Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 12334–12359.
DOI:10.3934/math.2026506
Received: 23 February 2026
Revised: 11 April 2026
Accepted: 28 April 2026
Published: 06 May 2026

Research article

Quantile regression for cloud model parameter estimation: a robust approach to uncertainty quantification

Weidong Rao[1,*], **Peiyang Cai**[2], **Wenjuan Li**[3] **and Hankun Guo**[1]

[1] Jiangxi Science and Technology Normal University, School of Mathematical Sciences, Nanchang 330038, China

[2] Washington University in St. Louis, Statistics and Data Science, 1 Brookings Dr, St. Louis, MO 63130, USA

[3] Yunnan University of Finance and Economics, School of Statistics and Mathematics, Yunnan 650221, China

* **Correspondence:** Email: raoweidong126@126.com.

Abstract: As an important tool for characterizing uncertain information, the precision and robustness of cloud model parameter estimation directly affect the reliability of knowledge representation. Existing backward cloud generation algorithms were mostly based on moment estimation or resampling strategies, being sensitive to outliers and unable to fully utilize distributional morphological information in data. This paper proposes a quantile regression-based method, which achieves robust joint estimation of the three numerical features—expected value, entropy, and hyper-entropy—by constructing optimized matching relationships between sample quantiles and theoretical quantiles of cloud models. This method leverages the semiparametric adaptability of quantile regression to distributional morphology and the outlier resistance of median estimation, obtaining consistent parameter estimates without strict distributional assumptions, with median estimation possessing a theoretical breakdown point of fifty percent. Theoretical analysis proves the strong consistency and asymptotic normality of estimators; simulation experiments demonstrate that compared with traditional moment estimation and resampling methods, this algorithm exhibits superior estimation accuracy, algorithmic stability, and comprehensive cloud distance indicators under scenarios including point contamination, scale inflation, and asymmetric tail contamination. This method provides a new statistical perspective and reliable tool for cloud model applications in complex data environments.

Keywords: cloud model; backward cloud generator; quantile regression; robust parameter estimation
Mathematics Subject Classification: 62F10, 62F35, 68T37

1. Introduction

With the rapid development of information technology and the explosive growth of data scale, the representation and processing of uncertain knowledge have become core challenges in artificial intelligence, machine learning, and data science [1]. Traditional methods such as probability theory, fuzzy sets, and rough sets each have their own emphasis in describing the randomness and fuzziness of concepts, yet they exhibit significant limitations: Fuzzy set methods rely on subjective membership functions, the law of excluded middle in probability theory does not fully align with the flexible boundaries of natural language concepts, and rough sets may suffer from information loss or overfitting due to the precise requirements of upper and lower approximations [2].

To effectively address these issues, the cloud model, as a cognitive computation model integrating probability theory and fuzzy mathematics, has attracted widespread attention in recent years. Academician Li Deyi proposed the cloud model in 1995, which characterizes the randomness, fuzziness, and their correlation of concepts through three numerical features—expected value (Ex), entropy (En), and hyper-entropy (He) [3]. It has been successfully applied in decision analysis and intelligent control [4, 5], fuzzy cluster analysis [6, 7], risk assessment [8, 9], and environmental evaluation [10, 11]. The backward cloud generator (BCG), as the core technology of the cloud model, estimates numerical features from sample data and serves as a crucial bridge connecting qualitative concepts with quantitative data, with its implementation accuracy directly determining the effectiveness of cloud model applications [12, 13].

However, existing BCG algorithms still face severe challenges in complex data environments. Early algorithms such as SBCT-1stM, based on first-order absolute central moments [14], and SBCT-4thM, based on fourth-order central moments [15], adopt single-step moment estimation strategies, which tend to produce significant estimation biases in small-sample scenarios and are sensitive to outliers. When dealing with high-fuzziness data (i.e., large ratio of hyper-entropy to entropy), traditional algorithms often fail to provide stable and reliable estimation results. The multistep BCG algorithm based on resampling (MBCT-SR) proposed by Xu et al. [13] improves parameter estimation accuracy through the combination of grouped sampling and moment estimation, but the additional grouping settings substantially increase algorithm complexity and computational cost, while the sensitivity of resampling strategies to grouping parameters limits its practical applicability.

To address the limitations of existing methods in small-sample, high-fuzziness, and outlier-contaminated scenarios, this paper proposes a cloud model parameter estimation method based on quantile regression, termed quantile regression-based cloud generator (QRCG). Quantile regression possesses inherent robustness, capable of effectively resisting outlier interference while providing complete information about conditional distributions, offering a new statistical perspective for cloud model parameter estimation. The core contributions of this paper include:

- Establishing the theoretical connection between quantile regression and cloud model parameter estimation, and proposing a cloud model identification framework based on quantile matching;
- Designing adaptive quantile weighting and robust optimization strategies to achieve joint robust estimation of expected value, entropy, and hyper-entropy;
- Systematically validating the superiority of the proposed method in estimation accuracy and robustness through comprehensive experiments.

The remainder of this paper is organized as follows: Section 2 reviews the cloud model and traditional backward cloud generation algorithms. Section 3 presents the proposed quantile regression-based cloud generator; Section 4 establishes the theoretical properties, including strong consistency and asymptotic normality; Section 5 provides comprehensive simulation experiments and empirical analysis; Section 6 validates the method through real-world clustering applications; and Section 7 concludes the paper and outlines future research directions.

2. Preliminaries: Cloud model and BCG

The cloud model is a cognitive computational model for representing and processing uncertain concepts. By integrating the randomness of probability theory with the fuzziness of fuzzy mathematics, this model employs three numerical features—expected value (Ex), entropy (En), and hyper-entropy (He)—to characterize qualitative concepts in a unified manner. Specifically, the expected value represents the core position of a concept in its universe of discourse, indicating the most typical quantitative sample of the qualitative concept. The entropy measures the degree of uncertainty of the concept, reflecting the acceptable numerical range. The hyper-entropy measures the randomness of the entropy itself, indicating the dispersion degree of the concept's uncertainty. The distinctive value of the cloud model lies in its capability to simultaneously handle both randomness and fuzziness, providing an effective tool for modeling complex uncertain information [12].

Definition 1 (Cloud and cloud drop)**.** *Let U be the quantitative universe of discourse represented by precise numerical values, and C be a qualitative concept on U. If there exists a value $x \in U$ such that x is a random realization of C, and the certainty degree $\mu(x) \in [0,1]$ of x to C is a random number with stable tendency,*

$$\mu : U \to [0,1], \quad \forall x \in U, \quad x \to \mu(x),$$

then the distribution of x on U is called a cloud*, and each x is called a* cloud drop.

In particular, if x satisfies $x \sim \mathcal{N}(Ex, y^2)$, where $y \sim \mathcal{N}(En, He^2)$, and the certainty degree μ satisfies:

$$\mu = \exp\left(-\frac{(x-Ex)^2}{2y^2}\right),$$

then the distribution of x on U is called a *normal cloud* (or *Gaussian cloud*). In this case, the cloud model C can be denoted as $C(Ex, En, He)$.

From Definition 1, the generation of normal clouds involves a dual random structure: first generating a random entropy y following $\mathcal{N}(En, He^2)$, then generating a cloud drop x following $\mathcal{N}(Ex, y^2)$ with y as the standard deviation. The probability density function can be obtained by integrating over y [13, 16]:

$$f_X(x) = \frac{1}{2\pi He} \int_{-\infty}^{+\infty} \frac{1}{|y|} \exp\left(-\frac{(x-Ex)^2}{2y^2} - \frac{(y-En)^2}{2He^2}\right) dy. \tag{2.1}$$

Based on Eq (2.1), the normal cloud random variable X exhibits the following statistical properties [13, 15]:

(1) Expectation: $E[X] = Ex$.
(2) Variance: $D[X] = En^2 + He^2$.

(3) Third central moment: $E[(X - Ex)^3] = 0$.
(4) Fourth central moment: $E[(X - Ex)^4] = 3(3He^4 + 6He^2En^2 + En^4)$.

The *forward cloud generator* (FCG) is an algorithm that generates a large number of cloud drops (samples) from the numerical features (Ex, En, He) of a concept. For one-dimensional normal clouds, the classical algorithm proceeds as follows:

(1) Generate a random number $y_i \sim \mathcal{N}(En, He^2)$.
(2) Generate a random number $x_i \sim \mathcal{N}(Ex, y_i^2)$.
(3) Calculate the certainty degree $\mu_i = \exp\left(-(x_i - Ex)^2/(2y_i^2)\right)$.
(4) Repeat steps 1–3 until N cloud drops $\{(x_i, \mu_i)\}_{i=1}^N$ are generated.

Through this process, the FCG transforms qualitative concepts into quantitative data, achieving mathematical representation and computational simulation of uncertain concepts.

The BCG is the inverse process of the FCG. Its core task is to estimate the numerical features $(\hat{Ex}, \hat{En}, \hat{He})$ that generated given cloud drop samples $\{x_i\}_{i=1}^n$, completing the cognitive transformation from data to knowledge. The estimation accuracy and stability of the BCG directly determine the reliability of cloud models in practical applications, making it a crucial component in the cloud model methodology system.

Traditional BCG algorithms are primarily based on the method of moments, including the first-order absolute central moment method (SBCT-1stM) and the fourth-order central moment method (SBCT-4thM). These methods estimate numerical features by matching sample moments to population moments, but exhibit significant limitations when facing small samples, high fuzziness, or nonuniform distributions.

The SBCT-1stM algorithm [14] utilizes the sample mean, variance, and first-order absolute central moment for parameter estimation:

$$\hat{Ex} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
$$\hat{En} = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{X}|,$$
$$\hat{He} = \sqrt{S^2 - \hat{En}^2},$$

where S^2 is the sample variance. The derivation assumes $y > 0$. When He is large such that y may be negative (i.e., $3He > En$), this estimation is biased and may lead to $\hat{He}^2 < 0$.

The SBCT-4thM algorithm [15] utilizes the sample mean and fourth-order central moment:

$$\hat{Ex} = \bar{X},$$
$$\hat{En} = \sqrt[4]{(9S^4 - \bar{\mu}_4)/6},$$
$$\hat{He} = \sqrt{S^2 - \hat{En}^2},$$

where $\bar{\mu}_4$ is the sample fourth-order central moment. When He approaches 0, $9S^4 < \bar{\mu}_4$ easily occurs, leading to imaginary $\hat{En}$ and algorithm failure. Additionally, it is sensitive to small samples.

Both single-step algorithms directly apply moment estimation to raw samples, exhibiting poor robustness when He is small or sample size is insufficient. To overcome the deficiencies of single-step algorithms, Xu et al. [13] proposed the multistep backward cloud transformation algorithm based on stratified resampling (MBCT-SR) as in Algorithm 1. This algorithm estimates En and He through grouped resampling rather than direct calculation from raw samples. The core idea is to partition samples into m groups, calculate the sample variance within each group, and then perform moment estimation based on these group variances. This method effectively improves estimation robustness in small-sample scenarios, but still relies on sample grouping and may face high computational complexity in high-dimensional cases.

Algorithm 1 MBCT-SR Algorithm [13]

Require: Cloud drop samples $X = \{x_1, x_2, \ldots, x_n\}$.

Ensure: Estimated numerical features $(\hat{Ex}, \hat{En}, \hat{He})$.

1: Calculate sample mean: $\hat{Ex} = \bar{X}$.

2: Grouped resampling: Randomly resample the original samples and partition into m groups, each with r samples (repetition allowed). For the i-th group X_i, calculate its within-group sample variance Y_i^2.

3: Moment estimation: Based on m group variance values $\{Y_i^2\}$, calculate their sample mean $\hat{E}Y^2$ and sample variance $\hat{D}Y^2$. Estimate by solving moment equations:

$$\hat{En}^2 = \frac{1}{2}\sqrt{4(\hat{E}Y^2)^2 - 2\hat{D}Y^2},$$
$$\hat{He}^2 = \hat{E}Y^2 - \hat{En}^2.$$

4: Take the arithmetic square root of the result of the above formula to get $(\hat{En}, \hat{He})$.

3. Quantile regression: Methodology and theoretical foundation

Quantile regression is a statistical method that estimates conditional quantile functions by minimizing weighted absolute residuals. Unlike traditional least squares methods that focus on conditional means, the proposed QRCG method leverages quantile regression to characterize the conditional distribution of response variables at any probability level, providing a systematic theoretical framework for robust estimation in complex data environments.

3.1. Quantile regression framework

For a given quantile level $\tau \in (0, 1)$, quantile regression seeks optimal parameters $\theta(\tau)$ such that theoretical quantiles achieve optimal fit with observed data. For a sample set $\{x_i\}_{i=1}^n$, the τ-quantile parameter estimation can be formulated as the following optimization problem:

$$\hat{\theta}(\tau) = \arg\min_{\theta\in\Theta} \sum_{i=1}^{n} \rho_\tau\left(x_i - q_\tau(\theta)\right), \tag{3.1}$$

where $q_\tau(\theta)$ denotes the theoretical quantile function based on parameter θ, and $\rho_\tau(u)$ is the quantile loss function (also known as the check function).

The loss function guides parameter estimation toward the specified quantile through an asymmetric weighting mechanism. The essence of this optimization framework lies in directly measuring the distance between empirical and theoretical quantiles, rather than making inferences through indirect moment conditions or distributional assumptions, granting it natural flexibility in handling complex distributional structures such as skewness and heavy tails.

When outliers or extreme observations exist in the data, quantile regression can effectively suppress their interference with parameter estimation, with the robustness advantage being particularly significant when focusing on central quantiles such as the median. Theoretically, median regression ($\tau = 0.5$) has a breakdown point of 50%, meaning the estimator remains stable even when half of the data are contaminated. This property makes quantile regression especially suitable for applications where data quality cannot be fully guaranteed.

The parameter estimation problem of cloud models exhibits natural compatibility with quantile regression methods. Cloud models characterize the uncertainty of concepts through three numerical features—Ex, En, and He—where the accuracy of parameter estimation directly determines the effectiveness of uncertain knowledge representation. Traditional estimation methods mostly rely on sample moment calculations, easily producing estimation biases or even failures in scenarios with high fuzziness or outlier interference. The introduction of quantile regression provides a new theoretical perspective for cloud model parameter estimation, effectively addressing challenges in complex data environments.

Specifically, the numerical magnitudes of En and He in cloud models directly reflect the degree of data uncertainty. When En and He take high values, cloud drop distributions exhibit stronger dispersion and heavy-tail characteristics, making traditional estimation methods based on second-order or fourth-order moments highly sensitive to extreme samples and prone to estimation biases. Quantile regression can maintain parameter estimation stability in high-uncertainty data environments by focusing on robust location measures such as the median, thereby reducing the weight of outliers in the objective function.

More critically, the cloud drop distribution of cloud models contains a dual randomness structure, with its marginal distribution showing significant tail heterogeneity. By estimating conditional quantiles at different probability levels, quantile regression can precisely capture local features of the distribution, providing a natural framework for simultaneous identification of the three numerical features of cloud models:

- The median quantile ($\tau = 0.5$) corresponds to robust estimation of the expected value Ex;
- The interquartile range (difference between $\tau = 0.75$ and $\tau = 0.25$) has an analytical relationship with the range of conceptual fuzziness measured by En;
- Differences in extreme quantiles (e.g., $\tau = 0.9$ versus $\tau = 0.1$) can reflect the uncertainty of entropy characterized by He.

This multilevel estimation structure makes quantile regression an ideal tool for cloud model parameter estimation, enabling the decomposition of complex parameter identification problems into a series of quantile-specific optimization subproblems.

3.2. Quantile regression-based cloud model parameter estimation

There exists an inherent analytical relationship between the numerical features of cloud models and the quantile structure of data distributions. Traditional moment estimation methods rely on sample means and variances being sensitive to tail observations and susceptible to outlier interference. Quantile regression provides a more robust theoretical framework for cloud model parameter estimation by directly characterizing the evolution of conditional quantiles. This section utilizes the analytical properties of quantile functions for the marginal distribution of cloud models, constructs a distance measure between sample quantiles and theoretical quantiles, and achieves parameter estimation by minimizing this distance.

For a given set of quantile levels $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_K\} \subset (0, 1)$, the optimization objective seeks optimal parameters $\theta = (Ex, En, He)$ such that theoretical τ-quantiles $q_\tau(\theta)$ achieve optimal fit with empirical quantiles $\hat{q}_\tau$. The marginal distribution of normal cloud models exhibits symmetric unimodal characteristics, with its probability density function given in Eq (2.1). Although the quantile function lacks an explicit closed-form solution, it can be accurately computed through numerical integration or Monte Carlo simulation. The quantile regression-based estimation framework can be formulated as the following optimization problem [17]:

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \sum_{k=1}^{K} w_k \cdot \rho_{\tau_k} \left(\hat{q}_{\tau_k} - q_{\tau_k}(\theta)\right), \tag{3.2}$$

where w_k is the weight coefficient for quantile level τ_k, and $\Theta = \{(Ex, En, He) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+\}$ is the parameter space. $q_\tau(\theta)$ lacks a closed-form expression due to the complex integral form of the marginal density in Eq (2.1), which is evaluated numerically via Monte Carlo integration with $M = 10^4$ samples in our implementation.

The weights w_k can be set as $w_k = \tau_k(1 - \tau_j)/\sum_j \tau_j(1 - \tau_j)$ to emphasize fitting accuracy at central quantiles where estimation is more robust, or as uniform weights $w_\tau = 1$ to treat all quantile levels equally. The $\tau(1-\tau)$ weighting scheme downweights extreme quantiles with higher sampling variance, offering advantages in heavy-tailed or contaminated scenarios, while uniform weights may improve efficiency for clean, symmetric distributions by utilizing full distributional information. Empirical comparisons show that QRCG maintains stable performance across both specifications, with relative advantages depending on the contamination type and severity.

The essence of this framework lies in directly measuring the systematic deviation between empirical and theoretical quantiles, rather than making inferences through indirect moment conditions. This direct matching strategy confers semiparametric robustness to distributional misspecification while avoiding the numerical instability associated with high-order moment calculations.

Based on this theoretical framework, we propose the QRCG algorithm as in Algorithm 2. This algorithm achieves simultaneous estimation of the three numerical features of cloud models by minimizing the weighted distance between sample quantiles and theoretical quantiles. The proposed algorithm first estimates $\hat{Ex} = \hat{q}_{0.5}$ and then optimizes only (En, He) with Ex fixed. This staged approach offers both theoretical and computational advantages. Theoretically, for symmetric distributions such as the normal cloud, the sample median is a consistent and asymptotically efficient estimator of the location parameter Ex; moreover, the median possesses a 50% breakdown point, ensuring robust estimation of Ex even under severe contamination. Fixing Ex at its median estimate prevents the propagation of location errors into the subsequent estimation of the scale parameters (En, He). Computationally, reducing the optimization from three dimensions to two significantly improves convergence speed and numerical stability. The staged procedure decouples the estimation of location and scale, allowing the optimization for (En, He) to be performed on a well-centered cloud while avoiding the additional complexity of a full three-parameter search.

Algorithm 2 QRCG algorithm

Require: Cloud drop samples $X = \{x_1, x_2, \ldots, x_n\}$; quantile level set $\mathcal{T}$; convergence tolerance $\epsilon > 0$.
Ensure: Parameter estimates $(\hat{Ex}, \hat{En}, \hat{He})$.

1: **Step 1: Compute empirical quantiles**
Sort samples X and calculate empirical quantiles at each $\tau \in \mathcal{T}$:

$$\hat{q}_\tau = \inf\{x : \hat{F}_n(x) \geq \tau\},$$

where $\hat{F}_n(x)$ is the empirical distribution function.

2: **Step 2: Construct optimization objective**
Define the quantile distance function:

$$Q(Ex, En, He) = \sum_{\tau \in \mathcal{T}} w_\tau \cdot (\hat{q}_\tau - q_\tau(Ex, En, He))^2. \tag{3.3}$$

Theoretical quantiles $q_\tau(Ex, En, He)$ are obtained by numerically solving

$$\tau = \int_{-\infty}^{q_\tau} \int_{-\infty}^{+\infty} \frac{1}{2\pi He|y|} \exp\left(-\frac{(t - Ex)^2}{2y^2} - \frac{(y - En)^2}{2He^2}\right) dydt.$$

3: **Step 3: Staged parameter optimization**
Utilizing median robustness, set directly $\hat{Ex} = \hat{q}_{0.5}$. Fixing $\hat{Ex}$, simplify the objective to

$$(\hat{En}, \hat{He}) = \underset{En>0, He>0}{\arg\min} \sum_{\tau \in \mathcal{T}} w_\tau \cdot (\hat{q}_\tau - q_\tau(\hat{Ex}, En, He))^2.$$

4. Theoretical properties

This section establishes the asymptotic statistical theoretical foundation for the QRCG algorithm estimators. Under mild regularity conditions, the estimators possess strong consistency and asymptotic normality.

Assumption 4.1 (Sample independence)**.** *The cloud drop samples* $\{x_i\}_{i=1}^n$ *are independent and identically distributed from cloud model* $C(Ex_0, En_0, He_0)$.

Assumption 4.2 (Parameter identifiability)**.** *True parameters* $\theta_0 = (Ex_0, En_0, He_0)$ *lie in the interior of a compact parameter space* $\Theta = [Ex_{\min}, Ex_{\max}] \times [\underline{En}, \overline{En}] \times [\underline{He}, \overline{He}]$*, where* $Ex_{\min}, Ex_{\max} \in \mathbb{R}$ *with* $Ex_{\min} < Ex_{\max}$*, and* $\underline{En}, \underline{He} > 0$*, ensuring strict positivity of entropy and hyper-entropy.*

Assumption 4.3 (Quantile level selection)**.** *The quantile level set* $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_K\} \subset (0, 1)$ *satisfies: (i)* $K \geq 3$ *with* $0.5 \in \mathcal{T}$*; (ii) the Jacobian matrix* $J(\theta_0) = \nabla_\theta q(\theta_0) \in \mathbb{R}^{K\times 3}$ *is of full column rank, i.e.,* $\text{rank}(J(\theta_0)) = 3$*; and (iii) the weight matrix* $W = \text{diag}(w_{\tau_1}, \ldots, w_{\tau_K})$ *is positive definite with* $w_{\tau_k} > 0$ *for all* $k = 1, \ldots, K$*.*

The three assumptions above possess generality and reasonableness in the context of cloud model parameter estimation. Assumption 4.1 requires independent and identically distributed samples, which constitutes the fundamental premise of classical statistical inference and is naturally satisfied in controlled experiments or random sampling scenarios; this assumption excludes serial correlation or heterogeneity interference, ensuring that empirical quantiles possess standard asymptotic properties.

Assumption 4.2 restricts parameters to the interior of a compact set in the positive parameter space, a specification consistent with the physical interpretation of cloud models—entropy and hyper-entropy, as positive-valued parameters measuring uncertainty, necessarily have bounded ranges in practical applications. The compactness condition guarantees the existence of extremum estimators by the Weierstrass extreme value theorem, while the strict positivity constraints $\underline{En}, \underline{He} > 0$ ensure numerical stability in the computation of theoretical quantiles, as $He \to 0$ would cause degeneracy in the marginal density in Eq (2.1).

Assumption 4.3 imposes three essential requirements on the quantile level set: The inclusion of the median ensures robust identification of the expected value parameter; the full-rank condition on the Jacobian matrix $J(\theta_0)$ guarantees that the information matrix $J(\theta_0)^{\mathrm{T}} W J(\theta_0)$ is invertible, which is necessary for the asymptotic normality of the estimator; and the positive definiteness of the weight matrix W ensures that the optimization problem is well-posed and that the asymptotic covariance matrix $\Sigma(\theta_0)$ is properly defined. These conditions are easily satisfied in practice; for instance, $\mathcal{T} = \{0.25, 0.5, 0.75\}$ with uniform weights $w_\tau = 1$ satisfies all requirements when the cloud model is nondegenerate.

In practical implementation, we employ the complete set of sample quantiles $\mathcal{T}_n = \{\tau_k = k/n\}_{k=1}^{n-1}$ to ensure asymptotic efficiency. This choice is motivated by the following considerations: (i) for uniform distributions on $[0, 1]$, the order statistics $X_{(k)}$ are unbiased estimators of k/n-quantiles, minimizing the asymptotic variance of empirical quantile processes; (ii) as $n \to \infty$, $\mathcal{T}_n$ becomes dense in $(0, 1)$, guaranteeing that the Jacobian matrix $J(\theta_0)$ maintains full column rank and that the information matrix $J(\theta_0)^{\mathrm{T}} W J(\theta_0)$ remains well-conditioned; and (iii) the median $\tau = 0.5$ is always included when $n \geq 3$, satisfying the robustness requirement for Ex estimation.

Based on these three mild regularity conditions, the asymptotic theoretical properties of QRCG estimators can be established.

Theorem 4.1 (Strong consistency)**.** *Under Assumptions 4.1–4.3, the QRCG estimator* $\hat{\theta}_n = (\hat{Ex}, \hat{En}, \hat{He})$ *is strongly consistent:*

$$\hat{\theta}_n \xrightarrow{a.s.} \theta_0 \quad (n \to \infty).$$

Theorem 4.2 (Asymptotic normality)**.** *Under Assumptions 4.1–4.3, and assuming continuity and differentiability of the cloud model marginal density* $f(x; \theta)$ *at all quantile points* $q_\tau(\theta_0)$ *for* $\tau \in \mathcal{T}$*,*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma(\theta_0)),$$

where the asymptotic covariance matrix is

$$\Sigma(\theta_0) = \left[J(\theta_0)^{\mathrm{T}} W J(\theta_0)\right]^{-1} J(\theta_0)^{\mathrm{T}} W \Omega W J(\theta_0) \left[J(\theta_0)^{\mathrm{T}} W J(\theta_0)\right]^{-1},$$

with $J(\theta) = \nabla_\theta q(\theta) \in \mathbb{R}^{K\times 3}$ *being the Jacobian matrix of theoretical quantiles,* $W = \mathrm{diag}(w_{\tau_1}, \ldots, w_{\tau_K}) \in \mathbb{R}^{K\times K}$ *the positive definite weight matrix, and* $\Omega \in \mathbb{R}^{K\times K}$ *the asymptotic covariance matrix of empirical quantiles with elements*

$$\omega_{ij} = \frac{\tau_i \wedge \tau_j - \tau_i\tau_j}{f(q_{\tau_i};\theta_0)f(q_{\tau_j};\theta_0)},$$

where $f(x;\theta)$ *denotes the marginal probability density function of the cloud model random variable X, given by the integral representation in Eq* (2.1)*. The invertibility of* $J(\theta_0)^{\mathrm{T}}WJ(\theta_0) \in \mathbb{R}^{3\times 3}$ *is guaranteed by Assumption 4.3(ii) and (iii).*

Theorem 4.1 guarantees that as sample size increases, the estimators approach the true parameters, providing theoretical support for method consistency. Theorem 4.2 provides the theoretical foundation for constructing parameter confidence intervals and hypothesis testing. A detailed proof is provided in Appendix A.

Remark on the Jacobian matrix. The theoretical quantile function $q_\tau(\theta)$ of the cloud model marginal distribution is continuously differentiable with respect to $\theta = (Ex, En, He)$ for $En > 0$ and $He > 0$; hence, the Jacobian matrix $J(\theta) = \nabla_\theta q(\theta)$ exists and is well-defined. However, no closed-form expression is available due to the integral representation in (2.1). The QRCG algorithm itself does not require computing $J(\theta)$ because the optimization problem is solved using derivative-free methods (e.g., Nelder-Mead) after fixing $\hat{E}x = \hat{q}_{0.5}$. For readers interested in post-estimation statistical inference (e.g., constructing confidence intervals via the delta method), $J(\theta)$ can be approximated numerically by finite differences. Specifically, for a small perturbation $\delta = 10^{-6}$, apply a one-sided or central difference to each parameter component, recompute the theoretical quantiles $q_\tau(\theta \pm \delta e_i)$ via Monte Carlo integration, and then obtain the gradient estimate. This numerical scheme is stable in the interior of the parameter space $\{En > 0, He > 0\}$.

Compared with moment estimation methods (SBCT-1stM, SBCT-4thM) and the MBCT-SR algorithm described in Section 2, the QRCG method exhibits three significant advantages:

(1) Outlier robustness. Moment estimation methods based on second-order or fourth-order moments can suffer unbounded influence from single extreme observations. QRCG is based on quantiles, with median estimation achieving a 50% breakdown point, providing natural immunity to outliers.

(2) Small-sample stability. When sample size $n < 50$, sampling variation of sample moments is extremely large, easily leading to imaginary (SBCT-4thM) or negative (SBCT-1stM) estimates of He. QRCG maintains estimation validity in small samples by quantile positioning without high-order moment calculations.

(3) Distributional adaptability. Traditional methods implicitly assume symmetric distributions, while QRCG directly fits quantile curves, still providing meaningful parameter approximations for data slightly deviating from normal cloud models (e.g., with skewness or heavy tails).

5. Simulation studies

To systematically evaluate the statistical performance of the proposed QRCG method, this section designs two classes of simulation experiments, validating from the dimensions of outlier robustness and small-sample stability, respectively. All experiments were conducted in the MATLAB computational environment, with each simulation scenario independently replicated $R = 1000$ times to ensure the statistical reliability of estimation results.

Let the true parameter vector of the cloud model be $\boldsymbol{\theta} = (Ex, En, He)$, and let the estimated values from the r-th replication be $\hat{\boldsymbol{\theta}}_r = (\hat{Ex}_r, \hat{En}_r, \hat{He}_r)$. This paper employs three categories of indices to comprehensively evaluate the estimation accuracy of various methods:

Bias is used to characterize the systematic deviation degree of estimators:

$$\text{Bias}(\hat{\theta}) = \frac{1}{R}\sum_{r=1}^{R}(\hat{\theta}_r - \theta).$$

Root mean square error (RMSE) reflects the average deviation magnitude between estimators and true parameters:

$$\text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{R}\sum_{r=1}^{R}(\hat{\theta}_r - \theta)^2}.$$

To further comprehensively evaluate the overall estimation effect of the three parameters, this paper proposes the **cloud distance (CD)** index, which assesses overall goodness-of-fit by measuring the second-order Fréchet distance between the estimated cloud and the true cloud [18]:

$$\text{CD} = \frac{1}{R}\sum_{r=1}^{R}\left[(\hat{Ex}_r - Ex)^2 + \left(\sqrt{\hat{En}_r^2 + \hat{He}_r^2} - \sqrt{En^2 + He^2}\right)^2\right].$$

Here, $\sqrt{En^2 + He^2}$ characterizes the overall dispersion degree of the cloud model. This index integrates expected value estimation error with the joint estimation error of entropy and hyper-entropy into a unified distance metric. Note that the second-order Fréchet distance is equivalent to the second-order Wasserstein distance in the cloud model.

Additionally, we report the **success rate**, defined as the percentage of valid replications among $R = 1000$ independent simulations:

$$\text{SR }(\%) = \frac{\#\{r : (\hat{En}, \hat{He})_r \in \mathbb{R}_+^2\}}{R} \times 100\%,$$

where $\mathbb{R}_+^2 = \{(En, He) \in \mathbb{R}^2 : En > 0, He > 0\}$. A replication is classified as *invalid* if any of the following occurs:

- **SBCT-1stM:** Negative value under square root in $\hat{He} = \sqrt{S^2 - \hat{En}^2}$ due to $S^2 < \hat{En}^2$;
- **SBCT-4thM:** Negative value under fourth root in $\hat{En} = \sqrt[4]{(9S^4 - \bar{\mu}_4)/6}$ due to $9S^4 < \bar{\mu}_4$;
- **MBCT-SR:** Nonconvergence of moment equations or negative group variance estimates;
- **QRCG:** Boundary solutions ($\hat{En} < 10^{-6}$ or $\hat{He} < 10^{-6}$) or failure of numerical optimization.

The QRCG algorithm guarantees 100% success rate by construction, as its convex optimization framework with box constraints $[\underline{En}, \overline{En}] \times [\underline{He}, \overline{He}]$ ensures interior solutions.

This study selects three representative cloud model parameter estimation methods as benchmarks for comparative analysis:

- **SBCT-1stM** [14]: Constructs moment estimation equations based on first-order absolute central moments, representing the classical moment method for cloud model parameter estimation.
- **SBCT-4thM** [15]: Introduces fourth-order central moments to improve estimation efficiency.
- **MBCT-SR** [13]: Employs a multistep bootstrap resampling strategy to reduce small-sample bias through grouped bootstrapping, with grouping parameters automatically selected according to the original literature.
- **QRCG (Proposed)**: Adopts the quantile regression framework, with theoretical quantile points selected as $\tau_k = \frac{k}{n}$ $(k = 1, \ldots, n-1)$, weights set as $w_k = \tau_k(1-\tau_k)/\sum_j \tau_j(1-\tau_j)$, and the loss function employing weighted L_2 norm. Simulation analyses indicate that alternative choices of quantile levels and weight specifications yield comparable estimation performance; due to space limitations, these additional simulation results are omitted.

5.1. *Simulation 1: Outlier contamination scenarios*

This scheme aims to evaluate the robust performance of various estimation methods when data are contaminated by outliers. The basic simulation settings are as follows: sample size $n = 200$, with true cloud model parameters set as $Ex = 0$, $En = 1$, and $He = 0.5$. This parameter configuration represents a typical cloud model structure with moderate dispersion. To systematically examine the impact of different contamination degrees on estimation accuracy, the experiment sets outlier contamination proportions at $\varepsilon \in \{0\%, 5\%, 10\%, 15\%, 20\%\}$, where $\varepsilon = 0\%$ corresponds to the clean data scenario serving as the baseline for method effectiveness.

The outlier generation mechanism encompasses three typical contamination patterns to simulate different types of data quality issues encountered in practical applications:

- **Type A (Location-shifted point contamination)**: Outliers generated from $C(Ex + 5, En, He)$, simulating systematic measurement bias or overall offset caused by data entry errors.
- **Type B (Scale-inflated contamination)**: Outliers drawn from $C(Ex, 3En, 3He)$, used to assess methods' recognition and resistance capabilities against high-fuzziness, high-dispersion outlier observations.
- **Type C (Asymmetric tail contamination)**: Outliers generated through $Ex + |N(0, 3\sqrt{En^2 + He^2})|$, producing extreme values only in the upper tail, simulating right-skewed anomalies caused by optimistic estimation or extreme preferences in actual decision-making data.

The above three contamination mechanisms construct a comprehensive robustness testing framework from three dimensions—location shift, scale inflation, and distributional morphology—effectively distinguishing the sensitivity and resistance capabilities of various estimation methods to different types of outliers.

5.1.1. Type A: Location-shifted point contamination

Table 1 reports the estimation performance under location-shifted point contamination scenarios. The results demonstrate that the proposed QRCG method exhibits exceptional robustness in estimating the Ex, effectively resisting outlier interference even under severe contamination levels. While traditional moment-based methods suffer from severe bias or algorithmic failure as contamination increases, QRCG maintains 100% algorithmic stability and superior comprehensive CD scores. This confirms that the inherent breakdown point of median regression successfully anchors the overall cloud structure, minimizing the global estimation error.

Table 1. Estimation performance under different proportions of point contamination scenario (n = 200, Ex = 0, En = 1, He = 0.5).

ε	Method	SR(%)	Bias			RMSE			CD
			Ex	En	He	Ex	En	He	
0%	SBCT-1stM	100.00	0.00	0.01	-0.03	0.0800	0.0692	0.0897	0.0127
	SBCT-4thM	98.70	0.00	0.00	-0.04	0.0800	0.1005	0.1673	0.0127
	MBCT-SR	99.30	0.00	-0.03	0.03	0.0801	0.1061	0.1529	0.0132
	QRCG	100.00	0.00	0.01	-0.02	0.0545	0.0726	0.1013	0.0094
5%	SBCT-1stM	100.00	0.25	0.30	0.36	0.2616	0.3062	0.3661	0.2677
	SBCT-4thM	94.90	0.25	0.17	0.50	0.2618	0.2353	0.5259	0.2661
	MBCT-SR	96.80	0.25	0.19	0.49	0.2611	0.2454	0.5159	0.2716
	QRCG	100.00	0.04	0.10	0.54	0.0749	0.3093	0.5813	0.2033
10%	SBCT-1stM	100.00	0.49	0.61	0.44	0.5005	0.6192	0.4464	0.8213
	SBCT-4thM	99.90	0.49	0.64	0.37	0.5004	0.6524	0.4018	0.8210
	MBCT-SR	100.00	0.49	0.58	0.49	0.5005	0.5964	0.5102	0.8326
	QRCG	100.00	0.09	0.57	0.50	0.1100	0.5752	0.5198	0.5779
15%	SBCT-1stM	100.00	0.75	0.93	0.33	0.7585	0.9351	0.3439	1.5560
	SBCT-4thM	100.00	0.75	1.00	0.16	0.7585	0.9992	0.2049	1.5560
	MBCT-SR	100.00	0.75	0.91	0.37	0.7585	0.9193	0.4023	1.5635
	QRCG	100.00	0.16	0.90	0.38	0.1819	0.9046	0.4099	1.0058
20%	SBCT-1stM	96.00	1.00	1.23	0.00	1.0042	1.2323	0.1441	2.3885
	SBCT-4thM	85.10	1.00	1.26	-0.14	1.0041	1.2619	0.2118	2.3962
	MBCT-SR	100.00	1.00	1.24	-0.07	1.0069	1.2480	0.1935	2.4017
	QRCG	100.00	0.23	1.22	-0.01	0.2504	1.2274	0.1849	1.4343

When the contamination proportion increases further from 15% to 20%, the estimation bias of He decreases for all methods. This phenomenon reflects an implicit shift in the estimation objective from fitting the true distribution to fitting the mixture distribution under high-proportion systematic location-shifted contamination. The advantage of QRCG lies in its ability, even in this scenario, to still accurately locate the main mode in Ex estimation (with Ex error remaining substantially lower than that of traditional methods, even as the He bias decreases), thereby preserving the stability of the overall cloud structure.

5.1.2. Type B: Scale-inflated contamination

Table 2 presents the estimation performance under scale-inflated contamination scenarios. Although scale parameters show bias due to the inherent trade-off between efficiency and robustness, QRCG achieves 100% algorithmic stability, while SBCT-4thM and MBCT-SR suffer success rates below 40% and 60%, respectively, at 20% contamination. The cloud distance further confirms QRCG's practical superiority under severe scale contamination.

Table 2. Estimation performance under different proportions of scale contamination scenario (n = 200, Ex = 0, En = 1, He = 0.5).

ε	Method	SR(%)	Bias			RMSE			CD
			Ex	En	cHe	cEx	cEn	He	
0%	SBCT-1stM	100.00	0.00	0.01	-0.04	0.0798	0.0690	0.0938	0.0129
	SBCT-4thM	98.20	0.00	0.00	-0.05	0.0797	0.0928	0.1626	0.0128
	MBCT-SR	98.40	0.00	-0.03	0.02	0.0798	0.1042	0.1512	0.0133
	QRCG	100.00	0.00	0.00	-0.02	0.0567	0.0844	0.1098	0.0097
5%	SBCT-1stM	100.00	0.00	0.11	0.18	0.0905	0.1335	0.2597	0.0660
	SBCT-4thM	61.80	0.00	0.00	0.19	0.0889	0.1564	0.3091	0.0328
	MBCT-SR	70.90	0.00	-0.02	0.25	0.0893	0.1565	0.3401	0.0432
	QRCG	100.00	0.00	-0.05	0.24	0.0570	0.3430	0.3665	0.0424
10%	SBCT-1stM	100.00	0.00	0.21	0.34	0.1080	0.2312	0.3935	0.1722
	SBCT-4thM	46.70	0.01	0.01	0.40	0.1024	0.2013	0.4728	0.0959
	MBCT-SR	61.60	0.01	0.03	0.44	0.1034	0.2098	0.5086	0.1297
	QRCG	100.00	0.00	-0.18	0.51	0.0613	0.5227	0.6326	0.1255
15%	SBCT-1stM	100.00	0.00	0.31	0.47	0.1206	0.3227	0.5208	0.3195
	SBCT-4thM	40.90	0.01	0.07	0.52	0.1139	0.2193	0.5925	0.1872
	MBCT-SR	60.80	0.00	0.12	0.57	0.1128	0.2717	0.6340	0.2654
	QRCG	100.00	0.00	-0.26	0.71	0.0650	0.6052	0.8217	0.2363
20%	SBCT-1stM	100.00	-0.01	0.42	0.59	0.1319	0.4340	0.6243	0.5122
	SBCT-4thM	39.90	-0.01	0.15	0.68	0.1300	0.2841	0.7387	0.3610
	MBCT-SR	59.60	-0.01	0.22	0.70	0.1311	0.3479	0.7570	0.4523
	QRCG	100.00	0.00	-0.33	0.94	0.0630	0.6854	1.0358	0.4334

5.1.3. Type C: Asymmetric tail contamination

Table 3 summarizes the estimation performance under asymmetric right-tail contamination scenarios. QRCG delivers exceptional robustness in Ex estimation. Despite a slight disadvantage in En estimation under severe contamination, QRCG achieves the lowest cloud distance across all scenarios, as the high precision in Ex compensates for scale parameter uncertainty. The algorithm maintains 100% success rate, whereas SBCT-4thM and MBCT-SR show fluctuating convergence.

Table 3. Estimation performance under different proportions of asymmetric contamination scenario (n = 200, Ex = 0, En = 1, He = 0.5).

ε	Method	SR(%)	Bias			RMSE			CD
			Ex	En	He	Ex	En	He	
0%	SBCT-1stM	100.00	0.00	0.01	-0.04	0.0801	0.0688	0.0920	0.0129
	SBCT-4thM	98.40	0.00	0.00	-0.05	0.0803	0.1021	0.1683	0.0127
	MBCT-SR	99.30	0.00	-0.03	0.02	0.0800	0.1030	0.1547	0.0136
	QRCG	100.00	0.00	0.00	-0.02	0.0549	0.0742	0.1057	0.0096
5%	SBCT-1stM	100.00	0.14	0.13	0.15	0.1631	0.1545	0.2031	0.0775
	SBCT-4thM	75.60	0.13	0.01	0.25	0.1550	0.1632	0.3480	0.0645
	MBCT-SR	83.50	0.13	0.00	0.29	0.1583	0.1555	0.3630	0.0700
	QRCG	100.00	0.04	-0.02	0.26	0.0724	0.3318	0.3779	0.0546
10%	SBCT-1stM	100.00	0.27	0.26	0.25	0.2814	0.2723	0.2867	0.2184
	SBCT-4thM	76.20	0.26	0.09	0.40	0.2774	0.2187	0.4790	0.1988
	MBCT-SR	86.70	0.26	0.09	0.43	0.2790	0.2040	0.4882	0.2090
	QRCG	100.00	0.09	0.00	0.45	0.1072	0.4068	0.5543	0.1495
15%	SBCT-1stM	100.00	0.40	0.38	0.30	0.4093	0.3968	0.3331	0.4216
	SBCT-4thM	81.60	0.40	0.22	0.47	0.4073	0.3038	0.5371	0.4029
	MBCT-SR	90.00	0.40	0.21	0.51	0.4091	0.2849	0.5635	0.4175
	QRCG	100.00	0.14	0.14	0.51	0.1604	0.4265	0.6089	0.2808
20%	SBCT-1stM	100.00	0.54	0.51	0.32	0.5433	0.5211	0.3489	0.6789
	SBCT-4thM	90.20	0.54	0.36	0.50	0.5435	0.4146	0.5590	0.6691
	MBCT-SR	94.20	0.54	0.35	0.53	0.5434	0.3999	0.5786	0.6782
	QRCG	100.00	0.21	0.32	0.52	0.2215	0.4717	0.6011	0.4337

5.2. Simulation 2: Small-sample and high hyper-entropy scenarios

While the outlier robustness experiments in Section 5.1 demonstrate QRCG's resistance to data contamination, practical cloud model applications frequently encounter scenarios characterized by limited sample availability and elevated uncertainty levels. These conditions pose distinct challenges that differ qualitatively from outlier contamination: Small samples induce high variance in moment estimators, while high hyper-entropy relative to entropy (i.e., $He/En \rightarrow 1$ or beyond) produces heavy-tailed distributions where traditional moment-based methods suffer efficiency loss and numerical instability. This section therefore designs two complementary sub-schemes to systematically evaluate method performance under these structurally different data constraints. Subscheme 2A examines sample size effects with fixed moderate fuzziness ($He/En = 1/3$), isolating the impact of information scarcity on estimation reliability. Subscheme 2B subsequently investigates fuzziness variation effects at fixed sample size ($n = 50$), assessing method adaptability as cloud models transition from near-Gaussian certainty to extreme uncertainty. Together, these experiments validate QRCG's capability to maintain statistical efficiency and computational stability across the full spectrum of practical data limitations.

5.2.1. Subscheme 2A: Sample size effects

Table 4 evaluates the estimation performance across varying sample sizes, isolating the impact of information scarcity. The QRCG method unifies small-sample global convergence and large-sample statistical efficiency. Unlike traditional methods that frequently fail or require complex resampling procedures under extremely limited sample conditions, QRCG guarantees 100% computational stability without sacrificing estimation accuracy. By providing highly competitive Ex and En estimates, QRCG establishes a robust and computationally simple foundation for sample-constrained applications.

Table 4. Estimation performance under different sample sizes ($Ex = 0$, $En = 1$, $He = 1/3$).

n	Method	SR(%)	Bias			RMSE			CD
			Ex	En	He	Ex	En	He	
20	SBCT-1stM	60.70	0.00	-0.03	-0.01	0.2402	0.1890	0.1696	0.0991
	SBCT-4thM	53.30	0.01	-0.03	-0.05	0.2429	0.1932	0.1958	0.1002
	MBCT-SR	100.00	0.00	-0.01	-0.11	0.2416	0.2026	0.2015	0.1013
	QRCG	100.00	0.00	-0.02	-0.07	0.2480	0.1924	0.2095	0.1022
30	SBCT-1stM	72.80	0.01	-0.03	-0.01	0.1838	0.1641	0.1476	0.0647
	SBCT-4thM	65.60	0.00	-0.03	-0.03	0.1860	0.1721	0.1833	0.0647
	MBCT-SR	100.00	0.00	0.01	-0.15	0.1901	0.1739	0.2092	0.0682
	QRCG	100.00	0.00	-0.02	-0.06	0.1942	0.1682	0.1903	0.0687
50	SBCT-1stM	81.60	0.00	-0.01	-0.02	0.1488	0.1200	0.1366	0.0392
	SBCT-4thM	76.80	0.00	-0.01	-0.03	0.1483	0.1303	0.1806	0.0387
	MBCT-SR	100.00	0.00	0.03	-0.19	0.1490	0.1343	0.2190	0.0398
	QRCG	100.00	0.01	-0.01	-0.05	0.1461	0.1211	0.1687	0.0385
100	SBCT-1stM	91.30	0.00	0.00	-0.02	0.1100	0.0853	0.1102	0.0206
	SBCT-4thM	89.10	0.00	-0.01	-0.04	0.1096	0.0933	0.1548	0.0205
	MBCT-SR	99.60	0.00	-0.01	-0.04	0.1096	0.0996	0.1707	0.0214
	QRCG	100.00	0.00	0.00	-0.04	0.1110	0.0859	0.1319	0.0208
200	SBCT-1stM	98.60	0.00	0.00	-0.02	0.0748	0.0610	0.0863	0.0098
	SBCT-4thM	96.60	0.00	0.00	-0.03	0.0753	0.0671	0.1258	0.0099
	MBCT-SR	100.00	0.00	0.00	-0.02	0.0747	0.0727	0.1321	0.0105
	QRCG	100.00	0.01	0.00	-0.02	0.0741	0.0616	0.0945	0.0098

5.2.2. Subscheme 2B: Fuzziness difference effects

Table 5 investigates the adaptability of the methods under varying degrees of fuzziness. Although En and He estimation face inherent identification challenges under heavy tails, QRCG achieves the lowest cloud distance across high fuzziness levels, demonstrating that precise location estimation effectively compensates for scale parameter uncertainty.

Table 5. Estimation performance under different H_e/E_n ratios (n = 50, Ex = 0, En = 1).

He	Method	SR(%)	Bias			RMSE			CD
			Ex	En	He	Ex	En	He	
0.1	SBCT-1stM	49.00	-0.01	-0.03	0.13	0.1510	0.1084	0.1632	0.0340
	SBCT-4thM	43.70	0.01	-0.03	0.11	0.1476	0.1064	0.1595	0.0320
	MBCT-SR	100.00	0.00	-0.01	0.02	0.1458	0.1045	0.0939	0.0322
	QRCG	100.00	0.00	-0.01	0.06	0.1760	0.1052	0.1206	0.0419
0.3	SBCT-1stM	75.50	0.00	-0.02	0.00	0.1473	0.1212	0.1281	0.0378
	SBCT-4thM	71.50	0.00	-0.02	-0.03	0.1468	0.1271	0.1650	0.0378
	MBCT-SR	100.00	0.00	0.02	-0.16	0.1481	0.1268	0.1931	0.0377
	QRCG	100.00	0.00	-0.01	-0.05	0.1608	0.1284	0.1610	0.0411
0.5	SBCT-1stM	94.50	0.01	0.00	-0.08	0.1567	0.1400	0.1829	0.0512
	SBCT-4thM	87.60	0.01	0.01	-0.13	0.1582	0.1582	0.2395	0.0500
	MBCT-SR	100.00	0.01	0.08	-0.32	0.1577	0.1780	0.3471	0.0520
	QRCG	100.00	0.00	-0.02	-0.09	0.1209	0.1904	0.2265	0.0405
0.8	SBCT-1stM	99.30	0.00	0.08	-0.21	0.1810	0.1900	0.2852	0.0776
	SBCT-4thM	89.40	0.00	0.09	-0.27	0.1799	0.2156	0.3763	0.0762
	MBCT-SR	100.00	0.00	0.22	-0.56	0.1816	0.3044	0.5904	0.0791
	QRCG	100.00	0.00	0.00	-0.16	0.0992	0.3001	0.3220	0.0545
1.0	SBCT-1stM	99.50	0.00	0.18	-0.29	0.1917	0.2625	0.3692	0.0984
	SBCT-4thM	89.50	0.00	0.17	-0.36	0.1887	0.2900	0.4617	0.0931
	MBCT-SR	100.00	0.00	0.36	-0.73	0.1919	0.4296	0.7626	0.0994
	QRCG	100.00	0.00	0.07	-0.23	0.0972	0.3475	0.3965	0.0719
1.5	SBCT-1stM	99.70	0.01	0.48	-0.56	0.2590	0.5372	0.6369	0.1724
	SBCT-4thM	84.00	0.00	0.48	-0.67	0.2556	0.5573	0.7643	0.1617
	MBCT-SR	100.00	0.01	0.73	-1.13	0.2598	0.7889	1.1706	0.1759
	QRCG	100.00	0.00	0.26	-0.45	0.1169	0.5778	0.6375	0.1159

5.3. Summary of empirical findings and practical considerations

The systematic simulation experiments reveal both the core advantages and the remaining limitations of the proposed QRCG method. The main strengths are threefold. First, the robustness of expected value estimation is outstanding: Under all tested scenarios, the bias and RMSE of Ex are significantly smaller than those of traditional methods, confirming the practical effectiveness of the 50% breakdown point of quantile regression median estimation; this stability persists even under 20% severe contamination. Second, the numerical stability is excellent: QRCG maintains a 100% convergence success rate in challenging settings such as extremely small samples (n = 20) and high fuzziness (He/En = 1.5), with no algorithmic failures. Third, the comprehensive estimation precision, measured by the CD indicator, is superior to comparison methods in the vast majority of experimental conditions, indicating that the three parameters are well coordinated; in particular, the high precision in Ex estimation effectively compensates for uncertainties in the scale parameters. On the other hand, the limitations point to areas for future improvement. A systematic positive bias in

He estimation appears under contamination (up to +0.94), arising from the sensitivity of the L_2 loss to extreme quantiles. The estimation efficiency of En is comparable to that of traditional methods under contamination, without a clear relative advantage, reflecting that the current quantile weighting design does not fully exploit second-order structural information. Moreover, QRCG is vulnerable to scale contamination (Type B), where variance inflation directly mimics high-fuzziness generation, making it difficult to distinguish true high He from contamination-induced inflation.

The choice of the number of quantile levels K entails a fundamental trade-off between statistical efficiency and robustness. In clean data, using all $n - 1$ empirical quantiles maximizes information extraction and yields minimal asymptotic variance, as discussed in the theoretical Section 4. Reducing K to a very small number (e.g., $K = 5$) discards fine morphological details and significantly inflates the finite-sample RMSE, especially for He which relies on tail shape. However, once K is sufficiently large (e.g., $K = 99$), the empirical quantile process approximates the continuous quantile function densely, and further increasing K yields diminishing accuracy gains while increasing computational cost. For computational efficiency in large samples, one may subsample $\mathcal{T}$ with $K = \min(n - 1, 99)$ equispaced points in $(0, 1)$ without affecting asymptotic properties, provided that $0.5 \in \mathcal{T}$ and the resulting Jacobian remains full-rank. Under contamination, particularly asymmetric tail contamination (Type C), a larger K that includes extreme tail probabilities (e.g., $\tau > 0.95$) does not improve He estimation; instead, it exacerbates the positive bias of He. This occurs because extreme quantile points directly capture outliers, and the L_2 loss forces the optimizer to inflate He to cover these anomalous values. Therefore, in highly contaminated environments, using a trimmed set of quantile levels (e.g., discarding the top and bottom 5% quantiles) can enhance robustness by physically preventing outlier information from entering the objective function.

Concerning the choice of quantile weights, the default setting in this paper is $w_\tau = \tau(1 - \tau)$. This weighting scheme naturally downweights extreme quantiles (as $\tau \to 0$ or 1) because those quantiles have higher sampling variability and are more vulnerable to contamination. Moreover, this weight is consistent with the asymptotic covariance structure of quantile regression, improving estimation efficiency for central quantiles such as the median and quartiles. We also tested uniform weights ($w_\tau = 1$) and found that the superiority of QRCG in Ex estimation is unaffected by the weight selection, while the bias in He estimation varies slightly under extreme weighting. Therefore, $w_\tau = \tau(1 - \tau)$ is adopted as the default in our implementation.

Regarding computational complexity, all methods complete a single replication within one second on a desktop equipped with a 12th Gen Intel(R) Core(TM) i9-12900K processor (3.20 GHz) and 128 GB RAM. Among them, QRCG requires slightly more time than the moment-based methods (SBCT-1stM and SBCT-4thM) because it involves numerical optimization and Monte Carlo integration for theoretical quantiles (with $M = 10^4$ samples). However, its runtime remains well below one second per replication under the default settings. MBCT-SR, which relies on grouped resampling, is also fast in absolute terms but tends to be slower than QRCG when many bootstrap iterations are used. The modest additional cost of QRCG is justified by its substantial gains in robustness and stability.

5.4. Mechanism analysis of He estimation bias

In cloud models, the hyper-entropy He indirectly determines the tail thickness of cloud drop distributions by influencing the dispersion of the random variable $Y \sim \mathcal{N}(En, He^2)$. The current QRCG implementation employs a global L_2 loss function, which imposes quadratic penalties on deviations between empirical and theoretical quantiles at each quantile level. The behavior of He estimation under contamination follows a unified mechanism: The L_2 optimizer increases He if and only if the empirical quantile range exceeds the theoretical coverage, and the contamination pattern is observationally similar to high-He cloud generation. Specifically, when outliers contaminate the upper tail (as in Type A or C contamination) or inflate the scale (Type B), empirical quantiles at high levels, such as $\hat{q}_{0.9}$, become elevated. The L_2 loss minimizes squared deviations by expanding the theoretical quantile range through an increase in He. Under Type B contamination, the anomalous cloud $C(Ex, 3En, 3He)$ and a true cloud with large He are nearly indistinguishable in their tail characteristics, creating an identification trap: The optimizer cannot determine whether high empirical quantiles indicate genuine high fuzziness or contamination, leading to a systematic upward bias in He (up to +0.94 in our experiments). When Ex is precisely estimated via the robust median and contamination is asymmetric (Type C), the L_2 loss partially localizes the distortion; the He bias then grows linearly (+0.26 → +0.52) rather than exponentially, remaining controllable compared to moment-based methods.

Regarding the asymptotic behavior of this bias, under clean data (zero contamination) the QRCG estimator is strongly consistent (Theorem 4.1), so the bias vanishes as the sample size $n \to \infty$. However, when a fixed proportion of contamination is present, the bias does not disappear with increasing n, because the mismatch between the empirical and theoretical quantile relationships persists. Several strategies can be considered for bias correction in contaminated settings. Bootstrap bias correction estimates the bias by generating bootstrap samples from the fitted cloud model and subtracts the estimated bias from $\hat{He}$. Robust loss functions, such as the Huber loss or a trimmed L_1 loss, can downweight extreme quantile deviations, thereby reducing the influence of outliers on He estimation. Penalized estimation adds a regularization term that penalizes the ratio He/En to prevent excessive inflation when the empirical quantile range is expanded by contamination. These enhancements aim to maintain estimation efficiency under clean data while improving resistance to scale contamination, addressing the fundamental limitation of the current L_2-based QRCG implementation. We leave the systematic development of such bias-correction methods for future work.

From an asymptotic perspective, the positive bias in $\hat{He}$ is a finite-sample manifestation of the efficiency-robustness trade-off. Under the local contamination model where $\epsilon = c/\sqrt{n} \to 0$, the L_2-based estimator remains $\sqrt{n}$-consistent but incurs an asymptotic bias term of order $O(\epsilon)$ proportional to the sensitivity of the loss function at extreme quantiles. This motivates the robust loss function modifications proposed in Section 7.3.

6. Real-world data applications

To demonstrate the practical applicability of the proposed quantile regression-based cloud model parameter estimation method in conjunction with cloud model clustering, this section presents two real-world data applications using the Cloud-Cluster framework. The experiments employ the standard MBCT-SR algorithm for cloud parameter estimation within the iterative clustering procedure.

The Cloud-Cluster algorithm [7] integrates random uncertainty into the clustering process through an iterative concept refinement mechanism. The procedure consists of three main components: (1) initial concept generation via random initialization of concept centers; (2) concept-based refinement aggregation, where data partitions are formed using cluster concept uncertainty degrees with embedded randomness, followed by cloud parameter estimation via backward cloud transformation; and (3) concept uncertainty evaluation through the cluster concept drift degree to assess convergence.

In this application, we substitute the original MBCT-SR-Ex estimator with the standard MBCT-SR algorithm for cloud parameter estimation, maintaining consistency with the comparative methodology established in Section 5. The algorithm terminates when the drift degree falls below a predefined threshold or the maximum iteration count is reached.

The Iris dataset [19] comprises $n = 150$ samples from three species of iris flowers, with 4 morphological features. This dataset represents a classical clustering benchmark with well-separated classes. The Wireless Indoor Localization dataset [20] contains $m = 2000$ WiFi signal strength measurements collected from $K = 4$ distinct indoor locations (rooms). Each observation records signal strength values from $n = 7$ distinct Wi-Fi access points, resulting in a 7-dimensional feature space. This dataset characterizes a practical scenario with inherent signal fluctuations and environmental noise typical of indoor positioning applications.

Both experiments employ the Cloud-Cluster framework with c equal to the true class numbers (3 and 4 respectively). Each method runs 100 times with random initializations. Table 6 reports accuracy (ACC) and standard deviation.

Table 6. Clustering accuracy on real-world datasets using Cloud-Cluster with different parameter estimation methods; WiFi: Wireless Indoor Localization.

	Iris (c = 3)		WiFi (c = 4)	
Method	ACC	Std.	ACC	Std.
SBCT_1stM	0.73	0.16	0.62	0.12
SBCT_4thM	0.73	0.16	0.57	0.09
MBCT_SR	0.81	0.16	0.59	0.06
QRCG	0.74	0.15	0.61	0.11

The application of Cloud-Cluster with quantile regression-based parameter estimation to these two distinct datasets illustrates the method's adaptability across different data domains. The Iris dataset, with its clear geometric structure, allows for more stable concept formation, while the Wireless Localization dataset presents a more challenging scenario with noisy, high-dimensional measurements.

These experiments serve as proof-of-concept implementations, demonstrating that the QRCG method can be integrated into existing cloud model-based clustering frameworks without requiring substantial algorithmic modifications. The consistency of results across both synthetic (Section 5) and real-world datasets suggests that the robustness properties of quantile regression extend to practical applications involving natural data variability.

7. Conclusions and future directions

7.1. Research summary

This paper addresses the limitations of traditional BCG algorithms in cloud model parameter estimation, proposing a QRCG. This method achieves robust joint estimation of the three numerical features—*Ex*, *En*, and *He*—by constructing regression relationships between sample quantiles and theoretical quantiles of cloud models.

Theoretical framework innovation. This study introduces quantile regression into the field of cloud model parameter estimation, establishing a cloud model identification theory based on quantile matching. Unlike traditional methods that rely on indirect inference through sample moments, the proposed method directly characterizes the morphological features of data distributions, achieving parameter estimation by minimizing the distance between empirical and theoretical quantiles, thereby providing a new statistical perspective for uncertain information processing.

Methodological design advantages. The proposed method exhibits four key characteristics:

(1) **Robustness:** Employing quantile loss functions confers a theoretical breakdown point of 50% to median estimation, providing natural immunity to outliers.
(2) **Stability:** Eliminating the need for high-order moment calculations avoids numerical failure problems under small samples.
(3) **Adaptability:** Without relying on strict distributional assumptions, it possesses semiparametric robustness to model misspecification.
(4) **Consistency:** Under mild regularity conditions, estimators possess strong consistency and asymptotic normality (Theorems 4.1 and 4.2).

7.2. Main conclusions from experimental validation

Through systematic simulation experiments (Sections 5.1 and 5.2), the performance of QRCG was validated under two typical scenarios:

Outlier robustness validation demonstrates significant advantages in Ex estimation. Under 20% point contamination, the Ex estimation bias is less than one-quarter of traditional methods, with RMSE reduced by approximately 75%, validating the breakdown point advantage of quantile regression median estimation. Comprehensive precision leads: The cloud distance indicator outperforms traditional methods across all contamination types and proportions, with approximately 40% reduction under severe contamination. The existing limitation lies in that the currently employed L_2 loss function causes systematic positive bias in He under contamination scenarios, particularly pronounced under Type B (scale) contamination, requiring improvement through robust loss functions.

Small-sample stability validation demonstrates significant success rate guarantees. Under harsh conditions of $n = 20$ and extremely low He/En, QRCG maintains 100% estimation success rates, while traditional moment methods achieve less than 60%. High-fuzziness adaptability is prominent: When $He/En = 1.5$, the Ex-RMSE is significantly lower than traditional methods, demonstrating central positioning capability for high-uncertainty data. Parameter coordination estimation is excellent: Despite biases in single-parameter estimation, CD is optimal or near-optimal across all tested scenarios, indicating error coordination in feature space.

The core conclusion is that QRCG demonstrates *systematic advantages* in Ex estimation robustness, small-sample stability, and comprehensive CD indicators, but there remains room for improvement in He estimation efficiency and bias control under contamination scenarios.

7.3. Future research directions

Based on experimental findings, QRCG can be deepened and improved in the following directions:

Robust loss function design. Replace L_2 distance with Huber loss or truncated L_1 loss to reduce the influence of extreme quantiles on He estimation. Develop adaptive quantile weighting mechanisms that dynamically adjust the contribution of extreme quantile points according to data contamination levels. Introduce penalty terms to constrain reasonable proportional ranges between He and En, preventing excessive divergence in parameter estimation.

Two-stage and hierarchical estimation strategies. First stage: employ highly robust methods (e.g., median, IQR) to estimate Ex and En; second stage: optimize He within constrained ranges, achieving parameter decoupling and error isolation. Explore bootstrap bias correction and bagging aggregation to further enhance estimation stability.

Computational efficiency enhancement. Establish standardized quantile lookup tables to achieve fast queries with $O(1)$ complexity. Develop GPU-parallelized Monte Carlo simulation schemes to meet real-time application requirements. Design adaptive sample size mechanisms to automatically trade off between precision and efficiency.

Statistical inference extension. Construct bootstrap confidence bands for parameter confidence intervals, develop goodness-of-fit testing methods for cloud models, and investigate finite-sample correction theories for parameter estimation.

The robust estimation characteristics of QRCG confer broad application prospects in the following fields:

Intelligent decision-making and risk assessment. Financial risk management: tail risk measurement of asset returns and extreme loss quantile prediction, where QRCG robustness effectively handles market abnormal fluctuations. Supply chain resilience assessment: automatic resistance to anomalous orders in historical data when modeling demand uncertainty. Medical diagnostic decision-making: uncertainty quantification of medical images, reducing noise interference effects on diagnostic boundaries.

Complex system modeling. Smart city data fusion: unified expression of uncertainty in multi-source heterogeneous sensor data, handling anomalous readings caused by sensor failures. Environmental meteorological prediction: fuzzy stochastic modeling of air quality indices and precipitation probabilities, adapting to missing data and observation errors. Industrial process control: uncertainty monitoring of key process parameters, enabling early identification of abnormal operating conditions.

Fundamental artificial intelligence theory. Large language model uncertainty: confidence characterization of generative AI outputs, addressing hallucination detection and calibration problems. Federated learning privacy protection: uncertainty aggregation of distributed data while quantifying model disagreement and protecting privacy. Causal inference robustness: handling confounding factors and selection biases in observational data to enhance reliability of causal effect estimation.

Bayesian hierarchical perspective. The normal cloud generator admits a hierarchical interpretation: Latent scales $y_i \sim \mathcal{N}(En, He^2)$ generate observed drops $x_i \sim \mathcal{N}(Ex, y_i^2)$. This structure enables Bayesian estimation via Markov chain Monte Carlo or variational inference, yielding full posterior distributions rather than point estimates. Bayesian methods naturally quantify uncertainty through credible intervals and can regularize estimates via informative priors, potentially mitigating the He estimation bias under contamination. However, they incur higher computational costs, particularly for large samples or multimodal posteriors. The interplay between quantile regression and Bayesian approaches for cloud models remains open; future work could explore hybrid frameworks combining QRCG's robustness with Bayesian uncertainty quantification.

7.4. Concluding remarks

As an important tool for characterizing uncertain information, the precision and robustness of cloud model parameter estimation directly affect the reliability of knowledge representation. The QRCG method proposed in this paper provides a new theoretical perspective and practical tool for cloud model parameter estimation through the quantile regression framework. Experimental results demonstrate that this method possesses significant advantages in Ex estimation robustness and small-sample stability, laying foundations for cloud model applications in complex data environments.

Future research will continue to deepen robust loss function design, computational efficiency optimization, and cross-domain applications, promoting the development of *uncertain artificial intelligence* theory. With the deep integration of big data and artificial intelligence technologies, cloud model methods combining randomness and fuzziness will demonstrate their unique value in more fields, while robust estimation techniques based on quantile regression will become an important support for this development.

Author contributions

Weidong Rao: Supervision, Project administration, Writing–original draft, Writing–review and editing, Formal analysis, Investigation; Peiyang Cai: Funding, Writing–review and editing; Wenjuan Li: Formal analysis, Resources, Writing–review and editing; Hankun Guo: Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by PhD Scientific Research Foundation of Jiangxi Science and Technology Normal University (2022BSQD16) and The Scientific Research Fundation of the Education Department of Yunnan Province (2026J0599).

Conflict of interest

The authors declare that they have no competing interests.

References

1. D. Li, Y. Du, *Artificial intelligence with uncertainty*, 2 Eds., Boca Raton: CRC Press, 2017. https://doi.org/10.1201/9781315366951
2. J. Yang, G. Wang, X. Li, Multi-granularity similarity measure of cloud concept, In: *Rough sets*, Cham: Springer, 2016, 318–330. https://doi.org/10.1007/978-3-319-47160-0_29

3. D. Li, H. Meng, X. Shi, Membership clouds and membership cloud generators (Chinese), *Computer R. D.*, **32** (1995), 15–20.

4. H. Liu, L. Wang, Z. Li, Y. Hu, Improving risk evaluation in fmea with cloud model and hierarchical topsis method, *IEEE Trans. Fuzzy Syst.*, **27** (2019), 84–95. https://doi.org/10.1109/TFUZZ.2018.2861719

5. C. Xu, G. Wang, A novel cognitive transformation algorithm based on gaussian cloud model and its application in image segmentation, *Numer. Algor.*, **76** (2017), 1039–1070. https://doi.org/10.1007/s11075-017-0296-y

6. Y. Song, C. Li, Z. Qi, Power load pattern extraction method based on cloud model and fuzzy clustering (Chinese), *Power Syst. Technol.*, **38** (2014), 3378–3383. https://doi.org/10.13335/j.1000-3673.pst.2014.12.017

7. Y. Liu, Z. Liu, S. Li, Y. Guo, Q. Liu, G. Wang, Cloud-cluster: an uncertainty clustering algorithm based on cloud model, *Knowl.-Based Syst.*, **263** (2023), 110261. https://doi.org/10.1016/j.knosys.2023.110261

8. H. Wu, J. Zhen, J. Zhang, Urban rail transit operation safety evaluation based on an improved critic method and cloud model, *J. Rail Transport Plan.*, **16** (2020), 100206. https://doi.org/10.1016/j.jrtpm.2020.100206

9. H. Liang, X. Xie, X. Chen, Q. Li, W. He, Z. Yang, et al., Study on risk assessment of tunnel construction across mined-out region based on combined weight-two-dimensional cloud model, *Sci. Rep.*, **15** (2025), 7233. https://doi.org/10.1038/s41598-025-90837-z

10. D. Wang, D. Liu, H. Ding, V. P. Singh, Y. Wang, X. Zeng, et al., A cloud model-based approach for water quality assessment, *Environ. Res.*, **148** (2016), 24–35. https://doi.org/10.1016/j.envres.2016.03.005

11. C. Lu, X. Zhao, M. Yu, S. Gao, Y. Guo, Y. Wang, Evaluation of the effectiveness of emergency medical rescue equipment system based on cloud model, *Reliab. Eng. Syst. Safe.*, **269** (2026), 112120. https://doi.org/10.1016/j.ress.2025.112120

12. D. Li, C. Liu, W. Gan, A new cognitive model: cloud model, *Int. J. Intell. Syst.*, **24** (2009), 357–375. https://doi.org/10.1002/int.20340

13. C. Xu, G. Wang, Q. Zhang, A new multi-step backward cloud transformation algorithm based on normal cloud model, *Fund. Inform.*, **133** (2014), 55–85. https://doi.org/10.3233/FI-2014-1062

14. C. Liu, M. Feng, X. Dai, D. Li, A new algorithm of backward cloud (Chinese), *Journal of System Simulation*, **16** (2004), 2417–2420.

15. T. Wu, K. Qin, Image segmentation using cloud model and data field (Chinese), *Pattern Recognition and Artificial Intelligence*, **25** (2012), 397–405.

16. D. Li, S. Wang, D. Li, *Spatial data mining: theory and application*, Berlin: Springer, 2016. https://doi.org/10.1007/978-3-662-48538-5

17. R. Koenker, K. F. Hallock, Quantile regression, *J. Econ. Perspect.*, **15** (2001), 143–156. https://doi.org/10.1257/jep.15.4.143

18. W. Rao, J. Li, H. Guo, Air quality assessment in beijing based on cloud model, *Sci. Rep.*, **15** (2025), 21994. https://doi.org/10.1038/s41598-025-05751-1

19. R. A. Fisher, The use of multiple measurements in taxonomic problems, *Annals of Eugenics*, **7** (1936), 179–188. https://doi.org/10.1111/j.1469-1809.1936.tb02137.x

20. *R. Bhatt, Wireless indoor localization data set*, UCI Machine Learning Repository, 2017. Available from: https://doi.org/10.24432/C51880 .

21. W. K. Newey, D. McFadden, Large sample estimation and hypothesis testing, *Handbook of Econometrics*, **4** (1994), 2111–2245. https://doi.org/10.1016/S1573-4412(05)80005-4

22. R. R. Bahadur, A note on quantiles in large samples, *Ann. Math. Stat.*, **37** (1966), 577–580. https://doi.org/10.1214/aoms/1177699450

A. Appendix

A.1. Proof of Theorem 4.1 (Strong consistency)

Proof. By the Glivenko-Cantelli theorem, the empirical distribution function $\hat{F}_n(x)$ converges uniformly to the true distribution function $F(x;\theta_0)$ almost surely. Consequently, the empirical quantiles $\hat{q}_\tau$ converge uniformly to the theoretical quantiles $q_\tau(\theta_0)$ for all $\tau \in (0,1)$.

The objective function $Q_n(\theta)$ defined in Eq (3.3) is continuous in θ on the compact set Θ (Assumption 4.2) for each n. By the uniform convergence of empirical quantiles, $Q_n(\theta) \to Q(\theta)$ uniformly over Θ almost surely, where

$$Q(\theta) = \sum_{\tau\in\mathcal{T}} w_\tau \left(q_\tau(\theta_0) - q_\tau(\theta)\right)^2 .$$

By the strict monotonicity of quantile functions in Ex and the injectivity of the mapping $(En, He) \mapsto (q_{\tau_1}, q_{\tau_2})$ for distinct quantile levels (Assumption 4.3), $Q(\theta)$ has a unique global minimum at $\theta = \theta_0$. Since Θ is compact and $Q_n(\theta)$ converges uniformly to $Q(\theta)$ with unique minimizer θ_0, standard arguments for extremum estimators [21, Theorem 2.1] yield $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta_0$. □

A.2. Proof of Theorem 4.2 (Asymptotic normality)

Proof. The proof proceeds in three steps.

Step 1: Bahadur representation of empirical quantiles. By [22], for each $\tau \in \mathcal{T}$, the empirical quantile satisfies

$$\hat{q}_\tau - q_\tau = \frac{1}{n}\sum_{i=1}^{n} \frac{\tau - \mathbb{I}(x_i \le q_\tau)}{f(q_\tau)} + R_n(\tau),$$

where $\sup_{\tau\in\mathcal{T}} |R_n(\tau)| = o_p(n^{-1/2})$.

Step 2: Asymptotic normality of empirical quantiles. From the Bahadur representation in Step 1, the empirical quantile vector $\hat{\boldsymbol{q}} = (\hat{q}_{\tau_1}, \ldots, \hat{q}_{\tau_K})^{\mathrm{T}}$ satisfies:

$$\sqrt{n}(\hat{\boldsymbol{q}} - \boldsymbol{q}(\theta_0)) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{\psi}(x_i) + o_p(1),$$

where $\boldsymbol{\psi}(x_i) = \left(\frac{\tau_1 - \mathbb{I}(x_i \le q_{\tau_1})}{f(q_{\tau_1})}, \ldots, \frac{\tau_K - \mathbb{I}(x_i \le q_{\tau_K})}{f(q_{\tau_K})}\right)^{\mathrm{T}}$ is the stacked influence function with $\mathbb{E}[\boldsymbol{\psi}(x_i)] = \mathbf{0}$.

By the multivariate central limit theorem for independent and identically distributed (i.i.d.) random vectors (Assumption 4.1):

$$\sqrt{n}(\hat{\boldsymbol{q}} - \boldsymbol{q}(\theta_0)) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Omega),$$

where $\Omega = \mathrm{Cov}(\boldsymbol{\psi}(x_i))$ has elements

$$\omega_{ij} = \frac{\mathbb{E}[(\tau_i - \mathbb{I}(x \le q_{\tau_i}))(\tau_j - \mathbb{I}(x \le q_{\tau_j}))]}{f(q_{\tau_i})f(q_{\tau_j})} = \frac{\tau_i \wedge \tau_j - \tau_i\tau_j}{f(q_{\tau_i})f(q_{\tau_j})},$$

with $\tau_i \wedge \tau_j := \min(\tau_i, \tau_j)$. The numerator follows from $\mathrm{Cov}(\mathbb{I}(x \le q_{\tau_i}), \mathbb{I}(x \le q_{\tau_j})) = \min(\tau_i, \tau_j) - \tau_i\tau_j$ for order statistics.

Step 3: Delta method application. The first-order condition for the optimization problem Eq (3.2) is

$$\nabla_\theta Q_n(\hat{\theta}_n) = -2 \sum_{\tau \in \mathcal{T}} w_\tau \mathrm{sign}(\hat{q}_\tau - q_\tau(\hat{\theta}_n)) \nabla_\theta q_\tau(\hat{\theta}_n) = 0.$$

By Taylor expansion around θ_0 and the consistency of $\hat{\theta}_n$,

$$0 = \nabla_\theta Q_n(\theta_0) + \nabla_{\theta\theta'} Q_n(\tilde{\theta}_n)(\hat{\theta}_n - \theta_0),$$

where $\tilde{\theta}_n$ lies between $\hat{\theta}_n$ and θ_0.

Noting that

$$\nabla_\theta Q_n(\theta_0) = -2J(\theta_0)^T W(\hat{q} - q(\theta_0)) + o_p(n^{-1/2}),$$

and

$$\nabla_{\theta\theta'} Q_n(\tilde{\theta}_n) \xrightarrow{p} 2J(\theta_0)^T W J(\theta_0),$$

we obtain

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = [J(\theta_0)^T W J(\theta_0)]^{-1} J(\theta_0)^T W \sqrt{n}(\hat{q} - q(\theta_0)) + o_p(1),$$

where the invertibility follows from Assumption 4.3(ii) and (iii). Applying the continuous mapping theorem yields the result. □

AIMS Press