{2}u'(t_{n-2})\right).
\end{aligned} \tag{3.5}
$$

Substituting the Taylor expansions for $u(t_{n+1})$, $u(t_{n-1})$, $u(t_{n-2})$, $u'(t_{n+1})$, $u'(t_{n-1})$, and $u'(t_{n-2})$, multiplying by the appropriate coefficient from Eq (3.5), and grouping and simplifying by powers of h term gives

$$
LTE = \frac{7\beta - 5}{9}h^3u'''(t_n) - \frac{1+13\beta}{18}h^4u^{iv}(t_n) + O(h^5).
$$

Based on the LTE analysis, the method is generally second-order accurate for an arbitrary parameter β but achieves third-order accuracy specifically when $\beta = 5/7$. For a third-order accurate method with $\beta = \frac{5}{7}$, the error constant is $-4/7$. In comparison, the error constant for the BDF3 method is $-3/22$ [1]. These values are consistent with the numerical experimental results presented in Tables 2 and 4.

3.3. Derivation of variable stepsize scheme

Extending time filters to variable time step requires a rigorous definition of discrete curvature. The translation of differential geometry to the discrete setting is an active area of research, yielding substantial literature on discrete curvature. For a three- point stencil, the most natural definitions are either algebraic, utilizing the discrete second difference, or geometric, taking the inverse radius of the unique interpolating circle. To calculate the discrete curvature, we define the unique quadratic interpolant $\varphi(t)$ over the nodes t_{n-1}, t_n, and t_{n+1}. Expressed via the standard Lagrange basis functions $\ell_i(t)$, the polynomial interpolating the points (t_i, y_i) for $i \in \{n-1, n, n+1\}$ takes the form

$$\varphi(t) = \sum_{i=n-1}^{n+1} y_i \ell_i(t).$$

Let h_n denote the size of the n-th time step such that the discrete time levels are given by $t_{n+1} = t_n + h_n$. We define the ratio of consecutive time steps as $\tau_n = h_n/h_{n-1}$, then

$$t_{n+1} - t_n = h_n, \quad t_n - t_{n-1} = h_{n-1}, \quad t_{n-1} - t_{n-2} = h_{n-2}, \quad \tau_{n-1} = \frac{h_{n-1}}{h_{n-2}}.$$

Definition 1. *The discrete curvature κ_{n-1} and κ_n at the points (t_{n-2}, y_{n-2}), (t_{n-1}, y_{n-1}), and (t_n, y_n) and (t_{n-1}, y_{n-1}), (t_n, y_n), and (t_{n+1}, y_{n+1}) are defined as the second derivative of the interpolating polynomial $\varphi(t)$ scaled by the stepsizes $h_{n-1}h_{n-2}$ and $h_n h_{n-1}$, which yields*

$$\kappa_{n-1} = h_{n-1}h_{n-2}\varphi''(t_n) = \frac{2h_{n-2}}{h_{n-1} + h_{n-2}}u_n - 2u_{n-1} + \frac{2h_{n-1}}{h_{n-1} + h_{n-2}}u_{n-2}.$$

Expressing this in terms of the stepsize ratio $\tau_{n-1} = h_{n-1}/h_{n-2}$ gives

$$\kappa_{n-1} = \frac{2}{\tau_{n-1} + 1}u_n - 2u_{n-1} + \frac{2\tau_{n-1}}{\tau_{n-1} + 1}u_{n-2},$$

and shifting $n \to n+1$ yields

$$\kappa_n = \frac{2}{\tau_n + 1}u_{n+1} - 2u_n + \frac{2\tau_n}{\tau_n + 1}u_{n-1}.$$

By extending the constant time-step algorithm (FBE) to accommodate a variable stepsize h_n and rearranging Eq (F), we obtain the generalized two-step method

$$\frac{3}{2}v_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = h_n f\Big(t_{n+1}, \frac{3}{2}v_{n+1} - u_n + \frac{1}{2}u_{n-1}\Big), \tag{vFBE}$$

$$u_{n+1} = v_{n+1} - \frac{\beta}{2}(v_{n+1} - 2u_n + u_{n-1}) + \frac{\beta}{2}(u_n - 2u_{n-1} + u_{n-2}). \tag{F}$$

Extension of the (F) step to variable time step (vF) is

$$u_{n+1} = v_{n+1} - \frac{\beta}{2}\Big(\frac{2}{\tau_n + 1}v_{n+1} - 2u_n + \frac{2\tau_n}{\tau_n + 1}u_{n-1}\Big) + \frac{\beta}{2}\Big(\frac{2}{\tau_{n-1} + 1}u_n - 2u_{n-1} + \frac{2\tau_{n-1}}{\tau_{n-1} + 1}u_{n-2}\Big). \tag{vF}$$

Proposition 2. *Let the numerical solution be updated using the filtering scheme defined in* (vF)*. If* κ_n^{old} *and* κ_n^{new} *denote the discrete curvature of the numerical solution before and after the filtering step, respectively, then the updated curvature satisfies the relation*

$$\kappa_n^{new} = \left(1 - \frac{\beta}{\tau_n + 1}\right)\kappa_n^{old} + \frac{\beta}{\tau_n + 1}\kappa_{n-1}^{old}. \tag{3.6}$$

Proof. Multiplying the (vF) step by $\frac{2}{\tau_n+1}$ and subsequently adding the terms $-2u_n + \frac{2\tau_n}{\tau_n+1}u_{n-1}$ to both sides leads to

$$\kappa_n^{new} = \left(1 - \frac{\beta}{\tau_n + 1}\right)\kappa_n^{old} + \frac{\beta}{\tau_n + 1}\kappa_{n-1}^{old}. \tag{3.7}$$

□

Remark 3. *We emphasize that, assuming* $0 \le \frac{\beta}{\tau_n+1} \le 1$*, the updated discrete curvature,* κ_n^{new}*, is formulated as a convex combination of* κ_n^{old} *and* κ_{n-1}^{old}*. Consequently, the filtered curvature is restricted to the closed interval spanned by these unfiltered values, yielding the rigorous boundedness condition:*

$$\min(\kappa_n^{old}, \kappa_{n-1}^{old}) \le \kappa_n^{new} \le \max(\kappa_n^{old}, \kappa_{n-1}^{old}). \tag{3.8}$$

This discrete maximum principle guarantees that the filtering operation is strictly dissipative with respect to curvature. By precluding the formation of new local extrema, the operator effectively dampens high-frequency numerical artifacts without amplifying any oscillations beyond the extremes established by the predictor.

Analysis of the variable time-step system is performed by applying the Dahlquist test equation (3.1) to express the scheme as an LMM method. Eliminating the intermediate stage variable v_{n+1} from (vFBE) and rearranging terms yields

$$v_{n+1} = \frac{4 - 2h_n\lambda}{3(1 - h_n\lambda)}u_n - \frac{1}{3}u_{n-1}.$$

Substituting the expression for v_{n+1} into (vF) results in

$$u_{n+1} = \frac{\tau_n + 1 - \beta}{\tau_n + 1}\left[\frac{4 - 2h_n\lambda}{3(1 - h_n\lambda)}u_n - \frac{1}{3}u_{n-1}\right] + \beta u_n - \frac{\beta\tau_n}{\tau_n + 1}u_{n-1} + \frac{\beta}{\tau_{n-1} + 1}u_n - \beta u_{n-1} + \frac{\beta\tau_{n-1}}{\tau_{n-1} + 1}u_{n-2}.$$

Collecting terms of u_n and grouping the dependence on $h_n\lambda$, the system can be characterized as an LMM of the form

$$u_{n+1}-\left[\frac{4(\tau_n+1-\beta)}{3(\tau_n+1)}+\frac{\beta(\tau_{n-1}+2)}{\tau_{n-1}+1}\right]u_n+\left[\frac{\tau_n+1+2\beta(3\tau_n+1)}{3(\tau_n+1)}\right]u_{n-1}-\left[\frac{\beta\tau_{n-1}}{\tau_{n-1}+1}\right]u_{n-2}$$
$$=h_n\cdot\lambda\left(u_{n+1}-\left[\frac{2(\tau_n+1-\beta)}{3(\tau_n+1)}+\frac{\beta(\tau_{n-1}+2)}{\tau_{n-1}+1}\right]u_n+\left[\frac{\tau_n+1+2\beta(3\tau_n+1)}{3(\tau_n+1)}\right]u_{n-1}-\left[\frac{\beta\tau_{n-1}}{\tau_{n-1}+1}\right]u_{n-2}\right).$$

By grouping terms corresponding to the solution values u_{n+1-j} and isolating the dependence on the discrete differential operator $h_n\lambda$, the preceding recurrence relation can be characterized as a variable-coefficient LMM. Specifically, the scheme is mapped onto the canonical form

$$\sum_{j=0}^{3}\alpha_j u_{n+1-j}=h_n\lambda\sum_{j=0}^{3}\beta_j u_{n+1-j}, \tag{3.9}$$

where the coefficients $\{\alpha_j,\beta_j\}$ are functions of the stepsize ratios $\tau_n=h_n/h_{n-1}$ and $\tau_{n-1}=h_{n-1}/h_{n-2}$, as well as the filter parameter β. Direct comparison between the expanded system and (3.9) allows us to identify the difference operator coefficients (α_j) and the derivative operator coefficients (β_j) as follows:

$$\alpha_0=1,\quad \alpha_1=-\left[\frac{4(\tau_n+1-\beta)}{3(\tau_n+1)}+\frac{\beta(\tau_{n-1}+2)}{\tau_{n-1}+1}\right],\quad \alpha_2=\frac{\tau_n+1+2\beta(3\tau_n+1)}{3(\tau_n+1)},\quad \alpha_3=-\frac{\beta\tau_{n-1}}{\tau_{n-1}+1},$$
$$\beta_0=1,\quad \beta_1=-\left[\frac{2(\tau_n+1-\beta)}{3(\tau_n+1)}+\frac{\beta(\tau_{n-1}+2)}{\tau_{n-1}+1}\right],\quad \beta_2=\frac{\tau_n+1+2\beta(3\tau_n+1)}{3(\tau_n+1)},\quad \beta_3=-\frac{\beta\tau_{n-1}}{\tau_{n-1}+1}.$$

Proposition 4. *For a given free parameter β, the LTE of the variable stepsize ($vLTE$) method* (3.9) *is*

$$vLTE=C_3h_n^3u'''(t_n)+C_4h_n^4u^{iv}+O(h_n^5),$$

where the third- and fourth-order error coefficients are defined respectively as

$$C_3=-\frac{1}{3}-\frac{(1+3\tau_n)[\tau_n+1+2\beta(3\tau_n+1)]}{18\tau_n^3(\tau_n+1)}+\frac{\beta(\tau_{n-1}+1)(1+\tau_{n-1}+3\tau_n\tau_{n-1})}{6\tau_n^3\tau_{n-1}^2},$$
$$C_4=-\frac{1}{8}+\frac{(1+4\tau_n)[\tau_n+1+2\beta(3\tau_n+1)]}{72\tau_n^4(\tau_n+1)}-\frac{\beta(\tau_{n-1}+1)^2(1+\tau_{n-1}+4\tau_n\tau_{n-1})}{24\tau_n^4\tau_{n-1}^3}.$$

Furthermore, the method is uniquely third-order accurate ($C_3=0$) if and only if the parameter is chosen at each step as

$$\beta=\frac{\tau_{n-1}^2(\tau_n+1)(6\tau_n^3+3\tau_n+1)}{3(\tau_n+1)(\tau_{n-1}+1)(1+\tau_{n-1}+3\tau_n\tau_{n-1})-2\tau_{n-1}^2(3\tau_n+1)^2}.$$

Proof. The proof follows directly from standard Taylor series expansions. □

Remark 5. *Restricting the variable stepsize to be constant by taking $\tau_n=\tau_{n-1}=1$ yields the standard LTE of the constant stepsize scheme.*

3.4. Stability analysis

We first need to check 0-stability of the four- point filter method. To do this, we must show that all roots of the first characteristic polynomial of (3.3) are simple* or strictly less than 1, i.e., the roots of

$$r^3 - \frac{8+5\beta}{6}r^2 + \frac{1+4\beta}{3}r - \frac{\beta}{2} = 0$$

are given by

$$\begin{aligned} r_1 &= 1, \\ r_{2,3} &= \frac{5\beta+2}{12} \mp \frac{\sqrt{(25\beta-2)(\beta-2)}}{12}, \end{aligned} \tag{3.10}$$

where r represents the shift operator ($u_{n+k} \to r^k$). Since $r_1 = 1$, we must find the interval for β, where the absolute value of r_2 and r_3 are strictly less than 1 or at most one of r_2 or r_3 is equal to -1.

Proposition 6. *The linear multistep FBEF method* (3.3) *is zero-stable for* $-1 \leq \beta < 2$.

Proof. We must show that the absolute value of r_2 and r_3 are found in (3.10) strictly less than 1. Since $|r_1| = 1$, we must find the values of β that satisfy the following system of inequalities:

$$|r_{2,3}| = \left| \frac{5\beta+2}{12} \mp \frac{\sqrt{(25\beta-2)(\beta-2)}}{12} \right| < 1. \tag{3.11}$$

Let the term inside the square root be defined as the radicand R:

$$R = (25\beta - 2)(\beta - 2).$$

The roots of $R = 0$ are $\beta = \frac{2}{25} = 0.08$ and $\beta = 2$. The sign of R determines whether the expression is real or complex. We analyze the solution in two cases.

Case 1: complex case ($R < 0$)

The radicand is negative when $\frac{2}{25} < \beta < 2$. In this interval, $\sqrt{R} = i\sqrt{-R}$ and the expressions become complex conjugates:

$$Z = \frac{5\beta+2}{12} \pm i\frac{\sqrt{-R}}{12}.$$

The condition $|Z| < 1$ is equivalent to $|Z|^2 < 1$:

$$\left(\frac{5\beta+2}{12}\right)^2 + \left(\frac{\sqrt{-R}}{12}\right)^2 < 1.$$

Multiply by 144:

$$(5\beta+2)^2 - R < 144.$$

Substitute $-R = -(25\beta^2 - 52\beta + 4) = -25\beta^2 + 52\beta - 4$:

$$(25\beta^2 + 20\beta + 4) + (-25\beta^2 + 52\beta - 4) < 144 \implies 72\beta < 144 \implies \beta < 2.$$

*Roots lying on the boundary of the unit circle are not repeated.

Case 2: real case ($R > 0$)

The R is non-negative when $\beta \leq \frac{2}{25}$ or $\beta \geq 2$. In this interval, $\sqrt{R}$ is a real number. For $\beta \geq 2$, consider the second inequality (3.11). Thus, both $\frac{5\beta+2}{12}$ and the square root term are non-negative. We check the value at the boundary $\beta = 2$:

$$r_3 = \frac{5(2)+2}{12} + \frac{\sqrt{0}}{12} = \frac{12}{12} = 1.$$

Since the inequality is strictly less than 1, $\beta = 2$ is not a solution. For any $\beta > 2$, the expression increases, so it remains ≥ 1. Thus, no solution exists for $\beta \geq 2$.

For $\beta \leq \frac{2}{25}$, in this interval, we check the lower bound condition for the smaller expression (3.11):

$$\frac{5\beta + 2 - \sqrt{R}}{12} \geq -1 \implies 5\beta + 2 - \sqrt{R} \geq -12 \implies 5\beta + 14 \geq \sqrt{(25\beta - 2)(\beta - 2)}.$$

Squaring both sides yields

$$(5\beta + 14)^2 \geq (25\beta - 2)(\beta - 2) \implies \beta \geq -1.$$

We combine the valid intervals from both cases: real case $[-1, 2/25]$ and complex case $[2/25, 2)$. The complete solution set is the union of these intervals:

$$-1 \leq \beta < 2.$$

□

We now consider the third-order case. By setting the parameter $\beta = 5/7$, the first characteristic polynomial associated with the multistep method (3.4) can be expressed as

$$r^3 - \frac{27}{14}r^2 + \frac{18}{14}r - \frac{5}{14} = 0.$$

This characteristic polynomial yields three roots

$$r_1 = 1$$

and the complex conjugate pair

$$r_{2,3} = 0.4643 \pm 0.3763i.$$

Calculating their magnitudes gives $|r_1| = 1$ and $|r_2| = |r_3| = 0.5976$. Since all roots lie within the unit circle ($|r_i| \leq 1$) and the root on the unit circle is simple, the third-order method is confirmed to be zero-stable.

To evaluate the stability [22–24] of the third-order method, we employ the boundary locus curve method. An LMM is considered A-stable if its region of absolute stability entirely contains the left half of the complex plane [20]. The stability region is defined as the set of all $z = h\lambda$ for which the roots r_i of the stability polynomial equation satisfy the root condition

$$\rho(r) - z\sigma(r) = 0 \implies |r_i| \leq 1,$$

where any roots with $|r_i| = 1$ must be simple. The boundary of this region is mapped by substituting $r = e^{i\theta}$ for $\theta \in [0, 2\pi]$ into the characteristic equation

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}.$$

Applying the method by substituting the specific characteristic polynomials derived from (3.3), we obtain the following expression for the boundary locus curve:

$$z(\theta) = \frac{e^{3i\theta} - \left(\frac{8+5\beta}{6}\right) e^{2i\theta} + \left(\frac{1+4\beta}{3}\right) e^{i\theta} - \frac{\beta}{2}}{e^{3i\theta} - \left(\frac{4+7\beta}{6}\right) e^{2i\theta} + \left(\frac{1+4\beta}{3}\right) e^{i\theta} - \frac{\beta}{2}}.$$

For the specific case when $\beta = 5/7$, simplified polynomials yield

$$z(\theta) = \frac{14e^{3i\theta} - 27e^{2i\theta} + 18e^{i\theta} - 5}{14e^{3i\theta} - 21e^{2i\theta} + 18e^{i\theta} - 5}.$$

Following the methodology in [20], we streamline the stability analysis by plotting the boundary locus curve in Figure 1 and identifying the stable region via the test point technique in Table 1.

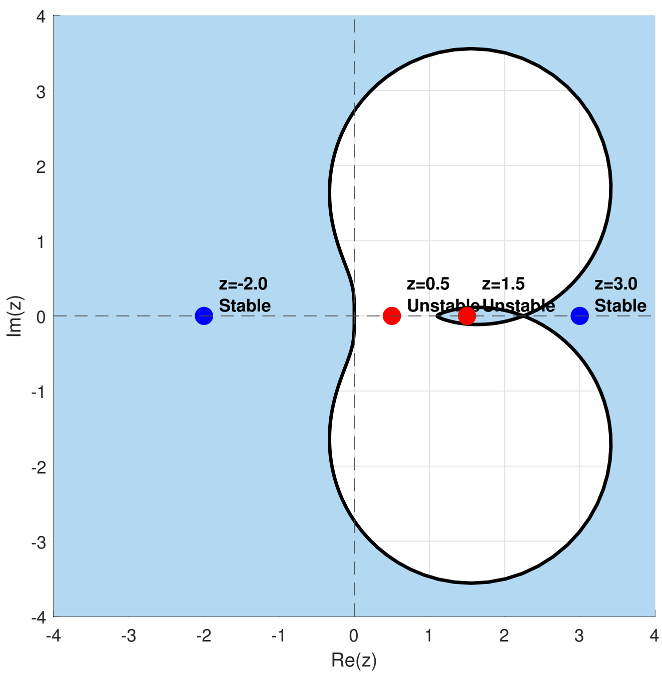

Figure 1. Stability region of the third-order FBEF. Blue points indicate stable region ($|z| \leq 1$), while red points indicate the unstable region.

Table 1. Stability polynomial analysis for various test points.

$h\lambda$	Stability polynomial	Roots	Stability
-2	$r^3 - \frac{23}{14}r^2 + \frac{18}{14}r - \frac{5}{14} = 0$	$0.5, 0.5714 \pm 0.6227i$	Stable
0.5	$\frac{1}{2}r^3 - \frac{33}{28}r^2 + \frac{9}{14}r - \frac{5}{28} = 0$	$1.7346, 0.3113 \pm 0.3302i$	Unstable
1.5	$\frac{1}{2}r^3 - \frac{9}{28}r^2 + \frac{9}{14}r - \frac{5}{28} = 0$	$0.3020, 0.1705 \pm 1.0741i$	Unstable
3	$2r^3 - \frac{18}{7}r^2 + \frac{18}{7}r - \frac{5}{7} = 0$	$0.3791, 0.4533 \pm 0.8582i$	Stable

The computed roots are evaluated against the strict root condition. This analysis reveals that all inner subregions violate the stability criteria; consequently, the valid stability region is identified as the outermost unbounded domain containing the test point $h\lambda = -2$ and 3. Figure 2 illustrates the contrasting stability domains for the proposed third-order FBEF method and the standard BDF3. The plotted curves delineate the stability boundaries in the complex plane. For both methods, the regions of absolute stability correspond to the exterior of their respective boundary curves. Notably, both methods exhibit large, unbounded stability domains that extend extensively into the left half-plane, confirming their suitability for integrating stiff systems of ODEs.

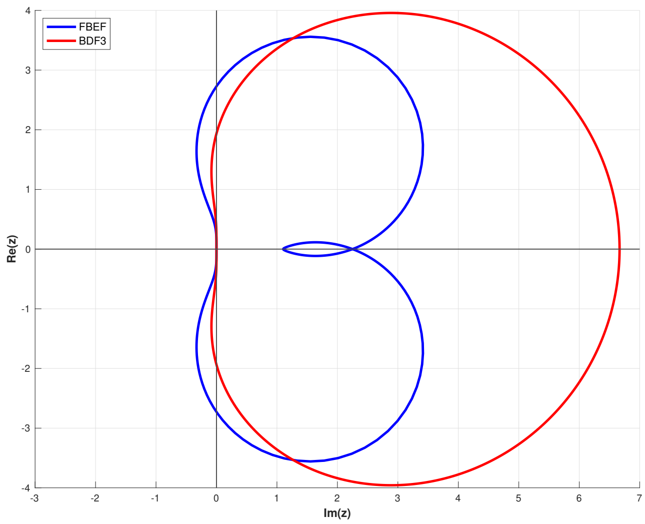

Figure 2. The absolute stability domains for the proposed third-order FBEF and the standard BDF3 method. The exterior of each bounded curve represents the region of absolute stability.

4. Numerical tests

In this section, we present three numerical experiments designed to validate the theoretical analysis of the FBEF method. These tests are conducted using a constant time step to explicitly demonstrate the predicted convergence behavior.

4.1. Oscillation system

Consider the following second-order ODE:

$$u'' + u = 0 \tag{4.1}$$

with initial conditions $u(0) = 1$ and $u'(0) = 0$, where one of the exact solutions is $u(t) = \cos(t)$.The startup procedure for the fourth-order recursive filter was facilitated by employing exact solution values for the first three time levels. This initialization strategy ensures that the starting values satisfy the necessary consistency requirements to support the theoretical third-order global accuracy of the filtered scheme. We run the constant step methods FBEF, a third order Implicit Euler with the same pre-filter, and then a post-filter(IE-Pre–Post-3) [25], and BDF3 with stepsizes $h = 0.1, 0.05, 0.025, 0.0125, 0.00625, 0.003125$. The results are presented in Table 2 and corresponding convergence rates given in Figure 3.

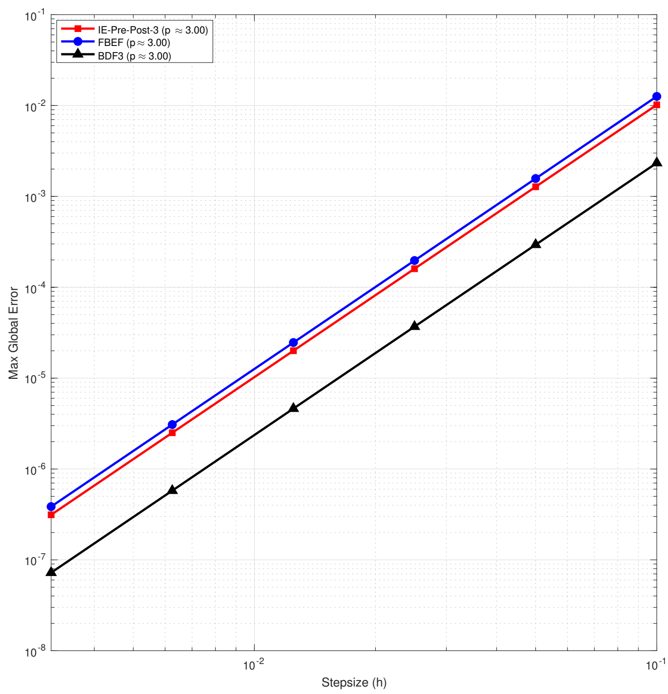

Figure 3. Convergence rates of FBEF, BDF3, and IE Pre-Post third-order methods.

Table 2. Comparison of maximum global errors for different stepsizes (h).

Step (h)	FBEF error	IE-Pre-Post-3 error	BDF3 error
0.100000	$1.2604e-02$	$1.0181e-02$	$2.3339e-03$
0.050000	$1.5732e-03$	$1.2765e-03$	$2.9409e-04$
0.025000	$1.9694e-04$	$1.5996e-04$	$3.6891e-05$
0.012500	$2.4647e-05$	$2.0024e-05$	$4.6196e-06$
0.006250	$3.0829e-06$	$2.5048e-06$	$5.7795e-07$
0.003125	$3.8550e-07$	$3.1322e-07$	$7.2275e-08$

4.2. Quasiperiodic

Consider the following second-order ODE:

$$u'''' + (\pi^2 + 1)u'' + \pi^2 u = 0 \tag{4.2}$$

with the initial conditions

$$u(0) = 2, \quad u'(0) = 0, \quad u''(0) = -(1 + \pi^2), \quad u'''(0) = 0. \tag{4.3}$$

The corresponding exact solution is $u(t) = \cos(t) + \cos(\pi t)$. The method was initialized using exact solution values for the first three time levels. The incommensurability of the component periods renders the solution quasiperiodic [25]; its time evolution is presented in Figure 4 and Table 3.

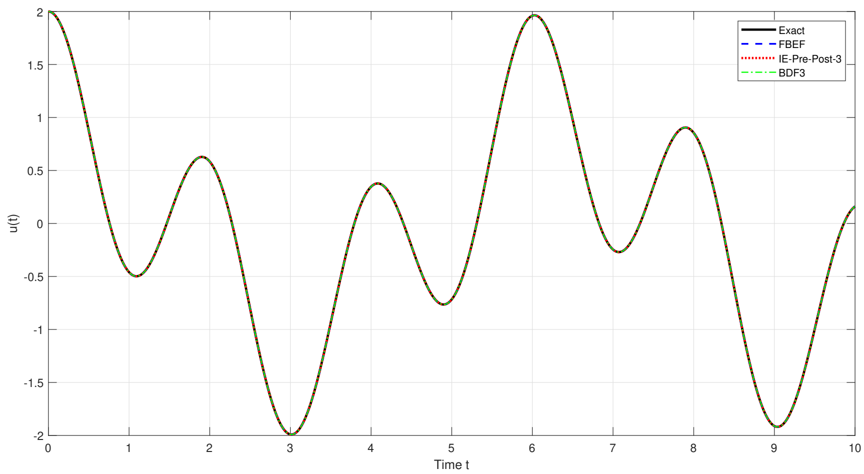

Figure 4. Comparison of FBEF, BDF3, and IE Pre-Post third-order method and exact solution.

Table 3. Comparison of maximum global errors for FBEF, IE-Pre-Post-3, and BDF3.

Step (h)	**FBEF error**	**IE-Pre-Post-3 error**	**BDF3 error**
0.10000	1.8251	1.3090	$2.3245e-01$
0.05000	$1.6669e-01$	$1.3217e-01$	$2.9543e-02$
0.02500	$2.0097e-02$	$1.6248e-02$	$3.7352e-03$
0.01250	$2.5073e-03$	$2.0351e-03$	$4.6952e-04$
0.00625	$3.1382e-04$	$2.5492e-04$	$5.8830e-05$

The numerical results presented in Tables 2 and 3 validate the theoretical derivation of the FBEF method, specifically confirming that the selection of $\beta = 5/7$ yields a global convergence rate of $O(h^3)$. In the oscillation test case ($u'' + u = 0$), the FBEF method successfully suppresses the excessive numerical dissipation inherent in the FBE scheme while avoiding the phase-lag errors typically associated with lower-order filters. As the time step h is halved, the error reduces by a factor of approximately 8, demonstrating the expected third-order behavior. Furthermore, compared to the BDF3 method, the FBEF scheme maintains a comparable error profile but offers a distinct implementation advantage: It avoids the multistep initialization complexity within the implicit solver itself, instead shifting the burden to a simple post-processing recursive filter. This confirms that the method is not only theoretically stable but also practically robust for oscillatory systems where long-term phase accuracy is paramount.

4.3. Simple pendulum

Consider the dynamics of a simple pendulum, governed by the following coupled nonlinear ODEs (see [18, 26]):

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{v}{L}, \\ \frac{dv}{dt} &= -g\sin\theta, \end{aligned} \tag{4.4}$$

where θ is the angular displacement, v is the velocity along the arc, L is the length of the pendulum, and g is the acceleration due to gravity. We set the physical parameters to $g = 9.8$ and $L = 49$ and initialize the system close to its unstable equilibrium point with the initial conditions $(\theta(0), v(0)) = (0.9\pi, 0)$. The system is integrated over the time interval [0, 100] using a constant time step. We compare the performance of three third-order numerical schemes: the FBEF, the IE-Pre-Post-3 method, and the standard BDF3 with time step $\Delta t = 0.2, 0.1, 0.05, 0, 025, 0.0125$.

To evaluate the accuracy of these methods, a high-fidelity reference solution is computed using MATLAB's adaptive ode15s solver, utilizing strict error tolerances (relative tolerance of 1e-10 and absolute tolerance of 1e-15). To isolate the global truncation error of the methods and prevent startup pollution, this reference solution is also used to generate the necessary historical values for the second and third steps required to initialize the multistep FBEF, IE-Pre-Post-3, and BDF3 schemes.

Table 4 presents the global truncation errors and empirical rates of convergence for the FBEF, IE-Pre-Post-3, and BDF3 methods when applied to the nonlinear simple pendulum system. As the time stepsize h is systematically halved from 0.20000 to 0.01250, the maximum absolute error for all three methods decreases by a factor of approximately eight at each refinement level. This consistent error reduction confirms that the FBEF scheme strictly maintains its theoretical third-order accuracy, yielding an estimated rate of $p = 3.00$. Furthermore, it is evident that the FBEF method exhibits competitive accuracy and identical asymptotic behavior $O(h^3)$ when compared against established third-order schemes like IE-Pre-Post-3 ($p = 3.01$) and BDF3 ($p = 3.01$), successfully navigating the nonlinearities of the pendulum dynamics without order reduction. To evaluate and compare the performance of the proposed FBEF scheme against the standard BDF3 method, we implement an adaptive stepsize algorithm utilizing a halving and doubling strategy. Within this framework, local error estimation is achieved by using the second-order FBE and second-order backward differentiation formula(BDF2) methods as predictors for their respective third-order correctors. To compute the necessary historical data points and initialize these multistep schemes, a third-order Runge-Kutta (RK3) method is employed for the starting steps. For the adaptive control mechanism, the local error tolerance is set to $1e - 8$. The integration begins with an initial stepsize of $h = 0.2$, which is dynamically bounded between a minimum allowable step of $h_{\min} = 1e - 5$ and a maximum limit of $h_{\max} = 0.4$ to maintain both computational stability and efficiency. The computational results of this adaptive comparative analysis are summarized in the following Table 5.

Table 4. Convergence analysis for the simple pendulum.

Step (h)	**FBEF error**	**IE-Pre-Post-3 error**	**BDF3 error**
0.20000	5.3849e+00	4.5750e+00	1.0652e+00
0.10000	7.1508e-01	5.7540e-01	1.3004e-01
0.05000	8.6769e-02	7.0362e-02	1.6072e-02
0.02500	1.0745e-02	8.7284e-03	1.9621e-03
0.01250	1.3602e-03	1.1018e-03	2.5985e-04
Estimated order (p): FBEF = 3.00, IE = 3.01, BDF3 = 3.01			

Table 5. Adaptive halving and doubling stepsize comparison for the simple pendulum.

Method	**Accepted steps**	**Rejected steps**	**Doubled steps**	**Rejection rate**
FBEF	18929	41	36	0.22%
BDF3	14249	42	37	0.29%

5. Conclusions

In this paper, we present and analyze a novel, non-intrusive methodology to elevate the FBE time discretization from second-order to third-order accuracy. By incorporating a recursive four-point time filter, we achieve third-order convergence without the necessity of multistage implicit solves or a fundamental architectural refactoring of the underlying solver. The resulting FBEF scheme demonstrates that higher-order temporal accuracy can be achieved with minimal computational overhead—requiring only the addition of algebraic lines of code and the storage of three previous time levels. Our theoretical derivation, which frames the filtered system as an equivalent one-leg multistep method, provides a robust foundation for evaluating stability and error characteristics. A key contribution of this work is the generalization of the framework to variable stepsize regimes.

By defining discrete curvature through quadratic interpolants, we establish that the filtering operation satisfies a discrete maximum principle. This proves that the updated curvature is a convex combination of previous values, rendering the scheme strictly dissipative and ensuring the suppression of high-frequency artifacts without the introduction of unphysical local extrema. Furthermore, through a meticulous analysis of the variable-coefficient LMM, we derive an analytical expression for a step-dependent filtering parameter $\beta(\tau_n, \tau_{n-1})$. This result ensures that the transition to third-order accuracy is maintained consistently across non-uniform temporal grids, providing a mathematically rigorous path for adaptive time-stepping. Stability analysis via the boundary locus method confirms that the FBEF method maintains a stability region that contains the negative real axis, making it highly suitable for the integration of stiff and oscillatory systems. This is empirically supported by our numerical results, where the FBEF method consistently matched the accuracy profiles of established benchmarks like BDF3 while exhibiting superior dissipative control. The modular and "plug-and-play" nature of the FBEF algorithm offers significant advantages for industrial legacy codes. Future research will focus on extending this filtering paradigm to higher-order BDF methods and exploring its performance in the context of large-scale fluid-structure interaction problems. In conclusion, the FBEF method represents a high-efficiency, "low-effort, high-reward" upgrade for modern computational frameworks seeking to balance numerical stability with high-order precision and geometric consistency.

Use of Generative-AI tools declaration

AI tools were employed exclusively for linguistic refinement and grammatical editing. All mathematical derivations, proofs, and results are the original work of the author, who assumes full responsibility for the technical integrity of the content.

Acknowledgments

This work has been supported by Batman University Scientific Research Projects Coordination Unit under grant number BTÜBAP-2026-YAYDEP-02.

Conflict of interest

The author declares that he has no conflict of interest.

References

1. J. D. Lambert, *Numerical methods for ordinary differential systems: the initial value problem*, Chichester, England: John Wiley & Sons, 1992.
2. K. E. Atkinson, *An introduction to numerical analysis*, 2 Eds., New York, NY: John Wiley & Sons, 1989.
3. C. W. Gear, *Numerical initial value problems in ordinary differential equations*, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1971.
4. B. N. Biswas, S. Chatterjee, S. Mukherjee, S. Pal, A discussion on Euler method: a review, *Electron. J. Math. Anal. Appl.*, **1** (2013), 294–317.

5. R. Asselin, Frequency filter for time integrations, *Mon. Weather Rev.*, **100** (1972), 487–490. https://doi.org/10.1175/1520-0493(1972)100%3C0487:FFFTI%3E2.3.CO;2

6. P. D. Williams, A proposed modification to the robert–asselin time filter, *Mon. Weather Rev.*, **137** (2009), 2538–2546. http://dx.doi.org/10.1175/2009MWR2724.1

7. A. Güzel, Analysis of forward Euler method with time filter, *TJNS*, **15** (2026), 97–103. http://dx.doi.org/10.46810/tdfd.1828215

8. A. Guzel, W. Layton, Time filters increase accuracy of the fully implicit method, *BIT Numer. Math.*, **58** (2018), 301–315. http://dx.doi.org/10.1007/s10543-018-0695-z

9. W. Layton, W. Pei, C. Trenchea, Refactorization of a variable step, unconditionally stable method of Dahlquist, Liniger and Nevanlinna, *Appl. Math. Lett.*, **125** (2022), 107789. http://dx.doi.org/10.1016/j.aml.2021.107789

10. W. Layton, W. Pei, Y. Qin, C. Trenchea, Analysis of the variable step method of Dahlquist, Liniger and Nevanlinna for fluid flow, *Numer. Methods Partial Differ. Equ.*, **38** (2022), 1713–1737. http://dx.doi.org/10.1002/num.22831

11. V. DeCaria, A. Guzel, W. Layton, Y. Li, A variable stepsize, variable order family of low complexity, *SIAM J. Sci. Comput.*, **43** (2021), A2130–A2160. http://dx.doi.org/10.1137/19M1258153

12. W. Layton, W. Pei, C. Trenchea, Time step adaptivity in the method of Dahlquist, Liniger and Nevanlinna, *Adv. Comput. Sci. Eng.*, **1** (2023), 320–350. http://dx.doi.org/10.3934/acse.2023014

13. A. Güzel, Halving the error in second order Adams-Bashforth methods via a simple time filter, *Black Sea J. Eng. Sci.*, **9** (2026), 887–893. http://dx.doi.org/10.34248/bsengineering.1870475

14. G. Dahlquist, On one-leg multistep methods, *SIAM J. Numer. Anal.*, **20** (1983), 1130–1138. http://dx.doi.org/10.1137/0720082

15. D. S. Watanabe, Q. M. Sheikh, One-leg formulas for stiff ordinary differential equations, *SIAM J. Sci. Statist. Comput.*, **5** (1984), 489–496. http://dx.doi.org/10.1137/0905036

16. G. Y. Kulikov, S. K. Shindin, One-leg integration of ordinary differential equations with global error control, *Comput. Methods Appl. Math.*, **5** (2005), 86–96. http://dx.doi.org/10.2478/cmam-2005-0004

17. A. Guzel, C. Trenchea, The Williams step increases the stability and accuracy of the hoRA time filter, *Appl. Numer. Math.*, **131** (2018), 158–173. http://dx.doi.org/10.1016/j.apnum.2018.05.003

18. Y. Li, C. Trenchea, A higher-order Robert–Asselin type time filter, *J. Comput. Phys.*, **259** (2014), 23–32. http://dx.doi.org/10.1016/j.jcp.2013.11.022

19. E. Hairer, G. Wanner, *Solving ordinary differential equations. II: Stiff and differential-algebraic problems*, 2 Eds., Heidelberg: Springer Berlin, 1996. http://dx.doi.org/10.1007/978-3-642-05221-7

20. D. F. Griffiths, D. J. Higham, *Numerical methods for ordinary differential equations: initial value problems*, London: Springer London, 2010. http://dx.doi.org/10.1007/978-0-85729-148-6

21. A. Güzel, Deriving the error of time filtered leapfrog scheme via modified equations, *Bitlis Eren Üniv. Fen Bilimleri Derg.*, **15** (2026), 480–491. http://dx.doi.org/10.17798/bitlisfen.1840686

22. A. Quarteroni, R. Sacco, F. Saleri, *Numerical mathematics*, 2 Eds., Heidelberg: Springer Berlin, 2007. http://dx.doi.org/10.1007/b98885

23. E. Hairer, G. Wanner, S. P. Nørsett, *Solving ordinary differential equations. I: Nonstiff problems*, 2 Eds., Heidelberg: Springer Berlin, 1993. http://dx.doi.org/10.1007/978-3-540-78862-1

24. N. Hurl, W. Layton, Y. Li, C. Trenchea, Stability analysis of the Crank–Nicolson-Leapfrog method with the Robert–Asselin–Williams time filter, *BIT Numer. Math.*, **54** (2014), 1009–1021. http://dx.doi.org/10.1007/s10543-014-0493-1

25. S. M. McGovern, Adaptive step selection for a filtered implicit method, *J. Sci. Comput.*, **103** (2025), 54. http://dx.doi.org/10.1007/s10915-025-02861-w

26. P. D. Williams, Achieving seventh-order amplitude accuracy in leapfrog integrations, *Mon. Weather Rev.*, **141** (2013), 3037–3051. http://dx.doi.org/10.1175/MWR-D-12-00303.1