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Research article

Infinitely many small energy solutions for the Schrödinger-Poisson equations with magnetic field

Huiling Niu[1,2], Junshan Liu[1], Jiayin Liu[1] and Jun Zheng[3,4,*]

[1] School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou, 730070, China
[2] School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China
[3] School of Mathematics, Southwest Jiaotong University, Chengdu, 611756, China
[4] Department of Electrical Engineering, Polytechnique Montreal, P.O. Box 6079, Station Centre-Ville, Montreal, QC, Canada H3T 1J4

* **Correspondence:** Email: zhengjun@swjtu.edu.cn.

Abstract: In this paper, we consider the following Schrödinger-Poisson equations with magnetic field

$$(-i\nabla - A(x))^2u + \theta(|x|^{-1} * |u|^2)u = f(|u|^2)u,\ u \in H^1(\mathbb{R}^3, \mathbb{C}),$$

where i is the imaginary unit and $\theta \geq 0$. The function $A : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a magnetic potential, and $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous potential. First, we establish the existence of ground state solutions without imposing the strict monotonicity condition and Ambrosetti-Rabinowitz condition. Then using the dual fountain theorem, we obtain the existence of infinitely many small energy solutions. Our results extend some recent work in the literature.

Keywords: Schrödinger-Poisson equation; magnetic field; ground state solution; infinitely many solutions
Mathematics Subject Classification: 35J10, 35J60, 35B38

1. Introduction

In this paper, we focus on the following Schrödinger-Poisson equation:

$$\begin{cases}(-i\nabla - A(x))^2u + V(x)u + \theta(|x|^{-1} * |u|^2)u = f(|u|^2)u,\\ u \in H^1(\mathbb{R}^3, \mathbb{C}),\end{cases} \tag{1.1}$$

with magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$, and $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous potential, where i is the imaginary unit and $\theta \geq 0$.

For $u : \mathbb{R}^3 \to \mathbb{C}$, we write $\nabla_A u := (-i\nabla - A(x))u$. The Schrödinger operator $(-i\nabla - A(x))^2$ is defined by

$$(-i\nabla - A(x))^2 := -\Delta + 2iA \cdot \nabla + |A|^2 + i\text{div}A.$$

When $\theta = 0$, Eq (1.1) reduces to the magnetic Schrödinger equation

$$(-i\nabla - A(x))^2 + V(x)u = f(|u|^2)u, \ u \in H^1(\mathbb{R}^3, \mathbb{C}). \tag{1.2}$$

Equation (1.2) is the nonlinear stationary version of the following Schrödinger equation:

$$i\hbar\frac{\partial \hbar}{\partial t} = (\frac{\hbar}{i}\nabla - A(x))^2\psi + U(z)\psi - f(|\psi|^2)\psi, \ z \in \mathbb{R}^3,$$

where $\hbar$ is the Planck constant, $U(z)$ is a real electric potential, and the function f takes values in $\mathbb{C}$.

On the other hand, when $\theta \neq 0$, the convolution potential $|x|^{-1} * |u|^2 u$ can be seen as a solution of the Poisson equation, which is used to simulate the motion of particles in their own gravitation field generated by the probability density of particles. Hence, Eq (1.1) can be regarded as a magnetic Schrödinger equation coupled with the Poisson equation. It is often referred to as the Schrödinger-Poisson equation, see [1–3] for example.

The case $A = 0$ leads to the Choquard equation. In the past decades, large quantities of excellent results, which concern the existence, multiplicity and dynamical behavior of solutions, have been obtained for the Schrödinger equation. We refer the reader to [4–6] for the case where both $A = 0$ and $\theta = 0$, to [7, 8] for works on the critical case, and to [9–11] for investigations into other qualitative properties of solutions. For a comprehensive account of how the topological properties of the underlying domain impact the number of solutions, we direct the reader to [12, 13]. In recent years, seminumerical work by Marangon, Ponno and Zanelli [14, 15] has uncovered a cascade structure of excited-state solutions for the non-magnetic Schrödinger-Poisson problem, while a series of deep theoretical results for Schrödinger-Poisson systems with critical growth have been established in [16–18].

In this paper, we focus on the case $A \neq 0$. The study of the Schrödinger equation with a magnetic field has aroused great interest among researchers in recent years, due to its wide relevance in semiconductor theory, condensed matter physics, and plasma physics. To the best of our knowledge, the first seminal result on the magnetic Schrödinger equation was established by Esteban and Lions [19] for the problem

$$(-ih\nabla - A(x))^2 u + V(x)u = |u|^{p-2}u \ \text{ in } \mathbb{R}^N. \tag{1.3}$$

They proved that the associated minimization problem is attained via the concentration-compactness principle, which yields the existence of solutions to problem (1.3). Since then, a wide range of problems involving magnetic potentials have been extensively studied. We refer the reader to [20–22] for Schrödinger equations with magnetic fields, [23–25] for Schrödinger equations with both nonlocal terms and magnetic fields, and [26–28] for other types of solutions, such as multi-bump solutions, vortex-type solutions, and semiclassical solutions.

In those works, only a few papers were concerned with the existence of infinity many solutions. We want to mention that in a recent paper [29] , Wen and Zhang obtained a ground state solution and infinitely many high energy solutions for problem (1.1) by using variational methods. More precisely, the authors supposed that $V(x), A(x)$, and $f(t)$ satisfy the following conditions:

(V1) $V \in C(\mathbb{R}^3, (0, \infty))$ and $V(x)$ is 1-periodic in x_1, x_2, and x_3;
(A) there exists $\psi_y \in H^1_{loc}(\mathbb{R}^3, \mathbb{R})$ such that $A(x + y) - A(x) = \nabla\psi_y$ *for all* $y \in \mathbb{Z}^3$;
(f1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{|t|\to 0} f(t) = 0$;
(f2) there exists $q \in (4, 6)$ such that $\lim_{t\to+\infty} \frac{f(t)}{t^{(q-2)/2}} = 0$;
(f3) $\frac{f(t)}{t}$ is nondecreasing in $(0, +\infty)$;
(f4) $\lim_{t\to+\infty} \frac{F(t)}{t^2} = +\infty$, where $F(t) = \int_0^t f(s)\mathrm{d}s$.

Under conditions $(V1)$, (A), and $(f1)$–$(f4)$, the authors obtained a ground state solution by the non-Nehari manifold method. In order to obtain infinitely many solutions, the authors supposed that $V(x)$ and $f(t)$ satisfy

$(V2)$ $V \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$, and for every $M > 0$, there holds

$$meas(\{x \in \mathbb{R}^3 : V(x) \leq M\}) < \infty,$$

where meas denotes the Lebesgue measure in $\mathbb{R}^3$;

$(f5)$ there exist constants $\tau > 0$ and $\alpha > 0$ such that

$$f(t)t - \tau F(t) + \alpha t \geq 0, \quad \forall t \in \mathbb{R}^+.$$

Under conditions $(V2)$ and $(f1)$–$(f5)$, the authors obtained infinitely many high energy solutions by using the fountain theorem [30].

Inspired by the above works, in this paper, we study problem (1.1) in the case $A \not\equiv 0$ and $\theta \neq 0$. First, we simplify the proof of the existence of a ground state solution of problem (1.1) obtained by Wen and Zhang [29]. Then, we investigate the existence of infinitely many small energy solutions of problem (1.1) by the dual fountain theorem [30]. The energy functional associated to problem (1.1) is given by $I : H^1_A(\mathbb{R}^3) \to \mathbb{R}$,

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3} [|(-i\nabla - A(x))u|]^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2\mathrm{d}x + \frac{\theta}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3} F(|u|^2)\mathrm{d}x.$$

The energy functional is well defined as a consequence of the Hardy-Littlewood-Sobolev inequality [31], and $I \in C^1(H^1_A(\mathbb{R}^3), \mathbb{R})$ (see Section 2 for details).

Now, we can state our main theorem as follows.

Theorem 1.1. *Assume that V, A, and f satisfy* $(V1), (A)$, *and* $(f1)$–$(f4)$. *Then Eq* (1.1) *has a ground state solution $\tilde{u} \in H^1_A(\mathbb{R}^3)$ such that $I(\tilde{u}) = \inf_{\mathcal{N}} I > 0$, where*

$$\mathcal{N} := \{u \in H^1_A(\mathbb{R}^3)\backslash\{0\} : \langle I'(u), u\rangle = 0\}.$$

Theorem 1.2. *Assume that V and f satisfy* $(V2)$ *and* $(f1)$–$(f4)$*. Then Eq* (1.1) *has infinitely many solutions* $\{u_k\}$ *satisfying*

$$I(u_k) \to 0^-, \ as\ k \to \infty.$$

Remark 1.3. *Here, we point out that the second result is different from Wen and Zhang [29]. On the one hand, Wen and Zhang obtained infinitely many high energy solutions in [29], while we obtained infinitely many small energy solutions for problem* (1.1)*. On the other hand, the condition* $(f5)$ *is not needed in our result, while in [29], the condition* $(f5)$ *is imposed to obtain infinitely many solutions.*

Notation: From now on in this paper, unless otherwise mentioned, we use the following notations:
• $B_r(x)$ denotes an open ball centered at x with radius $r > 0$.
• $C, C_1, C_2, ...$ denote positive constants that may be different in different places.
• $\|\cdot\|$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^N, \mathbb{C})$ for $p \in [1, +\infty]$.
• " $\to$ " and " $\rightharpoonup$ " denote the strong and weak convergence in the related function space, respectively.
• $\mathrm{R}e\{w\}$ denotes the real part of $w \in \mathbb{C}$, and $\overline{w}$ denotes its conjugate.

This paper is organized as follows. In Section 2, we give variational settings and preliminaries. In Section 3, we show the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

2. Preliminaries

We denote by $H_A^1(\mathbb{R}^3, \mathbb{C})$ the Hilbert space obtained as the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{C})$ with respect to the scalar product

$$\langle u, v\rangle_A := \mathrm{Re}\left\{\int_{\mathbb{R}^3} (\nabla_A u \overline{\nabla_A v} + V(x)u\overline{v})\mathrm{d}x\right\}.$$

The norm of $H_A^1(\mathbb{R}^3, \mathbb{C})$ is given by $\|u\|_A = \langle u, u\rangle_A^{1/2}$. Note that $H_A^1(\mathbb{R}^3, \mathbb{C})$ and $H(\mathbb{R}^3, \mathbb{C})$ are incomparable, that is, in general, $H_A^1(\mathbb{R}^3, \mathbb{C}) \not\subseteq H(\mathbb{R}^3, \mathbb{C})$ and $H_A^1(\mathbb{R}^3, \mathbb{C}) \not\supseteq H(\mathbb{R}^3, \mathbb{C})$. As proved by Lieb-Loss [31], for any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, there holds

$$|\nabla|u|| = \left|\mathrm{Re}\left(\nabla u \frac{\overline{u}}{|u|}\right)\right| = \left|\mathrm{Re}\left((\nabla u - iAu)\frac{\overline{u}}{|u|}\right)\right| \leq |\nabla_A u(x)|.$$

The above expression is the so-called diamagnetic inequality, by which $|u| \in H^1(\mathbb{R}^3, \mathbb{C})$ if $u \in H_A^1(\mathbb{R}^3, \mathbb{C})$. Moreover, the embedding $H_A^1(\mathbb{R}^3, \mathbb{C}) \hookrightarrow L^s(\mathbb{R}^3, C)$ is continuous for $s \in [2, 6]$ and locally compact for $s \in [2, 6)$.

Without loss of generality, we assume that $\theta = 1$, then the energy functional of (1.1) can be written as

$$I(u) = \frac{1}{2}\|u\|_A^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x - y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3} F(|u|^2)\mathrm{d}x. \tag{2.1}$$

It follows from $(f1)$ and $(f2)$ that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon + C_\varepsilon |t|^{\frac{q-2}{2}} \quad \text{and} \quad F(t) \leq \varepsilon|t| + C_\varepsilon|t|^{\frac{q}{2}}, \ \ \forall t \in \mathbb{R}. \tag{2.2}$$

Next, we recall the Hardy-Littlewood-Sobolev inequality that will be used in this paper.

Lemma 2.1. *[31] Let $r, s \in (1,\infty)$ and $\mu \in (0,N)$ with $\frac{1}{r}+\frac{\mu}{N}+\frac{1}{s}=2$. Let $f \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$. Then there exists a sharp constant $C(N,\mu,r)>0$, independent of f and h, such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\right| \le C(N,\mu,r)\|f\|_{L^r(\mathbb{R}^N)}\|h\|_{L^s(\mathbb{R}^N)}.$$

Remark 2.2. *As a consequence of Lemma 2.1, if $u \in L^{\frac{12}{5}}(\mathbb{R}^3,\mathbb{C})$, we have*

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y\right| \le C\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)}. \tag{2.3}$$

From this inequality, (2.1), and (2.2), we know that the functional I is well defined. By a standard argument, we have $I \in C^1(H^1_A(\mathbb{R}^3),\mathbb{R})$, and for any $u, v \in H^1_A(\mathbb{R}^3,\mathbb{C})$,

$$\langle I'(u),v\rangle = Re\left\{\langle u,v\rangle_A + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2u(y)\overline{v(y)}}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3} f(|u|^2)u\bar{v}\mathrm{d}x\right\}.$$

3. Proof of Theorem 1.1

Lemma 3.1. *If f satisfies $(f3)$, then $tf(t)-2F(t)$ is nondecreasing for $t>0$. In particular, $tf(t)-2F(t)\ge 0$ for all $t \ge 0$.*

Proof. Suppose $0<s<t$. Hence, we obtain

$$\begin{aligned} sf(s)-2F(s) &= \frac{f(s)}{s}s^2-2F(t)+2\int_s^t f(\tau)d\tau \\ &= \frac{f(s)}{s}s^2-2F(t)+2\int_s^t \frac{f(\tau)}{\tau}\tau d\tau \\ &\le \frac{f(t)}{t}s^2-2F(t)+\frac{f(t)}{t}(t^2-s^2) \\ &= tf(t)-2F(t), \end{aligned}$$

and this proves the lemma. □

Lemma 3.2. *Assume that (V1) and $(f1)$–$(f4)$ hold. Then for each $u \in H^1_A(\mathbb{R}^3,\mathbb{C})$, there exists a unique $t>0$ such that $tu \in \mathcal{N}$. Moreover, $I(u)>0$ for every $u \in \mathcal{N}$.*

Proof. First, we prove the existence of t. Given $u \in H^1_A(\mathbb{R}^3,\mathbb{C})\backslash\{0\}$, let $h_u(t)=I(tu)$. Then $tu \in \mathcal{N}$ if and only if $h'_u(t)=0$. From (2.2), we have

$$\begin{aligned} h_u(t) &= \frac{t^2}{2}\|u\|^2_A+\frac{t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y-\frac{1}{2}\int_{\mathbb{R}^3}F(t^2|u|^2)\mathrm{d}x \\ &\ge \frac{t^2}{2}\|u\|^2_A-\frac{\varepsilon t^2}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x-C_\varepsilon t^q\int_{\mathbb{R}^3}|u|^q\mathrm{d}x \\ &\ge \frac{t^2}{2}\|u\|^2_A-\frac{C_1\varepsilon t^2}{2}\|u\|^2_A-C_2C_\varepsilon t^q\|u\|^q_A. \end{aligned}$$

Let $\varepsilon > 0$ be so small that $\frac{1}{2} - \frac{C_1\varepsilon}{2} > 0$. Since $q > 4$, we have $h_u(t) > 0$ for all $t > 0$ sufficiently small. Now, from $(f4)$, we have

$$\begin{aligned}\frac{h_u(t)}{t^4} &= \frac{1}{2t^2}\|u\|_A^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3}\frac{F(t^2|u|^2)}{t^4}\mathrm{d}x\\ &= \frac{1}{2t^2}\|u\|_A^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3}\frac{F(t^2|u|^2)}{t^4|u|^4}|u|^4\mathrm{d}x.\end{aligned}$$

Hence, $\lim_{t\to+\infty} h_u(t) = -\infty$. Then, there exists at least one $t(u) > 0$ such that $h_u'(t) = 0$, i.e., $t(u)u \in \mathcal{N}$.

Next, we prove the uniqueness of $t(u)$. Since

$$\begin{aligned}h_u'(t) &= t\|u\|_A^2 + t^3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - t\int_{\mathbb{R}^3} f(t^2|u|^2)|u|^2\mathrm{d}x\\ &= t^3\left[\frac{1}{t^2}\|u\|_A^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\frac{f(t^2|u|^2)}{t^2|u|^2}|u|^4\mathrm{d}x\right],\end{aligned}$$

we conclude that $\frac{h_u'(t)}{t^3}$ is decreasing by using $(f3)$. Then, it vanishes exactly once, and consequently there is no other $t > 0$ such that $tu \in \mathcal{N}$. Note that $t(u)$ is a global maximum point of $h_u(t)$ and $h_u(t(u)) > 0$, i.e., $I(t(u)u) > 0$. Since $t(u) = 1$ if $u \in \mathcal{N}$, we deduce that $I(u) > 0$ for every $u \in \mathcal{N}$. □

Remark 3.3. *We prove the uniqueness of t by using condition* $(f3)$ *directly, while in [29], the authors first proved an inequality with respect to the energy functional* $I(u)$ *(see [29, Lemma 2.1] for details), then they used that inequality to prove the uniqueness of t (see [29, Lemma 2.2] for details).*

From Lemmas 3.1 and 3.2, we know that $\mathcal{N}$ is not empty and I restricted to $\mathcal{N}$ is bounded from below. Having this in mind, we can define the real number

$$m := \inf_{\mathcal{N}} I.$$

In the next result, we prove that minimizing sequences in $\mathcal{N}$ does not converge to 0.

Lemma 3.4. *There exists a constant* $C > 0$ *such that* $\|u\| \geq C > 0$ *for every* $u \in \mathcal{N}$.

Proof. For all $u \in \mathcal{N}$, we have $I'(u)u = 0$, that is,

$$\|u\|_A^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3} f(|u|^2)|u|^2\mathrm{d}x = 0.$$

In light of (2.2), we see that

$$\begin{aligned}\|u\|_A^2 &\leq \int_{\mathbb{R}^3} f(|u|^2)|u|^2\mathrm{d}x \leq \int_{\mathbb{R}^3}(\varepsilon + C_\varepsilon|u|^{q-2})|u|^2\mathrm{d}x\\ &= \varepsilon\int_{\mathbb{R}^3}|u|^2\mathrm{d}x + C_\varepsilon\int_{\mathbb{R}^3}|u|^q\mathrm{d}x\\ &\leq C_3\varepsilon\|u\|_A^2 + C_4C_\varepsilon\|u\|_A^q.\end{aligned}$$

We obtain

$$\|u\|_A^q \geq \frac{1 - C_3\varepsilon}{C_4C_\varepsilon}\|u\|_A^2,$$

hence

$$\|u\|_A^{q-2} \geq \frac{1 - C_3\varepsilon}{C_4 C_\varepsilon}.$$

Since $q > 4$, the result follows. □

In next lemma, we prove that all minimizing sequences in $\mathcal{N}$ are bounded in $H_A^1(\mathbb{R}^3)$.

Lemma 3.5. *If $\{u_n\} \subset \mathcal{N}$ is a minimizing sequence for I, then $\{u_n\}$ is bounded in $H_A^1(\mathbb{R}^3, \mathbb{C})$.*

Proof. Note that $I(u_n) \to m$ and $I'(u_n)u_n = 0$. Then, from Lemma 3.1, we have

$$\begin{aligned}
m + o(1) &= I(u_n) - \frac{1}{4} I'(u_n)u_n \\
&= \frac{1}{2}\|u_n\|_A^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3} F(|u_n|^2)\mathrm{d}x \\
&\quad -\frac{1}{4}\left(\|u_n\|_A^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2 \mathrm{d}x\right) \\
&= \frac{1}{4}\|u_n\|_A^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4} f(|u_n|^2)|u_n|^2 - \frac{1}{2}F(|u_n|^2)\mathrm{d}x\right) \\
&= \frac{1}{4}\|u_n\|_A^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left(f(|u_n|^2)|u_n|^2 - 2F(|u_n|^2)\mathrm{d}x\right) \\
&\geq \frac{1}{4}\|u_n\|_A^2,
\end{aligned}$$

and the result follows. □

The next tool in the proof of our results is a concentration-compactness lemma of Lions that we reformulate as an inequality [32, Lemma I.1] (see also [33, Lemma 1.21]).

Lemma 3.6. *Let $q \in [2, 2^*)$. Then for every $u \in H_A^1(\mathbb{R}^N, \mathbb{C})$, there exists $C > 0$ such that*

$$\int_{\mathbb{R}^N} |u|^q \mathrm{d}x \leq C\left(\sup_{x_0\in\mathbb{R}^N}\int_{B_1(x_0)} |u|^q \mathrm{d}x\right)^{1-\frac{2}{q}} \left(\int_{\mathbb{R}^N}\left(|\nabla_A u|^2 + V(x)|u|^2\right)\mathrm{d}x\right)^2.$$

Proof. Fix $u \in H_A^1(\mathbb{R}^N, \mathbb{C})$. If $q = 2$ then the proof is obvious. Let $q \in (2, 2^*)$. Exploiting the Hölder inequality, we obtain that

$$\begin{aligned}
\int_{B_1(x_0)} |u|^q \mathrm{d}x &= \int_{B_1(x_0)} |u|^{q-2} \cdot |u|^2 \mathrm{d}x \\
&\leq \left(\int_{B_1(x_0)} |u|^{\frac{N(q-2)}{2}} \mathrm{d}x\right)^{\frac{2}{N}} \left(\int_{B_1(x_0)} |u|^{\frac{2N}{N-2}} \mathrm{d}x\right)^{\frac{N-2}{N}} \\
&\leq C\left(\int_{B_1(x_0)} |u|^q \mathrm{d}x\right)^{\frac{N(q-2)}{2q}\cdot\frac{2}{N}} \left(\int_{B_1(x_0)}\left(|\nabla_A u|^2 + V(x)|u|^2\right)\mathrm{d}x\right)^2 \\
&\leq C\left(\sup_{x_0\in\mathbb{R}^N}\int_{B_1(x_0)} |u|^q \mathrm{d}x\right)^{1-\frac{2}{q}} \left(\int_{\mathbb{R}^N}\left(|\nabla_A u|^2 + V(x)|u|^2\right)\mathrm{d}x\right)^2.
\end{aligned}$$

Covering $\mathbb{R}^N$ by balls with radius 1 in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we deduce the conclusion. □

Now, we are ready to provide the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that $\{u_n\} \subset \mathcal{N}$ is a minimizing sequence for I, namely,

$$I(u_n) \to m, \quad \langle I'(u_n), u_n \rangle \to 0. \tag{3.1}$$

By Lemma 3.5, $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^3, \mathbb{C})$ and, up to a subsequence,

$$u_n \rightharpoonup u \ \text{ in } \ H^1_A(\mathbb{R}^3, \mathbb{C}).$$

Since $\{u_n\} \subset \mathcal{N}$, we have

$$\|u_n\|_A^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2 \mathrm{d}x. \tag{3.2}$$

Now, using (2.2), we have

$$\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2 \mathrm{d}x \le \varepsilon \int_{\mathbb{R}^3} |u_n|^2 \mathrm{d}x + C(\varepsilon) \int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x. \tag{3.3}$$

In view of that the embedding $H^1_A(\mathbb{R}^3, \mathbb{C}) \hookrightarrow L^s(\mathbb{R}^3, \mathbb{C})$ is continuous for $s \in [2, 6]$ and locally compact for $s \in [2, 6)$, we see that

$$\int_{B_1(x_0)} |u_n - u_0|^q \mathrm{d}x \to 0 \quad \text{as } \ n \to \infty.$$

If $u_0 \equiv 0$, we have $\int_{B_1(x_0)} |u_n|^q \mathrm{d}x \to 0$. By Lemmas 3.5 and 3.6, we have

$$\int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x \to 0, \ \ 2 \le q < 6.$$

This fact together with inequality (3.3) yields

$$\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2 \mathrm{d}x \to 0. \tag{3.4}$$

Combining (3.2) and (3.4), we see that

$$0 \le \|u_n\|_A^2 \le \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2 \mathrm{d}x \to 0.$$

Then we have $\|u_n\|_A \to 0$, contradicting Lemma 3.4. Thus, there exists a constant $\theta > 0$ and $\{y_n\} \subset \mathbb{R}^N$such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |u_n|^q \mathrm{d}x > \theta. \tag{3.5}$$

Next, by using similar arguments as in [29], we can assume that $\{y_n\} \subset \mathbb{Z}^N$. Let

$$v_n(x) := u_n(x + y_n) exp(-i\psi_{y_n}(x)).$$

It follows from (3.5) and the assumption (A) that

$$\limsup_{n\to\infty}\int_{B_{\sqrt{3}+1}(0)}|v_n|^q \mathrm{d}x > \theta. \tag{3.6}$$

Since $V(x)$ is periodic, we get

$$\|v_n\|_A = \|u_n\|_A,\;\; I(v_n) = I(u_n),\;\; \text{and } I'(v_n) \to 0 \text{ as } n \to \infty.$$

Therefore, there exists $v \in H^1_A(\mathbb{R}^3, \mathbb{C})$ such that

$$v_n \rightharpoonup v \;\text{ in }\; H^1_A(\mathbb{R}^3,\mathbb{C}), v_n \to v \text{ in } L^s_{loc}(\mathbb{R}^3,\mathbb{C}), 1 \le s < 6, \text{ and } v_n \to v\ a.e. \text{ on } \mathbb{R}^3.$$

Thus, (3.6) implies that $v \neq 0$.

Next, we show that

$$I'(v) = 0. \tag{3.7}$$

For any $\zeta \in C_0^\infty(\mathbb{R}^3,\mathbb{C})$ and setting $\Omega = \mathrm{supp}\zeta$, we have

$$\lim_{n\to\infty} Re\int_{\mathbb{R}^3}\nabla_A v_n\overline{\nabla_A\zeta}\mathrm{d}x = Re\int_{\mathbb{R}^3}\nabla_A v\overline{\nabla_A\zeta}\mathrm{d}x,$$

and

$$\begin{aligned}
& Re\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_n(x)|^2|v_n(y)|^2\overline{\zeta(y)}}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v(x)|^2|v(y)|^2\overline{\zeta(y)}}{|x-y|}\mathrm{d}x\mathrm{d}y\right]\\
\le\;& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_n(x)|^2|v_n(y)-v(y)||\zeta(y)| + (|v_n(x)|^2-|v(x)|^2)|v(y)||\zeta(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y\\
\le\;& C\Big[\|v_n\|^2_{L^{12/5}(\Omega)}\|v_n - v\|_{L^{12/5}(\Omega)}\|\zeta\|_{L^{12/5}(\Omega)}\\
& +\|v_n+v\|_{L^{12/5}(\Omega)}\,\|v_n - v\|_{L^{12/5}(\Omega)}\|v\|_{L^{12/5}(\Omega)}\|\zeta\|_{L^{12/5}(\Omega)}\Big] \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Furthermore, the subcritical growth of f implies that

$$\lim_{n\to\infty} Re\int_{\mathbb{R}^3} f(|v_n|^2)v_n\bar{\zeta}\mathrm{d}x = Re\int_{\mathbb{R}^3} f(|v|^2)v\bar{\zeta}\mathrm{d}x.$$

Therefore, we can derive

$$\langle I'(v),\zeta\rangle = \lim_{n\to\infty}\langle I'(v_n),\zeta\rangle = 0.$$

Hence, $I'(v) = 0$. This completes the proof of (3.7). Hence, $v \in \mathcal{N}$ is a nontrivial solution of Eq (1.1) and $I(v) \ge m$.

Note that $\|v\|^2 \le \liminf_{n\to\infty}\|v_n\|^2$. We will prove that

$$\|v\|^2 = \lim_{n\to\infty}\|v_n\|^2. \tag{3.8}$$

Suppose, by contraction, that (3.8) does not hold. Applying Lemma 3.1 and Fatou's lemma, we deduce that

$$m \;\le\; I(v) - \frac{1}{4}\langle I'(v, v\rangle$$

$$
\begin{aligned}
&< \liminf_{n\to\infty}\left[\frac{1}{4}\|v_n\|_A^2+\frac{1}{4}\int_{\mathbb{R}^3}[f(|v_n|^2)|v_n|^2-2F(|v_n|^2)]\mathrm{d}x\right]\\
&= \liminf_{n\to\infty}\left[I(v_n)-\frac{1}{4}\langle I'(v_n,v_n\rangle\right]=m,
\end{aligned}
$$

which is a contradiction. Therefore, equality (3.6) holds. Hence, $v_n \to v \neq 0$ in $H_A^1(\mathbb{R}^3)$, and consequently, $I(v)=m$. Namely, $v\in\mathcal{N}$ is a ground state solution of Eq (1.1).

4. Proof of Theorem 1.2

In this section, we prove the result about infinitely many solutions of Eq (1.1) under the assumptions of Theorem 1.2. We observe that the potential and nonlinearity are both sign-changing. By $(V2)$ and $(f4)$, there exists $\mu>0$ such that

$$
\tilde{V}(x):=V(x)+\mu>0,\quad \forall x\in\mathbb{R}^3,\ \text{and}\ F(t)+\mu t\geq 0,\quad \forall t\in\mathbb{R}^+. \tag{4.1}
$$

Define a new functional space as follows:

$$
E=\left\{u\in H^1(\mathbb{R}^3,\mathbb{C}):\int_{\mathbb{R}^3}\left(|\nabla_A u|^2+\tilde{V}(x)|u|^2\right)\mathrm{d}x<\infty\right\}.
$$

Then E is a Hilbert space equipped with the inner product

$$
\langle u,v\rangle_E=Re\int_{\mathbb{R}^3}\left(\nabla_A u\overline{\nabla_A v}+\tilde{V}(x)u\bar{v}\right)\mathrm{d}x,
$$

and the norm $\|u\|_E=\langle u,u\rangle_E^{1/2}$. Moreover, the embedding $E\hookrightarrow H^1(\mathbb{R}^3,\mathbb{C})$ is continous.

Lemma 4.1. *[29] Assume that (V2) holds, then $E\hookrightarrow L^1(\mathbb{R}^3,\mathbb{C})$ is compact for $2\leq s<6$.*

Next, we decompose the Hilbert space E as follows. Let $\{X_j\}$ be a sequence of finite-dimensional subspace for each $j\in\mathbb{N}$ and $E=\overline{\bigoplus_{j\in N}X_j}$. Set

$$
W_k=\bigoplus_{j=0}^{k}X_j,\qquad Z_k=\overline{\bigoplus_{j=k}^{\infty}X_j},
$$

and

$$
B_k=\{u\in W_k:\|u\|\leq\rho_k\},\qquad S_k=\{u\in Z_k:\|u\|=r_k\},
$$

where $\rho_k>r_k>0$. We study a family of functions $\Phi_\lambda:E\to\mathbb{R}$ of the form

$$
\begin{aligned}
\Phi_\lambda(u) &:= \frac{1}{2}\int_{\mathbb{R}^3}[|\nabla_A u|^2+V(x)|u|^2]\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y\\
&\quad -\frac{\lambda}{2}\int_{\mathbb{R}^3}[F(|u|^2)+\mu|u|^2]\mathrm{d}x,\quad \lambda\in[1,2]
\end{aligned} \tag{4.2}
$$

and let

$$
A(u):=\frac{1}{2}\int_{\mathbb{R}^3}[|\nabla_A u|^2+V(x)|u|^2]\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y,
$$

$$B(u) := \frac{1}{2}\int_{\mathbb{R}^3}[F(|u|^2) + \mu|u|^2]\mathrm{d}x.$$

By a standard argument, it easy to prove that $\Phi_\lambda(u) = A(u) - \lambda B(u)$ belongs to $C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$ and critical points of I_1 are the weak solutions of problem (1.1). To prove Theorem 1.2, we use the following dual fountain theorem [30, Theorem 3.2].

Theorem 4.2. *[30] Assume that*
(B_1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Further, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$;
(B2) $B(u) \geq 0; B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of E;
(B3) there exist $\rho_k > r_k > 0$ such that

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq 0,$$

$$b_k(\lambda) := \max_{u\in W_k, \|u\|=r_k} \Phi_\lambda(u) < 0$$

for all $\lambda \in [1, 2]$ and

$$d_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \to 0$$

as $k \to \infty$ uniformly for $\lambda \in [1, 2]$.
Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in W_n$ such that

$$\Phi'_{\lambda_n}|_{W_n}(u(\lambda_n)) = 0, \quad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k$$

as $n \to \infty$, where $c_k \in [d_k(2), b_k(1)]$. In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.

Similar to the proof of [33, Lemma 3.8], we obtain the following result.

Lemma 4.3. *For $2 \leq p < 6$ and $k \in \mathbb{N}$, define*

$$\beta_{k,p} = \sup\{\|u\|_p : \|u\|_E = 1, u \in Z_k\}.$$

Then $\lim\limits_{k\to\infty} \beta_{k,p} = 0.$

Proof of Theorem 1.2. By Theorem 4.2, it suffices to show that if k is large enough, then there exists $\rho_k > r_k > 0$ such that $(B1)-(B3)$ hold and $\{u(\lambda_n)\}$ is bounded in E. We divide the proof into serval steps.

(1) Verification of $(B1)$. The proof is exactly the same as in [29]. Here, we give the proof for the convenience of readers. Apparently, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Let $\{u_n\}$ be a bounded sequence in E, then it follows from (2.3), (4.2), and Lemma 4.1 that for any $\lambda \in [1, 2]$,

$$\begin{aligned}
\Phi_\lambda(u_n) &\leq \Phi_1(u_n) \\
&= \frac{1}{2}\|u_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3}\left[F(|u_n|^2) + \mu|u_n|^2\right]\mathrm{d}x \\
&\leq C_5\left(\|u_n\|_E^2 + \|u_n\|_E^4\right),
\end{aligned}$$

which implies $\{\Phi_\lambda(u_n)\}$ is bounded.

(2) Verification of $(B2)$. The conclusion follows from definitions of $A(u)$ and $B(u)$.

(3) Verification of $(B3)$. We first prove that $a_k(\lambda) := \inf_{u\in Z_k,\|u\|=\rho_k} \Phi_\lambda(u) \geq 0$. For any $u \in Z_k$ with $\|u\|_E < 1$, it follows from (2.2), Lemmas 4.1 and 4.3 that

$$\begin{aligned}
\Phi_\lambda(u) &\geq \Phi_2(u) \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla_A u|^2 + \tilde{V}(x)|u|^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y \\
&\quad -\int_{\mathbb{R}^3}\left(F(|u|^2) + \mu|u|^2\right)\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^3}\left(F(|u|^2) + \mu|u|^2\right)\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^3}|u|^2\mathrm{d}x - C_6\int_{\mathbb{R}^3}|u|^q\mathrm{d}x - \mu\int_{\mathbb{R}^3}|u|^2\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_E^2 - (1+\mu)\beta_{k,2}^2\|u\|_E^2 - C_6\beta_{k,q}^q\|u\|_E^q.
\end{aligned}$$

Since $\beta_{k,2} \to 0$ and $\beta_{k,q} \to 0$ as $k \to \infty$, there is $\beta > 0$ such that $\beta_{k,2} \leq \beta, \beta_{k,q} \leq \beta$ for all $k \in \mathbb{N}$ and

$$\rho_k := (1+\mu)\beta_{k,2}^2 \to 0 \ \text{ as } \ k \to \infty.$$

Hence, there exists $k_0 \in \mathbb{N}$ such that

$$\rho_k < \min\left\{\frac{1}{4}, (\frac{1}{4C_6\beta})^{\frac{1}{q-2}}\right\}, \quad \forall k > k_0.$$

Therefore, for all $u \in Z_k$ with $\|u\| = \rho_k,\ k \geq k_0$, we obtain

$$\Phi_\lambda(u) \geq \frac{1}{4}\rho_k^2 - C_6\beta\rho_k^q > 0.$$

Next, we show that

$$b_k(\lambda) = \max_{u\in W_k,\|u\|=r_k} \Phi_\lambda(u) < 0.$$

Combing (2.2), (2.3), and Lemma 4.1, we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{\lambda}{2}\int_{\mathbb{R}^3}\left(F(|u|^2) + \mu|u|^2\right)\mathrm{d}x \\
&\leq \frac{1}{2}\|u\|_E^2 + C\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} - \frac{1}{2}\int_{\mathbb{R}^3}F(|u|^2)\mathrm{d}x - \frac{\mu}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x \\
&\leq \frac{1}{2}\|u\|_E^2 + C_7\|u\|_E^4 - \mu C_8\|u\|_E^2 - \frac{1}{2}\int_{\mathbb{R}^3}F(|u|^2)\mathrm{d}x \\
&= \left(\frac{1}{2} - \mu C_8\right)\|u\|_E^2 + C_7\|u\|_E^4 + \varepsilon\int_{\mathbb{R}^3}|u|^2\mathrm{d}x + C_\varepsilon\int_{\mathbb{R}^3}|u|^q\mathrm{d}x
\end{aligned}$$

$$= \left(\frac{1}{2} - \mu C_8 + \varepsilon C_9\right)\|u\|_E^2 + C_7\|u\|_E^4 + C_\varepsilon \|u\|_E^q. \tag{4.3}$$

We may choose $\mu > 0$ large enough in (4.1) and $\varepsilon > 0$ small enough such that $\frac{1}{2} - \mu C_8 + \varepsilon C_9 < 0$. Note that $4 < q < 6$, then we can find $0 < r_k < \rho_k$ such that $\Phi_\lambda(u) < 0$ for all $u \in W_k$ with $\|u\| = r_k$.

Now, we prove

$$d_k(\lambda) = \inf_{u\in Z_k, \|u\|\leq \rho_k} \Phi_\lambda(u) \to 0 \tag{4.4}$$

as $k \to \infty$ uniformly for $\lambda \in [1, 2]$. First, we note that $d_k < 0$ for all $k \geq k_0$ since $W_k \bigcap Z_k \neq \emptyset$, $0 < r_k < \rho_k$, and $\Phi_\lambda(u) < 0$ for all $u \in W_k$ with $\|u\| = r_k$. From Lemma 4.3, we have

$$\|u\|_p \leq \beta_{k,p}\|u\|_E, \quad u \in Z_k. \tag{4.5}$$

Using (2.2) and (4.5), we have, for any $0 \leq t \leq \rho_k$ and $v \in Z_k$ with $\|v\| = 1$

$$\begin{aligned}
\Phi_\lambda(tv) &\geq \Phi_2(tv)\\
&\geq \frac{t^2}{2}\|v\|_E^2 + \frac{t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v(x)|^2|v(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\left(F(t^2|v|^2) + \mu t^2|v|^2\right)\mathrm{d}x\\
&\geq -\int_{\mathbb{R}^3} F(t^2|v|^2)\mathrm{d}x - \mu t^2\int_{\mathbb{R}^3}|v|^2\mathrm{d}x\\
&\geq -\varepsilon t^2\int_{\mathbb{R}^3}|v|^2\mathrm{d}x - C_\varepsilon t^q\int_{\mathbb{R}^3}|v|^q\mathrm{d}x - \mu t^2\int_{\mathbb{R}^3}|v|^2\mathrm{d}x\\
&\geq -\varepsilon\rho_k^2\int_{\mathbb{R}^3}|v|^2\mathrm{d}x - C_\varepsilon\rho_k^q\int_{\mathbb{R}^3}|v|^q\mathrm{d}x - \mu\rho_k^2\int_{\mathbb{R}^3}|v|^2\mathrm{d}x\\
&\geq -\varepsilon\rho_k^2\beta_{k,2}^2\|v\|_E^2 - C_\varepsilon\rho_k^q\beta_{k,q}^q\|v\|_E^q - \mu\rho_k^2\beta_{k,2}^2\|v\|_E^2\\
&= -\varepsilon\rho_k^2\beta_{k,2}^2 - C_\varepsilon\rho_k^q\beta_{k,q}^q - \mu\rho_k^2\beta_{k,2}^2,
\end{aligned}$$

which gives

$$-\varepsilon\rho_k^2\beta_{k,2}^2 - C_\varepsilon\rho_k^q\beta_{k,q}^q - \mu\rho_k^2\beta_{k,2}^2 \leq d_k < 0.$$

By Lemma 4.3, $\{\beta_{k,2}\}$ and $\{\beta_{k,q}\}$ are null sequences. Then $d_k(\lambda) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1, 2]$.

(4) By Theorem 4.2, we know that there exist $\lambda_n \to 1$, $u(\lambda_n) \in W_n$ such that

$$\Phi'_{\lambda_n}|_{W_n}(u(\lambda_n)) = 0, \quad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k, \quad \text{as } n \to \infty, \tag{4.6}$$

where $c_k \in [d_k(2), b_k(1)]$. Now, we show that the sequence $\{u(\lambda_n)\}$ is bounded in E.

Assume that $\|u(\lambda_n)\|_E \to \infty$ as $n \to \infty$. Setting $v_n = \frac{u(\lambda_n)}{\|u(\lambda_n)\|_E}$, one has $\|v_n\|_E = 1$, thus there exists $\tilde{v} \in E$ such that $v_n \rightharpoonup \tilde{v}$ in E, $v_n \to \tilde{v}$ in $L^s(\mathbb{R}^3, \mathbb{C})$ for $s \in [2, 6)$, and $v_n \to \tilde{v}$ a.e. on $\mathbb{R}^3$. It follows from (4.6) and Lemma 3.1 that

$$\begin{aligned}
o(1) + c_k &= \Phi_{\lambda_n}(u(\lambda_n)) - \frac{1}{4}\langle\Phi'_{\lambda_n}(u(\lambda_n)), u(\lambda_n)\rangle\\
&= \frac{1}{4}\|u(\lambda_n)\|_E^2 + \frac{\lambda_n}{4}\int_{\mathbb{R}^3}[f(|u(\lambda_n)|^2)|u(\lambda_n)|^2 - 2F(|u(\lambda_n)|^2)]\mathrm{d}x - \frac{\lambda_n}{4}\mu\int_{\mathbb{R}^3}|u(\lambda_n)|^2\mathrm{d}x
\end{aligned}$$

$$\geq \frac{1}{4}\|u(\lambda_n)\|_E^2 - \frac{\mu}{2}\int_{\mathbb{R}^3} |u(\lambda_n)|^2 \mathrm{d}x. \tag{4.7}$$

Multiplying (4.7) by $1/\|u(\lambda_n)\|_E^2$, we derive

$$\frac{\mu}{2}\|\tilde{v}\|_2^2 = \frac{\mu}{2}\lim_{n\to\infty}\|v_n\|_2^2 \geq \frac{1}{4} > 0.$$

Hence $\tilde{v} \neq 0$. Since $\tilde{v} \neq 0$, we have $\lim_{n\to\infty} |u(\lambda_n)| = \infty$. Hence, it follows from (2.3), $(f4)$ and Fatou's lemma that

$$\begin{aligned}
0 = \lim_{n\to\infty}\frac{\Phi_{\lambda_n}(u(\lambda_n))}{\|u(\lambda_n)\|_E^4} &\leq \lim_{n\to\infty}\frac{\Phi_1(u(\lambda_n))}{\|u(\lambda_n)\|_E^4} \\
&= \lim_{n\to\infty}\left(\frac{1}{2\|u(\lambda_n)\|_E^2} + C_{10} - \frac{1}{2}\int_{\mathbb{R}^3}\frac{F(|u(\lambda_n)|^2) + \mu|u(\lambda_n)|^2}{|u(\lambda_n)|^4}|v_n|^4\mathrm{d}x\right) \\
&\leq C_{10} - \frac{1}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\frac{F(|u(\lambda_n)|^2) + \mu|u(\lambda_n)|^2}{|u(\lambda_n)|^4}|v_n|^4\mathrm{d}x \\
&\leq C_{10} - \frac{1}{2}\int_{\mathbb{R}^3}\liminf_{n\to\infty}\frac{F(|u(\lambda_n)|^2) + \mu|u(\lambda_n)|^2}{|u(\lambda_n)|^4}|v_n|^4\mathrm{d}x = -\infty.
\end{aligned}$$

This contradiction implies $\{u(\lambda_n)\}$ is bounded in E. The proof is completed.

5. Conclusions

In this paper, we investigated the magnetic Schrödinger-Poisson equations, as given in Eq (1.1), which describe a coupled system of the Schrödinger equation and the Poisson equation. Using variational methods, we proved the existence of ground state solutions without imposing the strict monotonicity condition or the Ambrosetti–Rabinowitz condition. In addition, by applying the dual fountain theorem, we obtained infinitely many small energy solutions for Eq (1.1) without requiring condition $(f5)$. This improves upon the result in [29], where condition $(f5)$ was essential to ensure the existence of infinitely many solutions.

Author contributions

Huiling Niu: Conceptualization, Methodology, Validation, Investigation, Writing–original draft; Junshan Liu: Methodology, Validation, Investigation, Writing–review and editing; Jiayin Liu: Methodology, Validation, Investigation, Resources, Writing–review and editing; Jun Zheng: Methodology, Validation, Investigation, Resources, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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