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Research article

Wronskian solutions, Grammian solutions and lump molecules for a (2+1)-dimensional Yu-Toda-Sasa-Fukuyama equation

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Abstract: This paper investigates the exact solutions and dynamical properties of the (2+1)-dimensional Yu-Toda-Sasa-Fukuyama (YTSF) equation. First, using the bilinear method and complex linear systems, we theoretically prove that the equation admits Wronskian and Grammian determinant solutions, establishing the mathematical completeness of its solution structure. Second, based on the Grammian determinant solutions, we thoroughly analyze the complex localized wave behaviors exhibited by lump solutions during anomalous scattering. The results reveal that the system supports a novel bound state, lump molecules, i.e., stable composite structures formed by nonlinear interactions among multiple lump solutions, which subsequently propagate coherently. These findings not only uncover rich dynamical phenomena inherent to the YTSF equation, but also provide new theoretical insights into the formation and evolution mechanisms of multi-lump bound states in nonlinear partial differential equations.

Keywords: Wronskian solutions; Grammian solutions; lump molecules
Mathematics Subject Classification: 35Q51, 37K35

1. Introduction

Exact solutions of nonlinear partial differential equations play a crucial role in understanding complex wave dynamics in integrable systems [1–4]. Various methods have been developed to construct these solutions, including the auxiliary equation method [5], the Darboux transformation [6–9], and the multiple-scale approach [10, 11], etc. Among them, lump solutions are a class of nonsingular solutions characterized by localization and algebraic decay in all spatial directions, and are typically expressed as rational functions. As important spatially localized structures in integrable systems, lump solutions have attracted widespread attention in recent decades [12, 13]. Their discovery and analysis have enriched the understanding of wave localization in higher dimensions. Within the bilinear formalism, lump solutions are often constructed by applying a long-wave limit to soliton

solutions [14] or through direct ansatz methods [15].

More interestingly, the interaction dynamics between lump waves lead to fascinating scattering phenomena. Normal scattering refers to the elastic collision of lumps along straight lines, where they retain their individual characteristics after the interaction [16, 17]. In contrast, anomalous scattering describes significant, albeit transient, deformation during the interaction, which may involve curved trajectories, during which the lumps undergo more complex morphological changes before separating [18, 19]. Recent advances have further revealed the existence of lump molecules, bound states where two or more lumps, or lumps with solitons, breathers, etc., propagate together without dispersion over extended periods [20]. These molecules typically arise from specific balance conditions among parameters, representing a novel type of interacting structure. Systematically studying these normal and anomalous scattering behaviors and molecular states of lumps within the bilinear framework has been an active research area, as documented in foundational and recent works, but primarily focusing on normal scattering, with less exploration of molecular behavior associated with anomalous scattering [21].

The focus of our work is the (2+1)-dimensional Yu-Toda-Sasa-Fukuyama (YTSF) equation

$$-4u_{xt} - u_{xxxx} - 6u_x u_{xx} + 3u_{yy} = 0. \tag{1.1}$$

This equation is obtained by applying the transformation $\xi - z = x$ to its (3+1)-dimensional YTSF equation [22, 23]. Originating from nonlinear optics and plasma physics, this equation serves as a model describing certain types of wave propagation in optical fibers and magnetized plasmas. Significant work has been done by many researchers to unveil solutions of the (2+1)-dimensional YTSF equation. For instance, in a key study referenced in [24], the authors derived its bilinear form and systematically constructed its soliton solutions. They employed the standard Hirota method to provide explicit expressions for N-soliton solutions. This work laid an important foundation for further analysis. However, a notable gap was identified: While the expressions for the Wronskian and Grammian solutions are provided, a rigorous proof of their existence was not fully elaborated, leaving room for stronger mathematical consolidation. Expanding the perspective further, research in [25] delved into the anomalous scattering of lump solutions. By skillfully applying a long-wave limit method to the multi-soliton solutions of the YTSF equation, the authors successfully generated lump solutions exhibiting anomalous scattering. They then meticulously investigated the interactions between these lump solutions, revealing detailed scenarios of anomalous scattering. This work highlights the complex interaction dynamics possible within the YTSF model. Since the YTSF equation belongs to the KP hierarchy, recent studies on Wronskian solutions for other (2+1)-dimensional equations in the KP family are also relevant [26–28].

Despite these advances, several aspects remain underexplored. First, the powerful Wronskian and Grammian determinant techniques, systematic methods [29, 30] for generating large classes of solutions to bilinear equations, have not been widely applied to the (2+1)-dimensional YTSF equation. Rooted in Plücker relations and linear differential systems, these methods could offer a unified framework for solution generation and potentially streamline solution existence proofs. Second, although lump molecule phenomena have been observed in other systems such as the Kadomtsev-Petviashvili(KP)-I equation [21], their formation conditions, stability, and dynamical characteristics within the YTSF equation warrant dedicated study. For instance, can bound states of lumps arise from specific parameter constraints within the Wronskian or Grammian formulation? How do such lump

molecules interact with one another?

Accordingly, the main objectives of this paper are twofold: (1) To establish a systematic framework based on Wronskian and Grammian techniques for the (2+1)-dimensional YTSF equation, thereby consolidating and extending results suggested in earlier literature [24]. (2) To construct novel lump solutions within the Grammian formulation and systematically investigate the anomalous scattering phenomena they exhibit [25]. Through a general analytical approach, we aim to clarify the origins of lump molecules, complementing and extending prior findings. Special attention will be paid to identifying algebraic parameter conditions that lead to multi-lump bound states and to illustrating their propagation and interaction dynamics.

2. Wronskian and Grammian solutions

To prove that Eq (1.1) admits Wronskian and Grammian solutions, we first convert it into a bilinear form. Introducing the transformation

$$u = (2\ln f)_x, \tag{2.1}$$

Eq (1.1) can be transformed into the following bilinear equation:

$$\left(D_x^4 - 3D_y^2 + 4D_xD_t\right) f \cdot f = 0, \tag{2.2}$$

where D_x, D_y, and D_t are the Hirota bilinear operators [31]. We then work within this bilinear form to construct Wronskian and Grammian solutions.

Theorem 1. *Let the functions $\{\phi_i = \phi_i(x, y, t)\}$ be defined as solutions to the linear system*

$$\phi_{i,y} = I\phi_{i,xx}, \quad \phi_{i,t} = -\phi_{i,xxx}, \tag{2.3}$$

with $I^2 = -1$. The Wronskian determinant $f_N = |\widehat{(N-1)}|$, defined in [29], satisfies Eq (2.2)*.*

Proof. By applying Eq (2.3), we obtain the derivatives of the Wronskian determinant $f_N = |\widehat{(N-1)}|$ with respect to the variables x, y, and t in the following form:

$$\begin{aligned}
&f_{Nx} = |\widehat{(N-2)}, N|,\\
&f_{Nxx} = |\widehat{(N-2)}, N+1| + |\widehat{(N-3)}, N-1, N|,\\
&f_{Nxxx} = |\widehat{(N-2)}, N+2| + 2|\widehat{(N-3)}, N-1, N+1| + |\widehat{(N-4)}, N-2, N-1, N|,\\
&f_{Nxxxx} = |\widehat{(N-2)}, N+3| + 3|\widehat{(N-3)}, N-1, N+2| + 2|\widehat{(N-3)}, N, N+1|\\
&\qquad + 3|\widehat{(N-4)}, N-2, N-1, N+1| + |\widehat{(N-5)}, N-3, N-2, N-1, N|,\\
&f_{Ny} = I[|\widehat{(N-2)}, N+1| - |\widehat{(N-3)}, N-1, N|],\\
&f_{Nyy} = -[|\widehat{(N-2)}, N+3| + |\widehat{(N-3)}, N-1, N+2| - 2|\widehat{(N-3)}, N, N+1|\\
&\qquad + |\widehat{(N-4)}, N-2, N-1, N+1| - |\widehat{(N-5)}, N-3, N-2, N-1, N|],\\
&f_{Nt} = -|\widehat{(N-2)}, N+2| + |\widehat{(N-3)}, N-1, N+1| - |\widehat{(N-4)}, N-2, N-1, N|,\\
&f_{Nxt} = -|\widehat{(N-2)}, N+3| + |\widehat{(N-3)}, N, N+1| - |\widehat{(N-5)}, N-3, N-2, N-1, N|.
\end{aligned} \tag{2.4}$$

With the derivative formulas for $f_N = |(\widehat{N-1})|$ established above, we obtain

$$\begin{aligned}
&\left(D_x^4 - 3D_y^2 + 4D_xD_t\right) f_N \cdot f_N \\
&= 2\Big((f_{Nxxxx}f_N - 4f_{Nxxx}f_{Nx} + 3f_{Nxx}^2) - 3(f_{Nyy}f_N - f_{Ny}^2) + 4(f_{Nxt}f_N - f_{Nt}f_{Nx})\Big) \\
&= 2\Big((f_{Nxxxx} - 3f_{Nyy} + 4f_{Nxt})f_N - 4(f_{Nxxx} + f_{Nt})f_{Nx} + 3f_{Nxx}^2 + 3f_{Ny}^2\Big) \\
&= 24|(\widehat{N-3}), N, N+1|\,|(\widehat{N-1})| - 24|(\widehat{N-3}), N-1, N+1|\,|(\widehat{N-2}), N| \\
&\quad + 24|(\widehat{N-2}), N+1|\,|(\widehat{N-3}), N-1, N| \\
&= 12\begin{vmatrix} (\widehat{N-3}) & 0 & N-2 & N-1 & N & N+1 \\ 0 & (\widehat{N-3}) & N-2 & N-1 & N & N+1 \end{vmatrix} = 0.
\end{aligned} \tag{2.5}$$

The last expression vanishes identically due to the Plücker relations satisfied by Wronskian determinants. Hence,

$$\left(D_x^4 - 3D_y^2 + 4D_xD_t\right) f_N \cdot f_N = 0,$$

which shows that f_N satisfies Eq (2.2). □

Remark 1. *The linear system in Theorem 1 can be derived from the bilinear Bäcklund transformation of the YTSF equation. For the (3+1)-dimensional case, this connection has been established in [31,32]. Following the same procedure, the linear system for the (2+1)-dimensional case is obtained as Eq* (2.3)*. Alternatively, under the plane wave assumption $\phi_i = e^{k_i x + \omega_i y + \gamma_i t}$, where k_i is an arbitrary constant, the linear system corresponds to the dispersion relations $\omega_i = Ik_i^2$, $\gamma_i = -k_i^3$.*

Theorem 2. *Let a set of functions $\phi_i = \phi_i(x, y, t), \varphi_j = \varphi_j(x, y, t)$ satisfy the linear systems:*

$$\phi_{i,y} = I\phi_{i,xx}, \quad \phi_{i,t} = -\phi_{i,xxx}, \tag{2.6}$$

and

$$\varphi_{j,y} = -I\varphi_{j,xx}, \quad \varphi_{j,t} = -\varphi_{j,xxx}, \tag{2.7}$$

where $I^2 = -1$. Then, the Grammian determinant

$$\tau_N(x, y, t) = \det\Big(G_{ij}(x, y, t)\Big)_{N\times N}, \quad G_{ij} = c_{ij} + \int_{-\infty}^{x} \phi_i\varphi_j\, dx, \tag{2.8}$$

defined in [29] solves Eq (2.2)*.*

Proof. A Grammian determinant can be recast as a Pfaffian. Hence, we have

$$\tau_N = (1, 2, \ldots, N, N^*, \ldots, 2^*, 1^*), \tag{2.9}$$

where $(i, j) = (i^*, j^*) = 0$ and $(i, j^*) = a_{i,j} = c_{ij} + \int_{-\infty}^{x} \phi_i\varphi_j\, dx$. To compute derivatives of the entries G_{ij} and the Grammian determinant τ_N, we introduce new Pfaffian entries

$$(d_n,\ j^*) = \frac{\partial^n}{\partial x^n}\varphi_j, \qquad (d_m,\ d_n^*) = 0, \qquad (d_n^*,\ i) = \frac{\partial^n}{\partial x^n}\phi_i, \qquad (d_m^*,\ i^*) = (d_n,\ i) = 0, \tag{2.10}$$

$$m, n = 0, 1, 2, 3, \ldots.$$

In terms of these new entries Eq (2.10), the derivatives of the entries $G_{ij} = (i, j^*)$ are given, upon using Eqs (2.6) and (2.7), by

$$\begin{aligned}
&\frac{\partial}{\partial x}a_{ij} = \phi_i\varphi_j = (d_0,\ d_0^*,\ i,\ j^*),\\
&\frac{\partial}{\partial y}a_{ij} = \int_{-\infty}^{x}(\phi_{iy}\varphi_j + \phi_i\varphi_{jy})\,dx = I(\phi_{ix}\varphi_j - \phi_i\varphi_{jx}) = I\Big((d_0,\ d_1^*,\ i,\ j^*) - (d_1,\ d_0^*,\ i,\ j^*)\Big),\\
&\frac{\partial}{\partial t}a_{ij} = \int_{-\infty}^{x}(\phi_{it}\varphi_j + \phi_i\varphi_{jt})dx = -(\phi_{ixx}\varphi_j - \phi_{ix}\varphi_{jx} + \phi_i\varphi_{jxx}) = -\Big((d_0,d_2^*,i,j^*) - (d_1,d_1^*,i,j^*) + (d_2,d_0^*,i,j^*)\Big).
\end{aligned} \tag{2.11}$$

Using Eqs (2.10) and (2.11), the derivatives of the Grammian determinant τ_N with respect to x, y, and t are obtained as follows:

$$\begin{aligned}
&\tau_{Nx} = (d_0, d_0^*, \bullet),\\
&\tau_{Nxx} = (d_0, d_1^*, \bullet) + (d_1, d_0^*, \bullet),\\
&\tau_{Nxxx} = (d_0, d_2^*, \bullet) + (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet),\\
&\tau_{Nxxxx} = (d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet),\\
&\tau_{Ny} = I[(d_1, d_0^*, \bullet) - (d_0, d_1^*, \bullet)],\\
&\tau_{Nyy} = -(d_3, d_0^*, \bullet) + (d_1, d_2^*, \bullet) - (d_0, d_3^*, \bullet) + (d_2, d_1^*, \bullet) - 2(d_0, d_0^*, d_1, d_1^*, \bullet),\\
&\tau_{Nt} = (d_1, d_1^*, \bullet) - (d_0, d_2^*, \bullet) - (d_2, d_0^*, \bullet),\\
&\tau_{Nxt} = -(d_3, d_0^*, \bullet) + (d_0, d_0^*, d_1, d_1^*, \bullet) - (d_0, d_3^*, \bullet),
\end{aligned} \tag{2.12}$$

where the abbreviated symbol $\bullet$ denotes the common list of indices $1, 2, \ldots, N, N^*, \ldots, 2^*, 1^*$ appearing in each Pfaffian. With the derivative formulas for τ_N established above, we obtain

$$\begin{aligned}
\Big(D_x^4 - 3D_y^2 + 4D_xD_t\Big)\tau_N\cdot\tau_N &= 2\Big((\tau_{Nxxxx} - 3\tau_{Nyy} + 4\tau_{Nxt})\tau_N - 4(\tau_{Nxxx} + \tau_{Nt})\tau_{Nx} + 3\tau_{Nxx}^2 + 3\tau_{Ny}^2\Big)\\
&= 24(d_0, d_0^*, d_1, d_1^*, \bullet)(\bullet) - 24(d_1, d_1^*, \bullet)(d_0, d_0^*, \bullet) + 24(d_0, d_1^*, \bullet)(d_1, d_0^*, \bullet).
\end{aligned} \tag{2.13}$$

The last expression vanishes identically due to the Jacobi identity for determinants. Hence,

$$\Big(D_x^4 - 3D_y^2 + 4D_xD_t\Big)\tau_N\cdot\tau_N = 0,$$

which shows that τ_N satisfies Eq (2.2). □

Remark 2. *Similar to Remark 1, the linear system Eqs* (2.6) *and* (2.7) *in Theorem 2 can be derived. The details are omitted for brevity.*

3. Lump molecules

For the Grammian-form solutions given by Theorem 2, we restrict ourselves to the case $c_{ij} = 0$. One can see that $\{\phi_i = exp(\xi_i), \xi_i = \lambda_i x + I\lambda_i^2 y - \lambda_i^3 t + \alpha_i\lambda_i\}$ are solutions of the complex linear systems (2.6), and that $\phi_i = \frac{\partial^s}{\partial\lambda_i^s}exp(\xi_i)$ also satisfies systems (2.6). Similarly, it is observed that $\{\varphi_j = exp(\eta_j), \eta_j = \lambda_j x - I\lambda_j^2 y - \lambda_j^3 t + \alpha_j\lambda_j\}$ solves complex linear systems (2.7) and that $\varphi_\eta = \frac{\partial^s}{\partial\lambda_j^s}exp(\eta_j)$ likewise satisfies system (2.7).

3.1. 1-lump solution for N=1, s=1

To construct a 1-lump solution, assuming that τ_1 is of the form

$$\tau_1=\int_{-\infty}^{x}\phi_1\varphi_1 dx=2\lambda_1\Big(18\lambda_1^6t^2-12\alpha_1\lambda_1^4t-12\lambda_1^4tx+8\lambda_1^4y^2+2\alpha_1^2\lambda_1^2+4\alpha_1\lambda_1^2x+6\lambda_1^3t+2\lambda_1^2x^2-2\alpha_1\lambda_1-2\lambda_1x+1\Big), \quad (3.1)$$

and using Eq (2.1), we can obtain the 1-lump expression as follows:

$$u=\frac{72\lambda_1^3\left[\lambda_1^4t^2+\left(-\frac{2}{3}t(x+\alpha_1)+\frac{4}{9}y^2\right)\lambda_1^2+\frac{(x+\alpha_1)^2}{9}\right]}{1+18\lambda_1^6t^2+(-12t(x+\alpha_1)+8y^2)\lambda_1^4+6\lambda_1^3t+2(x+\alpha_1)^2\lambda_1^2-2(x+\alpha_1)\lambda_1}. \quad (3.2)$$

To study the trajectory of the 1-lump, we can investigate the motion of the extreme points of the 1-lump solution. Here, by using $\{u_x=0,u_y=0\}$, we can obtain that the extreme point corresponding to the minimum value 0 is at coordinates $(x=3\lambda_1^2t-\alpha_1,y=0)$, and the extreme point corresponding to the maximum value $8\lambda_1$ is at coordinates $(x=3\lambda_1^2t-\alpha_1+\dfrac{1}{\lambda_1},y=0)$. From the coordinates of the extreme points, we can see that the 1-lump moves along the straight line $y=0$.

3.2. 2-lump molecule for N=1, s=2

To construct a 2-lump solution, assuming that τ_2 is of the form

$$\begin{aligned}\tau_2=&\int_{-\infty}^{x}\frac{\partial^2}{\partial\lambda_1^2}exp(\xi_1)\frac{\partial^2}{\partial\lambda_1^2}exp(\eta_1)\,dx\\
=&324\lambda_1\Bigg[\frac{1}{54}+\lambda_1^{12}t^4-\frac{4}{3}(tx+t\alpha_1-\tfrac{2}{3}y^2)t^2\lambda_1^{10}-\frac{2}{3}\lambda_1^9t^3\\
&+\Big(\frac{2}{3}\alpha_1^2t^2+(\tfrac{4}{3}t^2x-\tfrac{16}{27}ty^2)\alpha_1+\frac{2}{3}t^2x^2+\frac{16}{81}y^4-\frac{16}{27}txy^2\Big)\lambda_1^8\\
&+\frac{2}{9}(tx+t\alpha_1+\tfrac{4}{3}y^2)t\lambda_1^7+\Big(-\frac{4}{27}t\alpha_1^3+(\tfrac{8}{81}y^2-\tfrac{4}{9}tx)\alpha_1^2+(-\tfrac{4}{9}tx^2\\
&+\tfrac{16}{81}xy^2)\alpha_1-\frac{4}{27}tx^3+\frac{8}{81}x^2y^2+\frac{1}{3}t^2\Big)\lambda_1^6+\frac{2}{27}(tx+t\alpha_1+\tfrac{4}{3}y^2)(x\\
&+\alpha_1)\lambda_1^5-\frac{2}{27}(x+\alpha_1)\Big(-\frac{1}{6}x^3-\frac{1}{2}x^2\alpha_1-\frac{1}{2}x\alpha_1^2-\frac{1}{6}\alpha_1^3+t\Big)\lambda_1^4\\
&+\Big(-\frac{2}{27}x^2\alpha_1-\frac{2}{27}x\alpha_1^2-\frac{2}{81}x^3-\frac{2}{81}\alpha_1^3+\frac{1}{27}t\Big)\lambda_1^3+\frac{1}{27}(x+\alpha_1)^2\lambda_1^2+\Big(-\frac{x}{27}-\frac{\alpha_1}{27}\Big)\lambda_1\Bigg].\end{aligned} \quad (3.3)$$

To graphically observe the behavior of 2-lump molecules, we set the parameters $\lambda_1=1$ and $\alpha_1=1$, obtaining

$$\tau_2=\frac{N(x,y,t)}{D(x,y,t)}, \quad (3.4)$$

where

$$\begin{aligned}N(x,y,t)=&\,648t^4+(-864x-1728)t^3+(432x^2+576y^2+1440x+1296)t^2\\
&-96(x+1)(x^2+4y^2+3x+2)t+8x^4+32x^3+(64y^2+48)x^2\end{aligned}$$

$$+ (256y^2 + 32)x + 128y^4 + 224y^2 + 8,$$

$$\begin{aligned} D(x,y,t) = {} & 162t^4 + (-216x - 324)t^3 + (108x^2 + 144y^2 + 252x + 198)t^2 \\ & - 24\Big(x^2 + 4y^2 + 2x + \frac{3}{2}\Big)\Big(x + \frac{1}{2}\Big)t + 2x^4 + 4x^3 + (16y^2 + 6)x^2 \\ & + (48y^2 + 2)x + 32y^4 + 32y^2 + 1. \end{aligned}$$

Remark 3. *Since the analytical expression of the 2-lump solution is cumbersome, directly examining its trajectory is challenging. Following the approach adopted in Subsection 3.1, we instead analyze the motion by locating the coordinates of its minima. Using the transformation* $u = 2(\ln \tau_2)_x$ *from Eq* (2.1) *and the expression for* τ_2 *from Eq* (3.4)*, solving the system*

$$u_x = 0, \qquad u_y = 0 \tag{3.5}$$

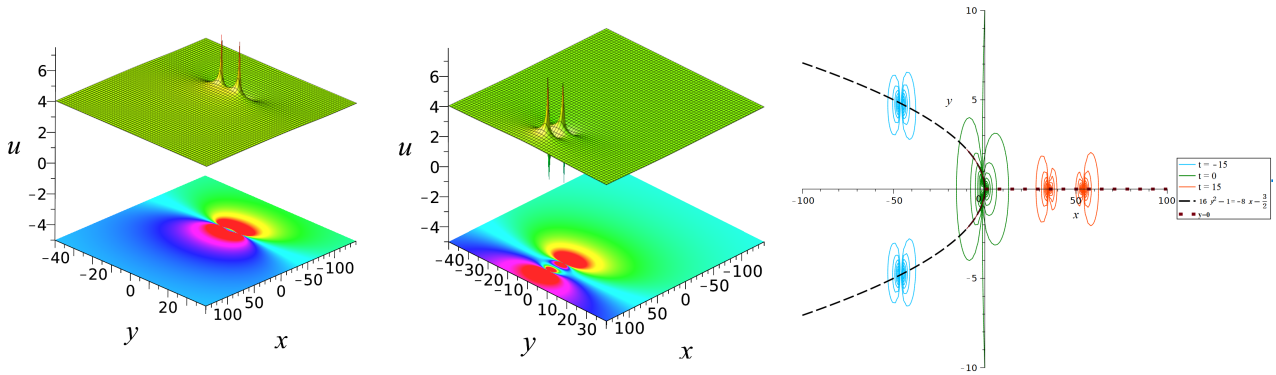
yields the positions of the zero minimum value:

$$\{x = 3t - \tfrac{3}{2},\ y = \pm\tfrac{1}{4}\sqrt{-24t + 1}\} \quad as\ t \to -\infty,$$

and

$$\{x = 3t - 1 \pm \sqrt{6t},\ y = 0\} \quad as\ t \to +\infty.$$

As $t \to -\infty$, the minimum value indicates that the lump propagates along the curve $16y^2 - 1 = -8x - \frac{3}{2}$. When $t \to +\infty$, the lump travels along the straight line $y = 0$. Moreover, as seen in Figure 1, the velocity of the extremal points approaches $[3, 0]^T$ both as $t \to -\infty$ and as $t \to +\infty$, reflecting a molecule-like behavior of the lump (see Figure 1). This trajectory differs from the normal lump-interaction process described in the literature [16, 17] and exhibits an anomalous scattering behavior.



(a) $t = -10$ **(b)** $t = 10$ **(c)**

Figure 1. (a) and (b) Three-dimensional visualization and density plot of the 2-lump molecule for parameters $\lambda_1 = 1, \alpha_1 = 1$. (c) Trajectory of the 2-lump molecule.

3.3. Interaction between 2-lump molecules for N=2, s=2

To construct the interaction between 2-lump molecules, assume that $\tau_{2'}$ is of the form

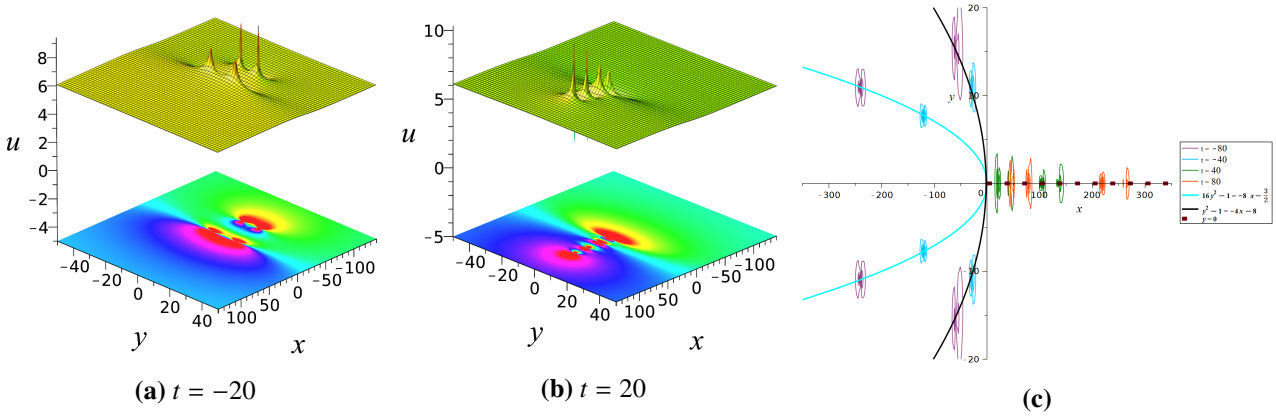
$$\tau_{2'} = \det\begin{pmatrix} \displaystyle\int_{-\infty}^{x}\Big[\frac{\partial^2}{\partial\lambda_1^2}e^{\xi_1}\Big]\Big[\frac{\partial^2}{\partial\lambda_1^2}e^{\eta_1}\Big]dx & \displaystyle\int_{-\infty}^{x}\Big[\frac{\partial^2}{\partial\lambda_1^2}e^{\xi_1}\Big]\Big[\frac{\partial^2}{\partial\lambda_2^2}e^{\eta_2}\Big]dx \\ \displaystyle\int_{-\infty}^{x}\Big[\frac{\partial^2}{\partial\lambda_2^2}e^{\xi_2}\Big]\Big[\frac{\partial^2}{\partial\lambda_1^2}e^{\eta_1}\Big]dx & \displaystyle\int_{-\infty}^{x}\Big[\frac{\partial^2}{\partial\lambda_2^2}e^{\xi_2}\Big]\Big[\frac{\partial^2}{\partial\lambda_2^2}e^{\eta_2}\Big]dx \end{pmatrix}. \tag{3.6}$$

Setting $\lambda_1 = \frac{1}{2}, \lambda_2 = 1, \alpha_1 = 1$, and $\alpha_2 = -1$ yields the interaction between two lump molecules.

Remark 4. *Due to the structural complexity of Eq* (3.6)*, it is not possible to directly derive its extremum points analytically from Eq* (3.6)*, nor can the extremum locations be determined using the system of equations* $\{\tau_{2'x} = 0, \tau_{2'y} = 0\}$ *as was done for the Kadomtsev–Petviashvili I equation in reference [21]. Therefore, this study proceeds by directly plotting the interaction image of two lump molecules. The trajectory equations*

$$16y^2 - 1 = -8x - \frac{3}{2}, \qquad y^2 - 1 = -4x - 8, y = 0$$

are obtained from the extremum condition of Eq (3.4) *with the parameters set as* $\{\lambda_1 = \frac{1}{2}, \lambda_2 = 1, \alpha_1 = 1, \alpha_2 = -1\}$*. The results show that the derived trajectory equations align perfectly with the image of the two lump molecules interacting (see Figure 2). We conjecture that the interaction between the two clusters of lump molecules may not affect the locations of their extrema, though the underlying mechanism requires further investigation. This also indicates that analytically determining the extrema for multi-lump expressions remains a challenging problem.*



(a) $t = -20$ **(b)** $t = 20$ **(c)**

Figure 2. (a) and (b) Three-dimensional visualization and density plot of the interaction between two 2-lump molecules for parameters $\lambda_1 = 1/2$, $\lambda_2 = 1$, $\alpha_1 = 1$, $\alpha_2 = -1$. (c) Trajectory of the interaction.

4. Conclusions

In this paper, based on the complex linear systems (2.3) and the Wronskian formulation (2.4), we first prove that Eq (1.1) admits Wronskian determinant solutions. Furthermore, with the aid of the complex linear systems (2.6) and (2.7) and the Pfaffian derivative formulae (2.12), we establish the corresponding Grammian determinant solutions for Eq (1.1).

Using the obtained Grammian solutions, we reveal a molecule-like phenomenon (i.e., multi-lump aggregation) in the lump solutions of Eq (1.1). In particular, the analysis in Sections 3.2 and 3.3 shows that the motion pattern of this lump molecule differs markedly from that of a classical lump solution: As $t \to -\infty$, it propagates along a curve rather than moving in a straight line. This result demonstrates richer geometric and dynamic behavior of lump solutions during interaction processes [23–25].

The Wronskian and Grammian solutions presented in Theorems 1 and 2 can be understood as reductions of general KP hierarchy solutions. Through this research, we expect to deepen the

understanding of the diversity of lump solutions and their interaction mechanisms in Eq (1.1), and to demonstrate the effectiveness of the Wronskian/Grammian methods in organizing and revealing its rich nonlinear wave structures.

Based on the results of this work, several directions can be explored in the future. First, the Grammian solutions that generate anomalous scattering lumps can be extended to other higher-dimensional nonlinear wave equations. Second, the interactions between anomalous scattering lumps derived from Grammian solutions and other types of solutions, such as solitons and breathers, deserve further investigation.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author confirms that he has no conflicts of interest related to the publication of this study.

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