



Research article

Stability and blow-up analysis for a Kirchhoff-type system with logarithmic dissipation mechanisms and logarithmic sources

Adel M. Al-Mahdi^{1,2,*}, Mohammad M. Al-Gharabli^{1,2}, Mohammad Kafini^{1,2} and Zaid Sawlan^{1,3}

¹ Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

² The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

³ Interdisciplinary Research Center for Refining & Advanced Chemicals, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

* **Correspondence:** Email: almahdi@kfupm.edu.sa.

Abstract: This paper investigated, for the first time, a coupled Kirchhoff-type wave system incorporating logarithmic damping and logarithmic source terms (external forces). We established both energy decay and finite-time blow-up results, emphasizing the novel interaction between the two logarithmic components acting as dissipative and driving mechanisms. In the stable region, we constructed a suitable Lyapunov functional and employed refined logarithmic estimates to derive a polynomial decay rate of the total energy. Conversely, for initial data belonging to the unstable set, we proved that the corresponding solutions blow up in finite time using a concavity argument. In addition, we provided some numerical examples to illustrate the stability and blow-up theoretical results. This study presents the first comprehensive analysis of a Kirchhoff-type system with logarithmic damping, revealing how the combined effects of the nonlocal Kirchhoff tension and logarithmic nonlinearities govern the transition between global stabilization and blow-up behavior.

Keywords: Kirchhoff-type system; logarithmic nonlinearity; multiplier method; finite difference method

Mathematics Subject Classification: 35L05, 35L20, 45K05, 93D23

1. Introduction

The study of Kirchhoff-type wave equations has attracted considerable attention since the pioneering work of Kirchhoff in 1876 [1], who extended the classical D'Alembert wave model by introducing a nonlocal term that depends on the total strain energy. This modification captures the

variation in string tension or length during vibration, offering a more realistic description of oscillations in elastic media such as strings, beams, and plates. From a mathematical point of view, Kirchhoff-type equations are appealing because of the rich interaction between nonlocal effects, nonlinear sources, and damping mechanisms, which together produce highly nontrivial dynamical behavior.

For the single Kirchhoff-type equation

$$\omega_{tt} - \Psi(\|\nabla\omega\|_2^2) \Delta\omega + h(\omega_t) = f(\omega), \quad (1.1)$$

many works have addressed issues of local and global well-posedness, long-time stability, and the blow-up of solutions. Early results on analytic solutions were obtained by Dickey [2]. Ono [3–5] later investigated global existence, exponential decay, and finite-time blow-up under various damping and source structures using the potential well method. Zeng and Mu [6] refined blow-up criteria for large initial energy, while Autuori and Pucci [7] established nonexistence results through concavity arguments. Further advances considered nonlinear dissipation [8, 9] and fractional Kirchhoff models [10, 11] providing a broader theoretical framework for this class of equations.

When the analysis extends to systems of Kirchhoff-type wave equations, the problem becomes more delicate due to the coupling effects between components. Santos et al., [12] studied a coupled Kirchhoff system with memory-type boundary conditions and showed that the energy decays at the same rate as the relaxation kernel. Wu [13] analyzed systems with power-type nonlinearities and local damping, establishing global well-posedness via the potential well theory. Park and Bae [14, 15] explored nonlinear damping effects and proved results on global existence and asymptotic stability. Liu and Wang [16] examined weak damping and polynomial sources, showing conditions for both global existence and blow-up. More recently, Mu and Ma [17] discussed stabilization and blow-up phenomena in systems with Balakrishnan–Taylor damping, highlighting the complexity of coupled Kirchhoff models. For various results concerning Kirchhoff-type wave and plate systems, we refer the reader to [18] for the study of global existence and blow-up in a parabolic problem of Kirchhoff type with logarithmic nonlinearity; [19] for Kirchhoff-type equations with Kelvin–Voigt damping and general nonlinearities; [20] for Kirchhoff-type wave problems involving the fractional Laplacian; [21] for semilinear wave equations with Kirchhoff-type nonlocal damping terms and logarithmic nonlinearities; [22] for the asymptotic behavior of a p -Kirchhoff-type hyperbolic equation with dynamic boundary conditions; [23] for the coupling of Kirchhoff and Euler–Bernoulli plates with logarithmic source terms under strong and weak damping of variable-exponent type; and [24] for optimal decay results of Kirchhoff plate equations with nonlinear damping and general relaxation functions. Related works have also considered logarithmic nonlinearities appearing as source terms (on the right-hand side) rather than in the damping; see, for instance, [25–27].

Beyond polynomial-type nonlinearities, increasing attention has been given to *logarithmic* nonlinearities of the form

$$f(\omega) = |\omega|^{\alpha-2} \omega \ln |\omega|,$$

which appear naturally in physics, including quantum mechanics, nuclear models, and cosmology [28–30]. These nonlinearities have also been studied in the context of partial differential equations [31, 32]. Compared with polynomial terms, logarithmic nonlinearities grow more slowly, introducing new mathematical difficulties. Wang et al., [33] analyzed a nonlinear Kirchhoff-type

system with logarithmic sources and weak damping, proving global existence, decay, and blow-up using potential well methods.

Logarithmic damping of the type $u_t \ln(1 + u_t^2)$ has recently emerged as a meaningful nonstandard dissipation law. It provides weak damping near equilibrium but strong dissipation for large velocities, capturing realistic physical behavior in complex media. Mathematically, it is more challenging than polynomial damping due to its non-polynomial, non-Lipschitz, and non-homogeneous structure. Li et al., [34] studied a wave equation with logarithmic damping and derived polynomial energy decay estimates, though existence issues were not fully addressed. Recently, Messaoudi and Al-Mahdi [35] proved the existence and established a polynomial energy decay of a Timoshenko system. Al-Mahdi and Al-Gharabli [36] proved the existence and polynomial decay for piezoelectric beams with magnetic effects and logarithmic damping. In addition, Al-Mahdi and Al-Gharabli [37] established the existence and long-time behavior of solutions to swelling models with a logarithmic damping mechanism and obtained an optimal polynomial decay rate.

Motivated by these developments, we investigate a coupled Kirchhoff-type system with both logarithmic damping and logarithmic source terms:

$$\begin{cases} \omega_{tt} - \Psi(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \Delta\omega + \omega_t \ln(1 + \omega_t^2) = |\omega|^{\alpha-2} \omega \ln |\omega|, & \text{in } \Omega \times (0, \infty), \\ \varphi_{tt} - \Psi(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \Delta\varphi + \varphi_t \ln(1 + \varphi_t^2) = |\varphi|^{\alpha-2} \varphi \ln |\varphi|, & \text{in } \Omega \times (0, \infty), \\ \omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x), & \text{in } \Omega, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), & \text{in } \Omega, \\ \omega(x, t) = 0, \quad \varphi(x, t) = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \quad (1.2)$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, and the Kirchhoff function Ψ is defined by

$$\Psi(r) = a + br^\gamma, \quad a \geq 1, b \geq 0, \gamma > 0. \quad (1.3)$$

The parameters α and γ satisfy

$$\alpha \geq 2\gamma + 2. \quad (1.4)$$

The main objectives of this work are summarized as follows:

- Prove energy decay rates via multiplier techniques and logarithmic inequalities.
- Show finite-time blow-up of solutions when $E(0) < 0$.
- Analyze the interaction between logarithmic damping and logarithmic source terms, which creates new analytical challenges in Kirchhoff-type systems.
- Provide numerical experiments that illustrate the theoretical results.

The novelty of this paper lies in introducing *logarithmic damping* into coupled Kirchhoff-type wave systems, which, to the best of our knowledge, has not been studied before. This damping form is physically realistic and captures complex dissipation phenomena beyond the usual linear or polynomial settings.

The analysis faces several mathematical challenges due to the peculiar nature of logarithmic terms. Classical damping laws such as

$$h(\omega_t) = \omega_t |\omega_t|^m \quad \text{or} \quad h(\varphi_t) = \varphi_t |\varphi_t|^m$$

are homogeneous, simplifying the analysis. In contrast, the logarithmic damping $\omega_t \ln(1 + \omega_t^2)$ is non-homogeneous and grows slower than any polynomial, which makes it too weak to directly control the energy. Hence, refined multiplier estimates and careful inequalities are required to establish stability.

It is worth noting the related studies on existence and stability results of systems involving logarithmic source terms with different damping mechanisms, such as the ones in [38–42]. The present work extends these ideas to Kirchhoff-type systems with both logarithmic damping and sources, thereby bridging the gap between existing single-equation models and coupled nonlocal frameworks.

2. Preliminaries

This section presents the materials required for stating and proving our result. Throughout this paper, we use C to denote a generic positive constant.

Remark 2.1. [34] *The following inequalities are essential for establishing the existence and the decay results.*

(i) For $|s| \leq 1$, $m \geq 1$, and $0 < n \leq 2$, there exist positive constants C_n and C_m such that

$$C_m |s|^{1+m} \leq |s| \ln(1 + |s|) \leq C_n |s|^n. \quad (2.1)$$

(ii) For $|s| \geq 1$, $0 < k < 1$, and $\rho > 0$, there exist positive constants C_k and C_ρ such that

$$C_k |s|^k \leq |s| \ln(1 + |s|) \leq C_\rho |s|^{1+\rho}. \quad (2.2)$$

The energy functional of system (1.2) is defined by

$$\begin{aligned} E(t) &= E(\omega, \varphi) \\ &= \frac{1}{2} (\|\omega_t\|_2^2 + \|\varphi_t\|_2^2) + \frac{a}{2} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \\ &\quad + \frac{b}{2(\gamma+1)} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^{\gamma+1} \\ &\quad - \frac{1}{\alpha} \left(\int_{\Omega} |\omega|^\alpha \ln |\omega| dx + \int_{\Omega} |\varphi|^\alpha \ln |\varphi| dx \right) + \frac{1}{\alpha^2} \left(\int_{\Omega} |\omega|^\alpha dx + \int_{\Omega} |\varphi|^\alpha dx \right). \end{aligned} \quad (2.3)$$

The potential energy

$$\begin{aligned} \mathcal{J}(t) &= \mathcal{J}(\omega, \varphi) \\ &= \frac{a}{2} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) + \frac{b}{2(\gamma+1)} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^{\gamma+1} \\ &\quad - \frac{1}{\alpha} \left(\int_{\Omega} |\omega|^\alpha \ln |\omega| + \int_{\Omega} |\varphi|^\alpha \ln |\varphi| \right) + \frac{1}{\alpha^2} \left(\int_{\Omega} |\omega|^\alpha + \int_{\Omega} |\varphi|^\alpha \right), \end{aligned} \quad (2.4)$$

and the Nehari functional

$$\begin{aligned}
\mathcal{I}(t) &= \mathcal{I}(\omega, \varphi) \\
&= a \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) + b \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} \\
&\quad - \left(\int |\omega|^\alpha \ln |\omega| + \int |\varphi|^\alpha \ln |\varphi| \right).
\end{aligned} \tag{2.5}$$

By (2.5), we define the potential well (stable set)

$$\mathcal{W} = \{(\omega, \varphi) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid \mathcal{I}(\omega, \varphi) > 0\} \cup \{0\},$$

the outer space of the potential well (unstable set)

$$\mathcal{V} = \{(\omega, \varphi) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid \mathcal{I}(\omega, \varphi) < 0\},$$

and the depth of the potential well

$$p := \inf_{(\omega, \varphi) \in \mathbb{N}} \mathcal{J}(\omega, \varphi), \tag{2.6}$$

where the Nehari manifold

$$\mathbb{N} := \{(\omega, \varphi) \in H_0^1(\Omega) \times H_0^1(\Omega) - \{(0, 0)\} \mid \mathcal{I}(\omega, \varphi) = 0\}.$$

We now state, without proof, the global existence result, which can be established using the Galerkin method in combination with the potential well framework, as in [33]. The treatment of the logarithmic damping nonlinearity requires higher regularity and can be handled in the same manner as in [35].

Theorem 2.2. *Let $(\omega_0, \varphi_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(\omega_1, \varphi_1) \in (H_0^1(\Omega))^2$. Assume that $E(0) < p$ and $\mathcal{I}(\omega_0, \varphi_0) > 0$ or $\|\nabla\omega_0\|_2^2 + \|\nabla\varphi_0\|_2^2 = 0$. Then system (1.2) admits a global weak solution.*

Lemma 2.3. [33] (The potential well depth). *The potential well depth p can be represented as*

$$p = \frac{\alpha - 2}{2\alpha} a \left(\frac{a}{\tilde{C}^{\alpha+1}} \right)^{\frac{2}{\alpha-1}} > 0,$$

where \tilde{C} is the Sobolev embedding constant and

$$\begin{cases} 1 \leq \alpha \leq \frac{n+2}{n-2}, & n \geq 3; \\ 1 \leq \alpha < +\infty, & n = 1, 2. \end{cases} \tag{2.7}$$

Proof. From $\mathcal{I}(\omega, \varphi) = 0$ and (2.5), it follows that

$$\begin{aligned}
a \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) &\leq a \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) + b \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} \\
&= \int_{\Omega} |\omega|^\alpha \ln |\omega| dx + \int_{\Omega} |\varphi|^\alpha \ln |\varphi| dx \\
&\leq \|\omega\|_{\alpha+1}^{\alpha+1} + \|\varphi\|_{\alpha+1}^{\alpha+1} \\
&\leq \tilde{C}^{\alpha+1} \left(\|\nabla\omega\|_2^{\alpha+1} + \|\nabla\varphi\|_2^{\alpha+1} \right) \\
&\leq \tilde{C}^{\alpha+1} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\frac{\alpha+1}{2}}
\end{aligned}$$

$$\leq \tilde{C}^{\alpha+1} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\frac{\alpha-1}{2}} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right). \quad (2.8)$$

This implies that

$$\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \geq \left(\frac{a}{\tilde{C}^{\alpha+1}} \right)^{\frac{2}{\alpha-1}}. \quad (2.9)$$

By $I(\omega, \varphi) = 0$ again, and using (2.4) and (2.5), we have, for $\alpha \geq 2\gamma + 2$,

$$\begin{aligned} \mathcal{J}(\omega, \varphi) &= \frac{1}{\alpha} I(\omega, \varphi) + \frac{\alpha-2}{2\alpha} a \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\ &\quad + \frac{(\alpha-2\gamma-2)}{2\gamma\alpha+2\alpha} b \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} + \frac{1}{\alpha^2} \left(\int_{\Omega} |\omega|^\alpha dx + \int_{\Omega} |\varphi|^\alpha dx \right) \\ &\geq \frac{\alpha-2}{2\alpha} a \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \geq \frac{\alpha-2}{2\alpha} a \left(\left(\frac{a}{\tilde{C}^{\alpha+1}} \right)^{\frac{2}{\alpha-1}} \right). \end{aligned} \quad (2.10)$$

Combining the above inequality with the definition of p in (2.6), we obtain

$$p := \frac{\alpha-2}{2\alpha} a \left(\frac{a}{\tilde{C}^{\alpha+1}} \right)^{\frac{2}{\alpha-1}}. \quad (2.11)$$

□

Lemma 2.4. *Under the assumptions of Theorem 2.2, we have*

$$\begin{aligned} \|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 &\leq \frac{2\alpha}{(\alpha-2)a} E(t) \\ &\leq \frac{2\alpha}{(\alpha-2)a} E(0), \text{ for } t \in [0, T]. \end{aligned} \quad (2.12)$$

Proof. The proof is direct from the definitions of the functionals in (2.4) and (2.5), and from the proof of the above lemma,

$$\frac{\alpha-2}{2\alpha} a \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \leq \mathcal{J}(\omega, \varphi) \leq E(t). \quad (2.13)$$

Hence, for $\alpha > 2$, the proof of this lemma is finished. □

3. The stability result

In this section, we establish a decay estimate for the energy associated with problem (1.2). To this end, we first present several auxiliary lemmas.

Lemma 3.1. *The energy functional of system (1.2) satisfies*

$$E'(t) = - \int_{\Omega} \omega_t^2 \ln(1 + \omega_t^2) dx - \int_{\Omega} \varphi_t^2 \ln(1 + \varphi_t^2) dx \leq 0. \quad (3.1)$$

Proof. By multiplying the first equation in (1.2) by ω_t and the second equation in (1.2) by φ_t , and subsequently integrating over Ω , we find that

$$\int_{\Omega} \left(\omega_t \omega_t - \Psi \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \Delta\omega\omega_t + \omega_t^2 \ln(1 + \omega_t^2) \right) dx = \omega_t |\omega|^{\alpha-2} \omega \ln |\omega|,$$

and

$$\int_{\Omega} (\varphi_{tt}\varphi_t - \Psi(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \Delta\varphi\varphi_t + \varphi_t^2 \ln(1 + \varphi_t^2)) dx = \varphi_t |\varphi|^{\alpha-2} \varphi \ln |\varphi|,$$

which gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\omega_t\|_2^2 + \frac{1}{2} (a + b(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^\gamma) \frac{d}{dt} \|\nabla\omega\|_2^2 + \int_{\Omega} \omega_t^2 \ln(1 + \omega_t^2) dx \\ &= \frac{d}{dt} \left(\frac{1}{\alpha} \int_{\Omega} |\omega|^\alpha \ln |\omega| dx - \frac{1}{\alpha^2} \int_{\Omega} |\omega|^\alpha dx \right), \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi_t\|_2^2 + \frac{1}{2} (a + b(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^\gamma) \frac{d}{dt} \|\nabla\varphi\|_2^2 + \int_{\Omega} \varphi_t^2 \ln(1 + \varphi_t^2) \\ &= \frac{d}{dt} \left(\frac{1}{\alpha} \int_{\Omega} |\varphi|^\alpha \ln |\varphi| dx - \frac{1}{\alpha^2} \int_{\Omega} |\varphi|^\alpha dx \right). \end{aligned} \quad (3.3)$$

By adding (3.2) and (3.3), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\omega_t\|_2^2 + \|\varphi_t\|_2^2) + \frac{1}{2} (a + b(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^\gamma) \frac{d}{dt} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \\ &+ \int_{\Omega} \omega_t^2 \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi_t^2 \ln(1 + \varphi_t^2) dx \\ &= \frac{d}{dt} \left(\frac{1}{\alpha} \int_{\Omega} (|\omega|^\alpha \ln |\omega| + |\varphi|^\alpha \ln |\varphi|) dx - \frac{1}{\alpha^2} \int_{\Omega} (|\omega|^\alpha + |\varphi|^\alpha) dx \right). \end{aligned} \quad (3.4)$$

Integrating (3.4) with respect to t over the interval $[0, t]$ leads us to

$$\begin{aligned} & \frac{1}{2} (\|\omega_t\|_2^2 + \|\varphi_t\|_2^2) + \frac{1}{2} \int_0^t (a + b(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^\gamma) \frac{d}{dt} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) dt \\ &+ \int_0^t \int_{\Omega} \omega_t^2 \ln(1 + \omega_t^2) dx dt + \int_0^t \int_{\Omega} \varphi_t^2 \ln(1 + \varphi_t^2) dx dt - \frac{1}{\alpha} \int_{\Omega} (|\omega|^\alpha \ln |\omega| + |\varphi|^\alpha \ln |\varphi|) dx \\ &+ \frac{1}{\alpha^2} \int_{\Omega} (|\omega|^\alpha + |\varphi|^\alpha) dx \\ &= \frac{1}{2} (\|\omega_1\|_2^2 + \|\varphi_1\|_2^2) - \frac{1}{\alpha} \int_{\Omega} (|\omega_0| \ln |\omega_0| + |\varphi_0| \ln |\varphi_0|) dx + \frac{1}{\alpha^2} \int_{\Omega} (|\omega_0|^\alpha + |\varphi_0|^\alpha) dx. \end{aligned} \quad (3.5)$$

We notice that

$$\begin{aligned} & \frac{1}{2} (a + b(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^\gamma) \frac{d}{dt} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \\ &= \frac{d}{dt} \left(\frac{a}{2} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) + \frac{b}{2\gamma + 2} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^{\gamma+1} \right). \end{aligned} \quad (3.6)$$

Hence, by using (3.6), we have

$$\begin{aligned} & \frac{1}{2} \int_0^t (a + b(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^\gamma) \frac{d}{dt} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) = \frac{a}{2} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) \\ &- \frac{a}{2} (\|\nabla\omega_0\|_2^2 + \|\nabla\varphi_0\|_2^2) + \frac{b}{2\gamma + 2} (\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^{\gamma+1} \\ &- \frac{b}{2\gamma + 2} (\|\nabla\omega_0\|_2^2 + \|\nabla\varphi_0\|_2^2)^{\gamma+1}. \end{aligned} \quad (3.7)$$

Combining (3.5) and (3.7), we get

$$\begin{aligned}
 & \frac{1}{2}(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2) + \frac{a}{2}(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2) + \frac{b}{2\gamma+2}(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2)^{\gamma+1} \\
 & + \int_0^t \int_{\Omega} \omega_t^2 \ln(1 + \omega_t^2) dx d\tau + \int_0^t \int_{\Omega} \varphi_t^2 \ln(1 + \varphi_t^2) dx d\tau \\
 & - \frac{1}{\alpha} \int_{\Omega} (|\omega|^\alpha \ln|\omega| + |\varphi|^\alpha \ln|\varphi|) dx + \frac{1}{\alpha^2} \int_{\Omega} (|\omega|^\alpha + |\varphi|^\alpha) dx \tag{3.8} \\
 & = \frac{1}{2}(\|\omega_0\|_2^2 + \|\varphi_0\|_2^2) + \frac{a}{2}(\|\nabla\omega_0\|_2^2 + \|\nabla\varphi_0\|_2^2) + \frac{b}{2\gamma+2}(\|\nabla\omega_0\|_2^2 + \|\nabla\varphi_0\|_2^2)^{\gamma+1} \\
 & - \frac{1}{\alpha} \int_{\Omega} (|\omega_0| \ln|\omega_0| + |\varphi_0| \ln|\varphi_0|) dx + \frac{1}{\alpha^2} \int_{\Omega} (|\omega_0|^\alpha + |\varphi_0|^\alpha) dx,
 \end{aligned}$$

that is,

$$E(t) + \int_0^t \int_{\Omega} \omega_t^2 \ln(1 + \omega_t^2) dx d\tau + \int_0^t \int_{\Omega} \varphi_t^2 \ln(1 + \varphi_t^2) dx d\tau = E(0),$$

which completes the proof of (3.1). \square

Lemma 3.2. *The solution of (1.2) satisfies*

$$\int_{\Omega} (\omega_t^2 + \varphi_t^2) dx \leq C(-E'(t))^{\frac{1}{2}} + C(-E'(t)). \tag{3.9}$$

Proof. We define the following partition of Ω :

$$\Omega^+ = \{x \in \Omega : |\omega_t| > 1\} \quad \text{and} \quad \Omega^- = \{x \in \Omega : |\omega_t| \leq 1\}. \tag{3.10}$$

By Hölder's inequality, and the algebraic inequality (2.1), with $m = 1$ and $s = \omega_t^2$, we obtain

$$\begin{aligned}
 \int_{\Omega^-} \omega_t^2 dx & \leq C \left(\int_{\Omega^-} \omega_t^4 dx \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega^-} \omega_t^2 \ln(1 + \omega_t^2) dx \right)^{\frac{1}{2}} \\
 & \leq C (-E'(t))^{\frac{1}{2}}.
 \end{aligned} \tag{3.11}$$

On the set Ω^+ , we have $\omega_t^2 > 1$, and therefore

$$\ln(1 + \omega_t^2) > \ln 2.$$

Consequently,

$$\frac{1}{\ln(1 + \omega_t^2)} \leq \frac{1}{\ln 2}.$$

Using this observation, we obtain

$$\begin{aligned}
 \int_{\Omega^+} \omega_t^2 dx & = \int_{\Omega^+} \omega_t^2 \cdot \frac{\ln(1 + \omega_t^2)}{\ln(1 + \omega_t^2)} dx \\
 & \leq \frac{1}{\ln 2} \int_{\Omega^+} \omega_t^2 \ln(1 + \omega_t^2) dx \\
 & \leq C(-E'(t)).
 \end{aligned} \tag{3.12}$$

$$\begin{aligned}\int_{\Omega^+} \omega_t^2 dx &= \int_{\Omega^+} \omega_t^2 \cdot \frac{\ln(1 + \omega_t^2)}{\ln(1 + \omega_t^2)} dx \\ &\leq \frac{1}{\ln 2} \int_{\Omega^+} \omega_t^2 \ln(1 + \omega_t^2) dx \leq C(-E'(t)).\end{aligned}\quad (3.13)$$

Hence,

$$\int_{\Omega} \omega_t^2 dx \leq C(-E'(t))^{\frac{1}{2}} + C(-E'(t)). \quad (3.14)$$

Similar arguments can be applied for $\int_{\Omega^+} \varphi^2$ and $\int_{\Omega^-} \varphi^2$ to obtain

$$\int_{\Omega} \varphi_t^2 dx \leq C(-E'(t))^{\frac{1}{2}} + C(-E'(t)). \quad (3.15)$$

This completes the proof of this lemma. □

Lemma 3.3. *The solution of (1.2) satisfies, for any $\delta > 0$,*

$$\begin{aligned}\int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx \\ \leq \delta c_p \frac{2\alpha}{(\alpha - 2)a} E(t) + C(\delta)(-E'(t)) + C(-E'(t))^{\frac{\gamma-1}{\gamma}},\end{aligned}$$

where c_p is Poincaré's constant.

Proof. Applying Young's and Poincaré's inequalities, we arrive at

$$\begin{aligned}\int_{\Omega^-} \omega \omega_t \ln(1 + \omega_t^2) dx &\leq \delta c_p \|\nabla \omega\|_2^2 + C(\delta) \int_{\Omega^-} \omega_t^2 \ln^2(1 + \omega_t^2) dx \\ &= \delta c_p \|\nabla \omega\|_2^2 + C(\delta) \int_{\Omega^-} [\omega_t^2 \ln(1 + \omega_t^2)] \ln(1 + \omega_t^2) dx \\ &\leq \delta c_p \|\nabla \omega\|_2^2 + C(\delta) \int_{\Omega^-} \omega_t^2 \ln(1 + \omega_t^2) dx \\ &= \delta c_p \|\nabla \omega\|_2^2 + C(\delta)(-E'(t)) \\ &\leq \frac{\delta}{2} c_p \frac{2\alpha}{(\alpha - 2)a} E(t) + C(\delta)(-E'(t)),\end{aligned}\quad (3.16)$$

By applying the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^\beta(\Omega)$ for

$$\begin{cases} 2 \leq \beta \leq \frac{2n}{n-2}, & n \geq 3, \\ 2 \leq \beta < +\infty, & n = 1, 2, \end{cases}\quad (3.17)$$

then, by means of Hölder's inequality and (2.2), with $s = \omega_t^2$ and $\rho = \frac{\beta}{2} - 1$, we have

$$\begin{aligned}
 \int_{\Omega^+} \omega \omega_t \ln(1 + \omega_t^2) dx &\leq \left(\int_{\Omega^+} \omega^\beta dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega^+} (|\omega_t| \ln(1 + \omega_t^2))^{\frac{\beta}{\beta-1}} dx \right)^{\frac{\beta-1}{\beta}} \\
 &= \left(\int_{\Omega^+} \omega^\beta dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega^+} (|\omega_t|)^{\frac{\beta-2}{\beta-1}} (|\omega_t|^2 \ln(1 + \omega_t^2))^{\frac{1}{\beta-1}} \ln(1 + \omega_t^2) dx \right)^{\frac{\beta-1}{\beta}} \\
 &\leq C \left(\int_{\Omega^+} \omega^\beta dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega^+} (|\omega_t|)^{\frac{\beta-2}{\beta-1}} ((\omega_t^2)^{1+\rho})^{\frac{1}{\beta-1}} \ln(1 + \omega_t^2) dx \right)^{\frac{\beta-1}{\beta}} \\
 &= C \left(\int_{\Omega^+} \omega^\beta dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega^+} \omega_t^2 \ln(1 + \omega_t^2) dx \right)^{\frac{\beta-1}{\beta}} \\
 &\leq C \|\omega\|_\beta (-E'(t))^{\frac{\beta-1}{\beta}} \\
 &\leq C \|\nabla \omega\|_2 (-E'(t))^{\frac{\beta-1}{\beta}} \\
 &\leq C \frac{4\alpha}{(\alpha-2)a} E(0) (-E'(t))^{\frac{\beta-1}{\beta}}.
 \end{aligned} \tag{3.18}$$

Similar calculations can be done for $\int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx$. Hence, (3.16) is established. \square

Lemma 3.4. *The functional defined by*

$$L(t) := N_1 \int_{\Omega} (\omega \omega_t + \varphi \varphi_t) dx + NE(t) \tag{3.19}$$

satisfies, for sufficiently small N_1 , positive constants N (sufficiently large), α_1 , α_2 , c_1 , and c_2 ,

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t) \tag{3.20}$$

and

$$\begin{aligned}
 L'(t) &\leq -c_1 E(t) + c_2 \left(\int_{\Omega} (\omega_t^2 + \omega \omega_t \ln(1 + \omega_t^2)) dx + \int_{\Omega} (\varphi_t^2 + \varphi \varphi_t \ln(1 + \varphi_t^2)) dx \right) \\
 &\quad + NE'(t).
 \end{aligned} \tag{3.21}$$

Proof. We will prove (3.21) as the proof of (3.20) is straightforward. Multiplying the first and second equations in (1.2) by ω and φ , respectively, and integrating the resulting expressions over Ω , we obtain

$$\int_{\Omega} \omega \omega_{tt} dx + \Psi(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2) \int_{\Omega} |\nabla \omega|^2 dx + \int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx = \int_{\Omega} |\omega|^\alpha \ln |\omega| dx, \tag{3.22}$$

and

$$\int_{\Omega} \varphi \varphi_{tt} dx + \Psi(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2) \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx = \int_{\Omega} |\varphi|^\alpha \ln |\varphi| dx. \tag{3.23}$$

Differentiating the functional $L(t)$ with respect to t and using (3.23), we get

$$\begin{aligned}
 L'(t) &= N_1 \left(\int_{\Omega} \omega_t^2 dx + \int_{\Omega} \omega \omega_{tt} dx + \int_{\Omega} \varphi_t^2 dx + \int_{\Omega} \varphi \varphi_{tt} dx \right) + NE'(t) \\
 &= N_1 \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right) + N_1 \left(\int_{\Omega} |\omega|^\alpha \ln |\omega| dx + \int_{\Omega} |\varphi|^\alpha \ln |\varphi| dx \right) \\
 &\quad - N_1 a \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right) - N_1 b \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right)^{\beta+1} \\
 &\quad - N_1 \left(\int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx \right) + NE'(t).
 \end{aligned} \tag{3.24}$$

After applying the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\alpha+1}(\Omega)$ for

$$\begin{cases} 2 \leq \alpha + 1 \leq \frac{2n}{n-2}, & n \geq 3, \\ 2 \leq \alpha + 1 < +\infty, & n = 1, 2, \end{cases} \tag{3.25}$$

and (2.12), we obtain

$$\begin{aligned}
 \int_{\Omega} |\omega|^\alpha \ln |\omega| dx + \int_{\Omega} |\varphi|^\alpha \ln |\varphi| dx &\leq \int_{\Omega} |\omega|^{\alpha+1} dx + \int_{\Omega} |\varphi|^{\alpha+1} dx \\
 &\leq \|\nabla \omega\|_{\alpha+1}^{\alpha+1} + \|\nabla \varphi\|_{\alpha+1}^{\alpha+1} \\
 &\leq \tilde{C}^{\alpha+1} \left(\|\nabla \omega\|_2^{\alpha+1} + \|\nabla \varphi\|_2^{\alpha+1} \right) \\
 &\leq \tilde{C}^{\alpha+1} \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right)^{\frac{\alpha+1}{2}} \\
 &\leq \tilde{C}^{k+1} \left(\frac{2\alpha}{a(\alpha-2)} E(0) \right)^{\frac{\alpha-1}{2}} \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right) \\
 &:= \xi_1 \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right),
 \end{aligned} \tag{3.26}$$

where $\xi_1 = \tilde{C}^{k+1} \left(\frac{2\alpha}{a(\alpha-2)} E(0) \right)^{\frac{\alpha-1}{2}}$. Combining (3.24) and (3.26) leads to

$$\begin{aligned}
 L'(t) &\leq N_1 \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right) + N_1 (\xi_1 - a) \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right) + NE'(t) \\
 &\quad - N_1 b \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right)^{\beta+1} + N_1 \left(\int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx \right).
 \end{aligned} \tag{3.27}$$

Taking into consideration the energy equation (2.3), and adding and subtracting the term $N_1 A E(t)$

to (3.27), where A is a positive constant to be determined later, gives

$$\begin{aligned}
 L'(t) &\leq N_1 \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right) + N_1(\xi_1 - a) \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\quad - N_1 b \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} \\
 &\quad + N_1 \left(\int_{\Omega} \omega\omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi\varphi_t \ln(1 + \varphi_t^2) dx \right) \\
 &\quad + \frac{N_1 A}{2} \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right) + \frac{N_1 a A}{2} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\quad + \frac{N_1 b A}{2(\gamma + 1)} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} + \frac{N_1 A \xi_1}{\alpha} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\quad + \frac{N_1 A}{\alpha^2} \left(\int_{\Omega} |\omega|^\alpha dx + \int_{\Omega} |\varphi|^\alpha dx \right) - N_1 A E(t) + N E'(t).
 \end{aligned} \tag{3.28}$$

Similar to the estimation in (3.26), we have

$$\begin{aligned}
 \int_{\Omega} |\omega|^\alpha dx + \int_{\Omega} |\varphi|^\alpha dx &\leq C_3^\alpha \left(\|\nabla\omega\|^\alpha + \|\nabla\varphi\|^\alpha \right) \\
 &\leq C_3^\alpha \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\frac{\alpha}{2}} \\
 &\leq C_3^\alpha \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\frac{\alpha}{2}-1} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\leq C_3^\alpha \left(\frac{2\alpha}{a(\alpha-2)} E(0) \right)^{\frac{\alpha-2}{2}} \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\leq \xi_2 \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right),
 \end{aligned} \tag{3.29}$$

where $\xi_2 = C_3^\alpha \left(\frac{2\alpha}{a(\alpha-2)} E(0) \right)^{\frac{\alpha-2}{2}}$. Combining (3.28) and (3.29), we obtain

$$\begin{aligned}
 L'(t) &\leq N_1 \left(1 + \frac{A}{2} \right) \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right) \\
 &\quad + N_1 \left(\xi_1 - a + \frac{aA}{2} + \frac{A\xi_1}{\alpha} + \frac{A\xi_2}{\alpha^2} \right) \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\quad + N_1 \left(\frac{bA}{2(\gamma + 1)} - b \right) \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} \\
 &\quad + N_1 \left(\int_{\Omega} \omega\omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi\varphi_t \ln(1 + \varphi_t^2) dx \right) - N_1 A E(t) + N E'(t).
 \end{aligned} \tag{3.30}$$

To complete the proof, we now specify the choice of the constants in a precise order. From (3.30), we have

$$\begin{aligned}
 L'(t) &\leq N_1 \left(1 + \frac{A}{2} \right) \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right) + N_1 M_1 \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right) \\
 &\quad + N_1 M_2 \left(\|\nabla\omega\|_2^2 + \|\nabla\varphi\|_2^2 \right)^{\gamma+1} \\
 &\quad + N_1 \left(\int_{\Omega} \omega\omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \varphi\varphi_t \ln(1 + \varphi_t^2) dx \right) - N_1 A E(t) + N E'(t),
 \end{aligned} \tag{3.31}$$

where

$$M_1 = \xi_1 - a + \frac{aA}{2} + \frac{A\xi_1}{\alpha} + \frac{A\xi_2}{\alpha^2}, \quad M_2 = \frac{bA}{2(\gamma + 1)} - b. \quad (3.32)$$

We first choose $A > 0$. Since $E(0) < p$, Lemma 2.3 implies that

$$\xi_1 = C_*^{\alpha+1} \left(\frac{2\alpha}{a(\alpha-2)} E(0) \right)^{\frac{\alpha-1}{2}} < a. \quad (3.33)$$

Hence $a - \xi_1 > 0$, and therefore $M_1 \leq 0$ whenever

$$0 < A \leq \frac{a - \xi_1}{\frac{a}{2} + \frac{\xi_1}{\alpha} + \frac{\xi_2}{\alpha^2}}. \quad (3.34)$$

Also, $M_2 \leq 0$ whenever

$$0 < A \leq 2(\gamma + 1). \quad (3.35)$$

Accordingly, we fix

$$0 < A \leq A_* := \min \left\{ \frac{a - \xi_1}{\frac{a}{2} + \frac{\xi_1}{\alpha} + \frac{\xi_2}{\alpha^2}}, 2(\gamma + 1) \right\}. \quad (3.36)$$

With this choice, both $M_1 \leq 0$ and $M_2 \leq 0$. Therefore, (3.31) yields

$$\begin{aligned} L'(t) &\leq N_1 \left(1 + \frac{A}{2} \right) (\|\omega_t\|_2^2 + \|\phi_t\|_2^2) \\ &\quad + N_1 \left(\int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx + \int_{\Omega} \phi \phi_t \ln(1 + \phi_t^2) dx \right) - N_1 A E(t) + N E'(t). \end{aligned} \quad (3.37)$$

Hence, we obtain, for some positive constants c_1, c_2 ,

$$\begin{aligned} L'(t) &\leq -c_1 E(t) + c_2 \left(\int_{\Omega} (\omega_t^2 + \omega \omega_t \ln(1 + \omega_t^2)) dx + \int_{\Omega} (\phi_t^2 + \phi \phi_t \ln(1 + \phi_t^2)) dx \right) \\ &\quad + N E'(t), \end{aligned} \quad (3.38)$$

which is exactly (3.21). This completes the proof. \square

Theorem 3.5. *Let $(\omega_0, \omega_1), (\varphi_0, \varphi_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Assume that $E(0) < p$ and the parameters α and γ satisfy (1.4) and the embedding condition (3.25). Then, there exists a positive constant κ , such that the energy functional of problem (1.2) satisfies, for all $t > 0$,*

$$E(t) \leq \frac{\kappa}{t}. \quad (3.39)$$

Proof. Combining (3.9), (3.21), and (3.16) leads to

$$\begin{aligned} L'(t) &\leq - \left(c_1 - \delta c_p \frac{2\alpha}{(\alpha-2)a} \right) E(t) + (N - c_2 C - c_2 C(\delta)) E'(t) + c_2 C (-E'(t))^{\frac{\beta-1}{\beta}} \\ &\quad + c_2 C (-E'(t))^{\frac{1}{2}}. \end{aligned} \quad (3.40)$$

We choose $\delta > 0$ small enough such that

$$m := c_1 - \delta c_p \frac{2\alpha}{(\alpha - 2)a} > 0,$$

and N large enough so that

$$N - c_2 C - c_2 C(\delta) \geq 0$$

and (3.20) remains valid. Therefore, we get

$$L'(t) \leq -mE(t) + C(-E'(t))^{\frac{1}{2}} + C(-E'(t))^{\frac{\beta-1}{\beta}}. \quad (3.41)$$

Multiply (3.41) by $E(t)$ and apply Young's inequality to get

$$\begin{aligned} E(t)L'(t) &\leq -mE^2(t) + \eta E^2(t) + C_\eta(-E'(t)) + \eta E^\beta(t) + C_\eta(-E'(t)) \\ &\leq -(m - \eta - \eta E^{\beta-2}(0))E^2(t) + C(-E'(t)). \end{aligned} \quad (3.42)$$

Choosing η small enough and letting $\mathcal{L} = EL + CE \sim E$, we have, for some positive constant m_0 ,

$$\mathcal{L}'(t) \leq -m_0 \mathcal{L}^2(t). \quad (3.43)$$

Simple integration gives, for some positive constant m_1 ,

$$\mathcal{L}(t) \leq \frac{m_1}{t}.$$

Then, recalling the equivalence $\mathcal{L} \sim E$, (3.39) is established. \square

Remark 3.6. *Theorem 3.5 yields a polynomial decay rate of order $O(1/t)$. In view of the logarithmic nature of the damping term, which provides only a weak dissipative effect, this rate appears to be the natural one in the present setting. We have explored the possibility of obtaining a faster polynomial decay under the same damping mechanism, but were not able to improve the rate $1/t$. For this reason, we conjecture that the decay obtained in Theorem 3.5 is optimal for this class of damping. Nevertheless, proving such an optimality result remains an open problem.*

4. The blow-up result

In this section, we establish sufficient conditions for the finite-time blow-up of solutions to system (1.2).

Lemma 4.1. [43] *Suppose that $\psi(t)$ is a twice continuously differentiable function satisfying, for $t > 0$,*

$$\begin{aligned} \psi''(t) + \psi'(t) &\geq C_0 \psi^{\mu+1}(t), \\ \psi(0) &> 0, \quad \text{and} \quad \psi'(0) \geq 0, \end{aligned} \quad (4.1)$$

where C_0 and μ are positive constants. Then $\psi(t)$ blows up in a finite time.

Theorem 4.2. Suppose that the initial data $(\omega_0, \varphi_0), (\omega_1, \varphi_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ satisfies $E(0) \leq 0$ and

$$\int_{\Omega} (\omega_0 \omega_1 + \varphi_0 \varphi_1) dx \geq 0.$$

Then the corresponding solution for system (1.2) blows up in a finite time.

Proof. To be able to apply Lemma 4.1, we define

$$\psi(t) := \frac{1}{2} \int_{\Omega} (\omega^2 + \varphi^2) dx. \quad (4.2)$$

Hence,

$$\psi'(t) = \int_{\Omega} (\omega \omega_t + \varphi \varphi_t) dx.$$

By using (1.2), we find that

$$\begin{aligned} \psi''(t) &= \int_{\Omega} (\omega_t^2 + \varphi_t^2 + \omega \omega_{tt} + \varphi \varphi_{tt}) dx \\ &= \|\omega_t\|_2^2 + \|\varphi_t\|_2^2 - a \|\nabla \omega\|_2^2 - a \|\nabla \varphi\|_2^2 \\ &\quad - b (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2)^\gamma \|\nabla \omega\|_2^2 - b (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2)^\gamma \|\nabla \varphi\|_2^2 \\ &\quad - \int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx - \int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx \\ &\quad + \int_{\Omega} (|\omega|^\alpha \ln |\omega| + |\varphi|^\alpha \ln |\varphi|) dx \\ &= \|\omega_t\|_2^2 + \|\varphi_t\|_2^2 - a (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2) - b (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2)^{\gamma+1} \\ &\quad - \int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx - \int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx \\ &\quad + \int_{\Omega} (|\omega|^\alpha \ln |\omega| + |\varphi|^\alpha \ln |\varphi|) dx. \end{aligned}$$

Recalling the partition (3.10), we find that $\ln(1 + s) \leq \ln 2$, and

$$\begin{aligned} \int_{\Omega^-} \omega \omega_t \ln(1 + \omega_t^2) dx &\leq \frac{1}{2} \int_{\Omega^-} \omega^2 dx + \frac{1}{2} \int_{\Omega^-} (\omega_t \ln(1 + \omega_t^2))^2 dx \\ &\leq \frac{1}{2} \int_{\Omega} \omega^2 dx + \frac{1}{2} \int_{\Omega} (\omega_t \ln 2)^2 dx \\ &\leq \frac{c_p}{2} \|\nabla \omega\|_2^2 + \frac{1}{2} \|\omega_t\|_2^2, \end{aligned}$$

where c_p is Poincaré's constant. On the other hand, by Hölder's inequality for $\alpha > 2$,

$$\begin{aligned} \int_{\Omega^+} \omega \omega_t \ln(1 + \omega_t^2) dx &\leq \left(\int_{\Omega^+} \omega^\alpha dx \right)^{\frac{1}{\alpha}} \left(\int_{\Omega^+} (\omega_t \ln(1 + \omega_t^2))^{\frac{\alpha}{\alpha-1}} dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\leq \left(\int_{\Omega^+} \omega^\alpha dx \right)^{\frac{1}{\alpha}} \left(\int_{\Omega^+} [\omega_t^{\gamma_1} (\ln(1 + \omega_t^2))^{\gamma_2}]^{\frac{\alpha}{\alpha-1}} dx \right)^{\frac{\alpha-1}{\alpha}} \end{aligned}$$

$$\begin{aligned} &\leq \left(\int_{\Omega^+} \omega^\alpha dx \right)^{\frac{1}{\alpha}} \left(\int_{\Omega^+} \left[\omega_t^{\frac{\alpha\gamma_1}{\alpha-1}} (\ln(1 + \omega_t^2))^{\frac{\alpha\gamma_2}{\alpha-1}} \right] dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\leq \left(\int_{\Omega^+} \omega^\alpha dx \right)^{\frac{1}{\alpha}} \left(\int_{\Omega^+} \left[2|\omega_t| (\sqrt{2}|\omega_t|)^{\frac{\rho}{\alpha-1}} \right] dx \right)^{\frac{\alpha-1}{\alpha}}, \end{aligned}$$

where we used $\gamma_1 = \frac{\alpha-1}{\alpha}$, $\gamma_2 = \frac{1}{\alpha}$, and the fact that

$$\ln |2\omega_t| \leq |2\omega_t|^\rho, \quad \rho = \alpha - 1 > 0.$$

Thus, Young's inequality gives

$$\begin{aligned} \int_{\Omega^+} \omega \omega_t \ln(1 + \omega_t^2) dx &\leq \left(\int_{\Omega^+} \omega^\alpha dx \right)^{\frac{1}{\alpha}} \left(2 \int_{\Omega^+} [|\omega_t| (\sqrt{2}|\omega_t|)] dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\leq \left(\int_{\Omega^+} \omega^\alpha dx \right)^{\frac{1}{\alpha}} \left(\int_{\Omega^+} 2\sqrt{2}\omega_t^2 dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\leq \frac{1}{\alpha} \|\omega\|_\alpha^\alpha + \frac{2\sqrt{2}(\alpha-1)}{\alpha} \|\omega_t\|_2^2. \end{aligned}$$

Hence

$$\int_{\Omega} \omega \omega_t \ln(1 + \omega_t^2) dx \leq \frac{c_p}{2} \|\nabla \omega\|_2^2 + \left(\frac{2\sqrt{2}(\alpha-1)}{\alpha} + \frac{1}{2} \right) \|\omega_t\|_2^2 + \frac{1}{\alpha} \|\omega\|_\alpha^\alpha.$$

Similarly,

$$\int_{\Omega} \varphi \varphi_t \ln(1 + \varphi_t^2) dx \leq \frac{c_p}{2} \|\nabla \varphi\|_2^2 + \left(\frac{2\sqrt{2}(\alpha-1)}{\alpha} + \frac{1}{2} \right) \|\varphi_t\|_2^2 + \frac{1}{\alpha} \|\varphi\|_\alpha^\alpha.$$

We estimate ψ' as follows:

$$\psi'(t) \leq \frac{c_p}{2} (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2) + \frac{1}{2} (\|\omega_t\|_2^2 + \|\varphi_t\|_2^2).$$

Thus, we find that

$$\begin{aligned} \psi'(t) + \psi''(t) &\geq -(a + c_p) (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2) - \left(\frac{2\sqrt{2}(\alpha-1)}{\alpha} + 1 \right) (\|\omega_t\|_2^2 + \|\varphi_t\|_2^2) \\ &\quad - b (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2)^{\gamma+1} - \frac{1}{\alpha} (\|\omega\|_\alpha^\alpha + \|\varphi\|_\alpha^\alpha) \\ &\quad + \int_{\Omega} (|\omega|^\alpha \ln |\omega| + |\varphi|^\alpha \ln |\varphi|) dx. \end{aligned}$$

By exploiting E to substitute the last term in the above equation, we get

$$\begin{aligned} &\int_{\Omega} (|\omega|^\alpha \ln |\omega| + |\varphi|^\alpha \ln |\varphi|) dx \\ &= -\alpha E(t) + \frac{\alpha}{2} (\|\omega_t\|_2^2 + \|\varphi_t\|_2^2) + \frac{a\alpha}{2} (\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2) \end{aligned}$$

$$+ \frac{\alpha b}{2(\gamma + 1)} \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right)^{\gamma+1} + \frac{1}{\alpha} \left(\int_{\Omega} (|\omega|^\alpha + |\varphi|^\alpha) dx \right).$$

Therefore, we arrive at

$$\begin{aligned} \psi'(t) + \psi''(t) &\geq -\alpha E(t) + \left(\frac{a\alpha}{2} - a - c_p \right) \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right) \\ &\quad + b \left(\frac{\alpha}{2(\gamma + 1)} - 1 \right) \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right)^{\gamma+1} \\ &\quad + \left[\frac{\alpha}{2} - \left(\frac{2\sqrt{2}(\alpha - 1)}{\alpha} + 1 \right) \right] \left(\|\omega_t\|_2^2 + \|\varphi_t\|_2^2 \right). \end{aligned}$$

As $E(t) < 0$, then $-\alpha E(t) > 0$, and for large α , we obtain

$$\psi'(t) + \psi''(t) \geq b \left(\frac{\alpha}{2(\gamma + 1)} - 1 \right) \left(\|\nabla \omega\|_2^2 + \|\nabla \varphi\|_2^2 \right)^{\gamma+1}.$$

It is well-known that

$$\int_{\Omega} |\nabla \omega|^2 dx \geq \frac{1}{c_p} \left(\int_{\Omega} |\omega|^2 dx \right) \quad \text{and} \quad \int_{\Omega} |\nabla \varphi|^2 dx \geq \frac{1}{c_p} \left(\int_{\Omega} |\varphi|^2 dx \right).$$

Then

$$\begin{aligned} \psi'(t) + \psi''(t) &\geq \frac{b}{c_p^{\gamma+1}} \left(\frac{\alpha}{2(\gamma + 1)} - 1 \right) \left(\int_{\Omega} (|\omega|^2 + |\varphi|^2) dx \right)^{\gamma+1} \\ &= b \left(\frac{2}{c_p} \right)^{\gamma+1} \left(\frac{\alpha}{2(\gamma + 1)} - 1 \right) \psi^{\gamma+1}(t). \end{aligned}$$

So, Lemma 4.1 is satisfied for $\mu = \gamma > 0$ and $C_0 = b \left(\frac{2}{c_p} \right)^{\gamma+1} \left(\frac{\alpha}{2(\gamma+1)} - 1 \right)$. This completes the proof.

5. Numerical results

In this section, we present numerical experiments that show the polynomial energy decay behavior for stable Kirchhoff systems and, for the unstable set, we show that the corresponding solutions blow up in finite time. The simulations verify the theoretical results for systems satisfying the stability condition $I(\omega_0, \phi_0) \geq 0$ with parameters $a \geq 1$, $b \geq 0$, $\gamma > 0$, and $\alpha \geq 2\gamma + 2$.

The numerical scheme is based on a finite difference discretization of the domain $[0, 1]$ with $N = 100$ grid points (so that $\Delta x = 1/(N-1) \approx 0.01$), time step $\Delta t = 0.001$, and final time $T = 6.0$ in order to capture the long-time polynomial decay behavior. The method is explicit and second-order accurate in both space and time: standard three-point central differences are used for the spatial derivatives, while the temporal discretization is a three-level leapfrog-type scheme for the second-order time derivative. The Kirchhoff nonlocal term Ψ is computed at each time step from the discrete gradient norms of ω and φ , and then treated as a scalar coefficient multiplying the discrete Laplacian. The logarithmic damping and source terms are evaluated pointwise and explicitly at the current time level.

5.1. Example 1

For the first example, we consider small amplitude initial conditions as follows: We choose the parameters: $a = 1.0$, $b = 0.5$, $\gamma = 1.0$, $\alpha = 4.0$, and the initial conditions:

$$\begin{aligned}\omega_0(x) &= \sin(\pi x) \\ \phi_0(x) &= 0.8 \sin(2\pi x) \\ \omega_1(x) &= 0.02 \cos(\pi x) \\ \phi_1(x) &= 0.01 \cos(2\pi x).\end{aligned}$$

Figure 1 shows the initial conditions and the solution profiles at $t = 6.0$. The stability functional evaluates to $I(\omega_0, \phi_0) = 1.514000 \geq 0$, confirming the system belongs to the stable set W . The initial energy is $E(0) = 1.3968$. The numerical simulation shows a polynomial energy decay with rate $p = 1.514$ as shown in Figure 2.

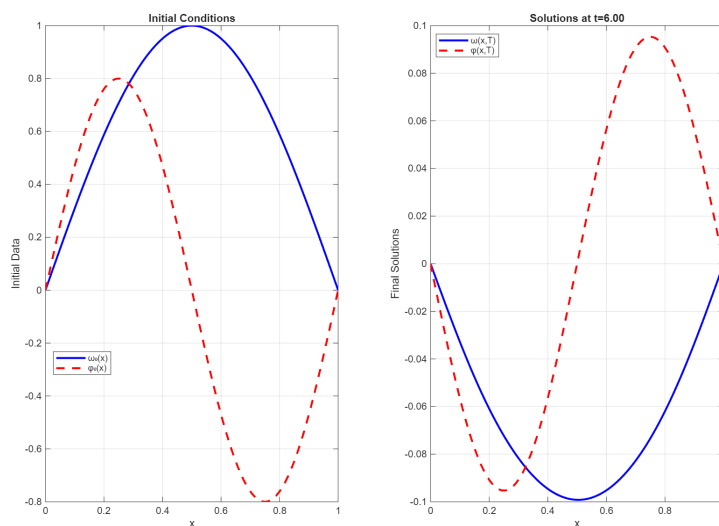


Figure 1. Example 1: Initial conditions (left), and final solutions at $t = 6.0$ (right).

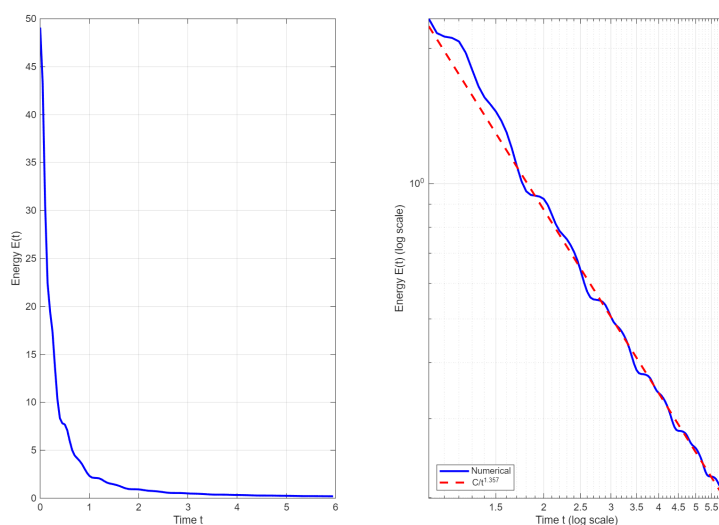


Figure 2. Example 1: Energy evolution on a linear scale (left) and log-log scale (right).

5.2. Example 2

The second example uses more complex initial data while maintaining stability. The initial conditions are:

$$\begin{aligned}\omega_0(x) &= \sin(3\pi x)e^{-2(x-0.5)^2} \\ \phi_0(x) &= \sin(2\pi x)(1-x^2) \\ \omega_1(x) &= 0.03 \cos(\pi x)e^{-(x-0.3)^2/0.1} \\ \phi_1(x) &= 0.02 \sin(4\pi x)x(1-x).\end{aligned}$$

Figure 3 shows the initial conditions and the solution profiles at $t = 6.0$. The stability analysis yields $I(\omega_0, \phi_0) = 1.357000 \geq 0$, confirming stability, with initial energy $E(0) = 1.2985$. Despite the increased complexity of the initial data, the system exhibits polynomial energy decay with rate $p = 1.357$ as shown in Figure 4.

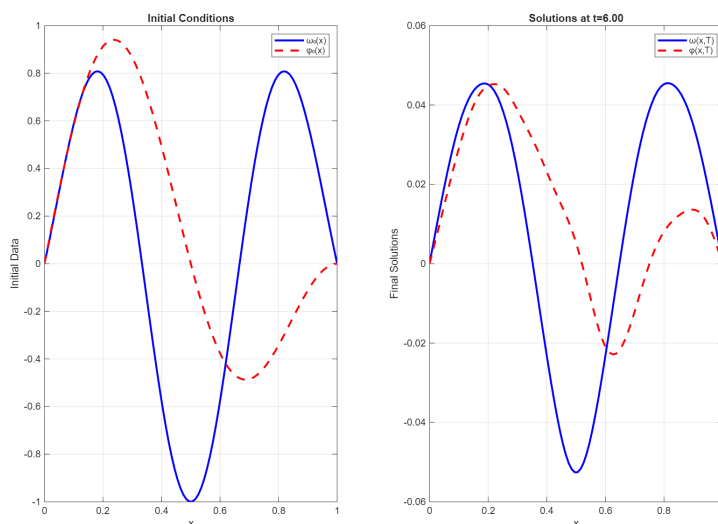


Figure 3. Example 2: Initial conditions (left), and final solutions at $t = 6.0$ (right).

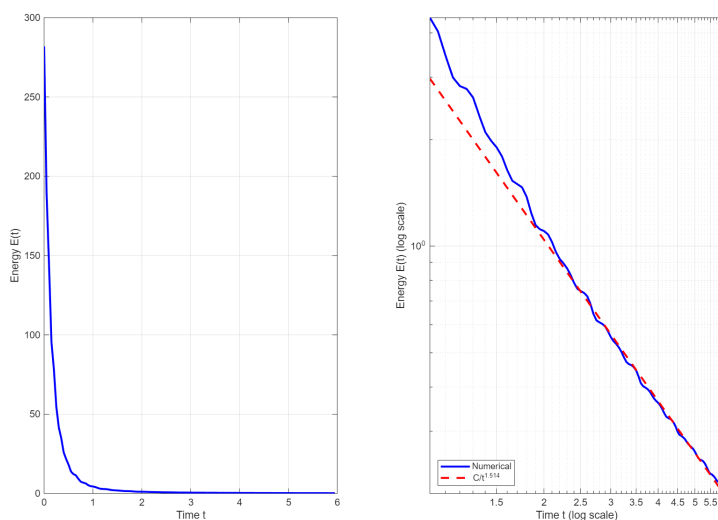


Figure 4. Example 2: Energy decay on a linear scale (left) and log-log scale (right).

Both examples demonstrate that initial data in the stable set W lead to polynomial energy decay of the form $E(t) \sim Ct^{-p}$ with positive decay rates $p > 0$.

5.3. Example 3

Here, we present a numerical example demonstrating finite-time blow-up as shown in Figure 5. We consider system (1.2) on $\Omega = (0, 1)$ with parameters $a = 1.0$, $b = 0.001$, $\gamma = 0.1$, and $\alpha = 2.5$. The small value of b is crucial for enabling the condition $I(\omega_0, \phi_0) < 0$. Using initial conditions $\omega_0(x) = 50 \sin(\pi x)$, $\phi_0(x) = 40 \sin(2\pi x)$, $\omega_1(x) = 3 \sin(\pi x)$, and $\phi_1(x) = 1.8 \sin(2\pi x)$, we obtain $I(\omega_0, \phi_0) = -2487.34 < 0$, which will make the system unstable.

Figure 5 shows the evolution of the blow-up solution: The left panel displays the maximum norm $\max\{|\omega(x, t)|, |\phi(x, t)|\}$, which increases rapidly and suggests an emergent singularity. The right panel tracks the Nehari functional $I(t)$, which stays negative for all computed times, starting from $I(0) = -2487.34$ and decreasing to about -1.3×10^4 at the final numerical time. This monotonic growth in $|I(t)|$ reflects that, as the amplitudes of ω and ϕ grow, the nonlinear logarithmic source terms in I dominate more and more strongly over the Kirchhoff contribution, so the trajectory moves deeper into the unstable set $V = \{(\omega, \phi) : I(\omega, \phi) < 0\}$. The computation is stopped when $|I(t)|$ becomes very large, and the explicit scheme can no longer be advanced reliably.

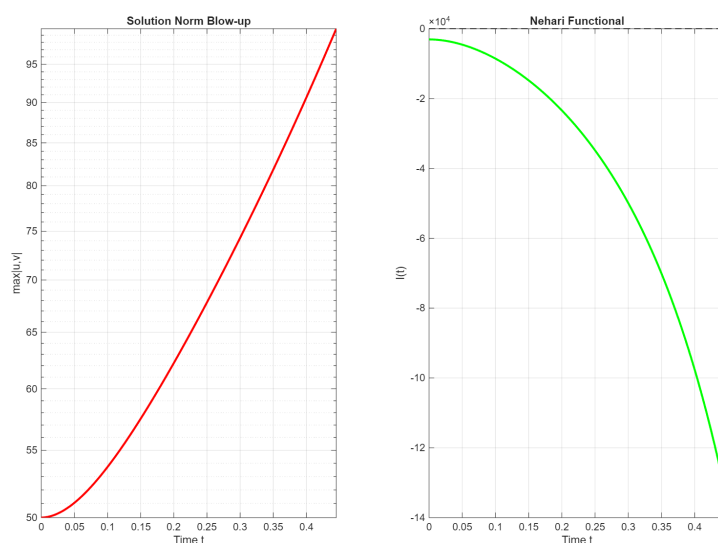


Figure 5. Example 3: Solution maximum norm evolution (left) and Nehari functional $I(t)$ (right).

Remark 5.1. *The numerical results show that the total energy satisfies $E(t) \sim Ct^{-p}$ with $p = 1.514$ in Example 1 and $p = 1.357$ in Example 2, that is, it decays only at a polynomial rate in time. This behavior is consistent with the fact that logarithmic damping is much weaker than classical nonlinear damping of polynomial type: It dissipates energy slowly, so that a non-negligible portion of the mechanical energy persists for long times and the oscillations decay only algebraically rather than exponentially.*

6. Conclusions

In this work, we have analyzed a coupled Kirchhoff-type wave system featuring both logarithmic damping and logarithmic source terms. The results highlight the delicate and nontrivial interplay between these two logarithmic effects, which act in opposition as dissipative and destabilizing mechanisms. By constructing an appropriate Lyapunov functional and employing refined analytical techniques adapted to logarithmic nonlinearities, we established a polynomial decay rate of the energy in the stable regime. On the other hand, through a concavity-based argument, we demonstrated that solutions arising from initial data in the unstable set undergo finite-time blow-up.

These findings contribute to a deeper understanding of how nonlocal Kirchhoff dynamics interact with weak (logarithmic) dissipation and growth terms, revealing a clear threshold behavior that separates global existence from singularity formation. The inclusion of numerical simulations further supports the theoretical analysis and provides insight into the qualitative behavior of solutions.

Overall, this study opens new directions for the investigation of wave systems with nonstandard damping and source mechanisms.

Author contributions

Adel M. Al-Mahdi contributed to the conception and design of the study, the development of the analytical framework, and the supervision of the research work. Mohammad Kafini contributed to the mathematical analysis, derivation of the main results, and drafting of the manuscript. Mohammad M. Al-Gharabli contributed to the stability analysis, verification of the proofs, and critical revision of the manuscript for important intellectual content. Zaid Sawlan contributed to the numerical simulations, interpretation of computational results, and preparation of the graphical illustrations.

All authors contributed to the interpretation of the results, participated in revising the manuscript critically for intellectual content, approved the final version of the manuscript to be published, and agreed to be accountable for all aspects of the work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia, is also greatly acknowledged. The authors sincerely thank the anonymous referees for their careful reading and valuable suggestions.

Funding

This work is funded by KFUPM under Project (INCB2613).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. G. R. Kirchhoff. *Vorlesungen über mathematische Physik: Mechanik*, Teubner, Leipzig, 1876.
2. R. W. Dickey. Infinite systems of nonlinear oscillation equations with linear damping, *SIAM J. Appl. Math.*, **19** (1970), 208–214. <https://doi.org/10.1137/0119019>
3. K. Ono, Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear kirchhoff strings, *J. Differ. Equ.*, **137** (1997), 273–301. <https://doi.org/10.1006/jdeq.1997.3263>
4. K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of kirchhoff type with a strong dissipation, *Math. Method. Appl. Sci.*, **20** (1997), 151–177. [https://doi.org/10.1002/\(SICI\)1099-1476\(19970125\)20:2<151::AID-MMA851>3.0.CO;2-0](https://doi.org/10.1002/(SICI)1099-1476(19970125)20:2<151::AID-MMA851>3.0.CO;2-0)
5. K. Ono, On global solutions and blow-up solutions of nonlinear kirchhoff strings with nonlinear dissipation, *J. Math. Anal. Appl.*, **216** (1997), 321–342. <https://doi.org/10.1006/jmaa.1997.5697>
6. R. Zeng, C. Mu, S. Zhou, A blow-up result for Kirchhoff-type equations with high energy, *Math. Method. Appl. Sci.*, **34** (2011), 479–486. <https://doi.org/10.1002/mma.1374>
7. G. Autuori, P. Pucci, M. C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, *Arch. Rational Mech. Anal.*, **196** (2010), 489–516. <https://doi.org/10.1007/s00205-009-0241-x>
8. M. M. Cavalcanti, V. N. D. Cavalcanti, J. A. Soriano, Global existence and uniform decay rates for the kirchhoff-carrier equation with nonlinear dissipation, *Adv. Differ. Equ.*, **6** (2001), 701–730. <https://doi.org/10.57262/ade/1357140586>
9. I. Chueshov. Long-time dynamics of kirchhoff wave models with strong nonlinear damping. *J. Differ. Equ.*, **252** (2012), 1229–1262. <https://doi.org/10.1016/j.jde.2011.08.022>
10. M. Xiang, V. D. Radulescu, B. Zhang, Nonlocal Kirchhoff diffusion problems: Local existence and blow-up of solutions, *Nonlinearity*, **31** (2018), 3228–3250. <https://doi.org/10.1088/1361-6544/aaba35>
11. M. Xiang, V. D. Radulescu, B. Zhang, Fractional kirchhoff problems with critical trudingger–moser nonlinearity, *Calc. Var.*, **58** (2019), 57. <https://doi.org/10.1007/s00526-019-1499-y>
12. J. Ferreira, M. L. Santos, Stability for a system of wave equations of kirchhoff with coupled nonlinear and boundary conditions of memory type, *Adv. Differ. Equ.*, **8** (2003), 873–896. <https://doi.org/10.57262/ade/1355926815>
13. S. T. Wu, On coupled nonlinear wave equations of Kirchhoff type with damping and source terms, *Taiwanese J. Math.*, **14** (2010), 585–610. <https://doi.org/10.11650/twjm/1500405808>
14. J. Y. Park, J. J. Bae, On existence of solutions of nondegenerate wave equations with nonlinear damping terms, *Nihonkai Math. J.*, **9** (1998), 27–46.
15. J. Y. Park, J. J. Bae, Variational inequality for quasilinear wave equations with nonlinear damping terms, *Nonlinear Anal.-Theor.*, **50** (2002), 1065–1083. [https://doi.org/10.1016/S0362-546X\(00\)00239-X](https://doi.org/10.1016/S0362-546X(00)00239-X)

16. L. Liu, M. Wang, Global existence and blow-up of solutions for some hyperbolic systems with damping and source terms, *Nonlinear Anal.-Theor.*, **64** (2006), 69–91. <https://doi.org/10.1016/j.na.2005.06.009>
17. C. Mu, J. Ma, On a system of nonlinear wave equations with balakrishnan–taylor damping, *Z. Angew. Math. Phys.*, **65** (2014), 91–113. <https://doi.org/10.1007/s00033-013-0324-2>
18. H. Ding, J. Zhou, Global existence and blow-up for a parabolic problem of kirchhoff type with logarithmic nonlinearity, *Appl. Math. Optim.*, **8** (2021), 1651–1707. <https://doi.org/10.1007/s00245-019-09603-z>
19. D. Hannia, Z. Khaled, New class of kirchhoff type equations with kelvin-voigt damping and general nonlinearity: Local existence and blow-up in solutions, *J. Partial. Differ. Equ.*, **34** (2021), 1–35.
20. Q. Lin, X. Tian, R. Xu, M. Zhang, Blow up and blow up time for degenerate kirchhoff-type wave problems involving the fractional laplacian with arbitrary positive initial energy, *Discrete Cont. Dyn-S*, **13** (2020), 2095–2107. <https://doi.org/10.3934/dcdss.2020160>
21. Y. Yang, Z. B. Fang. On a semilinear wave equation with Kirchhoff-type nonlocal damping terms and logarithmic nonlinearity, *Mediterr. J. Math.*, **20** (2023), 13. <https://doi.org/10.1007/s00009-022-02221-0>
22. B. Gheraibia, N. Boumaza, A. M. Al-Mahdi, M. M. Al-Gharabli, Asymptotic analysis of solutions for a p -Kirchhoff type hyperbolic equation with dynamic boundary conditions, *Mediterr. J. Math.*, **22** (2025), 164. <https://doi.org/10.1007/s00009-025-02934-y>
23. A. M. Al-Mahdi, The coupling system of Kirchhoff and Euler-Bernoulli plates with logarithmic source terms: Strong damping versus weak damping of variable-exponent type, *AIMS Mathematics*, **8** (2023), 27439–27459. <http://doi.org/10.3934/math.20231404>
24. A. M. Al-Mahdi, Optimal decay result for kirchhoff plate equations with nonlinear damping and very general type of relaxation functions, *Bound. Value Probl.*, **2019** (2019), 82. <https://doi.org/10.1186/s13661-019-1196-y>
25. S. H. Park, Global existence, energy decay and blow-up of solutions for wave equations with time delay and logarithmic source, *Adv. Differ. Equ.*, **2020** (2020), 631. <https://doi.org/10.1186/s13662-020-03037-6>
26. M. Kafini, M. M. Al-Gharabli, A. M. Al-Mahdi, Existence and stability results of nonlinear swelling equations with logarithmic source terms, *AIMS Mathematics*, **9** (2024), 12825–12851. <http://doi.org/10.3934/math.2024627>
27. A. M. Al-Mahdi, Stability result of a viscoelastic plate equation with past history and a logarithmic nonlinearity, *Bound. Value Probl.*, **2020** (2020), 84. <https://doi.org/10.1186/s13661-020-01382-9>
28. J. D. Barrow, P. Parsons, Inflationary models with logarithmic potentials, *Phys. Rev. D*, **52** (1995), 5576–5587. <https://doi.org/10.1103/physrevd.52.5576>
29. K. Enqvist, J. McDonald, Q-balls and baryogenesis in the MSSM, *Phys. Lett. B*, **425** (1998), 309–321. [https://doi.org/10.1016/s0370-2693\(98\)00271-8](https://doi.org/10.1016/s0370-2693(98)00271-8)
30. M. Bertsch, F. Smarrazzo, A. Tesei, Pseudoparabolic regularization of forward-backward parabolic equations: A logarithmic nonlinearity, *Anal. PDE*, **6** (2013), 1719–1754. <https://doi.org/10.2140/apde.2013.6.1719>

31. L. C. F. Ferreira, O. S. de Queiroz. A singular parabolic equation with logarithmic nonlinearity and l^p -initial data, *J. Differ. Equ.*, **249** (2010), 349–365. <https://doi.org/10.1016/j.jde.2010.03.019>
32. M. M. Al-Gharabli, A. Guesmia, S. Messaoudi, Existence and a general decay results for a viscoelastic plate equation with a logarithmic nonlinearity. *Commun. Pure Appl. Anal.*, **18** (2019), 159–180. <http://doi.org/10.3934/cpaa.2019009>
33. X. Wang, Y. Chen, Y. Yang, J. Li, R. Xu, Kirchhoff-type system with linear weak damping and logarithmic nonlinearities, *Nonlinear Anal.*, **188** (2019), 475–499. <https://doi.org/10.1016/j.na.2019.06.019>
34. D. Li, C. Zhang, H. Zhang, Energy decay estimates for the wave equation with logarithmic feedback, *Math. Method. Appl. Sci.*, **48** (2025), 10110–10113. <https://doi.org/10.1002/mma.10871>
35. S. A. Messaoudi, A. M. Al-Mahdi, Well-posedness and asymptotic stability of a one-dimensional timoshenko system with a logarithmic damping, *Discrete Cont. Dyn.-S*, **21** (2025), 1–16. <https://doi.org/10.3934/dcdss.2025155>
36. A. M. Al-Mahdi, M. M. Al-Gharabli. On the well-posedness and stability of piezoelectric beams with magnetic effects, logarithmic damping, and fourier-type thermal conduction. *J. Therm. Stresses*, **5** (2025), 712–727. <https://doi.org/10.1080/01495739.2025.2603426>
37. A. M. Al-Mahdi, M. M. Al-Gharabli, On the existence and long-time behavior of solutions to swelling models with weak logarithmic damping mechanism: Optimal polynomial decay rate, *Math. Method. Appl. Sci.*, 2026. <https://doi.org/10.1002/mma.70507>
38. M. M. Al-Gharabli, A. M. Al-Mahdi, M. Kafini, Global existence and new decay results of a viscoelastic wave equation with variable exponent and logarithmic nonlinearities, *AIMS Mathematics*, **6** (2021), 10105–10129. <https://doi.org/10.3934/math.2021587>
39. M. M. Al-Gharabli, A. M. Al-Mahdi, S. A. Messaoudi, Decay results for a viscoelastic problem with nonlinear boundary feedback and logarithmic source term, *J. Dyn. Control Syst.*, **28** (2022), 71–89. <https://doi.org/10.1007/s10883-020-09522-1>
40. A. M. Al-Mahdi, M. M. Al-Gharabli, N. E. Tatar, On a nonlinear system of plate equations with variable exponent nonlinearity and logarithmic source terms: Existence and stability results, *AIMS Mathematics*, **8** (2023), 19971–19992. <https://doi.org/10.3934/math.20231018>
41. M. M. Al-Gharabli, S. A. Messaoudi, Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term, *J. Evol. Equ.*, **18** (2018), 105–125. <https://doi.org/10.1007/s00028-017-0392-4>
42. E. Pişkin, S. Boulaaras, N. Irkil, Qualitative analysis of solutions for the p -laplacian hyperbolic equation with logarithmic nonlinearity, *Math. Method. Appl. Sci.*, **44** (2021), 4654–4672. <https://doi.org/10.1002/mma.7058>
43. Y. Zhou, Global existence and nonexistence for a nonlinear wave equation with damping and source terms, *Math. Nachr.*, **278** (2005), 1341–1358. <https://doi.org/10.1002/mana.200310310>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)