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Research article

Distance spectral radius and $[a, b]$-factor of graphs

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Abstract: Let H be a spanning subgraph of G. If every vertex $v \in V(G)$ satisfies $a \leq d_H(v) \leq b$, then H is called an $[a, b]$-*factor* of G. In 2005, Brouwer and Haemers pioneered the spectral approach for investigating 1-factors in regular graphs. Since this work, adjacency eigenvalue conditions for $[a, b]$-factors have been extensively studied. In this paper, we provided some conditions based on the distance spectral radius that ensured the existence of an $[a, b]$-factor in a connected graph and a balanced bipartite graph, respectively.

Keywords: perfect matching; $[a, b]$-factor; distance spectral radius
Mathematics Subject Classification: 05C50

1. Introduction

Let H be a spanning subgraph of G. If every vertex $v \in V(G)$ satisfies $a \leq d_H(v) \leq b$, then H is called an $[a, b]$-*factor* of G. Early foundational work on factors dates back to 1891, when Petersen [1] showed that any graph in which every vertex has even degree can be decomposed into a collection of edge-disjoint cycles. In the past few decades, many researchers have utilized adjacency eigenvalues to investigated $[a, b]$-factors. Brouwer and Haemers [2] used the third largest adjacency eigenvalue for regular graphs to admit a perfect matching, which was subsequently enhanced in [3–5]. Subsequently, O. Suil [6] generalized the work of Brouwer and Haemers to graphs that are not necessarily regular. He provided a spectral radius condition in connected graphs to admit a perfect matching. In 2010, Lu et al. [7] provided a third largest adjacency eigenvalue condition for a graph to contain an odd $[1, b]$-factor. Later, the result was further enhanced by Kim et al. [8]. Li and Miao [9] studied the edge condition and the spectral radius condition in a graph to contain an odd $[1, b]$-factor. Afterward, Fan et al. [10] extended the results in [9] by incorporating the minimum degree of a graph. For more results on spectral conditions for odd $[1, b]$-factors, we refer the reader to [7, 11]. In addition, many researchers have also investigated spectral conditions for the existence of other factors, such as k-factors [12, 13], parity factors [14, 15], and fractional $[a, b]$-factors [16].

Let $A(G)$ be the *adjacency matrix* of a graph G. The *spectral radius* of G is the largest eigenvalue of $A(G)$, denoted by $\rho(G)$. Suppose that $G_1 \vee G_2$ is a graph obtained from $G_1 \cup G_2$ by adding all possible edges between them. The following conjecture was proposed by Cho et al. [17] in 2021.

Conjecture 1.1 ([17]). *Suppose that $a \cdot n$ is an even integer with $n \geq a + 1$ and G is a graph of order n. If*

$$\rho(G) > \rho(K_{a-1} \vee (K_{n-a} \cup K_1)),$$

then G contains an $[a, b]$-factor.

Around this conjecture, Fan et al. [10] provided a partial result for order $n \geq b + 3a - 1$. Conjecture 1.1 was subsequently resolved by Wei and Zhang [18]. Recently, Hao and Li [19] demonstrated that $K_{a-1} \vee (K_{n-a} \cup K_1)$ in Conjecture 1.1 is essentially the best possible.

For $u_i, u_j \in V(G)$, let $d_G(u_i, u_j)$ (or d_{ij}) be the distance between u_i and u_j. Suppose that $D(G) = (d_{ij})$ is the distance matrix of G, whose (i, j)-entry is d_{ij}. The distance spectral radius of G is the largest eigenvalue of $D(G)$, denoted by $\rho_D(G)$. Motivated by the works in [10, 18, 19], it is natural to consider Conjecture 1.1 from the perspective of the distance spectral radius. In this paper, we first establish a condition on the distance spectral radius that guarantees a graph to admit an $[a, b]$-factor.

Theorem 1.2. *Suppose that a and b are two integers with $b \geq a \geq 1$. If G is a connected graph of order $n \geq 2a + 12$, and*

$$\rho_D(G) \leq \rho_D(K_{a-1} \vee (K_{n-a} \cup K_1)),$$

then G contains an $[a, b]$-factor, unless $G \cong K_{a-1} \vee (K_{n-a} \cup K_1)$.

An (X, Y)-bipartite graph with $|X| = |Y|$ is defined as *balanced*. Let $K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1})$ be the graph obtained from $K_{\frac{n}{2},\frac{n}{2}}$ by removing all edges of a star $K_{1,\frac{n}{2}-a+1}$, where $K_{\frac{n}{2},\frac{n}{2}}$ is a complete balanced bipartite graph of order n. In this paper, we also give an analogous result in bipartite graphs.

Theorem 1.3. *Suppose that a and b are two integers with $b \geq a \geq 1$. If G is a connected balanced bipartite graph of order $n \geq 6a + 4$, and*

$$\rho_D(G) \leq \rho_D(K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1})),$$

then G contains an $[a, b]$-factor, unless

$$G \cong K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1}).$$

2. Preliminaries

For subsequent proofs, we list some necessary lemmas in this section. The following result can be deduced by the Perron-Frobenius theorem.

Lemma 2.1. *Suppose that G is a connected graph. If $G - e$ is connected for $e \in E(G)$, then*

$$\rho_D(G - e) > \rho_D(G).$$

Assume that $V(G) = \{v_1, v_2, \cdots, v_n\}$. Thus, $W(G) = \sum_{i<j} d_G(v_i, v_j)$ is the Weiner index of a connected graph G, and

$$\rho_D(G) = \max_{\mathbf{x} \in R^n} \frac{\mathbf{x}^T D(G) \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geq \frac{\mathbf{1}^T D(G) \mathbf{1}}{\mathbf{1}^T \mathbf{1}} \geq \frac{2W(G)}{n}.$$

Lemma 2.2. *If a and b are two integers with $a \geq b \geq 2$, then*

$$\binom{b-1}{2} + \binom{a+1}{2} > \binom{b}{2} + \binom{a}{2}.$$

Proof. Since $a \geq b \geq 2$, we get

$$\binom{b-1}{2} + \binom{a+1}{2} - \left(\binom{b}{2} + \binom{a}{2}\right) = -(b-a) + 1 > 0,$$

as required. □

Lemma 2.3 ([20])**.** *Suppose that n and a are positive integers. If $n \geq 2a + 6$, then*

$$\rho_D(K_{a-1} \vee (K_{n-a} \cup K_1)) < n + 2.$$

Choose a subset S of $V(G)$, and let $G[S]$ be the subgraph of G induced by S and $G-S = G[V(G)\backslash S]$. For $u \in V(G)$ and $S \subseteq V(G)$, let $N_S(u) = N_G(u) \cap S$ and $d_S(u) = |N_S(u)|$. For $X, Y \subset V(G)$, let $E_G(X, Y)$ be the set of edges of G having one endpoint in X and the other in Y, where $e(S) = |E(G[S])|$ and $e_G(X, Y) = |E_G(X, Y)|$. In 1952, Tutte [21] got the following k-factor theorem.

Lemma 2.4 (k-factor theorem [21])**.** *Suppose that k is a positive integer. A graph G contains a k-factor if, and only if,*

$$\phi_G(S, T) = \sum_{u \in T} d_G(u) + k(|S| - |T|) - e_G(S, T) - \delta_G(S, T) \geq 0,$$

for all disjoint subsets $T, S \subseteq V(G)$, where $\delta_G(S, T)$ denotes the number of components C of $G-(S \cup T)$ such that $e_G(V(C), T) + k|C| \equiv 1 (\mathrm{mod}\ 2)$. Furthermore, $\phi_G(S, T) \equiv k|V(C)| (\mathrm{mod}\ 2)$.

The structure of (g, f)-factors was investigated by Lovász [22] in 1970.

Lemma 2.5 ((g, f)-factor theorem [22])**.** *Suppose that f and g are two integer-valued functions defined on the vertex set of a graph G satisfying $f(u) \geq g(u)$ for each $u \in V(G)$. Thus, G contains a (g, f)-factor if, and only if,*

$$\sum_{v \in S} f(v) + \sum_{u \in T} (d_G(u) - g(u)) - e_G(S, T) - \delta_G(S, T) \geq 0$$

for all disjoint subsets $T, S \subseteq V(G)$, where $\delta_G(S, T)$ is the number of components C of $G-(S \cup T)$ such that $f(u) = g(u)$ for each $u \in V(C)$ and $e_G(V(C), T) + \sum_{u \in V(C)} f(u) \equiv 1 (\mathrm{mod}\ 2)$.

When $0 < a < b$, if $f(u) = b$ and $g(u) = a$ for each $u \in V(G)$, then we can deduce the following result.

Corollary 2.6. *Suppose that a and b are two integers with $b > a \geq 1$. Thus, a graph G contains an $[a, b]$-factor if, and only if,*

$$\phi_G(S, T) = \sum_{u \in T} d_G(u) + b|S| - a|T| - e_G(S, T) \geq 0$$

for all disjoint subsets $T, S \subseteq V(G)$.

In 1970, Folkman and Fulkerson [23] gave the following conditions that guarantee a bipartite graph to contain a (g, f)-factor.

Lemma 2.7 ((g, f)-factor theorem in bipartite graphs [23]). *Suppose that f and g are two integer-valued functions defined on the vertex set of a bipartite graph G satisfying $f(u) \geq g(u)$ for each $u \in V(G)$. Thus, $G = (X, Y)$ contains a (g, f)-factor if, and only if,*

$$\sum_{u \in T} (d_G(u) - g(u)) + \sum_{u \in S} f(u) - e_G(S, T) \geq 0$$

and

$$\sum_{u \in S} (d_G(u) - g(u)) = \sum_{u \in T} f(u) - e_G(T, S) \geq 0$$

for all $S \subseteq X$ and $T \subseteq Y$.

Corollary 2.8. *Suppose that a and b are two integers with $b \geq a \geq 1$. Thus, a bipartite graph $G = (X, Y)$ contains an $[a, b]$-factor if, and only if, for all $S \subseteq X$ and $T \subseteq Y$,*

$$\sum_{u \in T} d_G(u) + b|S| - a|T| - e_G(S, T) \geq 0,$$

and

$$\sum_{u \in S} d_G(u) + b|T| - a|S| - e_G(S, T) \geq 0.$$

Lemma 2.9 ([20]). *Suppose that n and a are two positive integers. If $n \geq 4a + 4$, then*

$$\rho_D(K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1}) < \frac{3n}{2} + 1.$$

3. Proof of Theorem 1.2

Suppose, to the contrary, that G is a graph that contains no $[a, b]$-factors. Choose disjoint subsets S and T of $V(G)$. Let

$$\phi_G(S, T) = \begin{cases} \sum_{u \in T} d_{G-S}(u) + b|S| - a|T|, & \text{if } b > a \geq 1, \\ \sum_{u \in T} d_{G-S}(u) + b|S| - \delta_G(S, T) - a|T|, & \text{if } b = a, \end{cases}$$

where $\delta_G(S, T)$ is the number of components C of $G-(T \cup S)$ satisfying $e_G(V(C), T) + a|C| \equiv 1 (\bmod\ 2)$. Furthermore, $\phi_G(S, T) \equiv a|V(C)| (\bmod\ 2)$ when $a = b$. By Lemma 2.4 and Corollary 2.6, there are disjoint subsets S and T of $V(G)$ satisfying $|T \cup S|$ as large as possible with

$$\phi_G(S, T) \leq \begin{cases} -1, & \text{if } b > a \geq 1, \\ -2, & \text{if } b = a. \end{cases}$$

Suppose that there are $q_G(S, T)$ components of $G - (S \cup T)$. Let $q_G(S, T) = q$. Then, $\delta_G(S, T) \leq q$. Assume that $|S| = s$ and $|T| = t$. Thus,

$$\sum_{u \in T} d_{G-S}(u) \leq at - bs - 1 \tag{3.1}$$

for $b > a$ and

$$\sum_{u\in T} d_{G-S}(u) \le at - bs + q - 2 \tag{3.2}$$

for $a = b$.

According to Lemma 2.3, we have

$$\rho_D(G) \le \rho_D(K_{a-1} \vee (K_{n-a} \cup K_1)) < n + 2. \tag{3.3}$$

Note that

$$W(G) \ge \frac{1}{2} \sum_{u\in V(G)} (d_G(u) + 2(n - 1 - d_G(u))) = n^2 - n - e(G).$$

Then,

$$\rho_D(G) \ge \frac{2W(G)}{n} \ge 2n - 2 - \frac{2e(G)}{n}.$$

By (3.3), we have

$$e(G) > \frac{n^2 - 4n}{2}. \tag{3.4}$$

Claim 1. *$d_G(u) \ge a$ for each $u \in V(G)$.*

If not, there exists some vertex $z \in V(G)$ satisfying $d_G(z) \le a - 1$. Note that $K_{a-1} \vee (K_1 \cup K_{n-a})$ contains G as a spanning subgraph. By Lemma 2.1, we have

$$\rho_D(G) \ge \rho_D(K_{a-1} \vee (K_{n-a} \cup K_1)),$$

with equality holding if, and only if, $G \cong K_{a-1} \vee (K_{n-a} \cup K_1)$, which contradicts the condition of Theorem 1.2. Then, $d_G(u) \ge a$ for each $u \in V(G)$.

Claim 2. *$t \ge s + 1$.*

Otherwise, $t \le s$. If $b > a$, then

$$0 \le \sum_{u\in T} d_{G-S}(u) \le at - bs - 1 \le as - bs - 1 < 0$$

due to (3.1), which is impossible. If $b = a$, by (3.2), we have

$$0 \le \sum_{u\in T} d_{G-S}(u) \le at - as + q - 2,$$

and so

$$q \ge as - at + 2. \tag{3.5}$$

Suppose that $C_1, C_2, \cdots, C_q$ are the components of $G - (T \cup S)$. Thus,

$$\sum_{i=1}^{q} e_G(C_i, T) \le \sum_{u\in T} d_{G-S}(u) \le at - as + q - 2 \le q - 2$$

due to $t \leq s$. Therefore, there exist at least two components, say C_1 and C_2, such that $e_G(C_1, T) = 0$ and $e_G(C_2, T) = 0$. Assume that $|C_2| \geq |C_1|$. If $|C_1| = 1$, let $S' = S$ and $T' = T \cup V(C_1)$. Thus, $|S'| = s$, $|T'| = |T| + |C_1| = t + 1$ and $q_G(S', T') = q - 1$, and so

$$\begin{aligned}\sum_{u\in T'} d_{G-S'}(u) &= \sum_{u\in T} d_{G-S}(u)\\ &\leq at - as + q - 2\\ &= a(|T'| - |C_1|) - a|S'| + q_G(S', T') - 1\\ &= a(|T'| - |S'|) + q_G(S', T') - 2 - (a|C_1| - 1)\\ &\leq a(|T'| - |S'|) + q_G(S', T') - 2 \quad (\text{since } a \geq 1 \text{ and } |C_1| = 1).\end{aligned}$$

This is inconsistent with the maximality of $|T \cup S|$. If $|C_1| \geq 2$, we assert that $d_{G-S}(u) \geq a + 1$ for $u \in V(C_1 \cup C_2)$. Otherwise, there exists $x \in V(C_1 \cup C_2)$ satisfying $d_{G-S}(x) \leq a$. Suppose that $S' = S$ and $T' = T \cup \{x\}$. Thus, $|S'| = s$, $|T'| = t + 1$ and $q_G(S', T') = q$. Combining this with $d_{G-S}(x) \leq a$ and (3.2), we get

$$\begin{aligned}\sum_{u\in T'} d_{G-S'}(u) &= \sum_{u\in T} d_{G-S}(u) + d_{G-S}(x)\\ &\leq \sum_{u\in T} d_{G-S}(u) + a\\ &\leq at - as + q - 2 + a\\ &= a(|T'| - 1) - a|S'| + q_G(S', T') - 2 + a\\ &= a(|T'| - |S'|) + q_G(S', T') - 2,\end{aligned}$$

which also leads to a contradiction. It follows that $d_{G-S}(u) \geq a + 1$ for $u \in V(C_1 \cup C_2)$. Combining this with $e_G(C_i, T) = 0$, we have $d_{C_i}(u) \geq a + 1$ for $u \in V(C_i)$, where $i = 1, 2$. Thus, $|C_2| \geq |C_1| \geq a + 2$, and so $n = s + t + \sum_{i=1}^{q} |C_i| \geq s + t + 2a + q + 2$. By Lemma 2.2, $s \geq t$, $a = b$, and by (3.2), we get

$$\begin{aligned}e(G) &= e(S) + e_G(S, V(G)\backslash S) + \sum_{i=1}^{q} e(C_i) + \sum_{u\in T} d_{G-S}(u)\\ &\leq \binom{s}{2} + s(n - s) + \sum_{i=1}^{q} \binom{|C_i|}{2} + at - as + q - 2\\ &\leq \binom{s}{2} + s(n - s) + \binom{n-s-t-2a-q-1}{2} + 2\binom{a+2}{2} + at - as + q - 2\\ &= \frac{n^2 - 4n}{2} - \gamma(n),\end{aligned}$$

where

$$\begin{aligned}\gamma(n) &= \Big(2a+t+q-\frac{1}{2}\Big)n-1-\frac{t^2}{2}-\Big(3a+s+q+\frac{3}{2}\Big)t-s-\frac{q^2+5q}{2}-(3a+2q+s+6)a-sq\\ &\geq \frac{q^2}{2}+(t+2a-1)q-\frac{3s}{2}+at+\frac{t^2}{2}+a^2+sa-3a-2 \quad (\text{since } n \geq s + t + 2a + q + 2)\\ &\geq \frac{q^2}{2}+(t+2a-2)q + \Big(2a-\frac{3}{2}\Big)s+\frac{t^2}{2}+a^2 - 3a \quad (\text{by (3.5)})\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^2}{2}+(t+2a-2)q+\frac{3s(a-1)}{2}+\frac{t^2+as+2a^2-6a}{2}\\
&> 0 \quad (\text{since } t \geq a+1,\ s \geq t \text{ and } a \geq 1).
\end{aligned}
$$

Thus, $e(G) < \frac{n^2-4n}{2}$, and this contradicts (3.4). Therefore, $t \geq s+1$.

The following two cases are now presented.

Case 1. $b > a$.

We assert that $s \geq 1$. Otherwise, $s = 0$. By Claim 1 and (3.1), we obtain that

$$at \leq \sum_{u\in T} d_G(u) = \sum_{u\in T} d_{G-S}(u) \leq at - 1,$$

a contradiction. Thus, $s \geq 1$. Again by Claim 1, we have $d_{G-S}(u) \geq a - s$. According to (3.1), we get

$$(a-s)t \leq \sum_{u\in T} d_{G-S}(u) \leq at - bs - 1.$$

Then, $t \geq b + \frac{1}{s}$. Combining this with $s \geq 1$, we get $t \geq b+1$. Suppose that $U = V(G)\backslash(T \cup S)$. Thus, $\sum_{u\in U} d_T(u) \leq \sum_{u\in T} d_{G-S}(u)$, $d_G(u) \leq n-1$ for any $u \in S$ and $d_S(u) + d_U(u) \leq n-t-1$ for any $u \in U$. Combining this with (3.1), we get

$$
\begin{aligned}
2W(G) &\geq \sum_{u\in V(G)} (d_G(u) + 2(n-1-d_G(u)))\\
&= \sum_{u\in V(G)} (2n-2-d_G(u))\\
&= \sum_{u\in S}(2n-2-d_G(u))+\sum_{u\in T}(2n-2-d_G(u))+\sum_{u\in U}(2n-2-d_G(u))\\
&\geq s(n-1)+\sum_{u\in T}(2n-2-s-d_{G-S}(u))+\sum_{u\in U}(2n-2-(n-t-1)-d_T(u))\\
&= s(n-1)+(2n-2-s)t-\sum_{u\in T} d_{G-S}(u)+(n+t-1)(n-t-s)-\sum_{u\in U} d_T(u)\\
&\geq s(n-1)+(2n-2-s)t+(n+t-1)(n-t-s)-2\sum_{u\in T} d_{G-S}(u)\\
&\geq s(n-1)+(2n-2-s)t+(n+t-1)(n-(t+s))+2bs+2-2at\\
&= n(n+2)+(2t-3)n-2at+2sb-2ts-t^2-t+2.
\end{aligned}
$$

Now we show that $2W(G) > n(n+2)$. Let $f(n) = (2t-3)n - 2at + 2sb - 2ts - t^2 - t + 2$. If $t \geq 2b+5$, then

$$
\begin{aligned}
f(n) &= (2t-3)n-2at+2sb-2ts-t^2-t+2\\
&\geq (2t-3)(s+t)-2at+2sb-2ts-t^2-t+2\\
&\quad (\text{since } t \geq b+1,\ n \geq t+s \text{ and } b > a)\\
&= (t-(2a+4))t+(2b-3)s+2\\
&\geq (t-(2a+5))t+2(b-1)s+3 \quad (\text{since } t \geq s+1)\\
&> 0 \quad (\text{since } t \geq 2b+5,\ b > a \text{ and } s \geq 0).
\end{aligned}
$$

If $b+1 \le t \le 2b+4$, then

$$\begin{aligned} f(n) &= (2t-3)n - 2at - 2ts - t^2 - t + 2sb + 2 \\ &= (t-2)n + (t-1)n - 2at - 2ts - t^2 - t + 2sb + 2 \\ &\ge (t-2)(t+s) + 2(t-1)(a+6) - 2at - 2ts - t^2 - t + 2sb + 2 \\ &\quad (\text{since } n \ge \max\{s+t, 2a+12\},\ t \ge b+1 \text{ and } b > a \ge 1) \\ &= (2b-t)s + 9t - 2s - 2a - 10 \\ &= (2b+4-t)s + 6(t-s) + 3t - 2a - 10 \\ &\ge (2b+4-t)s + 3t - 2a - 4 \quad (\text{since } t \ge s+1) \\ &> 0 \quad (\text{since } b+1 \le t \le 2b+4 \text{ and } b > a \ge 1). \end{aligned}$$

Hence, $f(n) > 0$, and so $2W(G) > n^2 + 2n$. By Lemma 2.3, we get

$$\rho_D(G) \ge \frac{2W(G)}{n} > n+2 > \rho_D(K_{a-1} \vee (K_1 \cup K_{n-a})),$$

a contradiction.

Case 2. $a = b$.

If $q \le 1$, then by $a = b$ and (3.2), we get

$$\sum_{u \in T} d_{G-S}(u) \le at - bs + q - 2 \le at - as - 1.$$

By using the same analysis as Case 1, we can deduce a contradiction. Thus, we consider $q \ge 2$ in the following. Note that $e_G(C_i, C_j) = 0$ and $|C_i| \ge 1$ for $1 \le i \ne j \le q$. Then, $d_U(u) \le |U| - 1 - (q-1) = n - s - t - q$, and so $d_G(u) = d_T(u) + d_S(u) + d_U(u) \le d_T(u) + n - t - q$ for $u \in U$. Combining this with (3.2), $q \ge 2$, $a = b$, $\sum_{u \in U} d_T(u) \le \sum_{u \in T} d_{G-S}(u)$, and $n \ge s + t + q \ge s + t + 2$, we have

$$\begin{aligned} 2W(G) &\ge \sum_{u \in V(G)} (d_G(u) + 2(n-1-d_G(u))) \\ &= \sum_{u \in V(G)} (2n-2-d_G(u)) \\ &= \sum_{u \in S}(2n-2-d_G(u)) + \sum_{u \in T}(2n-2-d_G(u)) + \sum_{u \in U}(2n-2-d_G(u)) \\ &\ge s(n-1) + \sum_{u \in T}(2n-2-s-d_{G-S}(u)) + \sum_{u \in U}(2n-2-(n-t-q+d_T(u))) \\ &= s(n-1) + (2n-(s+2))t - \sum_{u \in T} d_{G-S}(u) + (n+q+t-2)(n-s-t) - \sum_{u \in U} d_T(u) \\ &\ge s(n-1) + (2n-(s+2))t + (n+t+q-2)(n-(t+s)) - 2\sum_{u \in T} d_{G-S}(u) \\ &\ge s(n-1) + (2n-(s+2))t + (n+t+q-2)(n-(t+s)) - 2(a(t-s)+q-2) \\ &= s(n-1) + (2n-(s+2))t + (n+t)(n-(t+s)) - 2a(t-s) + (q-2)(n-(s+t+2)) \\ &\ge s(n-1) + (2n-(s+2))t + (n+t)(n-(t+s)) - 2a(t-s) \quad (\text{since } n \ge t+s+2 \text{ and } q \ge 2) \\ &= n(n+2) + (2t-2)n - (2a+2s+t+2)t + (2a-1)s. \end{aligned}$$

Let $g(n) = (2t-2)n - (2a+2s+t+2)t + (2a-1)s$. If $t \geq 2a+3$, then

$$\begin{aligned} g(n) &= (2t-2)n - (2a+2s+t+2)t + (2a-1)s \\ &\geq 2(t-1)(t+s+2) - (2a+2s+t+2)t + (2a-1)s \quad (\text{since } n \geq t+s+2 \text{ and } t \geq 2a+3) \\ &= t^2 - 2at + 2as - 3s - 4 \\ &\geq 2(a-1)s + (t-2a-3)t + 2t - 3 \quad (\text{since } t \geq s+1) \\ &> 0 \quad (\text{since } t \geq 2a+3 \text{ and } a \geq 1). \end{aligned}$$

If $t \leq 2a+2$, then

$$\begin{aligned} g(n) &= (2t-2)n - 2at + 2as - 2ts - t^2 - 2t - s \\ &= (t-2)n + tn - 2at + 2as - 2ts - t^2 - 2t - s \\ &\geq (t-2)(s+t+2) + t(2a+12) - 2at + 2as - 2ts - t^2 - 2t - s \\ &\quad (\text{since } n \geq \max\{s+t+2, 2a+12\},\ t \geq a+1 \text{ and } a \geq 1) \\ &= 4(t-s-1) + s(2a+2-t) + 6t - s \\ &> 0 \quad (\text{since } t \geq s+1 \text{ and } t \leq 2a+2). \end{aligned}$$

Thus, $g(n) > 0$, and so $2W(G) > n^2 + 2n$. By Lemma 2.3, we have

$$\rho_D(G) \geq \frac{2W(G)}{n} > n+2 > \rho_D(K_{a-1} \vee (K_{n-a} \cup K_1)),$$

which also leads to a contradiction.

4. Proof of Theorem 1.3

Suppose, to the contrary, that $G = (X, Y)$ is a balanced bipartite graph that contains no $[a, b]$-factors. According to Corollary 2.8, there are two subsets $S \subseteq X$ and $T \subseteq Y$ with $|S| = s$ and $|T| = t$, satisfying

$$\Theta_1(S, T) = bs + \sum_{u \in T} d_{G-S}(u) - at \leq -1, \tag{4.1}$$

or

$$\Theta_2(S, T) = bt + \sum_{u \in S} d_{G-T}(u) - as \leq -1. \tag{4.2}$$

Claim 3. *$d_G(u) \geq a$ for each $u \in V(G)$.*

If not, there is a vertex $w \in V(G)$ satisfying $d_G(w) \leq a-1$. Note that $K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1})$ contains G as a spanning subgraph. By Lemma 2.1, we get

$$\rho_D(G) \geq \rho_D(K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1})),$$

with equality holding if, and only if, $G \cong K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1})$, a contradiction. It follows that $d_G(u) \geq a$ for each $u \in V(G)$. □

By Lemma 2.9, we have

$$\rho_D(K_{\frac{n}{2},\frac{n}{2}} \backslash E(K_{1,\frac{n}{2}-a+1})) < \frac{3n}{2} + 1. \tag{4.3}$$

Notice that

$$W(G) \geq \frac{1}{2}\sum_{i=1}^{n}\Big(d_i + 2\Big(\frac{n}{2}-1\Big) + 3\Big(\frac{n}{2}-d_i\Big)\Big) = \frac{1}{2}\Big(\frac{5n}{2}-2\Big)n - 2e(G).$$

Thus,

$$\rho_D(G) \geq \frac{2W(G)}{n} \geq \frac{5n}{2} - 2 - \frac{4e(G)}{n}.$$

By (4.3), we get

$$e(G) > \frac{n^2-3n}{4}. \tag{4.4}$$

If $\Theta_1(S,T) \leq -1$, then we can deduce the following claim.

Claim 4. (i) $1 \leq s \leq t-1$; (ii) $t \geq b+1$.

We first prove that $t \geq s+1$. Otherwise, $t \leq s$. By (4.1), we get

$$0 \leq \sum_{u\in T} d_{G-S}(u) \leq at - bs - 1 \leq (a-b)s - 1 \leq -1$$

due to $b \geq a$, a contradiction. It follows that $t \geq s+1$. If $s = 0$, by (4.1) and Claim 3, we get

$$at \leq \sum_{u\in T} d_G(u) = \sum_{u\in T} d_{G-S}(u) \leq at - 1,$$

which is impossible. Thus, $s \geq 1$, and so Claim 4 (i) is true. By Claim 3, $d_{G-S}(u) \geq a - s$ for $u \in T$. Combining this with (4.1), we deduce that

$$t(a-s) \leq \sum_{u\in T} d_{G-S}(u) \leq at - bs - 1,$$

and so $t \geq b + \frac{1}{s}$. Since $s \geq 1$, we have $t \geq b+1$. Thus, Claim 4 (ii) holds. □

Note that

$$\begin{aligned} e(G) &= e_G(X\backslash S, Y\backslash T) + e_G(S,Y) + e_G(X\backslash S, T) \\ &\leq \Big(\frac{n}{2}-s\Big)\Big(\frac{n}{2}-t\Big) + \frac{sn}{2} + at - bs - 1 \quad \text{(by (4.1))} \\ &= \frac{n^2-3n}{4} - \frac{1}{4}((2t-3)n - 4at - 4st + 4bs + 4). \end{aligned}$$

Let $h(n) = (2t-3)n - 4at - 4st + 4bs + 4$. If $s \leq 2a-1$, then

$$\begin{aligned} h(n) &= (2t-3)n - 4at - 4s(t-b) + 4 \\ &\geq (2t-3)(6a+4) - 4at - 4(2a-1)(t-b) + 4 \quad (\text{since } s \leq 2a-1,\ n \geq 6a+4 \text{ and } t \geq b+1) \\ &= 12t + 4ab + 4b(a-1) - 18a - 8 \\ &\geq 12b + 4 + 4ab + 4b(a-1) - 18a \quad (\text{since } t \geq b+1) \\ &> 0 \quad (\text{since } b \geq a \geq 1). \end{aligned}$$

Thus, $e(G) < \frac{n^2-3n}{4}$, and this contradicts (4.4). Therefore, $s \geq 2a$. Let $p = t - s$. Then, $p \geq 1$ by Claim 4. Since $T \subseteq Y$, we have $\frac{n}{2} \geq t$, and so $n \geq 2t$. If $b > a$, then

$$\begin{aligned} h(n) &= (2t-3)n - 4at - 4st + 4sb + 4 \\ &\geq (2s+2p-3)(2s+2p) - 4a(s+p) - 4s(s+p) + 4sb + 4 \quad (\text{since } n \geq 2t \text{ and } t = s+p) \\ &= 2s(p + 2b - 2a - 3) + 4p^2 - 6p + 4 + 2p(s - 2a) \\ &> 0 \quad (\text{since } b > a,\ s \geq 2a \text{ and } p \geq 1). \end{aligned}$$

If $a = b$, then we assert that $n \geq 2t + 2$. Otherwise, $n = 2t$ because n is even. Therefore, $t = \frac{n}{2}$, and so $T = Y$ and $|X \backslash S| = \frac{n}{2} - s = p$. By Claim 3, (4.1), and $a = b$, we get

$$ap \leq \sum_{u \in X \backslash S} d_G(u) = \sum_{u \in Y} d_{G-S}(u) = \sum_{u \in T} d_{G-S}(u) \leq at - bs - 1 = a(t-s) - 1 = ap - 1,$$

a contradiction. It follows that $n \geq 2t + 2$. Combining this with $a = b$, $t \geq s + 1$, and $s \geq 2a$, we get

$$\begin{aligned} h(n) &= (2t-3)n - 4at - 4st + 4sb + 4 \\ &\geq (2t-3)(2t+2) - 4at - 4(t-p)t + 4(t-p)a + 4 \\ &\quad (\text{since } n \geq 2t+2,\ t = s+p \text{ and } a = b) \\ &= t(2p-2) + 2p(t-2a) - 2 \\ &\geq 0 \quad (\text{since } p \geq 1,\ s \geq 2a \text{ and } t \geq s+1). \end{aligned}$$

Based on the above, we get $h(n) \geq 0$ for $b \geq a$. Thus, $e(G) \leq \frac{n^2-3n}{4}$, a contradiction.

If $\Theta_2(S, T) \leq -1$, by a similar analysis as above, we can also obtain the contradiction.

5. Conclusions

An $[a, b]$-factor of a graph is a spanning subgraph with vertex degrees bounded between a and b. Following Brouwer and Haemers' spectral approach to 1-factors in regular graphs, many existing works focus on adjacency eigenvalue conditions for $[a, b]$-factors. Utilizing the Weiner index, we provide distance spectral radius conditions to ensure the existence of a $[a, b]$-factor in connected graphs and balanced bipartite graphs, respectively.

Author contributions

The first author completed the main theoretical derivation and drafted the original manuscript. The corresponding author was responsible for research conceptualization, methodology design, as well as the critical review and revision of the manuscript. Other co-authors participated in partial theoretical reasoning and manuscript polishing. All authors have read and approved the final version of the manuscript for submission.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the composition of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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