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Research article

Lower bounds for the maximal Lyapunov exponent in one-parameter families of linear differential systems

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Abstract: We consider a two-dimensional, nonautonomous, homogeneous system of linear ordinary differential equations that depends on the real parameter μ. It is assumed that the Cauchy operator for each unit time interval is a product of a rotation matrix by an angle, the value of which is an affine function of μ, and of a diagonal matrix with a unit determinant, which is chosen to be close to a constant and whose norm is sufficiently large to guarantee the monotonicity with respect to μ of polar angle for any solution to the system. This class of systems contains an example of a non-almost-reducible linear system with limit-periodic coefficients constructed by V. M. Millionshchikov. We use his rotation method to establish the positivity of the maximal Lyapunov exponent in one-parameter family for some set of parameter values that has positive Lebesgue measure. To derive this result, we prove the monotonicity with respect to μ of angles in singular-value decomposition for Cauchy operator and moreover that its derivative is separated from zero. Further, the angle itself increases as a monotonic linear function of t. Both of these properties, by induction, give us a small average loss for the Cauchy operator norm on exponentially growing time intervals, which leads to its exponential growth as a function of t.

Keywords: Lyapunov exponent; linear differential system; real parameter; positive measure
Mathematics Subject Classification: 34A30, 34D08

1. Introduction

Consider a linear discrete system

$$x(n+1) = B_\mu(n)\, x(n), \quad x \in \mathbb{R}^2, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

with a coefficient matrix

$$B_\mu(n) := \begin{cases} \operatorname{diag}[e^{d_k(\mu)}, e^{-d_k(\mu)}], & n = 2k-2, \\ U(\mu + \gamma_k(\mu) + b_k), & n = 2k-1, \end{cases}$$

with $k \in \mathbb{N}$, the real parameter μ, the numbers $b_k \in \mathbb{R}$, and the continuous 2π-periodic functions $d_k(\cdot), \gamma_k(\cdot)\colon \mathbb{R} \to \mathbb{R}$. Here, by $U(\varphi) \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, we denote the matrix of the counterclockwise rotation by an angle $\varphi \in \mathbb{R}$.

M. Herman in [1] considered the system (1.1) in the case where for some functions $d(\cdot)$ and $\gamma(\cdot)$, the following conditions are satisfied:

$$d_k(\mu) = d(\mu + k\omega), \quad \gamma_k(\mu) = \gamma(\mu + k\omega), \quad b_k = k\omega, \quad k \in \mathbb{N}, \quad \mu \in \mathbb{R}, \quad \omega \in \mathbb{R}\backslash\mathbb{Q}, \tag{1.2}$$

$$d(\cdot) \equiv d \geq 1, \quad \gamma(\cdot) \equiv 0. \tag{1.3}$$

In this case, the Cauchy operator of the system (1.1) $Y_{B_\mu}(k,n) \stackrel{\text{def}}{=} B_\mu(k-1) \cdot \ldots \cdot B_\mu(n)$, $k > n \geq 0$, considered as a function of μ, is a linear cocycle (see the definition, for example, in [2, Section 5], or more a detailed definition in [3]).

If the conditions (1.2) hold, by the ergodic theorem, the maximal Lyapunov exponent of the system (1.1), defined by the formula

$$\lambda^+(B_\mu) \stackrel{\text{def}}{=} \overline{\lim_{n\to+\infty}} \frac{1}{n} \ln \|Y_{B_\mu}(n,0)\|,$$

is equal for almost all $\mu \in \mathbb{R}$ to

$$L(B) := \lim_{n\to+\infty} \frac{1}{2\pi n} \int_0^{2\pi} \ln \|Y_{B_\mu}(n,0)\|\, d\mu.$$

In [1, Section 4], under (1.2) and (1.3), using the sub-harmonicity of $L(B)$ on μ, the following estimate was proven

$$L(B) \geq \ln \frac{e^d + e^{-d}}{2}. \tag{1.4}$$

Sorets and Spencer [4] further developed Herman's complex-analytic method to show the positivity of the maximal Lyapunov exponent for the linear differential equation

$$\ddot{x} = -(K^2(\cos t + \cos(\omega t + \theta)) + E)\, x, \quad x \in \mathbb{R}^2, \quad t \geq 0$$

with any irrational $\omega \in \mathbb{R}$ and almost all $\theta \in \mathbb{R}$ on the set of values $E \geq 0$, whose relative Lebesgue measure tends to unity for large K.

Young [5, Corollary 1] considered the case when $\mu + \gamma(\mu)$ is a diffeomorphism of the unit circle, that is equivalent to the inequality

$$\gamma'(\mu) > -1, \quad \mu \in \mathbb{R}. \tag{1.5}$$

Provided that ω satisfies some Diophantine condition, which holds almost everywhere, she proved the approximation $L(B) \approx d$ for the maximal Lyapunov exponent of the system (1.1), (1.2), and (1.5) with sufficiently big values of $d(\cdot) \equiv d > 0$.

The method of [5] is in "spirit" of dynamical systems. It was applied in [6, 7] to estimate the maximal Lyapunov exponent for a one-dimensional discrete Schrödinger operator with quasi-periodic potential. In this paper, we use a similar technique.

Avila and Krikorian [8] studied so-called "monotonic" cocycles, i.e., (1.1) and (1.2), that the polar angle of any of their solutions is a monotonically increasing function of the parameter μ. This property can be represented by the inequality (here $(\cdot,\cdot)$ denotes the scalar product of vectors in $\mathbb{R}^2$)

$$\Big(B_\mu(2k)\,B_\mu(2k-1)\,y,\ U(\pi/2)\,(B_\mu(2k)\,B_\mu(2k-1)\,y)'_\mu\Big) \ge \varepsilon > 0, \quad 0 \ne y \in \mathbb{R}^2, \quad k \in \mathbb{N}. \tag{1.6}$$

Assuming the inclusion $d(\cdot), \gamma(\cdot) \in C^\infty$, they showed the smoothness of the maximal Lyapunov exponent considered as a function of μ. Clearly, each of the (1.3) or (1.5) implies the inequality (1.6). Therefore, the maximal Lyapunov exponent remains positive in some C^∞-neighborhood of systems (1.1)–(1.3).

The estimates for the maximal Lyapunov exponent of Schrödinger operators with mixed-type potentials were also obtained in recent papers [9–12].

In addition to the discrete system (1.1), we consider the linear differential system

$$\dot{x} = A_\mu(t)x, \quad x \in \mathbb{R}^2, \quad t \ge 0 \tag{1.7}$$

with matrices

$$A_\mu(t) = \begin{cases} d_k(\mu)\,\mathrm{diag}\,[1,-1], & 2k-2 \le t < 2k-1, \\ (\mu + \gamma_k(\mu) + b_k)\,J, & 2k-1 \le t < 2k, \end{cases}$$

where $k \in \mathbb{N}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and μ is a real parameter; the numbers b_k and the functions $d_k(\cdot), \gamma_k(\cdot)$ are assumed to be as those in (1.1).

The Cauchy operator $X_{A_\mu}(t,s)$, $t, s \ge 0$ of (1.7) for any $n \in \mathbb{N}$ satisfies the equality $X_{A_\mu}(n, n-1) = B_\mu(n-1)$. Thus, (1.1) and (1.7) are asymptotically equivalent.

By use of the singular value decomposition for $X_{C_\mu}(n+2, n)$, $n \in \mathbb{N}$, one can show that each system $\dot{x} = C_\mu(t)\,x$, whose coefficient matrix is uniformly bounded on t in $[0, +\infty)$, has zero trace $\mathrm{Tr}\,C_\mu(t) \equiv 0$, and is a continuous function of $\mu \in \mathbb{R}$, is asymptotically equivalent to some (1.7) with continuous functions $d_k(\cdot), \gamma_k(\cdot)\colon \mathbb{R} \to \mathbb{R}$.

As a consequence, the consideration can be restricted to the case (1.7) without a loss of generality.

The case

$$d_k(\mu) \equiv d_k \ge d > 0, \quad \gamma_k(\cdot) \equiv 0, \quad k \in \mathbb{N}$$

was considered in [13, 14].

In [13], under the constraint $d_k \equiv d > 4\ln 2$, it was shown that the maximal Lyapunov exponent of (1.7) is positive on a set of parameter values of positive Lebesgue measure. The result of [14] implies an absence of any upper bound for the norm of solutions of the system (1.7) that is uniform with respect to $\mu \in \mathbb{R}$ and $t \ge 0$.

For each natural number n, we denote by $\nu_2(n)$ the maximum power of 2 that divides n. Thus, the equality $n = 2^{\nu_2(n)}p(n)$ holds, where $p(n)$ is odd.

The goal of this article is to adopt the technique of [5–7], which requires only the differentiability of coefficients on a parameter, to systems beyond the quasi-periodic class.

The following theorem is the main result of the paper.

Theorem 1.1. *Let* $\{\alpha_m\}_{m=1}^{\infty}$ *be an arbitrary sequence of real numbers. The sequence* $\{b_k\}_{k=1}^{\infty}$ *is defined by the equality*

$$b_k = \alpha_{1+\nu_2(k)}, \quad k \in \mathbb{N}; \tag{1.8}$$

in addition, we assume that the functions

$$d_k(\mu) \equiv d(\cdot) > 0, \quad \gamma_k(\cdot) \equiv \gamma(\cdot), \quad k \in \mathbb{N}, \tag{1.9}$$

are differentiable on $\mathbb{R}$*,* π*-periodic, and such that the following inequalities hold:*

$$\gamma'(\mu) > 2|d'(\mu)| - 2^{-1}, \quad \mu \in \mathbb{R}, \tag{1.10}$$

$$\int_0^{\pi} d(\mu)\, d\mu > 2^{13}. \tag{1.11}$$

Then, the maximal Lyapunov exponent $\lambda_{\max}(A_\mu)$ *of the system (1.7) is positive on a set of parameter values of positive Lebesgue measure.*

For example, one can choose $\gamma \equiv 0$, $d(\mu) = p + r\cos(2\mu)$, where $p > 2^{13}$, $|r| < 2^{-3}$.

The inequality (1.11) assumed in the theorem is similar to the case of the big coupling constant considered in [6, 7].

The most restrictive condition (1.8) together with (1.10) imply the monotonicity on μ of the angles $\theta_{k,m}$, $\zeta_{k,m}$ defined by the equality

$$X_{A_\mu}(m2^k, (m-1)2^k) = U(\theta_{k,m})\,\mathrm{diag}\,[\eta_{k,m}, \eta_{k,m}^{-1}]\, U(\zeta_{k,m}),\ k, m \in \mathbb{N}.$$

This property fails in general without the assumption of (1.8). It may be that $\cos(\zeta_{k,2} + \theta_{k,1})$ is trapped in a narrow neighborhood of zero for a prevailing set of parameter values. Therefore, the formula $\eta_{k+1,1} \approx \eta_{k,2}\eta_{k,1}\cos(\zeta_{k,2} + \theta_{k,1})$ resulted in big losses in $\int_0^{\pi} |\ln \eta_{k+1,1}(\mu)|\, d\mu$.

It is plausible that this problem can be handled in a way analogous to [13], but the proof seems to be rather complicated. Thus, we leave the general case for further research.

It was proved in [15], that in the cases (1.8) and (1.9), where $d(\cdot)$ is an arbitrary continuous function, $\gamma(\cdot) \equiv 0$, there exists $\mu \in \mathbb{R}$, such that the corresponding system (1.7) is unstable.

If the conditions (1.8) and (1.9) hold, one can obtain the equality (it will be proved below)

$$X_{A_\mu}(2^{k+1}, 0) = U(\alpha_{k+1} - \alpha_k) X_{A_\mu}^2(2^k, 0), \quad k \in \mathbb{N}. \tag{1.12}$$

Systems, whose coefficients satisfy (1.8) and (1.9) allow us to construct one-parameter families with a various asymptotic properties. For example, it follows from the estimate (the formula (2.9) below)

$$\|A_\mu(t+m)2^{k-1}) - A_\mu(t)\| \le |\alpha_{k+\nu_2(m+1)} - \alpha_k|, \quad t \in [0, 2^{k-1}), \quad k, m \in \mathbb{N},$$

that if the sequence $\{\alpha_n\}_{n=1}^{\infty}$ converges, then the matrix $A_\mu(\cdot)$ is the uniform limit as $k \to +\infty$ of a sequence of 2^k-periodic matrices. Millionshchikov used these systems in [16, 17] to prove the existence of non-almost-reducible linear differential systems with limit-periodic and quasi-periodic coefficients.

Rakhimberdiev [18] constructed an example of a linear system with linear dependence on the real parameter μ, whose maximal Lyapunov exponent is discontinuous as a function of μ at all points of some interval.

Damanik proved in [19] an existence of a limit-periodic Scrödinger cocycle, whose maximal Lyapunov exponent is discontinuous as a function of energy on a set of positive Lebesgue measure. His result is based on estimates of Lyapunov exponents for almost-periodic cocycles obtained in [20] by use of a complex-analytic technique.

Proposition 1.2. *Let* $\{\beta_m\}_{m=1}^{\infty}$ *be an arbitrary sequence of real numbers, and let the conditions (1.8) and (1.9) hold.*

Then, there exists a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{R}$*, such that the system (1.7) is stable under Lyapunov for all* $\mu \in \{\beta_m\}_{m=1}^{\infty}$.

It follows from Theorem 1.1 and Proposition 1.2 that for an arbitrary countable set $M \subset \mathbb{R}$ of parameter values μ, there exists a matrix function $A_\mu(t)$, such that it and its derivative on μ are uniformly bounded on $t \in [0, +\infty)$, $\operatorname{Tr} A_\mu(t) \equiv 0$. The matrix $X_{A_\mu}(n, n-2)$, $n \in \mathbb{N}$ is nonanalytic on μ almost everywhere in $\mathbb{R}$, the system $\dot{x} = A_\mu(t)\, x$ is stable under Lyapunov in M, whereas its maximal Lyapunov exponent is positive on a set of positive Lebesgue measure.

Notice that one can choose the set of stability M to be dense in some interval $I \subset \mathbb{R}$. In this case, the maximal Lyapunov exponent of the corresponding system is discontinuous as a function of μ at all points of its positivity in I, when it is positive.

Thus, we have examples analogous to those in [18, 19], but coefficients of the system under consideration are not supposed to be analytic on μ. In the rest of our paper, we will assume that the conditions of Theorem 1.1 are satisfied.

Now, let us explain the method of its proof. We suppose $\gamma(\mu) \equiv 0$ for simplicity. For any fixed $0 \neq z \in \mathbb{R}^2$, let $x_\mu(t)$ denote the solution of (1.7) that satisfies the initial condition $x_\mu(0) = z$, and let $\varphi_\mu(t)$ be its polar angle, which is a continuous function of two variables t and μ (its values are not limited to any interval of length 2π).

On even time intervals of the form $(2k-1, 2k)$, $k \in \mathbb{N}$, any solution of (1.7) rotates by an angle proportional to μ. Therefore, for any μ_0, the difference between the polar angles of solutions with $\mu = \mu_0$ and $\mu = \mu_0 + \pi$ increases by π.

On odd intervals of the form $(2k, 2k+1)$, by the diagonality of the coefficient matrix, any solution does not leave a certain quadrant. Thus the difference between φ_μ and $\varphi_{\mu+\pi}$ remains the same, because of the coefficients of (1.7) are π-periodic with respect to μ, and therefore this difference is a multiple of π.

By induction, we obtain the equality

$$\varphi_{\mu+\pi}(2n) - \varphi_\mu(2n) = n\pi, \quad n \in \mathbb{N}. \tag{1.13}$$

Let us take two solutions of (1.7) for the parameter values $\mu = \mu_0$ and $\mu = \mu_0 + \Delta\mu$, and such that their norms at the moment $t = 2k$ are equal to unity.

For small $\Delta\mu$, they are close to each other, and the distance between them at $t = 2k+1$ will be not much bigger than the coefficients difference of corresponding systems (1.7) in the interval $(2k, 2k+1)$. The latter does not exceed approximately $|d'_\mu|\, \Delta\mu$. Hence, as a part, the difference between the polar angles of these solutions at this moment is not greater than $\approx |d'_\mu|\, \Delta\mu$.

Thus, because of $\varphi_{\mu+\Delta\mu} - \varphi_\mu$ increases by $\Delta\mu$ over the time interval $(2k-1, 2k)$, if $|d'_\mu| \ll 1$, then the monotonic growth of the polar angle with respect to μ, assumed at the moment $t = 2k-1$, will also hold at $t = 2k+1$.

Therefore, by induction, we obtain the inequality stated in Lemma 3.1.

$$\left(\varphi_\mu(k)\right)'_\mu > 0, \quad k \in \mathbb{N}. \tag{1.14}$$

Lemma 4.1 establishes the symmetry in singular value decomposition of the Cauchy operator $X_A(\theta, \tau)$ of the system (1.7) for the time interval from 0 to $2^k - 1$, namely, the equality of rotation angles in right and left sides. So, it can be represented as (here, $D_k = \text{diag}\,[r_k, r_k^{-1}]$ for some $0 \neq r_k \in \mathbb{R}$)

$$X_k := X_A(2^k - 1, 0) = U(\psi_k) D_k U(\psi_k), \quad \psi_k \in \mathbb{R}. \tag{1.15}$$

Similarly to (1.12), we prove the equality

$$X_{k+1} = X_k U(\mu + b_{2^k-1}) X_k. \tag{1.16}$$

If the relation (1.15) is valid at step k, then the mentioned symmetry together with (1.16) implies the same symmetry at the $k+1$-th step, which, by induction, gives the representation (1.15) for all k.

Denote $y = U(-\psi_{k,\mu})e_1$, $e_1 = (1, 0)^t$. The vectors $D_{k,\mu}U(\psi_{k,\mu})y$ and e_1 are obviously collinear. Hence, in the case $\left(\psi_{k,\mu}\right)'_\mu < 0$, if ν is little bigger than μ, by the representation (1.15), it is easy to get the inclusion of the vector $X_{k,\nu}y$ to the right-hand neighborhood of $U(\psi_{k,\mu})e_1$. On the other hand, (1.14) implies that the vector $X_{k,\nu}y$ belongs to the left-hand neighborhood of the vector $X_{k,\mu}y \overset{(1.15)}{=} U(\psi_{k,\mu})e_1$.

So, we have the contradiction that proves the inequality

$$\left(\psi_{k,\mu}\right)'_\mu > 0. \tag{1.17}$$

Further, if $\psi_{k,\mu}$ is a multiple of $\frac{\pi}{2}$, then the matrix X_k is diagonal. Thus, the polar angle φ of the vector $X_k e_1$ satisfies for some integer m the equality

$$\varphi(X_k e_1) = m\pi. \tag{1.18}$$

Because of (1.17), the value of ψ_k strongly increases with respect to μ. Therefore the segment $[0, \pi]$ is divided into intervals, at whose boundary points the angles $\psi_{k,\mu}$ are some multiplies of $\frac{\pi}{2}$. It follows from (1.18) that their quantity does not exceed the number of vector $X_{k,\mu}e_1$ intersections with the Ox_1 axis. The equality (1.13) means that the latter is not greater than 2^{k-1}.

Thus, we have asserted by Lemma 4.2 the estimate

$$\psi_{k,\pi} - \psi_{k,0} \leq 2^{k-2}. \tag{1.19}$$

Denote $\theta_\mu = 2\psi_k + b_{2^k-1} + \mu$. By use of (1.16), we have

$$\|X_{k+1}\| \overset{(1.15),(1.16)}{=} \|D_k U(\theta_\mu) D_k\| = \begin{pmatrix} r_k^2 \cos\theta_\mu & -\sin\theta_\mu \\ \sin\theta_\mu & r_k^{-2}\cos\theta_\mu \end{pmatrix} \approx r_k^2 |\cos\theta_\mu|. \tag{1.20}$$

Lemma 4.2 (namely (1.17) and (1.19)) gives the continuity and monotonic growth of θ_μ with respect to μ, together with the inequality $\theta_{\mu+\pi} - \theta_\mu \leq n\pi$, where $n := 2^{k-1} + 1$.

Therefore, there exist no more than n intervals I_j, where $|\cos\theta_\mu| < \frac{1}{n}$.

The estimate (1.17) implies that $\theta'_\mu \geq 1$. Hence, the length of each I_j does not substantially exceed $\frac{1}{n}$. Moreover, by the same reason, we have approximately $|\cos\theta_\mu| \approx \frac{1}{n}$ for a prevailing set of points in I_j.

Thus, in integral average, a similar inequality will hold on the entire interval $[0,\pi]$:

$$|\cos\theta_\mu| \geq \varkappa_n, \quad \varkappa_n \approx \frac{1}{n}. \tag{1.21}$$

It gives us the estimates

$$L_{k+1} := \int_0^\pi \ln\|X_{k+1}\|\,d\mu \overset{(1.20)}{\geq} 2\int_0^\pi \ln r_{k,\mu}\,d\mu + \int_0^\pi \ln|\cos\theta_\mu|\,d\mu \overset{(1.15),(1.21)}{\geq} 2L_k - \ln\varkappa_n \approx 2L_k - k. \tag{1.22}$$

Increasing r_1, one can choose the norm L_1 so big that, by (1.22), L_{k+1} is not significantly less than $2L_k$. This, by induction, leads to the positivity of the integral for the maximal Lyapunov exponent of the system (1.7).

2. Proof of Proposition 1.2

Fix $k \in \mathbb{N}$.

For any natural numbers $j < 2^{k-1}$ and m, there exists an odd $p(j)$ such that

$$j + m2^{k-1} = 2^{\nu_2(j)}p(j) + m2^{k-1} = 2^{\nu_2(j)}(p(j) + m2^{k-1-\nu_2(j)}). \tag{2.1}$$

Because $k-1 > \nu_2(j)$, and $p(j)$ is odd, the number $p(j)+2^{k-1-\nu_2(j)}$ is also odd, that is the formula holds:

$$\nu_2(p(j) + m2^{k-1-\nu_2(j)}) = 0. \tag{2.2}$$

Together with (2.1), that imply that

$$\nu_2(j+m2^{k-1}) \overset{(2.1)}{=} \nu_2(2^{\nu_2(j)}(p(j)+m2^{k-1-\nu_2(j)})) = \nu_2(2^{\nu_2(j)}) + \nu_2(p(j)+m2^{k-1-\nu_2(j)}) \overset{(2.2)}{=} \nu_2(j). \tag{2.3}$$

Hence, because of (1.8), we obtain the equalities

$$b_{j+m2^{k-1}} \overset{(1.8)}{=} \alpha_{1+\nu_2(j+m2^{k-1})} \overset{(2.3)}{=} \alpha_{1+\nu_2(j)} \overset{(1.8)}{=} b_j. \tag{2.4}$$

Thus, for each $t \in [2j-1, 2j)$, we have

$$A_\mu(t+m2^{k-1}) - A_\mu(t) \overset{(1.9)}{=} b_{j+m2^{k-1}} - b_j \overset{(2.4)}{=} 0. \tag{2.5}$$

In the case $t \in [2l-1, 2l)$, $l \in \mathbb{N}$, the formula (1.9) implies that

$$A_\mu(t+m2^{k-1}) - A_\mu(t) \overset{(1.9)}{=} 0. \tag{2.6}$$

The equalities then hold at

$$b_{m2^{k-1}} \overset{(1.8)}{=} \alpha_{1+\nu_2(m2^{k-1})} = \alpha_{1+\nu_2(m)+\nu_2(2^{k-1})} = \alpha_{k+\nu_2(m)}. \tag{2.7}$$

Hence, for each $t \in [2^{k-1} - 1, 2^{k-1})$, we have

$$A_\mu(t + m)2^{k-1}) - A_\mu(t) = b_{(m+1)2^{k-1}} J - b_{2^{k-1}} J = (\alpha_{k+\nu_2(m+1)} - \alpha_k)J. \tag{2.8}$$

So, by (2.5), (2.6), and (2.8), we obtain

$$\|A_\mu(t + m)2^{k-1}) - A_\mu(t)\| \le |\alpha_{k+\nu_2(m+1)} - \alpha_k|, \quad t \in [0, 2^{k-1}). \tag{2.9}$$

For all $l, k \in \mathbb{N}$, there exist $q = q(l, k), m = m(l, k) \in \mathbb{N} \cup \{0\}$, $q \in [0, 2^{k-1})$ such that $l = m2^{k-1} + q$.

We will use in the rest of the paper a notation $\prod$ for the product of matrices B_j, $j = \overline{1, n}$, assuming $\prod_{j=1}^{n} B_j \overset{\text{def}}{=} B_1 \cdot B_2 \cdot \ldots \cdot B_n$. The following formulas hold:

$$\begin{aligned}
&X_{A_\mu}(2q + 1, 0)\\
&= \left(\prod_{j=2^{k-1}-q}^{2^{k-1}-1} X_{A_\mu}(2^k + 1 - 2j, 2^k - 2j)\, X_{A_\mu}(2^k - 2j, 2^k - 2j - 1) \right) X_{A_\mu}(1, 0)\\
&= \left(\prod_{j=2^{k-1}-q}^{2^{k-1}-1} \operatorname{diag}\,[e^{d(\mu)}, e^{-d(\mu)}]\; U(\mu + \gamma(\mu) + b_{2^{k-1}-j}) \right) \operatorname{diag}\,([e^{d(\mu)}, e^{-d(\mu)}]\\
&\overset{(2.4)}{=} \left(\prod_{j=2^{k-1}-q}^{2^{k-1}-1} X_{A_\mu}((m + 1)2^k + 1 - 2j, (m + 1)2^k - 2j)\; U\Big(\mu + \gamma(\mu) + b_{(m+1)2^{k-1}-j}\Big) \right) X_{A_\mu}(1 + m2^k, m2^k)\\
&= X_{A_\mu}(2l + 1, m2^k).
\end{aligned} \tag{2.10}$$

As a corollary, we have

$$X_{A_\mu}(2^k - 1, 0) = X_{A_\mu}((m + 1)2^k - 1, m2^k), \quad m \in \mathbb{N}. \tag{2.11}$$

Thus, because of the equalities

$$\begin{aligned}
X_{A_\mu}(2^{k+1}, 2^{k+1} - 1) &= U(\mu + \gamma(\mu) + b_{2^k}) = U(\mu + \gamma(\mu) + \alpha_{k+1})\\
&= U(\alpha_{k+1} - \alpha_k)\, U(\mu + \gamma(\mu) + b_{2^{k-1}}) = U(\alpha_{k+1} - \alpha_k)\, X_{A_\mu}(2^k, 2^k - 1),
\end{aligned} \tag{2.12}$$

we obtain that

$$\begin{aligned}
X_{A_\mu}(2^{k+1}, 0) &= X_{A_\mu}(2^{k+1}, 2^{k+1} - 1)\, X_{A_\mu}(2^{k+1} - 1, 2^k)\, X_{A_\mu}(2^k, 0)\\
&\overset{(2.11),(2.12)}{=} U(\alpha_{k+1} - \alpha_k)\, X_{A_\mu}(2^k, 2^k - 1)\, X_{A_\mu}(2^k - 1, 0)\, X_{A_\mu}(2^k, 0) = U(\alpha_{k+1} - \alpha_k) X^2_{A_\mu}(2^k, 0).
\end{aligned}$$

So, the formula (1.12) is proved.

Denote $D(\varkappa) = \operatorname{diag}\,[\varkappa, \varkappa^{-1}]$, $\varkappa \ne 0$. Let $\alpha_1 = 0$.

According to the singular value decomposition theorem, there exist a real $\theta_{1,k}(\mu)$, $\theta_{2,k}(\mu)$, $\varkappa_k(\mu)$ such that

$$X^2_{A_\mu}(2^k, 0) = U(\theta_{1,k}(\mu))\, D(\varkappa_k(\mu))\, U(\theta_{2,k}(\mu)), \quad \mu \in \mathbb{R}. \tag{2.13}$$

We put $\alpha_{k+1} := \alpha_k + \dfrac{\pi}{2} - \theta_{1,k}(\beta_k) - \theta_{2,k}(\beta_k)$.

The following equalities hold:

$$\begin{aligned}&D(\varkappa_k(\beta_k))\,U(\theta_{2,k}(\beta_k)+\alpha_{k+1}-\alpha_k+\theta_{1,k}(\beta_k))\,D(\varkappa_k(\beta_k))\\&=D(\varkappa_k(\beta_k))\,J\,D(\varkappa_k(\beta_k))=J\,D^{-1}(\varkappa_k(\beta_k)),\qquad D(\varkappa_k(\beta_k))=J.\end{aligned}\tag{2.14}$$

They imply the formulas

$$\begin{aligned}&X_{A_{\beta_k}}(2^{k+2},0)\overset{(1.12)}{=}U(\alpha_{k+2}-\alpha_{k+1})\,X^2_{A_{\beta_k}}(2^{k+1},0)\\&\overset{(1.12)}{=}U(\alpha_{k+2}-\alpha_{k+1})\,U(\alpha_{k+1}-\alpha_k)\,X^2_{A_{\beta_k}}(2^k,0)\,U(\alpha_{k+1}-\alpha_k)\,X^2_{A_{\beta_k}}(2^k,0)\\&\overset{(2.13)}{=}U(\alpha_{k+2}-\alpha_k)\,U(\theta_{1,k}(\beta_k))\,D(\varkappa_k(\beta_k))\,U(\theta_{2,k}(\beta_k)+\alpha_{k+1}-\alpha_k+\theta_{1,k}(\beta_k))\,D(\varkappa_k(\beta_k))\,U(\theta_{2,k}(\beta_k))\\&\overset{(2.14)}{=}U\left(\alpha_{k+2}-\alpha_k+\theta_{1,k}(\beta_k)+\frac{\pi}{2}+\theta_{2,k}(\beta_k)\right).\end{aligned}\tag{2.15}$$

There exists $p(m,k)\in\mathbb{N}$ such that $X_{A_\mu}(m2^k,m2^k-1)=U(\alpha_{p(m,k)}+\gamma(\mu))$.
Hence,

$$\begin{aligned}\|X_{A_\mu}(m2^k,(m-1)2^k)\|&=\|X_{A_\mu}(m2^k,m2^k-1)\,X_{A_\mu}(m2^k-1,(m-1)2^k)\|\\&=\|U(\alpha_{p(m,k)}+\gamma(\mu))\,X_{A_\mu}(m2^k-1,(m-1)2^k)\|\\&\overset{(2.11)}{=}\|U(\alpha_{p(1,k)}+\gamma(\mu))\,X_{A_\mu}(2^k-1,0)\|\\&=\|X_{A_\mu}(2^k,2^k-1)\,X_{A_\mu}(2^k-1,0)\|=\|X_{A_\mu}(2^k,0)\|.\end{aligned}\tag{2.16}$$

The value of $\|X_{A_\mu}(t,0)\|$ is a continuous function of $t\geq 0$. Therefore, there exists $C_{k,\mu}:=\max\left\{\|X_{A_\mu}(t,0)\|:\ t\in[0,2^k]\right\}$.
For all integers $k\geq 3$, the following formulas hold:

$$\begin{aligned}\|X_{A_{\beta_k}}(2l+1,0)\|&=\left\|X_{A_{\beta_k}}(2l+1,m2^k)\prod_{j=0}^{m-1}X_{A_{\beta_k}}((m-j)2^k,(m-j-1)2^k)\right\|\\&\leq\left\|X_{A_{\beta_k}}(2l+1,m2^k)\right\|\prod_{j=0}^{m-1}\left\|X_{A_{\beta_k}}((m-j)2^k,(m-j-1)2^k)\right\|\\&\overset{(2.10),(2.16)}{=}\left\|X_{A_{\beta_k}}(2q+1,0)\right\|\left\|X_{A_{\beta_k}}(2^k,0)\right\|^m\overset{(2.15)}{=}\left\|X_{A_{\beta_k}}(2q+1,0)\right\|<C_{k,\beta_k}.\end{aligned}\tag{2.17}$$

For each $t\geq 0$, there exist $l(t)\in\mathbb{N}\cup\{0\}$, $r(t)\in[0,2)$ such that $t=2l(t)+1+r(t)$.
If $r(t)\leq 1$, then

$$\|X_{A_\mu}(t,2l(t)+1)\|=\left\|U(r(t)(\mu+\gamma(\mu)+b_{l(t)+1}))\right\|=1.\tag{2.18}$$

Otherwise, we have the estimates

$$\begin{aligned}\|X_{A_\mu}(t,2l(t)+1)\|&=\|X_{A_{\beta_k}}(t,2l(t)+2)\,X_{A_{\beta_k}}(2l(t)+2,2l(t)+1)\|\\&=\left\|\operatorname{diag}[e^{(r(t)-1)d(\mu)},e^{-(r(t)-1)d(\mu)}]\,U(\mu+\gamma(\mu)+b_{l(t)+1})\right\|\\&=\left\|\operatorname{diag}[e^{(r(t)-1)d(\mu)},e^{-(r(t)-1)d(\mu)}]\right\|=e^{(r(t)-1)d(\mu)}\leq e^{d(\mu)}.\end{aligned}\tag{2.19}$$

Thus, we obtain for each $t \geq 0$ the inequalities

$$\|X_{A_{\beta_k}}(t,0)\| = \|X_{A_{\beta_k}}(t,2l(t)+1)\, X_{A_{\beta_k}}(2l(t)+1,0)\|$$
$$\leq \|X_{A_{\beta_k}}(t,2l(t)+1)\|\, \|X_{A_{\beta_k}}(2l(t)+1,0)\| \overset{(2.17),(2.18),(2.19)}{\leq} e^{d(\beta_k)} C_{k,\beta_k}.$$

Proposition 1.2 is proved.

3. The estimate of the polar angle and of its derivative for any solution

Let $[\cdot]$ denote the integer part of a number.

For all $\zeta, \mu \in \mathbb{R}$, let $\mathfrak{g}_\zeta(0,\mu) = \zeta$. Let us define the function $\mathfrak{g}_\zeta(\cdot,\mu) : \mathbb{N} \to \mathbb{R}$, setting for each $n \in \mathbb{N}\cup\{0\}$ $\mathfrak{g}_\zeta(2n+1,\mu) = \pi[\pi^{-1}\mathfrak{g}_\zeta(2n,\mu)] + \operatorname{arcctg} e^{2d(\mu)} \operatorname{ctg} \mathfrak{g}_\zeta(2n,\mu)$ in the case where $\pi^{-1}\mathfrak{g}_\zeta(2n,\mu) \notin \mathbb{Z}$, and $\mathfrak{g}_\zeta(2n+1,\mu) = \mathfrak{g}_\zeta(2n,\mu)$ otherwise. Let also $\mathfrak{g}_\zeta(2n+2,\mu) = \gamma(\mu) + \mu + b_{n+1} + \mathfrak{g}_\zeta(2n+1,\mu)$.

Lemma 3.1. *For all $\zeta \in \mathbb{R}$ and $n \in \mathbb{N}$, the following equalities hold:*

$$\mathfrak{g}_0(2n+1,\pi) - \mathfrak{g}_0(2n+1,0) = \mathfrak{g}_0(2n,\pi) - \mathfrak{g}_0(2n,0) = n\pi. \tag{3.1}$$

In addition, the function $\mathfrak{g}_\zeta(n,\mu)$ is differentiable with respect to μ on $\mathbb{R}$, and for all $\mu \in \mathbb{R}$ in the case where $m > 1$ we have the estimate

$$(\mathfrak{g}_\zeta(m,\mu))'_\mu > \frac{1}{2} e^{-2d(\mu)}. \tag{3.2}$$

Furthermore, the solution $x_\zeta(t,\mu)$ of the system (1.7) with the initial condition $x_\zeta(0,\mu) = U(\zeta)e_1$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, satisfies the equality

$$x_\zeta(n,\mu) = \|x_\zeta(n,\mu)\| U(\mathfrak{g}_\zeta(n,\mu)) e_1. \tag{3.3}$$

Proof. The formulas (2_n) and (3) in [13] define the functions $g_n, \tilde{g}_n \colon [0,1] \to \mathbb{R}$ for all $\tilde{d}, \tilde{b}_k \in \mathbb{R}$, $k, n \in \mathbb{N}$, $\tilde{d} \neq 0$, by the equalities $g_0(\mu) \equiv 0$,

$$\tilde{g}_n(\mu) = \tilde{g}_n(\mu,\tilde{d},\tilde{b}_1,\ldots,\tilde{b}_n) = \tilde{b}_n + \pi\mu + g_{n-1}(\mu),$$
$$g_n(\mu) = g_n(\mu,\tilde{d},\tilde{b}_1,\ldots,\tilde{b}_n) = \tilde{g}_n(\mu) \quad \text{if} \quad \pi^{-1}\tilde{g}_n(\mu) \in \mathbb{Z},$$
$$g_n(\mu) = \pi[\pi^{-1}\tilde{g}_n(\mu)] + \operatorname{arcctg} \tilde{d}^2 \operatorname{ctg} \tilde{g}_n(\mu) \quad \text{otherwise.}$$

The matrices D and H_n are also defined in [13] by the formulas

$$D = D(\tilde{d}) = \operatorname{diag}[\tilde{d}, \tilde{d}^{-1}] \quad \text{and} \quad H_n(\mu) = H_n(\mu,\tilde{d},\tilde{b}_1,\ldots,\tilde{b}_n) = \prod_{k=0}^{n-1} (U(\tilde{b}_{n-k} + \pi\mu)\, D(\tilde{d}))$$

(function $H_n(\mu)$ in [13] was mistakenly multiplied from the left by D, and the order of the factors in its definition was reversed).

Let us denote by $\{\cdot\}$ the fractional part of the number, $\hat{b}_{k+1}(\mu) = b_k + \gamma(\mu) + \pi[\pi^{-1}\mu]$, $k \in \mathbb{N}$, $\hat{b}_1 := \zeta - \mu$; $\Theta_n(\mu) = (\{\pi^{-1}\mu\}, e^{d(\mu)}, \hat{b}_1(\mu), \ldots, \hat{b}_n(\mu))$.

Define the vector $\tilde{x}(n,\mu) = \tilde{x}(n,\mu,\tilde{d},\tilde{b}_1,\ldots,\tilde{b}_n)$, $\mu,\tilde{d},\tilde{b}_k \in \mathbb{R}$, $k,n \in \mathbb{N}$, by the equality $\tilde{x}(2n+\delta,\mu) = D^{\delta} H_n(\mu) e_1$, $\delta \in \{0,1\}$.

The following equalities hold:

$$\begin{aligned} x_\zeta(2n+\delta,\mu) &= X_{A_\mu}(2n+\delta,0)\, x_\zeta(0,\mu) \\ &= X^{\delta}_{A_\mu}(2n+1,2n)\left(\prod_{j=0}^{n-1} X_{A_\mu}(2n-2j,2n-1-2j)\, X_{A_\mu}(2n-1-2j,2n-2-2j)\right) U(\zeta)\, e_1 \\ &= e^{-d(\mu)} D^{\delta}(e^{d(\mu)})\left(\prod_{j=0}^{n-1} U(b_{n-j}+\mu+\gamma(\mu))\, D(e^{d(\mu)})\right) U(\zeta)\, D(e^{d(\mu)})\, e_1 \\ &= e^{-d(\mu)} D^{\delta}(e^{d(\mu)})\, H_{n+1}(\Theta_{n+1}(\mu))\, e_1 = e^{-d(\mu)} \tilde{x}(2n+2+\delta,\Theta_{n+1}(\mu)). \end{aligned} \tag{3.4}$$

According to the formulas (6_n) and (7_n) from [13], we have for all $\Xi \in [0,1]\times(0,+\infty)\times\mathbb{R}^n$ the equalities

$$\tilde{x}(2n+1,\Xi_n) = \|\tilde{x}(2n+1,\Xi_n)\|\, U(g_n(\Xi_n))\, e_1, \quad \tilde{x}(2n,\Xi_n) = \|\tilde{x}(2n,\Xi_n)\|\, U(\tilde{g}_n(\Xi_n))\, e_1, \quad n \in \mathbb{N}. \tag{3.5}$$

The following relations hold:

$$\mathfrak{g}_\zeta(2n+1,\mu) = g_{n+1}(\Theta_{n+1}(\mu)), \quad \mathfrak{g}_\zeta(2n,\mu) = \tilde{g}_{n+1}(\Theta_{n+1}(\mu)). \tag{3.6}$$

Let us denote $\hat{g}_{n,1} = g_n$, $\hat{g}_{n,0} = \tilde{g}_n$.

The formulas (3.4)–(3.6) imply for all $\delta \in \{0,1\}$ the equalities

$$\begin{aligned} x_\zeta(2n+\delta,\mu) &\overset{(3.4)}{=} e^{-d(\mu)} \tilde{x}(2n+2+\delta,\Theta_{n+1}(\mu)) \\ &\overset{(3.5)}{=} e^{-d(\mu)} \|\tilde{x}(2n+2+\delta,\Theta_{n+1}(\mu))\|\, U(\hat{g}_{n,\delta}(\Theta_{n+1}(\mu))) \\ &\overset{(3.4),(3.6)}{=} \|x_\zeta(2n+\delta,\mu)\|\, U(\mathfrak{g}_\zeta(2n+\delta,\mu)). \end{aligned}$$

Thus, we obtain (3.3) for all $n \in \mathbb{N}$.

Let us denote $\tilde{b}_k(\mu) = b_k + \gamma(\mu)$, $k \in \mathbb{N}$, $\tilde{\Theta}_n(\mu) = (\pi^{-1}\mu, e^{d(\mu)}, \tilde{b}_1, \ldots, \tilde{b}_n)$.

The following formulas hold:

$$\mathfrak{g}_0(2n+1,\mu) = g_n(\tilde{\Theta}_n(\mu)), \quad \mathfrak{g}_0(2n,\mu) = \tilde{g}_n(\tilde{\Theta}_n(\mu)). \tag{3.7}$$

According to (4_n) in [13], we have the equality

$$g_n(1,\tilde{d},\tilde{b}_1,\ldots,\tilde{b}_n) - g_n(0,\tilde{d},\tilde{b}_1,\ldots,\tilde{b}_n) = n\pi, \quad \tilde{d},\tilde{b}_k \in \mathbb{R}, \quad k,n \in \mathbb{N}. \tag{3.8}$$

We put $\tilde{d} = e^{d(0)}$, $\tilde{b}_k = b_k + \gamma(\pi)$, $k \in \mathbb{N}$, in (3.8). By (3.7) and because of the π-periodicity of the functions $\gamma(\cdot)$ and $d(\cdot)$, we obtain the equalities

$$\begin{aligned} \mathfrak{g}_0(2n+1,\pi) &\overset{(3.7)}{=} g_n(1,e^{d(\pi)},b_1+\gamma(\pi),\ldots,b_n+\gamma(\pi)) = g_n(1,e^{d(0)},b_1+\gamma(0),\ldots,b_n+\gamma(0)) \\ &\overset{(3.8)}{=} \pi n + g_n(0,e^{d(0)},b_1+\gamma(0),\ldots,b_n+\gamma(0)) \overset{(3.7)}{=} \pi n + \mathfrak{g}_0(2n+1,0). \end{aligned} \tag{3.9}$$

Similarly, (3.7) and the formula $\tilde{g}_n(1) - \tilde{g}_n(0) = n\pi$, given in [13, Proof of Lemma 1], imply that $\mathfrak{g}_0(2n, \pi) = \pi n + \mathfrak{g}_0(2n, 0)$. This equality together with (3.9) are equivalent to (3.1).

For any differentiable functions $h(\cdot)$ and $\omega(\cdot)$, in the case where $\sin\omega(\mu) \neq 0$, we have

$$\left(\operatorname{arccotg}\left(h^2(\mu)\operatorname{cotg}\omega(\mu)\right)\right)'_\mu = \frac{\omega'(\mu) - (\ln h)'(\mu)\sin(2\omega(\mu))}{h^2(\mu)\cos^2\omega(\mu) + h^{-2}(\mu)\sin^2\omega(\mu)}, \quad \sin\omega(\mu) \neq 0. \tag{3.10}$$

Also, by (3.3), the following equalities hold:

$$\frac{\|x_\zeta(2n+1,\mu)\|^2}{\|x_\zeta(2n,\mu)\|^2} = \frac{\|D(e^{d(\mu)})\, x_\zeta(2n,\mu)\|^2}{\|x_\zeta(2n,\mu)\|^2} \overset{(3.3)}{=} e^{2d(\mu)}\cos^2\mathfrak{g}_\zeta(2n,\mu) + e^{-2d(\mu)}\sin^2\mathfrak{g}_\zeta(2n,\mu). \tag{3.11}$$

Suppose the function $\mathfrak{g}_\zeta(2n-1,\mu)$ to be differentiable with respect to μ for some $n \in \mathbb{N}$. In addition, assume the inequality

$$(\mathfrak{g}_\zeta(2n-1,\mu))'_\mu \geq -|d'(\mu)|. \tag{3.12}$$

Thus, the function $\mathfrak{g}_\zeta(2n,\mu)$ is also differentiable with respect to μ. Hence, in the case where $\sin\mathfrak{g}_\zeta(2n,\mu) \neq 0$, by (3.10) and (3.11), we obtain that the function $\mathfrak{g}_\zeta(2n+1,\cdot)$ is differentiable, and the equality then holds:

$$(\mathfrak{g}_\zeta(2n+1,\mu))'_\mu \overset{(3.10),(3.11)}{=} \frac{\|x_\zeta(2n,\mu)\|^2}{\|x_\zeta(2n+1,\mu)\|^2}(-d'(\mu)\sin(2\mathfrak{g}_\zeta(2n,\mu)) + (\mathfrak{g}_\zeta(2n,\mu))'_\mu), \text{ if } \mathfrak{g}_\zeta(2n,\mu) \neq 0. \tag{3.13}$$

It follows from (3.13) that for each μ_0, such that $\mathfrak{g}_\zeta(2n,\mu_0) = l\pi$, $l \in \mathbb{Z}$, there exists a limit

$$\lim_{\mu_0 \neq \mu \to \mu_0} (\mathfrak{g}_\zeta(2n+1,\mu))'_\mu \overset{(3.11),(3.13)}{=} \frac{(\mathfrak{g}_\zeta(2n,\mu_0))'_\mu}{e^{2d(\mu_0)}}.$$

Hence, by an obvious continuity of the function $\mathfrak{g}_\zeta(2n+1,\cdot)$ at the point μ_0, as it was shown in the proof of the lemma in [21, p. 232], we have its differentiability at this point and the equality

$$(\mathfrak{g}_\zeta(2n+1,\mu))'_\mu = \frac{(\mathfrak{g}_\zeta(2n,\mu))'_\mu}{e^{2d(\mu)}}, \quad \text{if} \quad \sin\mathfrak{g}_\zeta(2n,\mu) = 0. \tag{3.14}$$

Then, by the formulas

$$\frac{\|x_\zeta(1,\mu)\|^2}{\|x_\zeta(0,\mu)\|^2} \overset{(3.11)}{=} e^{2d(\mu)}\cos^2\zeta + e^{-2d(\mu)}\sin^2\zeta \geq |\sin(2\zeta)| \tag{3.15}$$

and the estimates

$$\begin{aligned}\frac{\|x_\zeta(1,\mu)\|^2}{\|x_\zeta(0,\mu)\|^2}(\mathfrak{g}_\zeta(1,\mu))'_\mu &\overset{(3.14),(3.15)}{=} d'(\mu)\sin(2\mathfrak{g}_\zeta(0,\mu)) + (\mathfrak{g}_\zeta(0,\mu))'_\mu \\ &= d'(\mu)\sin(2\zeta) + \zeta'_\mu \geq -|d'(\mu)|\,|\sin(2\zeta)|,\end{aligned} \tag{3.16}$$

we get the inequalities

$$(\mathfrak{g}_\zeta(1,\mu))'_\mu \overset{(3.16)}{\geq} -\frac{\|x_\zeta(0,\mu)\|^2}{\|x_\zeta(1,\mu)\|^2}|d'(\mu)|\,|\sin(2\zeta)| \overset{(3.15)}{\geq} -|d'(\mu)|. \tag{3.17}$$

The formula (3.12) implies the estimates

$$(\mathfrak{g}_\zeta(2n,\mu))'_\mu = 1 + \gamma'(\mu) + (\mathfrak{g}_\zeta(2n-1,\mu))'_\mu \overset{(3.12)}{\geq} 1 + \gamma'(\mu) - |d'(\mu)| \overset{(1.10)}{>} \frac{1}{2}. \tag{3.18}$$

From them, by (3.13) and (3.14), we obtain

$$\begin{aligned} e^{2d(\mu)}(\mathfrak{g}_\zeta(2n+1,\mu))'_\mu &\overset{(3.11)}{\geq} \frac{\|x_\zeta(2n+1,\mu)\|^2}{\|x_\zeta(2n,\mu)\|^2}(\mathfrak{g}_\zeta(2n+1,\mu))'_\mu \\ &\overset{(3.13),(3.14),(3.18)}{\geq} -|d'(\mu)| + 1 + \gamma'(\mu) - |d'(\mu)| \overset{(1.10)}{>} \frac{1}{2}. \end{aligned}$$

They together with (3.18) imply the inequality (3.2) in the cases where $m = 2n$ or $m = 2n+1$.

Thus, also taking into consideration (3.17), by induction we obtain the estimates (3.2) for all $1 < m \in \mathbb{N}$.

The lemma is thus proved.

4. The estimate of singular angles and of their derivative for Cauchy operator

For all $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$, we define recursively the real numbers $\eta_k = \eta_k(\mu) \geq 1$ and $\psi_k = \psi_k(\mu)$ as follows. Let us denote $\eta_1(\mu) = e^{d(\mu)}$, $\psi_1(\mu) :\equiv 0$, $\xi_k = \xi_k(\mu) = 2\psi_k(\mu) + \alpha_k + \mu + \gamma(\mu)$, $q_k(\mu) = \pi[\pi^{-1}\xi_k(\mu)]$. Because the functions $\sinh\ln(\cdot)\colon (0,+\infty) \to \mathbb{R}$ and $\cot(\cdot)\colon (0,\pi) \to \mathbb{R}$ are bijective, which follows from their strict monotonicity and unboundedness, there exist unique $0 < \eta_{k+1} \in \mathbb{R}$ and $\varphi_k = \varphi_k(\mu) \in [q_k(\mu), q_k(\mu)+\pi)$, such that the following equalities hold:

$$\sinh\ln\eta_{k+1} = (\sinh(2\ln\eta_k))\cos\xi_k, \tag{4.1}$$

$$\cot\varphi_k = \sqrt{2}\,(\cosh(2\ln\eta_k))\cot\xi_k \ \text{ if } \ \sin\xi_k \neq 0, \quad \varphi_k = \xi_k \text{ in the case when } \sin\xi_k = 0. \tag{4.2}$$

Finally, we set $\psi_{k+1}(\mu) = \psi_k(\mu) + 2^{-1}\varphi_k(\mu)$.

For any set $S \subset \mathbb{R}$, we denote by $\mathcal{CM}(S)$ (respectively, $\mathcal{D}(S)$) the set of all continuous, monotonically increasing (respectively, differentiable), real-valued functions defined on S.

Lemma 4.1. *For all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$, the functions $\eta_n(\mu)$ and $\psi_n(\mu)$ are differentiable with respect to μ, and the condition holds*

$$Y_{n,\mu} := X_{A_\mu}(2^n-1,0) = U(\psi_n(\mu))\begin{pmatrix} \eta_n(\mu) & 0 \\ 0 & \eta_n(\mu)^{-1} \end{pmatrix} U(\psi_n(\mu)). \tag{4.3}$$

Proof. Let $\operatorname{sgn}(\cdot)$ denote the sign of a number.

Because the inclusions $\xi_k - \pi q_k$, $\varphi_k - \pi q_k \in [0,\pi)$, the following estimates hold:

$$\sin\xi_k\sin\varphi_k = \cos^2(\pi q_k)\,\sin(\xi_k - \pi q_k)\,\sin(\varphi_k - \pi q_k) \geq 0.$$

From them, we get the equality

$$\operatorname{sgn}\,\sin\xi_k = \operatorname{sgn}\,\sin\varphi_k. \tag{4.4}$$

Thus, we have

$$\operatorname{sgn}\,\cos\varphi_k = (\operatorname{sgn}\,\sin\varphi_k)\operatorname{sgn}\operatorname{cotg}\varphi_k \overset{(4.2),(4.4)}{=} (\operatorname{sgn}\,\sin\xi_k)\operatorname{sgn}\operatorname{cotg}\xi_k = \operatorname{sgn}\,\cos\xi_k. \tag{4.5}$$

In the case where $\cos\varphi_k \neq 0$, the following formulas hold:

$$2\left(\frac{\cosh(2\ln\eta_k)}{\cos\varphi_k}\right)^2 = 2(\cosh^2(2\ln\eta_k))(1+\operatorname{tg}^2\varphi_k) \overset{(4.2)}{=} 2^{-1}(\eta_k^2+\eta_k^{-2})^2 + (\operatorname{tg}^2\xi_k)$$

$$= 2^{-1}(\eta_k^2-\eta_k^{-2})^2 + (1+\operatorname{tg}^2\xi_k) \overset{(4.1)}{=} 2\left(\frac{\sinh\ln\eta_{k+1}}{\cos\xi_k}\right)^2 + \frac{1}{\cos^2\xi_k} = 2\left(\frac{\cosh\ln\eta_{k+1}}{\cos\xi_k}\right)^2. \tag{4.6}$$

Hence, by (4.5), we obtain that

$$(\cosh\ln\eta_{k+1})\cos\varphi_k = (\operatorname{sgn}\,\cos\varphi_k)\,|\cos\varphi_k|\,\cosh\ln\eta_{k+1}$$

$$\overset{(4.5),(4.6)}{=} (\operatorname{sgn}\,\cos\xi_k)\,|\cos\xi_k|\,\cosh(2\ln\eta_k) = (\cosh(2\ln\eta_k))\cos\xi_k. \tag{4.7}$$

This formula implies the equalities

$$((\cosh\ln\eta_{k+1})\sin\varphi_k)^2 = (\cosh^2\ln\eta_{k+1})(1-\cos^2\varphi_k) \overset{(4.7)}{=} 1 + (\sinh^2\ln\eta_{k+1}) - (\cosh^2(2\ln\eta_k))\cos^2\xi_k$$

$$\overset{(4.1)}{=} 1 + (\cos^2\xi_k)((\sinh^2(2\ln\eta_k)) - \cosh^2(2\ln\eta_k)) = \sin^2\xi_k. \tag{4.8}$$

From them, by the use of (4.4), we have

$$(\cosh\ln\eta_{k+1})\sin\varphi_k \overset{(4.4),(4.8)}{=} \sin\xi_k, \quad k\in\mathbb{N}. \tag{4.9}$$

Because of (4.1) and (4.7), for all $\delta\in\{-1,1\}$ the following equalities hold:

$$\begin{aligned}
\eta_k^{2\delta}\cos\xi_k &= (\cos\xi_k)(\delta\sinh(2\ln\eta_k) + \cosh(2\ln\eta_k)) \\
&\overset{(4.1),(4.7)}{=} \delta\sinh\ln\eta_{k+1} + (\cosh\ln\eta_{k+1})(\cos\varphi_k) \\
&= \cos^2(\varphi_k/2)\,(\delta\sinh\ln\eta_{k+1} + \cosh\ln\eta_{k+1}) + \sin^2(\varphi_k/2)\,(\delta\sinh\ln\eta_{k+1} - \cosh\ln\eta_{k+1}) \\
&= \eta_{k+1}^{\delta}\cos^2(\varphi_k/2) - \eta_{k+1}^{-\delta}\sin^2(\varphi_k/2).
\end{aligned} \tag{4.10}$$

The formulas $Y_1 = X_{A_\mu}(1,0) = \operatorname{diag}[e^{d(\mu)}, e^{-d(\mu)}] = \operatorname{diag}[\eta_1, \eta_1^{-1}]$ imply the equality (4.3) in the case where $n=1$.

Now assume that (4.3) is true for some $n=k\in\mathbb{N}$. Then, by use of (2.11) together with the formula

$$b_{2^{k-1}} \overset{(1.8)}{=} \alpha_{1+\nu_2(2^{k-1})} = \alpha_k, \tag{4.11}$$

we have the equalities

$$\begin{aligned}
Y_{k+1} &= X_{A_\mu}(2^{k+1}-1,0) = X_{A_\mu}(2^{k+1}-1,2^k)\,X_{A_\mu}(2^k,2^k-1)\,X_{A_\mu}(2^k-1,0) \\
&\overset{(2.11)}{=} X_{A_\mu}(2^k-1,0)\,U(\mu+\gamma(\mu)+b_{2^{k-1}})\;X_{A_\mu}(2^k-1,0) \\
&\overset{(4.3_k),(4.11)}{=} U(\psi_k)\operatorname{diag}[\eta_k,\eta_k^{-1}]\,U(2\psi_k+\mu+\alpha_k+\gamma(\mu))\operatorname{diag}[\eta_k,\eta_k^{-1}]\,U(\psi_k).
\end{aligned} \tag{4.12}$$

The formulas (4.9), (4.10), and (4.12) imply that

$$U(-\psi_k)\,Y_{k+1}\,U(-\psi_k) \overset{(4.12)}{=} \begin{pmatrix} \eta_k & 0 \\ 0 & \eta_k^{-1} \end{pmatrix} U(\xi_k) \begin{pmatrix} \eta_k & 0 \\ 0 & \eta_k^{-1} \end{pmatrix} = \begin{pmatrix} \eta_k^2 \cos\xi_k & -\sin\xi_k \\ \sin\xi_k & \eta_k^{-2}\cos\xi_k \end{pmatrix}$$

$$\overset{(4.9),(4.10)}{=} \begin{pmatrix} \eta_{k+1}\cos^2(\varphi_k/2) - \eta_{k+1}^{-1}\sin^2(\varphi_k/2) & -2^{-1}(\eta_{k+1}+\eta_{k+1}^{-1})\sin\varphi_k \\ 2^{-1}(\eta_{k+1}+\eta_{k+1}^{-1})\sin\varphi_k & \eta_{k+1}^{-1}\cos^2(\varphi_k/2) - \eta_{k+1}\sin^2(\varphi_k/2) \end{pmatrix}$$

$$= U\Big(\frac{\varphi_k}{2}\Big) \begin{pmatrix} \eta_{k+1} & 0 \\ 0 & \eta_{k+1}^{-1} \end{pmatrix} U\Big(\frac{\varphi_k}{2}\Big) = U(\psi_{k+1}-\psi_k) \begin{pmatrix} \eta_{k+1} & 0 \\ 0 & \eta_{k+1}^{-1} \end{pmatrix} U(\psi_{k+1}-\psi_k).$$

Thus, the equality (4.3) holds in the case where $n = k+1$. Hence, by induction, (4.3) is true for any $n \in \mathbb{N}$.

Now, suppose the inclusion

$$\psi_k, \eta_k \in \mathcal{D}(\mathbb{R}) \tag{4.13}$$

holds for some $k \in \mathbb{N}$.

The formula (4.1) implies the equality

$$\eta_{k+1}(\mu) = \exp(\operatorname{arsh}(\sinh(2\ln\eta_k(\mu))\,\cos\xi_k(\mu))). \tag{4.14}$$

In the case where $\sin\xi_k \neq 0$, by (4.2), the following formula holds:

$$\cot(\varphi_k - \pi q_k) = \cot\varphi_k \overset{(4.2)}{=} \sqrt{2}\,(\cosh(2\ln\eta_k))\cot\xi_k.$$

Therefore, and because of the inclusion $\varphi_k - \pi q_k \in (0, \pi)$, we obtain that

$$\varphi_k - \pi q_k = \operatorname{arcctg}(\sqrt{2}(\cosh(2\ln\eta_k))\cot\xi_k).$$

As a result, we have

$$\psi_{k+1}(\mu) = \psi_k(\mu) + 2^{-1}\pi q_k(\mu) + 2^{-1}\operatorname{arcctg}(\sqrt{2}(\cosh(2\ln\eta_k(\mu)))\cot\xi_k(\mu)) \text{ if } \sin\xi_k(\mu) \neq 0. \tag{4.15}$$

Let us denote $\tilde{q}_k(\mu) = [\pi^{-1}\xi_k(\mu) + 2^{-1}]$.

In the case where $\cos\xi_k(\mu) \neq 0$, the inclusion $\varphi_k(\mu) - \pi\tilde{q}_k(\mu) \in (-2^{-1}\pi, 2^{-1}\pi)$, together with (4.2) imply that

$$\tan(\varphi_k - \pi\tilde{q}_k) = \tan\varphi_k \overset{(4.2)}{=} \sqrt{2}^{-1}\,(\cosh(2\ln\eta_k))^{-1}\tan\xi_k.$$

Thus, we obtain the equality

$$\varphi_k - \pi\tilde{q}_k = \operatorname{arctg}(\sqrt{2}^{-1}(\cosh(2\ln\eta_k))^{-1}\tan\xi_k).$$

Hence, the following formula holds:

$$\begin{aligned}\psi_{k+1}(\mu) =& \psi_k(\mu) + \pi\tilde{q}_k(\mu) + 4^{-1}\pi \\ &+ 2^{-1}\arctan(\sqrt{2}^{-1}(\cosh(2\ln\eta_k(\mu)))^{-1}\tan\xi_k(\mu)) \text{ if } \cos\xi_k(\mu) \neq 0.\end{aligned} \tag{4.16}$$

The inclusion (4.13) in the case $k = n \in \mathbb{N}$ implies the relation $\xi_k \in \mathcal{D}(\mathbb{R})$. Therefore, by (4.13)–(4.16), we have the inclusions (4.13) where $k = n+1$.

By induction, taking into consideration also the obvious case $k = 1$, we obtain the formulas (4.13) for all $k \in \mathbb{N}$. The lemma is proved.

Lemma 4.2. *For all $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, we have the equality*

$$\psi_k(\pi + \mu) - \psi_k(\mu) = (2^{k-2} - 2^{-1})\pi. \tag{4.17}$$

In addition, in the case where $k \neq 1$, the following estimate holds:

$$(\psi_k(\mu))'_\mu > \frac{1}{e^{2d(\mu)}(1 + \eta_k(\mu)^2)}. \tag{4.18}$$

Proof. Because of the representation (4.3), we have the equalities

$$\begin{aligned} y(\zeta, k, \mu) &= U(-\psi_k(\mu))\, x_\zeta(2^k - 1, \mu) \overset{(4.3)}{=} \mathrm{diag}[\eta_k(\mu), \eta_k(\mu)^{-1}]\, U(\psi_k(\mu))\, U(\zeta) e_1 \\ &= (\eta_k(\mu) \cos(\psi_k(\mu) + \zeta), \eta_k(\mu)^{-1} \sin(\psi_k(\mu) + \zeta))^{\mathrm{T}}. \end{aligned} \tag{4.19}$$

The formula (3.3), where $n = 2^k - 1$, implies the relation

$$y(\zeta, k, \mu) \overset{(3.3)}{=} \|y(\zeta, k, \mu)\| U(\mathfrak{g}_\zeta(2^k - 1, \mu) - \psi_k(\mu)) e_1. \tag{4.20}$$

By formula (4.19), the equality

$$\frac{y_1(\zeta, k, \mu)}{y_2(\zeta, k, \mu)} = \eta_k^2(\mu) \cot(\psi_k(\mu) + \zeta)$$

holds. Thus, because of (4.20), in the case where $y_2(\zeta, k, \mu) \neq 0$, we obtain for some integer $n(\zeta, k, \mu)$ the equality

$$\mathfrak{g}_\zeta(2^k - 1, \mu) - \psi_k(\mu) \overset{(4.19),(4.20)}{=} n(\zeta, k, \mu) + \mathrm{arcctg}\left(\eta_k(\mu)^2 \mathrm{ctg}(\psi_k(\mu) + \zeta)\right). \tag{4.21}$$

Let us fix an arbitrary $\mu_0 \in \mathbb{R}$ and put $\zeta = 2^{-1}\pi - \psi_k(\mu_0)$.

By formula (4.19), the following equalities hold:

$$y_2(\zeta, k, \mu_0) \overset{(4.19)}{=} \eta_k(\mu_0)^{-1} \sin(\psi_k(\mu_0) + \zeta) = \eta_k(\mu_0)^{-1} \neq 0. \tag{4.22}$$

Lemmas 3.1 and 4.1 imply the functions $\psi_k(\cdot)$, $\eta_k(\cdot)$ and $\mathfrak{g}_\zeta(2^k - 1, \cdot)$ to be continuous on $\mathbb{R}$. Therefore, because of the inequality in (4.22), the formula $\sin(\psi_k(\mu) + \zeta) \neq 0$ holds for all μ in some neighborhood $V(\mu_0) \subset \mathbb{R}$ of the point μ_0.

This property together with the equality (4.21) imply the continuity of $n(\zeta, k, \mu)$ considered as a function of μ on the interval $V(\mu_0)$. Hence, by the inclusion $n(\zeta, k, \mu) \in \mathbb{Z}$ we have that the function $n(\zeta, k, \cdot)$ is constant on $V(\mu_0)$. Therefore, the following equality holds:

$$n(\zeta, k, \mu) \equiv n(\zeta, k, \mu_0), \quad \mu \in V(\mu_0). \tag{4.23}$$

Thus, according to (4.21) and (4.23), the formula is true

$$\mathfrak{g}_\zeta(2^k - 1, \mu) \overset{(4.21),(4.23)}{=} n(\zeta, k, \mu_0) + \psi_k(\mu) + \mathrm{arcctg}\left(\eta_k(\mu)^2 \mathrm{ctg}(\psi_k(\mu) + \zeta)\right), \quad \mu \in V(\mu_0). \tag{4.24}$$

By Lemma 4.1, the functions $\eta_k(\cdot)$ and $\psi_k(\cdot)$ are differentiable on $\mathbb{R}$. Therefore, from (3.10), (4.24), because of the formula $\psi_k(\mu_0) + \zeta = 2^{-1}\pi$, the equalities follow

$$(\mathfrak{g}_\zeta(2^k-1,\mu))'_\mu|_{\mu=\mu_0} \overset{(4.24)}{=} \psi_k'(\mu_0) + \left(\operatorname{arcctg}\left(\eta_k(\mu)^2 \operatorname{ctg}(\psi_k(\mu)+\zeta)\right)\right)'_\mu|_{\mu=\mu_0}$$
$$\overset{(3.10)}{=} \psi_k'(\mu_0) + \frac{(\psi_k(\mu)+\zeta)'_\mu|_{\mu=\mu_0} + (\ln \eta_k(\mu)^2)'_\mu|_{\mu=\mu_0} \sin(2(\psi_k(\mu_0)+\zeta))}{\eta_k(\mu_0)^2\cos^2(\psi_k(\mu_0)+\zeta) + \eta_k(\mu_0)^{-2}\sin^2(\psi_k(\mu_0)+\zeta)}$$
$$= \psi_k'(\mu_0)(1+\eta_k(\mu_0)^2). \tag{4.25}$$

As a result, by (3.2), we have the estimates

$$\psi_k'(\mu_0) \overset{(4.25)}{=} \frac{(\mathfrak{g}_\zeta(2^k-1,\mu))'_\mu|_{\mu=\mu_0}}{1+\eta_k(\mu_0)^2} \overset{(3.2)}{\geq} \frac{1}{e^{2d(\mu)}(1+\eta_k(\mu_0)^2)}.$$

Thus, the inequality (4.18) holds.

Fix an arbitrary $\tilde{\mu} \in \mathbb{R}$.

According to Lemma 3.1, the function $\hat{\mathfrak{g}}(\cdot) = \mathfrak{g}_0(2^k-1,\cdot)$ is continuous and, because of the inequality (3.2), where $m = 2^k - 1$, monotonically increases on $\mathbb{R}$. Hence, the inclusion holds:

$$\hat{\mathfrak{g}}(\cdot) \in C\mathcal{M}(\mathbb{R}). \tag{4.26}$$

Therefore, it maps the set $\mathbb{R}$ one-to-one onto its image $\hat{\mathfrak{g}}(\mathbb{R})$. The estimate (3.2) implies the unboundedness of the function $\hat{\mathfrak{g}}(\cdot)$ on $\mathbb{R}$. Thus, by (4.26), we have the equality $\hat{\mathfrak{g}}(\mathbb{R}) = \mathbb{R}$. Hence, there exists the inverse to $\hat{\mathfrak{g}}(\cdot)$ function $\hat{\mathfrak{g}}^{-1}(\cdot) : \mathbb{R} \to \mathbb{R}$.

Moreover, relations (3.1), where $n = 2^{k-1} - 1$, and (4.26) imply the equalities

$$[\hat{\mathfrak{g}}(\tilde{\mu}), \hat{\mathfrak{g}}(\tilde{\mu}) + (2^{k-1}-1)\pi] \overset{(3.1)}{=} [\hat{\mathfrak{g}}(\tilde{\mu}), \hat{\mathfrak{g}}(\pi+\tilde{\mu})] \overset{(4.26)}{=} \hat{\mathfrak{g}}([\tilde{\mu}, \pi+\tilde{\mu}]). \tag{4.27}$$

By (3.3), where $n = 2^k - 1$, and because of (4.3) the formulas hold:

$$\|x_0(2^k-1,\mu)\| \begin{pmatrix} \cos\hat{\mathfrak{g}}(\mu) \\ \sin\hat{\mathfrak{g}}(\mu) \end{pmatrix} \overset{(3.3)}{=} x_0(2^k-1,\mu) = X_{A_\mu}(2^k-1,0)e_1$$
$$\overset{(4.3)}{=} \begin{pmatrix} \eta_k(\mu)^{\delta_k(\mu)}\cos^2\psi_k(\mu) - \eta_k(\mu)^{-\delta_k(\mu)}\sin^2\psi_k(\mu) \\ (\eta_k(\mu)^{-1}+\eta_k(\mu))\cos\psi_k(\mu)\sin\psi_k(\mu) \end{pmatrix}. \tag{4.28}$$

Hence, we obtain that

$$\sin\hat{\mathfrak{g}}(\mu) \overset{(4.28)}{=} \frac{\eta_k(\mu)^{-1}+\eta_k(\mu)}{2\,\|x_0(2^k-1,\mu)\|}\sin(2\psi_k(\mu)). \tag{4.29}$$

This gives

$$\{\hat{\mathfrak{g}}^{-1}(\pi j) \,:\, j \in \mathbb{Z}\} = \{\mu \in \mathbb{R} \,:\, \sin\hat{\mathfrak{g}}(\mu) = 0\} \overset{(4.29)}{=} \{\mu \in \mathbb{R} \,:\, \sin(2\psi_k(\mu)) = 0\}. \tag{4.30}$$

Because of (2.18) and (2.19), the estimate holds:

$$\|X_{A_\mu}(t,0)\| \leq e^{td(\mu)}, \quad t \geq 0. \tag{4.31}$$

Hence, by (4.3), we have

$$\max\{\eta_n(\mu), \eta_n^{-1}(\mu)\} = \|U(\psi_n(\mu))\|\,\|\operatorname{diag}[\eta_n(\mu), \eta_n(\mu)^{-1}]\|\,\|U(\psi_n(\mu))\| \overset{(4.3)}{=} \|X_{A_\mu}(2^n-1,0)\| \overset{(4.31)}{\leq} e^{2^n d(\mu)}. \tag{4.32}$$

Denote $n_0 = \left[\frac{\hat{\mathfrak{g}}(\tilde{\mu})}{\pi}\right]$, $\mu_j = \hat{\mathfrak{g}}^{-1}(\pi(n_0 + j))$, $j \in \mathbb{Z}$.

By (4.26), the function $\hat{\mathfrak{g}}^{-1}(\cdot)$ increases monotonically on $\mathbb{R}$. Therefore, we have the inclusion

$$\hat{\mathfrak{g}}^{-1}(\cdot) \in \mathcal{CM}(\mathbb{R}). \tag{4.33}$$

Thus, the inequality holds

$$\mu_j < \mu_{j+1}, \quad j \in \mathbb{Z}. \tag{4.34}$$

Because of the function $d(\cdot)$ is continuous and π-periodic, the number $\mathfrak{m} = \max_{\mu\in\mathbb{R}} d(\mu)$ is defined.

By Lemma 4.1, the function $\psi_k(\cdot)$ is continuous and, by the estimates in (4.18) and (4.32), we have

$$(\psi_k(\mu))'_\mu \overset{(4.18)}{\geq} \frac{1}{e^{2\mathfrak{m}}(1 + \eta_k(\mu)^2)} \overset{(4.32)}{\geq} \frac{1}{e^{2\mathfrak{m}}(1 + e^{2^{n+1}\mathfrak{m}})}.$$

Hence, $\psi_k(\cdot)$ increases monotonically and is unbounded on $\mathbb{R}$. Thus, the inclusion holds:

$$\psi_k(\cdot) \in \mathcal{CM}(\mathbb{R}). \tag{4.35}$$

Therefore, the inverse to $\psi_k(\cdot)$ function $\psi_k^{-1}(\cdot) : \mathbb{R} \to \mathbb{R}$ exists.

Because of (4.33) together with the estimates

$$\hat{\mathfrak{g}}(\mu_0) = \pi n_0 \leq \hat{\mathfrak{g}}(\tilde{\mu}) < \pi(n_0 + 1) = \hat{\mathfrak{g}}(\mu_1), \tag{4.36}$$

we obtain the inequalities

$$\mu_0 \overset{(4.33),(4.36)}{\leq} \tilde{\mu} \overset{(4.33),(4.36)}{<} \mu_1. \tag{4.37}$$

Similarly, the equalities (3.1), where $n = 2^{k-1} - 1$, imply the estimates

$$\begin{aligned}\hat{\mathfrak{g}}(\mu_{2^{k-1}-1}) &= \pi(n_0 + 2^{k-1} - 1) \leq \hat{\mathfrak{g}}(\tilde{\mu}) + \pi(2^{k-1} - 1) \overset{(3.1)}{=} \hat{\mathfrak{g}}(\pi + \tilde{\mu}) \\ &\overset{(3.1)}{=} \hat{\mathfrak{g}}(\tilde{\mu}) + \pi(2^{k-1} - 1) < \pi(n_0 + 2^{k-1}) = \hat{\mathfrak{g}}(\mu_{2^{k-1}}).\end{aligned} \tag{4.38}$$

Thus, by the inclusion (4.33), we have the inequalities

$$\mu_{2^{k-1}-1} \overset{(4.33),(4.38)}{\leq} \pi + \tilde{\mu} \overset{(4.33),(4.38)}{<} \mu_{2^{k-1}}. \tag{4.39}$$

Because of (4.30) the equalities hold:

$$\{\psi_k(\mu_j)\}_{j\in\mathbb{Z}} \overset{(4.30)}{=} \{\beta \in \mathbb{R} : \sin(2\beta) = 0\} = \{2^{-1}\pi j\}_{j\in\mathbb{Z}}. \tag{4.40}$$

Formulas (4.34) and (4.35) imply that the sequence $\{\psi_k(\mu_j)\}_{j\in\mathbb{Z}}$ is monotonically increasing. Hence, from (4.40) we obtain the equality

$$\psi_k(\mu_{j+1}) - \psi_k(\mu_j) \overset{(4.40)}{=} 2^{-1}\pi, \quad j \in \mathbb{Z}. \tag{4.41}$$

The latter by (4.35), (4.37) and (4.39) gives us the inequalities

$$\psi_k(\mu_1) - 2^{-1}\pi \overset{(4.41)}{=} \psi_k(\mu_0) \overset{(4.35),(4.37)}{\leq} \psi_k(\tilde{\mu}) \overset{(4.35),(4.37)}{<} \psi_k(\mu_1), \tag{4.42}$$

$$\psi_k(\mu_{2^{k-1}}) - 2^{-1}\pi \overset{(4.41)}{=} \psi_k(\mu_{2^{k-1}-1}) \overset{(4.35),(4.39)}{\leq} \psi_k(\pi+\tilde{\mu}) \overset{(4.35),(4.39)}{<} \psi_k(\mu_{2^{k-1}}). \tag{4.43}$$

Subtracting the estimates (4.42) from (4.43), we have

$$\psi_k(\mu_{2^{k-1}}) - \psi_k(\mu_1) - 2^{-1}\pi \overset{(4.42),(4.43)}{\leq} \psi_k(\pi+\tilde{\mu}) - \psi_k(\tilde{\mu}) \overset{(4.42),(4.43)}{<} \psi_k(\mu_{2^{k-1}}) - \psi_k(\mu_1) + 2^{-1}\pi. \tag{4.44}$$

Next, we restrict ourselves by considering only the case where $k > 1$ (in the remaining case $k = 1$, the required equality (4.17) is obvious).

The following formula holds:

$$X_{A_\mu}(2^k-1,0) = \left(\prod_{j=1}^{2^{k-1}-1} \Big(\operatorname{diag}\,[e^{d(\mu)}, e^{-d(\mu)}]\, X_{A_\mu}(2^k-2j, 2^k-2j-1)\Big)\right) \operatorname{diag}\,([e^{d(\mu)}, e^{-d(\mu)}], \tag{4.45}$$

and by π-periodicity of functions $\gamma(\cdot)$, we have

$$X_{A_{\pi+\tilde{\mu}}}(2j,2j-1) = U(b_j+\pi+\tilde{\mu}+\gamma(\pi)) = U(\pi)\,U(b_j+\tilde{\mu}+\gamma(0)) = -X_{A_{\tilde{\mu}}}(2j,2j-1). \tag{4.46}$$

Because of π-periodicity of the function $d(\cdot)$, (4.45) and (4.46) imply the formula $X_{A_{\pi+\tilde{\mu}}}(2^k-1,0) = -X_{A_{\tilde{\mu}}}(2^k-1,0)$.

Thus, by the representation (4.3), we have the comparison

$$\psi_k(\pi+\tilde{\mu}) \equiv \psi_k(\tilde{\mu}) + 2^{-1}\pi (\operatorname{mod}\pi). \tag{4.47}$$

From (4.41) the equalities follow

$$\psi_k(\mu_{2^{k-1}}) - \psi_k(\mu_1) = \sum_{j=1}^{2^{k-1}-1} (\psi_k(\mu_{j+1}) - \psi_k(\mu_j)) \overset{(4.41)}{=} 2^{-1}\pi(2^{k-1}-1) = (2^{k-2}-2^{-1})\pi. \tag{4.48}$$

Hence, by use of the comparison (4.47) we obtain

$$\psi_k(\pi) - \psi_k(0) - \psi_k(\mu_{2^{k-1}}) + \psi_k(\mu_1) \overset{(4.47),(4.48)}{\equiv} 2^{-1}\pi - (2^{k-2}-2^{-1})\pi = \left(1-2^{k-2}\right)\pi \equiv 0 (\operatorname{mod}\pi). \tag{4.49}$$

Then, because of (4.44) and (4.48), the equality holds

$$\psi_k(\pi+\tilde{\mu}) - \psi_k(\tilde{\mu}) \overset{(4.44),(4.49)}{=} \psi_k(\mu_{2^{k-1}}) - \psi_k(\mu_1) \overset{(4.48)}{=} (2^{k-2}-2^{-1})\pi.$$

The lemma is thus proved. □

5. The estimate of the Cauchy operator norm

Lemma 5.1. *For any differentiable function $f(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfying the inequality*

$$f(\pi) - f(0) \geq \pi \tag{5.1}$$

and, for some $r > 0$, the condition

$$f'(\mu) \geq r, \quad \mu \in \mathbb{R}, \tag{5.2}$$

the integral $\int_0^\pi \ln|\cos f(\mu)|\, d\mu$ is defined, and we have the estimate

$$\int_0^\pi \ln|\cos f(\mu)|\, d\mu \geq -\pi - \frac{2^4}{r} \ln(e(f(\pi) - f(0))). \tag{5.3}$$

Proof. The function $f(\cdot)$ is continuous and, by (5.19), strictly monotonic and unbounded on $\mathbb{R}$. Therefore, it is bijective on $\mathbb{R}$ and, consequently, the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is defined.

In the case where $\gamma \in \mathbb{R}$ satisfies the condition $\cos\gamma \neq 0$, because of the formula $|\cos\gamma| \leq 1$, the inequality holds:

$$\ln|\cos\gamma| \leq 0, \quad \gamma \neq 2^{-1}\pi + \pi n, \quad n \in \mathbb{Z}. \tag{5.4}$$

The formula (5.2) implies for all $\gamma \in \mathbb{R}$ the estimates

$$0 \overset{(5.2)}{<} \frac{1}{f'(f^{-1}(\gamma))} = (f^{-1})'(\gamma) = \frac{1}{f'(f^{-1}(\gamma))} \overset{(5.2)}{\leq} \frac{1}{r}. \tag{5.5}$$

Hence, by (5.4), we obtain the inequalities

$$0 \overset{(5.4),(5.5)}{\geq} (f^{-1})'(\gamma)\ln|\cos\gamma| \overset{(5.4),(5.5)}{\geq} r^{-1}\ln|\cos\gamma|, \quad \gamma \neq 2^{-1}\pi + \pi n, \quad n \in \mathbb{Z}. \tag{5.6}$$

The function $\ln(\cdot)$ increases monotonically on $(0, +\infty)$. Thus, for all $\beta \in [-2^{-1}\pi, 2^{-1}\pi]$, because of the inequality $|\sin\beta| \geq \frac{2}{\pi}|\beta|$, we have

$$\ln|\sin\beta| \geq \ln\left(\frac{2}{\pi}|\beta|\right). \tag{5.7}$$

The condition (5.2) implies that $f(\pi) - f(0) = \int_0^\pi f'(\gamma)\,d\gamma \overset{(5.2)}{\geq} \pi r > 0$. Therefore, the estimate holds

$$p := (f(\pi) - f(0))^{-1} > 0. \tag{5.8}$$

Fix an arbitrary $n \in \mathbb{Z}$.

For all $\varepsilon \in (0, p]$ by inequalities (5.1), (5.6) and (5.7), we obtain the formulas

$$r\left(\int\limits_{\pi(n+2^{-1})-p}^{\pi(n+2^{-1})-\varepsilon} + \int\limits_{\pi(n+2^{-1})+\varepsilon}^{\pi(n+2^{-1})+p}\right)(f^{-1})'(\gamma)\ln|\cos\gamma|\,d\gamma \overset{(5.6)}{\geq} \left(\int\limits_{\pi(n+2^{-1})-p}^{\pi(n+2^{-1})-\varepsilon} + \int\limits_{\pi(n+2^{-1})+\varepsilon}^{\pi(n+2^{-1})+p}\right)\ln|\cos\gamma|\,d\gamma$$

(here we make a change of variables $\beta = \gamma + \pi(n + 2^{-1})$)

$$= \left(\int_{-p}^{-\varepsilon} + \int_{\varepsilon}^{p}\right)\ln|\sin\beta|\,d\beta \overset{(5.1),(5.7)}{\geq} \left(\int_{-p}^{-\varepsilon} + \int_{\varepsilon}^{p}\right)\ln\left(\frac{2}{\pi}|\beta|\right)d\beta$$

$$= \ln\frac{2}{\pi}\left(\int_{-p}^{-\varepsilon} + \int_{\varepsilon}^{p}\right)d\beta + 2\int_{\varepsilon}^{p}\ln\beta\,d\beta = 2(p-\varepsilon)\ln\frac{2}{\pi} + 2p\ln\frac{p}{e} - 2\varepsilon\ln\frac{\varepsilon}{e}. \tag{5.9}$$

Obviously, there exists the limit of the right-hand side of the last formula where $\varepsilon \to 0$. Hence, we have

$$2p\ln\frac{2}{\pi} + 2p\ln\frac{p}{e} = \lim_{\varepsilon\to 0}(2(p-\varepsilon)\ln\frac{2}{\pi} + 2p\ln\frac{p}{e} - 2\varepsilon\ln\frac{\varepsilon}{e}$$

$$\overset{(5.9)}{\leq} r\,\underline{\lim}_{\varepsilon\to 0}\left(\int\limits_{\pi(n+2^{-1})-p}^{\pi(n+2^{-1})-\varepsilon} + \int\limits_{\pi(n+2^{-1})+\varepsilon}^{\pi(n+2^{-1})+p}\right)(f^{-1})'(\gamma)\ln|\cos\gamma|\,d\gamma$$

$$\stackrel{(5.6)}{=} r \int\limits_{\pi(n+2^{-1})-p}^{\pi(n+2^{-1})+p} (f^{-1})'(\gamma) \ln|\cos\gamma| \, d\gamma. \tag{5.10}$$

By the inequality (5.1), the estimates hold:

$$\begin{aligned} \pi n < \pi n + \frac{\pi}{2} - \frac{1}{\pi} &\stackrel{(5.1)}{\leq} \pi(n+2^{-1}) - p \stackrel{(5.8)}{<} \pi(n+2^{-1}) \\ &\stackrel{(5.8)}{<} \pi(n+2^{-1}) + p \stackrel{(5.1)}{\leq} \pi(n+1) - \frac{\pi}{2} + \frac{1}{\pi} < \pi(n+1). \end{aligned}$$

They imply the inequality

$$\begin{aligned} &\ln|\cos\gamma| \geq \ln\cos p, \\ &\gamma \in [\pi n, \pi(n+2^{-1}) - p] \cup [\pi(n+2^{-1}) + p, \pi(n+1)]. \end{aligned} \tag{5.11}$$

Thus, we have

$$\begin{aligned} &\left(\int_{\pi n}^{\pi(n+2^{-1})-p} + \int_{\pi(n+2^{-1})+p}^{\pi(n+1)} \right) (f^{-1})'(\gamma) \ln|\cos\gamma| \, d\gamma \\ &\stackrel{(5.5),(5.11)}{\geq} \ln\cos p \left(\int_{\pi n}^{\pi(n+2^{-1})-p} + \int_{\pi(n+2^{-1})+p}^{\pi(n+1)} \right) (f^{-1})'(\gamma) \, d\gamma \\ &\stackrel{(5.4),(5.5)}{\geq} \ln\cos p \int_{\pi n}^{\pi(n+1)} (f^{-1})'(\gamma) \, d\gamma \\ &\qquad \text{(here we make a change of variables } \mu = f^{-1}(\gamma)) \\ &= \ln\cos p \int_{f^{-1}(\pi n)}^{f^{-1}(\pi(n+1))} d\mu = (f^{-1}(\pi(n+1)) - f^{-1}(\pi(n))) \ln\cos p. \end{aligned} \tag{5.12}$$

Therefore, the estimates hold:

$$\begin{aligned} &(f^{-1}(\pi(n+1)) - f^{-1}(\pi(n))) \ln\cos p + \frac{2p}{r} \ln\frac{2p}{e\pi} \\ &\stackrel{(5.10),(5.12)}{\leq} \left(\int_{\pi n}^{\pi(n+2^{-1})-p} + \int_{\pi(n+2^{-1})-p}^{\pi(n+2^{-1})+p} + \int_{\pi(n+2^{-1})+p}^{\pi(n+1)} \right) (f^{-1})'(\gamma) \ln|\cos\gamma| \, d\gamma \\ &= \int_{\pi n}^{\pi(n+1)} (f^{-1})'(\gamma) \ln|\cos\gamma| \, d\gamma = \text{(we make the change of variables } \ \gamma = f(\mu)) \\ &= \int_{f^{-1}(\pi n)}^{f^{-1}(\pi(n+1))} \ln|\cos f(\mu)| \, d\mu. \end{aligned} \tag{5.13}$$

Denote $n_0 = \sup(\mathbb{Z} \cap (-\infty, \pi^{-1} f(0)])$, $n_1 = \inf(\mathbb{Z} \cap [\pi^{-1} f(\pi), +\infty))$.

Obviously, the inclusions are true $f(0) \in [\pi n_0, \pi(1+n_0))$, $f(\pi) \in (\pi(n_1 - 1), \pi n_1]$. They imply the inequalities

$$\pi n_1 - \pi n_0 \leq 2\pi + f(\pi) - f(0), \tag{5.14}$$

$$0 \overset{(5.5)}{\leq} (\pi n_1 - f(\pi)) \min_{f(\pi)\leq q\leq \pi n_1} (f^{-1})'(q) \leq f^{-1}(\pi n_1) - \pi$$
$$\leq (\pi n_1 - f(\pi)) \max_{f(\pi)\leq q\leq \pi n_1} (f^{-1})'(q) \overset{(5.5)}{\leq} r^{-1}(\pi n_1 - f(\pi)) < r^{-1}\pi, \tag{5.15}$$
$$0 \overset{(5.5)}{\leq} (f(0) - \pi n_0) \min_{\pi n_0\leq q\leq f(0)} (f^{-1})'(q) \leq -f^{-1}(\pi n_0)$$
$$\leq (f(0) - \pi n_0) \max_{\pi n_0\leq q\leq f(0)} (f^{-1})'(q) \overset{(5.5)}{\leq} r^{-1}(f(0) - \pi n_0) < r^{-1}\pi. \tag{5.16}$$

From them, we get the inclusion

$$[0,\pi] \overset{(5.15),(5.16)}{\subset} [f^{-1}(\pi n_0), f^{-1}(\pi n_1)]. \tag{5.17}$$

By formula (5.14), we have

$$p(n_1 - n_0) \overset{(5.14)}{\leq} p(2\pi + (f(\pi) - f(0)) = 2p\pi + 1 \overset{(5.1)}{\leq} 3. \tag{5.18}$$

From the inequalities (5.1) and (5.8), we obtain the inclusion $p \in (0, \pi^{-1})$, which implies the estimates $\ln\cos p > \ln\cos\frac{1}{\pi} > \ln\cos\frac{\pi}{4} = -\frac{1}{2}\ln 2 > -\frac{1}{2}$.

Then, because of (5.4), (5.15), and (5.16), the estimates hold:

$$(f^{-1}(\pi n_1) - f^{-1}(\pi n_0))\ln\cos p \overset{(5.4),(5.15),(5.16)}{\geq} \left(\frac{2}{r}\pi + \pi\right)\ln\cos p > -\left(\frac{1}{r} + \frac{1}{2}\right)\pi. \tag{5.19}$$

The condition (5.1) gives us the inequalities

$$\ln\frac{2p}{e\pi} \overset{(5.1)}{<} \ln\frac{1}{2e} < 0. \tag{5.20}$$

By (5.13), there exists the integral $\int_0^\pi \ln|\cos f(\mu)|\,d\mu$.

Therefore, by (5.4), (5.13) and (5.17)–(5.20), we have the estimates

$$\int_0^\pi \ln|\cos f(\mu)|\,d\mu \overset{(5.4),(5.17)}{\geq} \sum_{n=n_0}^{n_1-1} \int_{f^{-1}(\pi n)}^{f^{-1}(\pi(n+1))} \ln|\cos f(\mu)|\,d\mu$$
$$\overset{(5.13)}{\geq} \sum_{n=n_0}^{n_1-1}\left((f^{-1}(\pi(n+1)) - f^{-1}(\pi n))\ln\cos p + \frac{2p}{r}\ln\frac{2p}{e\pi}\right)$$
$$= (f^{-1}(\pi n_1) - f^{-1}(\pi n_0))\ln\cos p + (n_1 - n_0)\frac{2p}{r}\ln\frac{2p}{e\pi}$$
$$\overset{(5.18)-(5.20)}{\geq} -\left(\frac{1}{r} + \frac{1}{2}\right)\pi + \frac{6}{r}\ln\frac{2p}{e\pi} > -\left(\frac{1}{r} + \frac{1}{2}\right)\pi - \frac{12}{r} + \frac{6}{r}\ln p$$
$$> -\pi - \frac{2^4}{r} - \frac{6}{r}\ln(f(\pi) - f(0))$$
$$\overset{(5.1)}{>} -\pi - \frac{2^4}{r}\ln(e(f(\pi) - f(0))).$$

The lemma is thus proved. □

Proof of Theorem 1.1. By (4.17) together with π-periodicity of the function $\gamma(\cdot)$, the equalities hold

$$\xi_k(\pi) = 2\psi_k(\pi) + \pi + \alpha_k + \gamma(\pi) \overset{(4.17)}{=} 2^k\pi + \xi_k(0). \tag{5.21}$$

The inequality (4.18) implies the inclusion $\xi_k \in D(\mathbb{R})$. Thus, we have the estimates

$$\xi_k'(\mu) = 2\psi_k'(\mu) + 1 + \gamma'(\mu) \overset{(4.18)}{>} 1 + \gamma'(\mu) \overset{(1.10)}{>} 2^{-1}, \quad \mu \in \mathbb{R}.$$

Hence, the function $f(\mu) = \xi_k(\mu)$ satisfies conditions of Lemma 5.1, where $r = 2^{-1}$. As a result, by (5.21), we obtain the inequalities

$$\begin{aligned}\int_0^\pi \ln|\cos\xi_k(\mu)|\,d\mu &\overset{(5.3)}{\geq} -\pi - 2^5\ln(2e(\xi_k(\pi) - \xi_k(0)))\\ &\overset{(5.21)}{\geq} -\pi - 2^5\ln(e2^{k+1}\pi) > -\pi - 2^5(\ln(e\pi) + (k+1)\ln 2) > -2^6(k+4).\end{aligned} \tag{5.22}$$

For all $\mu \in [0, \pi]$, the estimates hold:

$$\begin{aligned}|\eta_k^2(\mu) - \eta_k^{-2}(\mu)| &= \eta_k^2(\mu) + \eta_k^{-2}(\mu) - 2\min\{\eta_k^2(\mu), \eta_k^{-2}(\mu)\}\\ &\geq \eta_k^2(\mu) + \eta_k^{-2}(\mu) - 2 = (\eta_k(\mu) - \eta_k^{-1}(\mu))^2.\end{aligned} \tag{5.23}$$

Thus, by (4.1) and (5.22), we have

$$\begin{aligned}S_{k+1} &:= \int_0^\pi \ln|\eta_{k+1}(\mu) - \eta_{k+1}^{-1}(\mu)|\,d\mu \overset{(4.1)}{=} \int_0^\pi \ln(|\eta_k^2(\mu) - \eta_k^{-2}(\mu)||\cos\xi_k(\mu)|)\,d\mu\\ &= \int_0^\pi \ln|\eta_k^2(\mu) - \eta_k^{-2}(\mu)|\,d\mu + \int_0^\pi \ln|\cos\xi_k(\mu)|\,d\mu\\ &\overset{(5.23)}{\geq} \int_0^\pi \ln|\eta_k(\mu) - \eta_k^{-1}(\mu)|^2\,d\mu + \int_0^\pi \ln|\cos\xi_k(\mu)|\,d\mu \overset{(5.22)}{\geq} 2S_k - 2^6(k+4).\end{aligned} \tag{5.24}$$

Let us denote $\hat{C} = -2^6(k+4)$.

Suppose there exists $n = k \in \mathbb{N}$, such that the inequality holds

$$S_n \geq 2^{11}(2^n + 2^{n/2}). \tag{5.25}$$

The estimates

$$2^{11}(\sqrt{2} - 1)2^{(k+1)/2} > 2^{10}2^{k/2} > 2^9(1+k) > -\hat{C} \tag{5.26}$$

together with (5.24) and (5.25) imply that

$$S_{k+1} \overset{(5.24),(5.25)}{\geq} 2^{11}(2^{k+1} + 2^{1+k/2}) + \hat{C} \overset{(5.26)}{\geq} 2^{11}(2^{k+1} + 2^{(k+1)/2}).$$

Thus, (5.25) holds in the case $n = k+1$. In addition, the condition (1.11) gives us the formula (5.25) where $n = 1$. So, by induction we obtain the estimate (5.25) for all $n \in \mathbb{N}$.

Denote $M_n = \{\mu \in [0,\pi]\colon 2^{-n}\ln|\eta_n(\mu) - \eta_n^{-1}(\mu)| \ge 2^{-1}\pi^{-1}\}$.

According to Lemma 4.1, the function $\eta_n(\cdot)$ is continuous. Therefore, the function $|\eta_n(\mu) - \eta_n^{-1}(\mu)|$ is also continuous. This implies that the set M_n is measurable. Hence, the set $\hat{M} := \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} M_n = \operatorname*{Lim}_{k\to+\infty}\bigcup_{n=k}^{\infty} M_n$ is also measurable, and its Lebesgue measure satisfies the estimates

$$\operatorname{mes}\hat{M} = \lim_{k\to+\infty}\operatorname{mes}\bigcup_{n=k}^{\infty} M_n \ge \varlimsup_{k\to+\infty}\operatorname{mes} M_k. \tag{5.27}$$

The function $d(\cdot)$ is continuous. Thus, there exists the number $\mathfrak{d} = \max_{0\le\mu\le\pi} d(\mu)$.

By (4.32), the formulas hold:

$$|\eta_n(\mu) - \eta_n^{-1}(\mu)| \le \max\{\eta_n(\mu), \eta_n^{-1}(\mu)\} \overset{(4.32)}{\le} e^{2^n d(\mu)}. \tag{5.28}$$

They imply the inequalities

$$\int\limits_{M_n} 2^{-n}\ln|\eta_n(\mu) - \eta_n^{-1}(\mu)|\,d\mu \overset{(5.28)}{\le} \int\limits_{M_n} 2^{-n}\ln e^{2^n d(\mu)}\,d\mu = \int\limits_{M_n} d(\mu)\,d\mu \le \mathfrak{d}\operatorname{mes} M_n. \tag{5.29}$$

Moreover, we have the estimates

$$\int\limits_{[0,\pi]\setminus M_n} 2^{-n}\ln|\eta_n(\mu) - \eta_n^{-1}(\mu)|\,d\mu \le \int\limits_{[0,\pi]\setminus M_n}\sup_{\mu\in[0,\pi]\setminus M_n}(2^{-n}\ln|\eta_n(\mu) - \eta_n^{-1}(\mu)|)\,d\mu \le$$
$$\le 2^{-1}\pi^{-1}\operatorname{mes}([0,\pi]\setminus M_n) \le 2^{-1}\pi^{-1}\operatorname{mes}[0,\pi] = 2^{-1}. \tag{5.30}$$

Therefore, by (5.25), we obtain the formulas

$$1 \overset{(5.25)}{\le} 2^{-n}S_n = \left(\int\limits_{M_n} + \int\limits_{[0,\pi]\setminus M_n}\right) 2^{-n}\ln|\eta_n(\mu) - \eta_n^{-1}(\mu)|\,d\mu \overset{(5.29),(5.30)}{\le} \mathfrak{d}\operatorname{mes} M_n + 2^{-1}. \tag{5.31}$$

It follows from (4.3) that

$$\tilde{M} := \{\mu \in [0,\pi]\colon \lambda_{\max}(A_\mu) \ge 2^{-1}\pi^{-1}\} \supset$$
$$\supset \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\mu \in [0,\pi]\colon 2^{-n}\ln\|X_{A_\mu}(2^n - 1, 0)\| \ge 2^{-1}\pi^{-1}\} \overset{(4.3)}{\supset} \hat{M}. \tag{5.32}$$

Hence, because of (5.27) and (5.31), we have

$$\underline{\operatorname{mes}}\,\tilde{M} \overset{(5.32)}{\ge} \operatorname{mes}\hat{M} \overset{(5.27)}{\ge} \varlimsup_{n\to+\infty}\operatorname{mes} M_n \overset{(5.31)}{\ge} 2^{-1}\mathfrak{d}^{-1}.$$

The latter gives us the assertion of the theorem. □

6. Conclusions

In this article, we discussed the estimates for the maximal Lyapunov exponent of linear differential systems depending on a real parameter. We proved its positivity on a set of positive Lebesgue measure. The result was obtained under the condition which implies the monotonicity of angles in the singular value decomposition of the Cauchy operator.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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